# Physical effects of the Immirzi parameter in loop quantum gravity

Alejandro Perez* and Carlo Rovelli†

*Centre de Physique Theorique de Luminy, F-13288 Marseille, France, European Union*

(Received 26 September 2005; published 8 February 2006)

The Immirzi parameter is a constant appearing in the general-relativity action used as a starting point for the loop quantization of gravity. The parameter is commonly believed not to appear in the equations of motion and not to have any physical effect besides nonperturbatrive quantum gravity. We show that this is not true in general: in the presence of minimally coupled fermions, the parameter appears in the equations of motion: it determines the coupling constant of a four-fermion interaction. Under some general assumptions, there is therefore a relation between the Immirzi parameter and physical effects that are observable in principle, independently from nonperturbative quantum gravity

DOI: 10.1103/PhysRevD.73.044013 PACS numbers: 04.60.Pp

In the context of the attempts to find a nonperturbative quantum theory of gravity, an important role is played by formulations of general relativity in terms of a Yang-Mills-like field, instead of the metric field used by Einstein. In the eighties, Ashtekar realized that the gravitational field can be effectively described in terms of a self-dual $SL(2, \mathbb{C})$ Yang-Mills-like connection and its conjugate electric field [1]. Loop quantum gravity [2] started shortly after as a canonical quantization of general relativity using these self-dual variables [3]. To circumvent difficulties related to the implementation of the reality conditions in the quantum theory and the noncompactness of $SL(2, \mathbb{C})$, the attention had later shifted to a real $SU(2)$ version of the Ashtekar connection known as the Barbero connection [4], characterized by a real parameter $\gamma$, called the Immirzi parameter [5]. An action functional that leads to this formalism is

$$S[e, A] = \frac{1}{16\pi G}\bigg(\int d^4x\, e e_I^a e_J^b\, F_{ab}^{IJ} - \frac{1}{\gamma}\int d^4x\, e e_I^a e_J^{b*}\, F_{ab}^{IJ}\bigg), \tag{1}$$

called the Holst action [6]. Here $I, J \ldots = 0, 1, 2, 3$ are internal Lorentz indices and $a, b \ldots = 0, 1, 2, 3$ are spacetime indices. The field $e_a^I$ is the tetrad field, $e$ is its determinant, and $e_I^a$ its inverse; $A_a^{IJ}$ is a Lorentz connection. $F$ is the curvature of $A$ and $*F$ is its dual, defined by $*F^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}F^{KL}$. The choice $\gamma = i$ for the Immirzi parameter leads to the self-dual Ashtekar canonical formalism, while a real $\gamma$ leads to the $SU(2)$ Barbero connection.

The first term in (1) is the well-known tetrad-Palatini action of general relativity. The second term does not affect the equations of motion. Therefore, as it is often stressed, the Immirzi parameter $\gamma$ does not appear to have any effect on classical physics. The parameter $\gamma$, on the other hand, plays an important role in loop quantum gravity. It determines the form of the momentum conjugate to the tetrad, which, in the loop quantization, is (proportional to) the connection that defines the holonomies in terms of which the quantum theory is defined. As a consequence, the spectrum of quantum geometry operators is modulated by its value. For instance, the area of a surface and the volume of a space region are quantized in units of $\gamma\ell_p^2$ and $\gamma^{3/2}\ell_p^3$, respectively. Furthermore, the nonperturbative calculation of the entropy of a black hole appears to yield a result compatible with Hawking's semiclassical formula only for a specific value of $\gamma$. See [7] for recent evaluations and references. The role of the Immirzi parameter is often compared with the role of the $\Theta$ angle in QCD—which also appears as a constant in front of a term in the action with no effect on the equations of motion: a parameter that governs only nonperturbative quantum effects. See for instance [8]. In this paper we point out that this is in fact not the case in general; if fermions are minimally coupled to gravity, then the Immirzi parameter determines the strength of a four-fermion interaction.

The catch is the following: The equation of motion obtained varying the connection in (1) is

$$D_{[a}e_{b]}^I = 0 \tag{2}$$

(Cartan's first structure equation), where $D_a$ is the covariant derivative defined by $A$. The solution of this equation is $A = \omega[e]$, where $\omega[e]$ is the torsion-free spin connection of the tetrad field $e$. If we thus replace $A$ by $\omega[e]$ in (1), the first term becomes the tetrad expression of the Einstein-Hilbert action, while the second term is identically zero, due to the Bianchi identity $R_{[abc]d} = 0$. Stationarity with respect to the variation of the tetrad yields then the Einstein equations. Now, Eq. (2) is modified by the presence of fermions (or, more in general, matter fields that couple to the connection, see [9,10]). In the presence of a fermion field, (1) becomes

$$S[e, A, \psi] = S[e, A] + \frac{i}{2}\int d^4x\, e(\overline{\psi}\gamma^I e_I^a D_a\psi - \overline{D_a\psi}\gamma^I e_I^a\psi), \tag{3}$$

*Electronic address: perez@cpt.univ-mrs.fr
†Electronic address: rovelli@cpt.univ-mrs.fr

 

where $\gamma^I$ are the Dirac matrices, and (2) becomes

$$D_{[a}e^I_{b]} = \text{fermion current}. \tag{4}$$

The fermion current acts as a source for a torsion component in the connection, and the second term in (1) does not vanish.

In the following we compute the equations of motion of (3), namely, the equations of motion of general relativity minimally coupled with a fermionic field in the presence of a nontrivial Immirzi parameter. We solve the equations for the connection explicitly. By inserting the solution into the action we obtain an effective action that contains a four-fermion interaction. The coupling constant that determines the strength of this interaction depends explicitly on the Immirzi parameter.

Let us start by introducing the tensor

$$p_{IJ}{}^{KL} = \frac{1}{2}(\delta^K_I\delta^L_J - \delta^L_I\delta^K_J) - \frac{1}{2\gamma}\epsilon_{IJ}{}^{KL} \tag{5}$$

and its inverse

$$p^{-1IJ}{}_{KL} = \frac{\gamma^2}{\gamma^2+1}\left(\frac{1}{2}(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K) + \frac{1}{2\gamma}\epsilon_{KL}{}^{IJ}\right). \tag{6}$$

Using this, the action (3) can be written in the form

$$S[e,A,\psi] = \frac{1}{16\pi G}\int d^4x\, e e^a_I e^b_J\, p^{IJ}{}_{KL}F^{KL}_{ab} + \frac{1}{2}\int d^4x\, e(i\overline{\psi}\gamma^I e^a_I D_a\psi + \text{c.c.}), \tag{7}$$

and the equation of motion for the connection reads

$$p^{IJ}{}_{KL}D_b(ee^b_Ie^a_J) = 8\pi G e J_{KL}{}^a, \tag{8}$$

where the fermion current $eJ_{KL}{}^a$ is the variation of the fermionic action with respect to the connection. Recalling that $D_a\psi = \partial_a\psi - 1/4A^{KL}_a\gamma_K\gamma_L\psi$, and using the identity $\gamma^A\gamma^{[B}\gamma^{C]} = -i\epsilon^{ABCD}\gamma_5\gamma_D + 2\eta^{A[B}\gamma^{C]}$, we obtain

$$J_{KL}{}^a = \tfrac{1}{8}(2ie^a_{[K}j_{vL]} + e^a_I\epsilon^I{}_{KLJ}j^J_a + \text{c.c.}) = \tfrac{1}{4}e^a_I\epsilon^I{}_{KLJ}j^J_a, \tag{9}$$

where $j^K_v = \overline{\psi}\gamma^K\psi$ and $j^K_a = \overline{\psi}\gamma_5\gamma^K\psi$ are the vector and the axial fermion currents, and we have used the fact that they are real. Using the inverse tensor (6) Eq. (8) gives

$$D_b(ee^b_Ie^a_J) = 8\pi G e\, p^{-1}{}_{IJ}{}^{KL}J_{KL}{}^a. \tag{10}$$

This equation can be solved for the connection. For this, we write the connection in the form $A^{IJ}_a = \omega[e]^{IJ}_a + C_a{}^{IJ}$, where $\omega[e]$ is the torsion-free spin connection determined by $e$, namely, the solution of (2), and $C$ is the torsion. Using this, (10) gives

$$C_{b[K}{}^b e_{L]}{}^a + C_{[KL]}{}^a = 8\pi G e\, p^{-1}{}_{IJ}{}^{KL}J_{KL}{}^a. \tag{11}$$

Notice that we transform internal and spacetime indices into one another, using the tetrad field, and preserving the horizontal order of the indices. This equation can be solved by contracting the indices, and then summing terms with cyclical permutation of the indices: it is easy to verify that the solution is

$$C_a{}^{IJ} = -2\pi G\frac{\gamma}{\gamma^2+1}(2e^{[I}_a j^{J]}_a - \gamma e^K_a\epsilon_K{}^{IJ}{}_L j^L_a). \tag{12}$$

Notice that the torsion $C$ depends on $\gamma$.

We can now obtain an equivalent action by replacing $A$ with $\omega[e]+C$ in (3). It is easy to see that the terms linear in the fermion current are total derivatives, leaving

$$S[e,\psi] = S[e] + S_f[e,\psi] + S_{\text{int}}[e,\psi], \tag{13}$$

where the first two terms are the standard second-order tetrad action of general relativity with fermions,

$$S[e] + S_f[e,\psi] = \frac{1}{16\pi G}\int d^4x\, ee^a_Ie^b_J F^{IJ}_{ab}[\omega[e]] + i\int d^4x\, e\overline{\psi}\gamma^I e^a_I D_a[\omega[e]]\psi, \tag{14}$$

and the interaction term can be obtained by a tedious but straightforward calculation as

$$S_{\text{int}}[e,\psi] = -\frac{3}{2}\pi G\frac{\gamma^2}{\gamma^2+1}\int d^4x\, e(\overline{\psi}\gamma_5\gamma_A\psi)(\overline{\psi}\gamma_5\gamma^A\psi). \tag{15}$$

This term describes a four-fermion interaction mediated by a nonpropagating torsion. An interaction of this form is well known: it is predicted by the Einstein-Cartan theory [11]. It preserves parity. It is weak, because it is suppressed by one power of the Newton constant. It has never been observed, but it is compatible with all present observations and it might be observed in the future; the empirical observability of this interaction is discussed in [12]. Here we see that the coupling constant of this interaction depends on the Immirzi parameter. In the limit $\gamma \to \infty$, we recover the standard coupling of the Einstein-Cartan theory. Equivalently, the Einstein-Dirac theory *with* a four-fermion term defined by the action (13)–(15) can be rewritten as the Holst theory with minimally coupled fermions, given by (1)–(3).

In summary, general relativity admits the natural formulation given by the action (3), widely used as a starting point for the nonperturbative quantization of the theory. This formulation predicts a four-fermion interaction mediated by the torsion, whose strength is determined by the Immirzi parameter. The interaction is present also on a flat spacetime. If we give up minimal coupling, the same four-fermion interaction also can be obtained by explicitly adding a four-fermion term to the action (3). In this case, the strength of the interaction would be determined by a combination of the Immirzi parameter and the coupling constant of the four-fermion term. Consequently, for instance, an eventual strict experimental bound on the four-fermion interaction strength could be interpreted *either* as a bound on the Immirzi parameter itself, *or* as a cancellation between the effect of the Immirzi parameter and one of the four-fermion terms in the action. A nonminimally-coupled fermion field also can have a parity-violating four-fermion interaction [13]. In any case, the analogy with the $\Theta$ angle of QCD is, in this regard, misleading (see also [14]). The Immirzi parameter is a coupling constant that contributes to the strength of a four-fermion interaction, and in a minimally coupled theory, its value is observable in principle, independently from its effect on the nonperturbative quantum theory.

We close with a disclaimer and an analogy. Our conclusion relies on the assumption that the action governing the quantum theory is just the Holst action, and the quantum representation is the one defined by the polarization this action determines (without additional hidden boundary terms), so that $\gamma$ in (1) is the Immirzi parameter. Notice that classically equivalent actions can lead to distinct physics in two situations. First, they can define distinct quantum theories: in the functional integral formalism, terms in the action that do not affect the classical equations of motion can have observable effects; in the canonical framework, different actions define different canonical pairs and therefore different phase space polarizations, which may lead to nonequivalent quantum theories. Second, actions that give the same equations of motion may couple differently to the same interaction terms. An example of all this is the mass of a free particle, which presents intriguing analogies with the situation analyzed in this paper. Consider a free particle on a circle. Its classical motion is fully determined by the equation of motion $\ddot{\alpha} = 0$, where there is no mass. Hence we cannot observe the mass by observing the free particle. Its quantum physics, however, depends on the mass; for instance, its velocity is quantized in units determined by the mass. Thus the two actions $S = \frac{m}{2} \int dt \dot{\alpha}^2$ and $S = \frac{m'}{2} \int dt \dot{\alpha}^2$, with $m \neq m'$ are classically equivalent but quantum mechanically nonequivalent. We can say that in this particular case the mass is a purely quantum parameter, that can be observed only in the quantum theory of the particle. But let us now add an interaction, in the form of a potential $S_{\text{int}} = -\int dt V(\alpha)$. Then the two actions $S + S_{\text{int}}$ and $S' + S_{\text{int}}$ lead to *different* classical equations of motion. We can say that the mass of the particle, which was a purely quantum parameter as long as the particle was free, can be observed classically as soon as we observe the interaction of the particle with a potential $V(\alpha)$. The same is true for the Newton constant: it does not enter the classical Einstein equations of the pure gravitational field, but we expect it to affect the quantum theory of pure gravity (for instance, it enters the commutation relation between a 3-metric and extrinsic curvature); hence it is a purely quantum parameter for pure gravity. However, it becomes a classically observable parameter as soon as matter is coupled to gravity. Analogously, the Immirzi parameter can be viewed as a purely quantum parameter as long as the gravitational field does not interact with fermions, but, under the hypotheses considered, it might be measured classically as soon as we can observe the effect of the interaction of the gravitational field with a fermion.

We thank I. B. Khriplovich and A. A. Pomeransky, as well as L. Freidel, for pointing out to us the need of adding the complex conjugate term to the action, and the PRL divisional associate editor for valuable criticisms.

[1] A. Ashtekar, *Non-perturbative Canonical Gravity* (World Scientific, Singapore, 1991).
[2] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2004); T. Thiemann, *Modern Canonical General Relativity* (Cambridge University Press, Cambridge, England, 2005); A. Perez, gr-qc/0409061.
[3] C. Rovelli and L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nucl. Phys. **B331**, 80 (1990).
[4] F. Barbero, Phys. Rev. D **51**, 5498 (1995); Phys. Rev. D **51** 5507 (1995).
[5] G. Immirzi, Nucl. Phys. B, Proc. Suppl. **57** 65 (1997); Classical Quantum Gravity **14** L177 (1997).
[6] S. Holst, Phys. Rev. D **53** 5966 (1996).
[7] M. Domagala and J. Lewandowski, Classical Quantum Gravity **21**, 5233 (2004); S. Alexandrov, gr-qc/0408033; K. A. Meissner, Classical Quantum Gravity **21**, 5245 (2004).
[8] C. Rovelli and T. Thiemann, Phys. Rev. D **57**, 1009 (1998).
[9] T. Jacobson, Classical Quantum Gravity **5**, L143 (1988).
[10] A. Ashtekar, J. D. Romano, and R. S. Tate, Phys. Rev. D **40**, 2572 (1989).
[11] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).
[12] L. Shapiro, Phys. Rep. **357**, 113 (2002).
[13] L. Freidel, D. Minic, and T. Takeuchi, Phys. Rev. D **72**, 104002 (2005).
[14] S. Alexander, Phys. Lett. B **629**, 53 (2005).