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Tilted cosmological models of Bianchi type V

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Cosmological models of Bianchi types V and I containing a perfect fluid with a linear equation of state plus cosmological constant are investigated. In general, these spacetimes are tilted and describe fluids with expansion, shear, and vorticity. We use a tetrad approach where our variables are the Riemann tensor, the Ricci rotation coefficients, and a subset of the tetrad vector components. This set, called S, describes a spacetime when its elements are constrained by certain integrability conditions and due to a theorem by Cartan S gives a complete local description of the spacetime. With the help of the Lie algebra, the full line element is constructed up to quadratures in terms of the elements in S. The system obtained by imposing the integrability conditions and Einstein's equations on the elements in S can be reduced to an integrable system of five coupled first order ordinary differential equations. In general, exact solutions to this system are hard to find, but the linearized equations around the open Friedmann models are easily integrated. The full system is also studied numerically and the perturbative solutions agree well with the numerical ones in the appropriate domains. We also give some examples of numerical solutions in the nonperturbative regime.

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I. INTRODUCTION

In this paper, general Bianchi types V and I perfect fluids with linear equation of state and cosmological constant are studied. In general, these spacetimes are tilted and, in particular, there are solutions with rotating matter. It has been difficult to find exact solutions with both expansion and nonzero rotation of the matter flow. To our knowledge the only known exact homogeneous perfect fluid solution with rotation and expansion is the self-similar radiation-filled Bianchi type VI_0 found by Rosquist [1].

A number of rotating imperfect fluid solutions with heatflow are known, see e.g. [2]. Since for rotating matter the hypersurfaces of homogeneity are tilted with respect to the restframe of matter, local space will not look homogeneous. Hence, heatflow is expected and for some solutions the heatflow can be related to a temperature gradient [3], but often with unrealistic coefficient of conductivity. Normally the heat conductivity is negligibly small and a perfect fluid approximation should work well.

For treatments of the properties of homogeneous rotating models in general, see [4,5], and for some perturbative calculations see [6,7] (homogeneous perturbations) and [8,9] (inhomogeneous perturbations). In [10–12] the influences of anisotropies on the background radiation are studied.

The qualitative behavior of locally rotationally symmetric (LRS) Bianchi V solutions is analyzed in [13–15] and, in particular, expressions for different quantities at early and late times are given in [15]. In [16,17] Bianchi cosmologies of type I and V are found to isotropize under rather general conditions. There are a number of works on tilted Bianchi cosmologies using a dynamical system approach. The irrotational subcase of type V was studied in [18]. Recently, the late time behavior of tilted Bianchi models including type V was considered in [19]. The stability of nontilted models against tilt was studied in [20].

To find the solutions we use a method for construction of solutions to Einstein's equations [21–24], based on the invariant classification scheme by Cartan and Karlhede [25,26]. The method is shortly described in Sec. II. In Sec. III the method is applied to Bianchi V and I models. First, choice of frame and the set (called S) of quantities needed to specify the spacetime are given. Then the structure of the isometry group is imposed, giving relations among the elements in S. Next, the integrability conditions for the set S to describe a geometry are imposed together with Einstein's equations. The general system is reduced to an integrable system of five coupled first order ordinary differential equations. With the help of standard bases for Bianchi V and I the full line element is found up to quadratures in terms of the elements in S. The subclass of orthogonal solutions is easily solved, but all these solutions are well known. For LRS dust the equations were integrated in [27,28].

In Sec. IV we consider first order perturbations around the open Friedmann universe. The general first order solution depends on five constants of integration, the same number as for the general exact solution, and has nonzero expansion, rotation, and shear. Finally, in Sec. V the general system is studied numerically and the results agree well with the perturbative calculations in the allowed range. Examples of numerical solutions in the nonperturbative regime are given.

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II. CONSTRUCTION OF SOLUTIONS TO EINSTEIN'S EQUATIONS IN TERMS OF CURVATURE INVARIANTS

A brief summary of the method is given here. For more details see [24,29].

According to a theorem by Cartan, spacetimes are locally completely determined by a set, R^{p+1}, consisting of the components of the Riemann tensor and a finite number, $p+1$, of its covariant derivatives in a frame with constant metric η_{ij} [25,26]. Here $p+1$ is such that all the elements in R^{p+1} are functionally dependent on those in R^p as functions on $F(M)$, the bundle of frames on the manifold M.

Assume that we have symmetries such that R^{p+1} only depends on x^α, $\alpha = 1, 2, \ldots, l < n =$ dimension of spacetime (in some canonical coordinates) and rotations in the ab-planes, $\{^a{}_b\} = 1, \ldots, m < n(n-1)/2$ (with frames adopted to the rotational symmetries). Here $l = n - \dim(\text{orbits})$ and $m = n(n-1)/2 - \dim(\text{isotropy group})$. The set R^{p+1} is completely determined by the smaller set $S = \{R^p{}_{qkl}, \gamma^a{}_{bi}, x^\alpha{}_{|i}\}$ where $x^\alpha_{|i} \equiv X_i(x^\alpha) = X_i{}^\mu \partial x^\alpha / \partial x^\mu$ are the derivatives with respect to the frame vectors and $\gamma^i{}_{jk}$ are the Ricci rotation coefficients. Here the numbering is such that $\{^p{}_q\} = m+1, \ldots, n(n-1)/2$ are the complementary rotations to the $\{^a{}_b\}$ ones, i.e., those that keep the set R^{p+1} unchanged.

A set R^{p+1} together with a constant frame metric η_{ij} describes a geometry iff certain integrability conditions, being equivalent to the Ricci identities and part of the Bianchi identities, are satisfied [21–24]. In a fixed frame the Ricci identities split into the commutators for the essential coordinates x^α

$$x^\alpha{}_{[|i,\beta} x^\beta{}_{|j]} + x^\alpha{}_{|m} \gamma^m{}_{[ij]} = 0, \tag{1}$$

and the Riemann equations for rotations in the ab-planes

$$R^a{}_{bij} = 2\gamma^a{}_{b[j,\alpha} x^\alpha{}_{|i]} + 2\gamma^{ak}{}_{[j} \gamma_{bki]} + 2\gamma^a{}_{bk} \gamma^k{}_{[ij]} \tag{2}$$

(the antisymmetrizations are only over ij). Since not all commutators or Riemann equations are used when the spacetime has symmetries, some more integrability conditions are needed. They are parts of the cyclic and Bianchi identities

$$R^t{}_{[ijk]} = 0, \qquad t = l+1, \ldots, n, \tag{3}$$

$$R^p{}_{q[ij;k]} = 0, \qquad \{^p{}_q\} = m+1, \ldots, n(n-1)/2, \tag{4}$$

where $t = l+1, \ldots, n$ numbers the symmetry directions (in a suitable numbering of the frame vectors).

The above description in terms of R^{p+1} can be used to find new solutions to Einstein's equations. First a set R^{p+1} (or equivalently S) is chosen. Some of the elements in R^p or functions of them are used as coordinates. Einstein's equations are imposed by restricting the Ricci tensor. The integrability conditions (1)–(4) are then imposed leading to a set of first order differential equations (together with algebraic constraints) for the elements in R^{p+1} (S). Solving this set of equations gives R^{p+1} and, hence, a complete local description of the geometry. If the geometry does not have any symmetries, one can solve for all the 1-forms from $dx^\mu = x^\mu{}_{|i}\omega^i$ and hence get the full line-element $ds^2 = \eta_{ij}\omega^i\omega^j$.

When there are symmetries one only obtains part of the 1-forms, but one may determine the Lie algebra of the isometry group [30], and from this it is often possible to make an ansatz for the remaining 1-forms (see Sec. III D).

III. BIANCHI V AND I

In this section we consider homogeneous cosmological models of Bianchi types V and I, i.e. those characterized by the symmetric matrix in the Ellis-MacCallum scheme [31] being zero. We assume that matter can be described as a perfect fluid. First the preliminaries, like choice of frame and the elements in the set S (R^{p+1}) are given. Einstein's equations with a cosmological constant are used. Then it is illustrated how one imposes the isometry group. After this we give the integrability conditions and reduce them to an integrable system of five first order ordinary differential equations.

A. Preliminaries

As energy-momentum tensor we take that of a perfect fluid

$$T_{ij} = (\rho + p)u_i u_j - p\eta_{ij}$$

with linear equation of state $p = (\gamma - 1)\rho$. Here ρ is the restframe density, p is the isotropic pressure, and u^i the 4-velocity of matter. Since homogeneity is assumed the elements in the set S will depend on only one timelike coordinate, that we choose as the density ρ. Sometimes, especially in problems with more than one independent coordinate, it can be advantageous to specify the coordinates at a later stage to simplify the equations, see [29].

A Lorentz frame ω^i will be used. We choose a comoving frame, i.e., the 4-velocity is given by $u = \delta^0_i \omega^i = \omega^0$. In general, the normals of the hypersurfaces, $d\rho = \rho_{|i}\omega^i$, will be tilted relative to the 4-velocity. The 1-direction is chosen to be in the direction of the spatial part of the density gradient, i.e., $\rho_{|2} = \rho_{|3} = 0$. From these equations we see that, once $\rho_{|0}$ and $\rho_{|1}$ are determined, one can solve for ω^0 as

$$\omega^0 = \frac{d\rho}{\rho_{|0}} - \frac{\rho_{|1}}{\rho_{|0}}\omega^1 \quad \text{or} \quad X_0 = \rho_{|0}\frac{\partial}{\partial\rho}.$$

This choice of frame means that we deviate from the usual approach of adopting the frame to the hypersurfaces of homogeneity.

Since there is only one essential coordinate, ρ, this is the only 1-form that we will be able to solve for from R^{p+1}. The frame is finally fixed by requiring that the vorticity (rotation) vector of the fluid is in the 12-plane, i.e., $\Omega^3 \equiv \frac{1}{2}\epsilon^{3ijk}\omega_{ij}u_k = 0$, corresponding to $\omega_{12} = 0$ (see below for the definition of the vorticity tensor).

The general model in this class will not have any isotropies and one can from the set consisting of $\rho_{|0}$, $\rho_{|1}$, and all γ^i_{jk} construct the full set R^{p+1}. [The set could in principle be even more reduced since the equations (1) give relations among the quantities.] Since we want to impose Einstein's equations

$$G_{ij} = T_{ij} + \Lambda\eta_{ij}$$

for a perfect fluid and are using a comoving frame, the Einstein tensor should be given by

$$G_{00} = \rho + \Lambda, \qquad G_{11} = G_{22} = G_{33} = p - \Lambda,$$

where Λ is the cosmological constant. The cyclic identity must also be imposed. The nonzero elements of the Riemann tensor are then given by

$$\begin{gathered}
R_{0101} = C_1 - \tfrac{1}{2}p - \tfrac{1}{6}\rho + \tfrac{1}{3}\Lambda, \qquad R_{0102} = R_{1323} = C_2,\\
R_{0103} = -R_{1223} = C_3, \qquad R_{0112} = R_{0323} = C_4,\\
R_{0113} = -R_{0223} = C_5, \qquad R_{0123} = C_6,\\
R_{0202} = C_7 - \tfrac{1}{2}p - \tfrac{1}{6}\rho + \tfrac{1}{3}\Lambda, \qquad R_{0203} = R_{1213} = C_8,\\
R_{0212} = -R_{0313} = C_9, \qquad R_{0213} = C_{10},\\
R_{0303} = -C_1 - C_7 - \tfrac{1}{2}p - \tfrac{1}{6}\rho + \tfrac{1}{3}\Lambda,\\
R_{0312} = C_{10} - C_6, \qquad R_{1212} = C_1 + C_7 - \tfrac{1}{3}\rho - \tfrac{1}{3}\Lambda,\\
R_{1313} = -C_7 - \tfrac{1}{3}\rho - \tfrac{1}{3}\Lambda, \qquad R_{2323} = -C_1 - \tfrac{1}{3}\rho - \tfrac{1}{3}\Lambda,
\end{gathered} \tag{5}$$

where C_i are the ten independent components of the Weyl tensor.

Some of the rotation coefficients are expressible in terms of the kinematic quantities shear σ_{ij}, vorticity ω_{ij}, expansion θ, and acceleration a_i:

$$\gamma_{0ij} = -u_{i;j} = -\omega_{ij} - \sigma_{ij} + \tfrac{1}{3}h_{ij}\theta - a_i u_j,$$

where $\sigma_{ij} = h_i{}^k h_j{}^l(u_{(k;l)} + \frac{1}{3}h_{kl}\theta)$, $\omega_{ij} = h_i{}^k h_j{}^l u_{[k;l]}$, $\theta = u^i_{;i}$, and $a_i = u_{i;j}u^j$ and $h_{ij} = u_i u_j - \eta_{ij}$ is the projection operator onto the space perpendicular to the 4-velocity. In the next subsection we impose the requirement that the isometry group is of Bianchi types V or I. This will give six restrictions on the rotation coefficients.

B. Symmetry group

The Lie algebra of the isometry group can be determined by projection of Cartan's equations onto the orbits [30]. Cartan's equations look the same in $F(M)$, the bundle of frames on M, as in M

$$d\omega^i = \omega^j \wedge \omega^i{}_j, \tag{6}$$

$$d\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j + \tfrac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l, \tag{7}$$

but now $d = d_x + d_\xi$, where x^μ are the coordinates on M and ξ^A the parameters of the orthogonal group. The connection forms $\omega^i{}_j$ will now be linearly independent of the 1-forms ω^i, and can be written as

$$\omega^i{}_j = \gamma^i{}_{jk}\omega^k + \tau^i{}_j \equiv \gamma^i{}_{jk}\omega^k + \tau^i{}_{jA}d\xi^A, \tag{8}$$

where $\tau^i{}_j$ are the generators of the orthogonal group. Since we do not have any isotropies in this case, the orbits will be the hypersurfaces of homogeneity $d\rho = 0$ in M. From $d\rho = 0$, we get

$$d\rho = \rho_{|0}\omega^0| + \rho_{|\alpha}\omega^\alpha| = 0 \quad \text{or} \quad \omega^0| = -\frac{\rho_{|\alpha}}{\rho_{|0}}\omega^\alpha|,$$

where $\alpha = 1, 2, 3$ are the spatial indices and a vertical bar | indicates projection onto $d\rho = 0$. From the requirement of no isotropy one has that $\tau^i{}_j| = 0$ holds on the orbits and (8) then gives

$$\omega^i{}_j| = \gamma^i{}_{jk}\omega^k|.$$

Hence the orbits are spanned by $\{\omega^1|, \omega^2|, \omega^3|\}$. By projecting the first pair of Cartan's equations (6) (the second pair will not give anything new in this case), one obtains

$$d\omega^\alpha| = \omega^j| \wedge \omega^\alpha{}_j| \equiv \tfrac{1}{2}C^\alpha{}_{\beta\delta}\omega^\beta| \wedge \omega^\delta|,$$

where the structure constants are given by

$$\tfrac{1}{2}C^\alpha{}_{\beta\delta} = \gamma^\alpha{}_{[\beta\delta]} + \frac{\rho_{|\beta}}{\rho_{|0}}\gamma^\alpha{}_{[\delta 0]} - \frac{\rho_{|\delta}}{\rho_{|0}}\gamma^\alpha{}_{[\beta 0]}. \tag{9}$$

The structure constants are hence given in terms of elements in R^{p+1}, and since they are functions only of ρ the $C^\alpha{}_{\beta\delta}$ are constants on the orbits.

From the Ellis-MacCallum scheme for the Bianchi classes [32], we can decompose the structure constants into one symmetric 3×3 matrix $N^{\alpha\beta}$ and a 3-vector A^α according to

$$\begin{aligned}
\tfrac{1}{2}C^\alpha{}_{\beta\delta}\epsilon^{\beta\delta\gamma} = N^{\alpha\gamma} + \epsilon^{\alpha\gamma\beta}A_\beta \text{ giving } N^{\alpha\delta}\\
= \tfrac{1}{2}[C^\alpha{}_{\beta\gamma}\epsilon^{\beta\gamma\delta} - C^\gamma{}_{\beta\gamma}\epsilon^{\beta\alpha\delta}].
\end{aligned} \tag{10}$$

Bianchi classes V and I are characterized by $N^{\alpha\delta} = 0$. This will give six relations among the γ^i_{jk}. Note that we cannot simply use the standard form of the structure constants in (9) and obtain nine relations for the Ricci coefficients this way since our choice of frame (giving other relations among the elements in S) might be inconsistent with that giving the structure constants in standard form. The non-zero Ricci rotation coefficients are then given by ($\alpha, \beta = 1, 2, 3$ and $\omega_{12} = 0$)

$$\begin{aligned} &\gamma_{0\alpha 0} = -a_\alpha, \qquad \gamma_{0\alpha\beta} = -\sigma_{\alpha\beta} - \omega_{\alpha\beta}(\alpha \neq \beta), \\ &\gamma_{0\alpha\alpha} = \frac{1}{3}\theta - \sigma_{\alpha\alpha}, \qquad \gamma_{123} = \gamma_{132} = -\frac{\rho_{|1}}{\rho_{|0}}\sigma_{23}, \\ &\gamma_{133} = \gamma_{122} + \frac{\rho_{|1}}{\rho_{|0}}(\sigma_{11} + 2\sigma_{22}), \\ &\gamma_{231} = \frac{\rho_{|1}}{\rho_{|0}}(\gamma_{230} - \omega_{23}), \\ &\gamma_{232} = \gamma_{131} + \frac{\rho_{|1}}{\rho_{|0}}(\sigma_{13} + \omega_{13} - \gamma_{130}), \gamma_{130}, \gamma_{131}, \gamma_{230}, \\ &\gamma_{233} = -\gamma_{121} + \frac{\rho_{|1}}{\rho_{|0}}(-\sigma_{12} + \gamma_{120}), \gamma_{120}, \gamma_{121}, \gamma_{122}. \end{aligned} \tag{11}$$

C. Integrability conditions

The commutator equations (1) give

$$\frac{d\rho_{|i}}{d\rho}\rho_{|j} - \frac{d\rho_{|j}}{d\rho}\rho_{|i} + \rho_{|k}(\gamma^k{}_{ji} - \gamma^k{}_{ij}) = 0 \tag{12}$$

and the Riemann equations (2)

$$R^i{}_{jkl} = 2\gamma^i{}_{j[l,\rho}\rho_{|k]} + 2\gamma^{im}{}_{[l}\gamma_{jmk]} + 2\gamma^i{}_{jm}\gamma^m{}_{[kl]} \tag{13}$$

(antisymmetrization only over kl). The cyclic identity is already imposed due to the choice (5) of the Riemann tensor and the Bianchi identities need not be imposed due to the lack of isotropies. Hence, the system (12) and (13) is the complete system. It is a set of first order ordinary differential equations, where the independent variable is ρ, and algebraic constraints. Some are easily solved and give

$$\begin{aligned} &a_2 = a_3 = \gamma_{121} = \omega_{23} = 0, \qquad \gamma_{120} = \sigma_{12}, \\ &\gamma_{130} = \sigma_{13} + \omega_{13}, \qquad \gamma_{230} = -\sigma_{23} (\text{or} \gamma_{131} = 0), \\ &\rho_{|0} = -\gamma\rho\theta, \qquad a_1 = (1-\gamma)v\theta, \qquad \omega_{13} = \tfrac{1}{2}v\gamma_{131}, \end{aligned} \tag{14}$$

where we have introduced the tilt

$$v \equiv \frac{\rho_{|1}}{\rho_{|0}}. \tag{15}$$

The case $\gamma_{131} = 0$ will be treated separately in Sec. III F. We see that the vorticity given by

$$\omega_{13} = \tfrac{1}{2}v\gamma_{131} \quad \text{or} \quad \Omega_2 = -\tfrac{1}{2}v\gamma_{131} \tag{16}$$

is perpendicular to the acceleration, a result already found in [4]. The ten components, C_i, of the Weyl tensor are also given by (13).

The system then reduces to nine first order differential equations

$$\dot{f}_i(\rho) = F_i(f_j(\rho), \rho), \tag{17}$$

where $\dot{f} \equiv df/d\rho$, for the functions $f_i(\rho)$

$$f_i \in \{v, \theta, \sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \sigma_{23}, \gamma_{131}, B\},$$

where we use the variable

$$B \equiv \gamma_{122} - \tfrac{1}{3}v\theta + v\sigma_{22} \tag{18}$$

instead of γ_{122}, and four algebraic constraints

$$G_a(f_j(\rho), \rho) = 0. \tag{19}$$

It turns out that the differentiated constraints, when using (17), all are satisfied, i.e.,

$$\dot{G}_a = \frac{\partial G_a}{\partial f_j}\dot{f}_j + \frac{\partial G_a}{\partial \rho} = \frac{\partial G_a}{\partial f_j}F_j + \frac{\partial G_a}{\partial \rho} = 0.$$

Hence, the system can be reduced to five differential equations and 4 constraints [23]. The constraints are given by

$$\begin{aligned} &\gamma_{131}[-2v^2(3\sigma_{11} + (3\gamma - 4)\theta) - 3vB - 18(\sigma_{11} + \sigma_{22})(1 - v^2)] - 18B\sigma_{13} = 0, \\ &B\sigma_{12} - \gamma_{131}\sigma_{23}(1 - v^2) = 0, \\ &9\gamma_{131}(\gamma_{131}v + 2\sigma_{13})(1 - v^2) + 6v(2B^2 + \gamma\rho) - 2v^2B[2(3\gamma - 4)\theta - 3\sigma_{11}] - 18B\sigma_{11} = 0, \\ &9\gamma_{131}[\gamma_{131}(12 - 5v^2) - 8v\sigma_{13}] + 48vB\theta + 36[\sigma_{11}^2 + \sigma_{11}\sigma_{22} + \sigma_{22}^2 + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 + \Lambda + 3B^2 \\ &\quad + \rho - v^2(\sigma_{11}\sigma_{22} + \sigma_{22}^2 + \sigma_{23}^2)] + 4\theta^2[(6\gamma - 5)v^2 - 3] + 12\sigma_{11}[6vB + (3\gamma - 2)v^2\theta] = 0 \end{aligned} \tag{20}$$

that are easily solved for σ_{13}, σ_{23}, and σ_{11} in which the three first constraints are linear. Finally the last constraint gives a second order polynomial in σ_{22}. The differential equations are

$$\dot{v} = v\left(\frac{-\sigma_{11}}{\gamma\rho\theta} + \left(\frac{4}{3\gamma} - 1\right)\frac{1}{\rho}\right), \qquad \dot{\gamma}_{131} = \gamma_{131}\left(\frac{(\sigma_{11}+\sigma_{22})}{\gamma\rho\theta} + \frac{1}{3\gamma\rho}\right),$$
$$\dot{\sigma}_{12} = \sigma_{12}\left(\frac{(\sigma_{11}-\sigma_{22})}{\gamma\rho\theta} + \frac{1}{\gamma\rho}\right) - \frac{2v\gamma_{131}\sigma_{23}}{\gamma\rho\theta}, \qquad \dot{B} = B\left(-\frac{\sigma_{11}}{\gamma\rho\theta} + \frac{1}{3\gamma\rho}\right) + \frac{2\gamma_{131}\sigma_{13}}{\gamma\rho\theta} \tag{21}$$
$$(\theta^2)^{\cdot} = [3\gamma_{131}^2 v^2 + 12(\gamma-1)v\theta(B+v\sigma_{11}) - 2\theta^2 + 2(\gamma-1)(6\gamma-5)v^2\theta^2 - 3((3\gamma-2)\rho - 2\Lambda) - 12(\sigma_{11}^2 + \sigma_{11}\sigma_{22} + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{22}^2 + \sigma_{23}^2)]/[3\gamma\rho((\gamma-1)v^2 - 1)].$$

If one instead wants to use proper time for a comoving observer as independent coordinate the equation

$$\frac{d\rho}{d\tau} = \rho_{|0} = -\gamma\rho\theta, \tag{22}$$

obtained by putting $dx = dy = dz = 0$ in the line-element [see Eq. (25) below], should be added to the system (21). The derivatives in (21) are then expressed as

$$\dot{f}_i \equiv \frac{df_i}{d\rho} = \frac{df_i}{d\tau}\frac{d\tau}{d\rho} = -\frac{df_i}{d\tau}\frac{1}{\gamma\rho\theta}.$$

The system has a unique solution for given initial conditions provided F_i for the reduced system (21) satisfy a Lipschitz condition (for example if F_i are C^1 in a compact convex domain). The general solution hence depends on five constants of integration and in general its matter flow has both expansion, shear, and vorticity.

One could of course choose to solve for four other quantities than $\sigma_{13}, \sigma_{23}, \sigma_{11}$, and σ_{22} from (20). When doing a first order perturbative calculation around an isotropic universe, as in Sec. IV, both σ_{11} and σ_{22} cannot be solved from (20) since the last constraint will be quadratic in σ_{22} or σ_{11}. Hence, we solve for θ from the last constraint and use the differential equation for σ_{22}:

$$\dot{\sigma}_{22} = \frac{\dot{\theta}}{3} + \Bigg[(\gamma-2)\rho - 4(\sigma_{12}^2 - (v^2-1)\sigma_{23}^2) - 2\Lambda + 2\left(\sigma_{22} - \frac{\theta}{3}\right)\left(v^2\left(2\sigma_{11} + (6\gamma-5)\frac{\theta}{3}\right) - \theta\right) - 4B^2 + 2vB(2(\sigma_{22}-\sigma_{11}) - \gamma\theta) + 4\gamma_{131}(v\sigma_{13} - \gamma_{131})\Bigg]\Big/[2\gamma\rho\theta(v^2-1)] \tag{23}$$

instead of the one for θ.

This system is not in a suitable form for a dynamical system analysis. The system should then be made autonomous and compact dimensionless variables should be introduced. In [18] irrotational tilted Bianchi type V cosmologies were studied with this method. The field equations are derived in terms of expansion-normalized variables making the state space compact. The existence of a monotonic function shows that the dynamics to a large extent is determined by the invariant subset of locally rotationally symmetric models. A complete analysis of the orbits with nonextreme tilt was obtained. In [19] tilted Bianchi models of solvable type, including Bianchi type V, were considered with emphasis on the late-time behavior. The equilibrium points were given in [18], but here the stability analysis was performed in the full state space. It was found that for $2/3 < \gamma < 2$ Bianchi type V models approach the Milne universe in the asymptotic future. To complete the analysis a study of the behavior for early times (high densities) would be of interest.

D. The metric

From equations (9) for the structure constants we find the Lie algebra of the isometry group to be

$$\begin{aligned} d\omega^1| &= \gamma_{131}\omega^1| \wedge \omega^3|, \\ d\omega^2| &= B\omega^1| \wedge \omega^2| + \gamma_{131}\omega^2| \wedge \omega^3|, \\ d\omega^3| &= B\omega^1| \wedge \omega^3|, \end{aligned} \tag{24}$$

where B is given by (18). As expected it is not in the standard form for Bianchi V, unless $B = 0$. Guided by the above algebra and standard 1-forms for Bianchi V, we make the following simplified, but sufficient, ansatz for a basis of 1-forms for the full spacetime

$$\begin{aligned} \tilde{\omega}^0 &= -\frac{d\rho}{\gamma\rho\theta} - v\tilde{\omega}^1, \qquad \tilde{\omega}^1 = f_1 e^{-x}dy, \\ \tilde{\omega}^2 &= g_1 e^{-x}dy + g_2 e^{-x}dz + g_3 dx, \\ \tilde{\omega}^3 &= \frac{1}{\gamma_{131}}dx + h_1\frac{B}{\gamma_{131}}e^{-x}dy, \end{aligned} \tag{25}$$

where f_1, g_1, g_2, g_3, and h_1 are functions of ρ to be determined. (The case $\gamma_{131} = 0$ is treated separately in Sec. III F.) From (25) we calculate the set $\tilde{S} = \{\tilde{\rho}_{|i}, \tilde{\gamma}_{ijk}\}$. A comparison with the set S gives

$$\begin{aligned} f_1 = h_1 &= C_1\rho^{(1/\gamma)-1}v, \qquad g_1 = \gamma_{131}\rho^{1-(2/\gamma)}vI_1, \\ g_2 &= C_2\gamma_{131}\rho^{1-(2/\gamma)}v, \qquad g_3 = \gamma_{131}\rho^{1-(2/\gamma)}vI_2, \end{aligned} \tag{26}$$

where C_1 and C_2 are (nonzero) constants of integration that can be absorbed in the definitions of the coordinates y and z and the integrals I_1 and I_2 are given by

$$I_1 = \frac{2C_1}{\gamma}\int \frac{\rho^{(3/\gamma)-2}}{v^2\rho\theta\gamma_{131}^2}[\sigma_{12}\gamma_{131} + \sigma_{23}B]d\rho,$$
$$I_2 = \frac{2}{\gamma}\int \frac{\sigma_{23}\rho^{(2/\gamma)-1}}{v\rho\theta\gamma_{131}^2}d\rho, \tag{27}$$

respectively. From the 1-forms the metric is given by $ds^2 = \eta_{ij}\omega^i\omega^j$. The full metric is hence given in terms of quadratures once the set S has been constructed. The solution depends on some arbitrary constants of integration and an even more general ansatz than (25) could have been made, but all these metrics give the same set S and hence they are locally equivalent.

E. Orthogonal solutions

For the orthogonal case $v = 0$, when also the vorticity is zero, it is better to use a frame where the shear tensor is diagonal, $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$. The obtained system is easily solved and all solutions are well known. Essentially only two types of solutions appear. These are the Bianchi V solutions

$$\sigma_{11} = -\sigma_{22} = -k_1\rho^{1/\gamma}, \qquad \gamma_{131} = \gamma_{232} = k_2\rho^{1/(3\gamma)},$$
$$\theta = \pm\sqrt{3}\sqrt{3k_2^2\rho^{2/(3\gamma)} + k_1^2\rho^{2/\gamma} + \rho + \Lambda} \tag{28}$$

and the Bianchi class I solutions

$$\sigma_{11} = k_1\rho^{1/\gamma}, \qquad \sigma_{22} = k_2\rho^{1/\gamma},$$
$$\theta = \pm\sqrt{3}\sqrt{(k_1^2 + k_2^2 + k_1k_2)\rho^{2/\gamma} + \rho + \Lambda}. \tag{29}$$

All orthogonal solutions can be found from these interchanging the 1-, 2-, and 3-directions. The corresponding metrics are

$$\omega^0 = -\frac{d\rho}{\gamma\rho\theta}, \qquad \omega^1 = c_1e^{-\int[\gamma_{011}/(\gamma\rho\theta)]d\rho}e^{-x}dy,$$
$$\omega^2 = c_2e^{-\int[\gamma_{022}/(\gamma\rho\theta)]d\rho}e^{-x}dz, \qquad \omega^3 = \frac{1}{k_2}\rho^{-[1/(3\gamma)]}dx$$

and

$$\omega^0 = -\frac{d\rho}{\gamma\rho\theta}, \quad \omega^1 = c_1e^{-\int[\gamma_{011}/(\gamma\rho\theta)]d\rho}dx,$$
$$\omega^2 = c_2e^{-\int[\gamma_{022}/(\gamma\rho\theta)]d\rho}dy, \quad \omega^3 = c_3e^{-\int[\gamma_{033}/(\gamma\rho\theta)]d\rho}dz,$$

respectively. The integrals can be performed for specific values of γ.

F. Irrotational tilted solutions

The irrotational solutions are given by $\gamma_{131} = 0$, which implies that the vorticity vanishes and the frame cannot be fixed by demanding $\Omega^3 = 0$. Instead the frame is fixed by putting $\sigma_{23} = 0$. Some care should be taken in using Eqs. (20) and (21) directly since an extra constraint is introduced and they were derived assuming $\gamma_{131} \neq 0$. Yet the only nontrivial cases are those obtained from this system with $\gamma_{131} = \sigma_{23} = 0$ and $B \neq 0$. From the two first constraints $\sigma_{12} = \sigma_{13} = 0$ is obtained. σ_{11} and σ_{22} can be solved from the two others. The system of differential equations is reduced to a system for $\dot{v}$, $\dot{B}$, and $\dot{\theta}$. This can be reduced to a system of two differential equations since the equations for $\dot{v}$ and $\dot{B}$ in this case can be combined to

$$\frac{\dot{v}}{v} = \frac{\dot{B}}{B} + \left(\frac{1}{\gamma} - 1\right)\frac{1}{\rho}$$

with solution

$$B = C_1v\rho^{(\gamma-1)/\gamma}, \tag{30}$$

where C_1 is a constant of integration. The metric is in this case given by

$$\omega^0 = -\frac{d\rho}{\gamma\rho\theta} - v\omega^1, \qquad \omega^1 = -\frac{1}{B}dx,$$
$$\omega^2 = k_2\rho^{-[1/(3\gamma)]}e^{\int[\sigma_{22}/(\gamma\rho\theta)]d\rho}e^{-x}dy,$$
$$\omega^3 = k_3\rho^{-[1/(3\gamma)]}e^{-\int[(\sigma_{11}+\sigma_{22})/(\gamma\rho\theta)]d\rho}e^{-x}dz.$$

1. LRS solutions

This subset was studied in detail in [15] for different values of γ. There are solutions for which the tilt always is less than one and hence the hypersurfaces of homogeneity remain spacelike when going backwards in time. However, it may approach one for large times for certain values of γ including the radiation case $\gamma = 4/3$. There are also solutions for which the tilt gets larger than 1 for small times and hence the hypersurfaces change from being spacelike to being timelike. For this class singularities may occur for finite densities. For example, in the $\gamma = 4/3$ case some of the kinematic quantities and the Weyl tensor diverge for $v = \sqrt{3}$.

LRS solutions are obtained by putting $\sigma_{11} = -2\sigma_{22}$ in the above equations (the LRS symmetry lies in the 23-plane). If one solves the constraints for σ_{22} and θ, the system is reduced to one differential equation for v.

For $\gamma = 1$ and $\Lambda = 0$ the field equations were completely integrated in [27,28]. The solution in our variables is given by

$$\sigma_{11} = -2\sigma_{22} = \pm\frac{\kappa^2(3C\kappa + 2)}{3(D - \kappa\sqrt{1 + C\kappa})},$$
$$\theta = \mp\frac{3D\kappa\sqrt{1 + C\kappa} - \kappa^2(2 + \frac{3}{2}C\kappa)}{D - \kappa\sqrt{1 + C\kappa}}, \tag{31}$$
$$v = \mp\frac{\kappa}{D - \kappa\sqrt{1 + C\kappa}}, \qquad \gamma_{122} = -\kappa,$$

where C and D are constants of integration and κ and ρ are related as

$$\rho = \frac{3CD\kappa^3}{D - \kappa\sqrt{1 + C\kappa}}. \tag{32}$$

The basis 1-forms are

$$\begin{aligned} \omega^0 &= \pm\frac{d\kappa}{\kappa^2\sqrt{1+C\kappa}} \mp dx, \\ \omega^1 &= -\frac{D - \kappa\sqrt{1+C\kappa}}{\kappa}dx, \\ \omega^2 &= \frac{e^{Dx}}{\kappa}dy, \\ \omega^3 &= \frac{e^{Dx}}{\kappa}dz. \end{aligned} \tag{33}$$

From (32) one finds essentially two types of solutions. If both C and D are positive, the density rises from zero to infinity when κ goes from zero to κ_1, where κ_1 is given by

$$D - \kappa_1\sqrt{1 + C\kappa_1} = 0.$$

The tilt v then also increases from zero to infinity, and hence the hypersurfaces of homogeneity change from being spacelike to being timelike.

A positive density is also obtained for $C > 0$ and $D < 0$, and in this case the density goes from zero to infinity when κ goes from zero to infinity, but now the tilt varies from zero to zero in this interval. The maximum value is obtained for κ_2

$$C\kappa_2^2 + 2D\sqrt{1 + C\kappa_2} = 0,$$

giving

$$|v_{\max}| = \frac{\sqrt{1 + C\kappa_2}}{1 + \frac{3}{2}C\kappa_2}$$

that is less than one and hence the hypersurfaces of homogeneity remain spacelike for all times.

A perturbative calculation (see IV for the perturbative method) with a small $\gamma_{131} = \epsilon\gamma_{131}^{(1)}$ around the exact solution gives

$$\gamma_{131}^{(1)} = F\frac{\kappa}{D - \kappa\sqrt{1 + C\kappa}}, \tag{34}$$

where F is a constant of integration. Hence, the perturbation grows as the other quantities for the first case ($C, D > 0$) as $\kappa \to \kappa_1$. However, for the second case ($D < 0$) it goes to zero when $\kappa \to \infty$.

G. Solutions with a timelike homothetic motion

A spacetime has a homothetic Killing vector ξ if

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2cg_{\mu\nu}$$

is satisfied for some constant c. This implies that quantities of the same dimension scale in the same way in the direction of ξ. For a timelike homothetic Killing vector it is hence easy to show that for perfect fluids all components of the Riemann tensor, R_{ijkl}, are proportional to ρ, the Ricci rotation coefficients to $\rho^{1/2}$ and $\rho_{|i}$ to $\rho^{3/2}$, see e.g. [33]. The field equations are then reduced to algebraic equations.

Unfortunately this limitation is quite restrictive and the only solution of this type is the flat Friedmann universe (of Bianchi type I).

IV. PERTURBATIVE SOLUTIONS

In this section we consider perturbative solutions to the system (20) and (21). In general perturbative solutions to systems of nonlinear equations with constraints need not correspond to exact solutions, but in the present case the system can be reduced to (21), that under reasonable assumptions is known to be integrable with five constants of integration. Hence a perturbative solution, obtained by solving the Taylor expanded equations, should agree with the Taylor expansion of some exact solution. This is also favored by comparing the perturbative solutions with a numerical solving of the full system, see Sec. V.

We here first briefly recall some basic results of perturbation theory. Calling the four functions solved from the algebraic constraints (19) g_a, $a, b, \ldots = 1, \ldots, 4$, the reduced system can be written as

$$\begin{aligned} \dot{f}_i &= F_i(f_j, g_a, \rho), \qquad i, j, \ldots = 1, \ldots, 5, \\ 0 &= G_a(f_j, g_b, \rho). \end{aligned} \tag{35}$$

The functions f_i and g_a are expanded around the solution $f_i^{(0)}$ and $g_a^{(0)}$ of (35) in the small parameter ϵ as

$$\begin{aligned} \Delta f_i &\equiv f_i - f_i^{(0)} = \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \cdots, \\ \Delta g_a &\equiv g_a - g_a^{(0)} = \epsilon g_a^{(1)} + \epsilon^2 g_a^{(2)} + \cdots \end{aligned} \tag{36}$$

giving, with H_p equal to F_i or G_a,

$$\begin{aligned} H_p &= H_p(f_j^{(0)}, g_a^{(0)}, \rho) + \frac{\partial H_p}{\partial f_j}\Delta f_j + \frac{\partial H_p}{\partial g_a}\Delta g_a + \frac{\partial^2 H_p}{\partial f_j \partial g_a}\Delta f_j \Delta g_a + \frac{1}{2}\frac{\partial^2 H_p}{\partial f_j \partial f_k}\Delta f_j \Delta f_k + \frac{1}{2}\frac{\partial^2 H_p}{\partial g_a \partial g_b}\Delta g_a \Delta g_b + \cdots \\ &= H_p(f_j^{(0)}, g_a^{(0)}, \rho) + \epsilon\left(\frac{\partial H_p}{\partial f_j}f_j^{(1)} + \frac{\partial H_p}{\partial g_a}g_a^{(1)}\right) + \epsilon^2\left(\frac{\partial H_p}{\partial f_j}f_j^{(2)} + \frac{\partial H_p}{\partial g_a}g_a^{(2)}\right) \\ &\quad + \epsilon^2\left(\frac{1}{2}\frac{\partial^2 H_p}{\partial f_j \partial f_k}f_j^{(1)}f_k^{(1)} + \frac{1}{2}\frac{\partial^2 H_p}{\partial g_a \partial g_b}g_a^{(1)}g_b^{(1)} + \frac{\partial^2 H_p}{\partial f_j \partial g_a}f_j^{(1)}g_a^{(1)}\right) + \cdots, \end{aligned} \tag{37}$$

where the partial derivatives are evaluated at $f_i = f_i^{(0)}$ and $g_a = g_a^{(0)}$. Identifying equal powers of ϵ in (35), using (36) and (37) with $H_p = G_a$, one can solve for $g_a^{(n)}$ in terms of $f_i^{(n)}$ and lower order quantities as

$$g_a^{(n)} = -(G^{-1})_a^b \frac{\partial G_b}{\partial f_j} f_j^{(n)} + \mathcal{G}_a^{(n-1)}, \qquad (38)$$

where $(G^{-1})_a^b$ is the inverse of $\frac{\partial G_a}{\partial g_b}$ (assuming that it is invertible) and $\mathcal{G}_a^{(n-1)}$ only depends on $f_i^{(m)}$ and $g_a^{(m)}$ up to order $n-1$, i.e. $m \leq n-1$. Substitution of (36)–(38) with $H_p = F_i$ into (35) and identification of equal powers of ϵ now gives

$$\dot{f}_i^{(n)} - \left(\frac{\partial F_i}{\partial f_j} - \frac{\partial F_i}{\partial g_a}(G^{-1})_a^b \frac{\partial G_b}{\partial f_j}\right) f_j^{(n)} = \mathcal{F}_i^{(n-1)}, \qquad (39)$$

where $\mathcal{F}_i^{(n-1)}$ only depends on $f_i^{(m)}$ and $g_a^{(m)}$ up to order $n-1$ and for $n=1$ reduces to $\mathcal{F}_i^{(n-1)} = \mathcal{F}_i^{(0)} = 0$. From this equation we recall the result that the homogeneous parts of the differential equations are the same to each order in ϵ, and hence the integration constants add up as $k_i = \epsilon k_i^{(1)} + \epsilon^2 k_i^{(2)} + \cdots$, so that the correct number of constants is maintained to any order.

A. First order perturbations

We here consider perturbations to first order in the small parameter ϵ around the open Friedmann universes. It would be of interest to also consider perturbations around the general orthogonal solutions given by (28) and (29). However, the perturbative solutions can then only be given in terms of quadratures in the general case.

The nonzero elements in the set S for the Friedmann universe are given by

$$\theta^{(0)} = \pm\sqrt{3}\sqrt{3(k_1^{(0)})^2\rho^{2/(3\gamma)} + \rho + \Lambda}, \qquad \gamma_{133}^{(0)} = \gamma_{122}^{(0)},$$
$$\gamma_{131}^{(0)} = \gamma_{232}^{(0)} = \pm\sqrt{(k_1^{(0)})^2\rho^{2/(3\gamma)} - (\gamma_{122}^{(0)})^2}, \qquad (40)$$

where $k_1^{(0)}$ is a constant of integration. In the following we choose $\gamma_{122}^{(0)} = 0$, so that $\gamma_{131}^{(0)} = \gamma_{232}^{(0)} = k_1^{(0)}\rho^{1/(3\gamma)}$. The freedom in one of the Ricci rotation coefficients is due to that only the γ_{0ij} appear in S for the Friedmann universe due to the isotropy. Note, however, that the resulting perturbed solutions are depending on this choice. For example, if we instead had chosen $\gamma_{131}^{(0)} = 0$, the first order perturbations would have had zero vorticity.

Instead of solving for σ_{22} from the last of Eqs. (20), we solve this equation for θ, and hence the differential equation for θ in (21) is replaced by the corresponding for σ_{22} (23).

To first order we write the elements in S as

$$v = \epsilon v^{(1)}, \qquad \gamma_{ijk} = \gamma_{ijk}^{(0)} + \epsilon\gamma_{ijk}^{(1)}.$$

Expanding the system (20) and (21) [with the last equation replaced by (23)] to first order then gives

$$a_1 = \epsilon k_3 \theta \rho^{[4/(3\gamma)]-1}(1-\gamma),$$
$$\sigma_{11} = -\sigma_{22} = \epsilon k_2 \rho^{1/\gamma}, \quad \sigma_{12} = \epsilon k_4 \rho^{1/\gamma},$$
$$\sigma_{13} = -\epsilon\frac{\gamma k_3}{3k_1}\rho^{1/\gamma} - \epsilon\frac{k_1 k_3}{2}\rho^{[5/(3\gamma)]-1},$$
$$\omega_{13} = \epsilon\frac{k_1 k_3}{2}\rho^{[5/(3\gamma)]-1}, \quad \gamma_{131} = k_1\rho^{1/(3\gamma)},$$
$$\gamma_{122} = \epsilon\gamma_{122}^{(1)}, \qquad (41)$$

where $\gamma_{122}^{(1)} = k_5\rho^{1/(3\gamma)} - \frac{k_3}{6}\rho^{1/(3\gamma)}$

$$\times \int\left[\frac{2}{\gamma}(\gamma-1)\theta\rho^{(1/\gamma)-2} + \frac{\rho^{(1/\gamma)-1}}{\theta}\right]d\rho,$$
$$\theta = \pm\sqrt{3}\sqrt{3k_1^2\rho^{2/(3\gamma)} + \rho + \Lambda},$$
$$v = \epsilon k_3 \rho^{[4/(3\gamma)]-1},$$

where $k_1 = k_1^{(0)} + \epsilon k_1^{(1)}$, $k_2 = k_2^{(1)}$, $k_3 = k_3^{(1)}$, $k_4 = k_4^{(1)}$, and $k_5 = k_5^{(1)}$ are the five constants of integration.

The integral in γ_{122} can be evaluated for certain values of γ if $\Lambda = 0$. For $\gamma = 1$ (dust) one gets

$$\gamma_{122}^{(1)} = k_5\rho^{1/3} \pm \frac{k_3\theta}{9}(\rho^{1/3} - 6k_1^2),$$

for $\gamma = \frac{4}{3}$ (radiation)

$$\gamma_{122}^{(1)} = k_5\rho^{1/4} - k_3\rho^{1/4}\left[\frac{5}{9}\theta\rho^{-(1/4)} \pm k_1 \ln\left(\frac{\rho^{1/4} \pm \frac{1}{\sqrt{3}}\theta\rho^{-(1/4)} - \sqrt{3}k_1}{\rho^{1/4} \pm \frac{1}{\sqrt{3}}\theta\rho^{-(1/4)} + \sqrt{3}k_1}\right)\right],$$

and for $\gamma = 2$ (stiff matter), with $\tilde{k}_5 = k_5 \pm k_3/2$,

$$\gamma_{122}^{(1)} = \tilde{k}_5\rho^{1/6} \mp \frac{\sqrt{3}k_3\rho^{1/6}}{6}\left[4\ln\left(\frac{\rho^{1/3}}{\sqrt{3}k_1} \pm \frac{\rho^{-(1/6)}\theta}{3k_1}\right) - \frac{9\sqrt{3}k_1^2\rho^{-(1/6)}}{\sqrt{3}\rho^{1/2} \pm \theta}\right].$$

The integral can also be evaluated with nonzero Λ for $\gamma = 1$ if $k_1 = 0$, corresponding to that the background metric is of Bianchi type I (but the perturbed one is of type V), giving

$$\gamma_{122}^{(1)} = k_5\rho^{1/3} \mp \frac{k_3}{3\sqrt{3}}\rho^{1/3}\sqrt{\rho + \Lambda}.$$

To see in what regions the solutions are valid, we take the ratios of the first order elements $\gamma_{ijk}^{(1)}$ and the zeroth order value $\theta^{(0)}$ and check if they and v remain small. When approaching the initial singularity, i.e., when $\rho \to \infty$, the ratios $\sigma_{\alpha\beta}/\theta^{(0)}$ diverge. Hence, the perturbations cannot be valid for very early times. For $\gamma < 4/3$ also v diverges. However, the normalized vorticity, $\omega_{13}/\theta^{(0)}$, goes to zero for $\gamma > 10/9$.

For large times, i.e., when $\rho \to 0$, all ratios $\gamma_{ijk}^{(1)}/\theta^{(0)}$ and v remain finite for $\gamma \leq 4/3$, including the dust and radiation cases. If $\gamma < 4/3$ all these quantities go to zero (except for $\gamma^{(1)}_{122}/\theta^{(0)}$ that goes to a constant value, but this is related to the choice of frame as discussed above) in accordance with the results that Bianchi I and V universes isotropize [17]. One should of course be careful in drawing conclusions, since higher order perturbations could in principle dominate over first order perturbations when going far from the point around which the Taylor expansion is done.

B. The metric to first order

The form of the metric (25)–(27) is not suitable for a perturbative calculation since some of the coefficients diverge when $\epsilon \rightarrow 0$. This can be avoided by introducing new coordinates according to

$$y = \epsilon\tilde{y}, \qquad z = \frac{\tilde{z}}{\epsilon}. \tag{42}$$

When expanding the 1-forms to first order in ϵ the tilt v to second order, given in IV C, will be needed. To first order in ϵ one then obtains the following 1-forms (with $C_1 = C_2 = 1$)

$$\begin{aligned}
\omega^0 &= -\frac{d\rho}{\gamma\rho\theta} - \epsilon\rho^{(1/\gamma)-1}e^{-x}d\tilde{y},\\
\omega^1 &= \frac{1}{k_3}\rho^{-[1/(3\gamma)]}e^{-x}(1-\epsilon A_1)d\tilde{y},\\
\omega^2 &= -\epsilon\frac{2k_4}{k_2k_3}A_1\rho^{-[1/(3\gamma)]}e^{-x}d\tilde{y}\\
&\quad + k_1k_3\rho^{-[1/(3\gamma)]}e^{-x}(1+\epsilon A_1)d\tilde{z},\\
\omega^3 &= \frac{1}{k_1}\rho^{-[1/(3\gamma)]}dx\\
&\quad + \frac{\epsilon}{k_1}e^{-x}\left(\frac{1}{k_3}\rho^{-[2/(3\gamma)]}\gamma^{(1)}_{122} - \frac{1}{3}\rho^{[2/(3\gamma)]-1}\theta\right)d\tilde{y},
\end{aligned} \tag{43}$$

where θ and $\gamma^{(1)}_{122}$ are given by (41) and A_1 by

$$A_1(\rho) = -k_2\int\frac{\rho^{(1/\gamma)-1}}{\gamma\theta}d\rho. \tag{44}$$

For dust ($\gamma = 1$) and $\Lambda = 0$ A_1 becomes

$$A_1 = -\frac{2k_2\theta}{3}(1 - 6k_1^2\rho^{-(1/3)})$$

and with $\Lambda \neq 0$ and $k_1 = 0$

$$A_1 = -\frac{2k_2\theta}{3}.$$

Radiation ($\gamma = 4/3$) with $\Lambda = 0$ gives

$$A_1 = -k_2\theta\rho^{-1/2}.$$

The relation to proper time for a comoving observer is obtained by putting $dx = dy = dz = 0$ in the line-element and is given by $d\tau = -\frac{d\rho}{\gamma\rho\theta}$. For example, for an expanding universe with $\gamma = 1$, we have $\theta = \sqrt{3}\sqrt{3k_1^2\rho^{2/3} + \rho}$. Integration gives [with integration constant such that $\tau(\rho = \infty) = 0$]

$$\tau = \frac{\theta\rho^{-(2/3)}}{3k_1^2} - \frac{1}{6k_1^3}\ln\left(\frac{\theta + 3k_1\rho^{1/3}}{\theta - 3k_1\rho^{1/3}}\right).$$

Note that deviation from zeroth order quantities in the expression for proper time will appear first to second order.

C. Second order perturbations

From (38) we see that the second order perturbations can be obtained up to quadratures. A full second order calculation will be done in a forthcoming paper. Here we focus on the tilt, v, whose nth order equations decouple from other nth order quantities, and also since it will be needed to second order to obtain the metric to first order.

To second order one obtains

$$v = \epsilon k_3\rho^{[4/(3\gamma)]-1}\left(1 - \frac{\epsilon k_2}{\gamma}\int\frac{\rho^{(1/\gamma)-1}}{\theta}d\rho\right), \tag{45}$$

where now $k_3 = k_3^{(1)} + \epsilon k_3^{(2)}$. For $\Lambda = 0$ and $\gamma = 1$ we have

$$\begin{aligned}
v &= \epsilon k_3\rho^{1/3}\left(1 - \frac{2\epsilon k_2}{3}\theta(1 - 6k_1^2\rho^{-(1/3)})\right)\\
&= \epsilon k_3\rho^{1/3}\left(1 \mp \frac{2\epsilon k_2}{\sqrt{3}}\sqrt{\rho^{1/3} + 3k_1^2}(\rho^{1/3} - 6k_1^2)\right)
\end{aligned}$$

and for $\Lambda \neq 0$, $\gamma = 1$, and $k_1 = 0$

$$v = \epsilon k_3\rho^{1/3}\left(1 \mp \frac{2\epsilon k_2}{\sqrt{3}}\sqrt{\rho + \Lambda}\right).$$

For $\Lambda = 0$ and $\gamma = 4/3$ the tilt is given by

$$\begin{aligned}
v &= \epsilon k_3(1 - \epsilon k_2\theta\rho^{-(1/4)})\\
&= \epsilon k_3(1 \mp \epsilon k_2\sqrt{3}\sqrt{\rho^{1/2} + 3k_1^2}).
\end{aligned}$$

As seen, depending on sign of k_2, the second order contributions can either enforce the growth of the tilt or make it turn and start decreasing. This result is in accordance with the exact equation (21) for v that shows that the behavior is determined by the sign of σ_{11}/θ ($\sigma_{11} = \epsilon k_2\rho^{1/\gamma}$).

V. NUMERICAL SOLUTIONS

In this section we solve the system (20) and (21) numerically using the Runge-Kutta method with a truncation error of order h^4 in the step length h. A comparison between the code and the exact solutions (28) gives agreement to high accuracy.

A. Comparison with perturbative calculations

To check the perturbative calculation in Sec. IV (or conversely the numerical method), we have solved the system numerically for small deviations (at start) from the Friedmann models and compared with the perturbative solutions. The constants of integration are chosen so that the numerical and perturbative solutions agree for the starting value of the independent variable ρ. In the two following figures, first and second order perturbations together with the numerical solutions for v with $\gamma = 1$ (Fig. 1) and $\gamma = 4/3$ (Fig. 2) are depicted. The agreement is similar for the other quantities in S and other values of γ.

B. Asymptotic behavior

The numerical runs confirm the result of [19], where a dynamical system analysis was used to find that for $2/3 < \gamma < 2$ Bianchi type V models with zero cosmological constant approach the Milne universe in the asymptotic future (for low densities). Figure 3 shows the components of the shear normalized with expansion (σ_{ij}/θ) and the tilt for $\gamma = 1$ for the case with a nonzero cosmological constant $\Lambda = 1$, and they all vanish in the limit $\rho \longrightarrow 0$. In this case the de Sitter universe is obtained in the asymptotic future.

As found for the LRS case in [15], anisotropies may remain for late times with $\gamma \neq 1$. This is not in conflict with the result of [19]. An anisotropy remains in the matter fields, but since matter is getting infinitely diluted spacetime still approaches isotropy. In Fig. 4 ω_{13}/θ and v are plotted for $\gamma = 4/3$. As seen they do not approach zero for small ρ.

For the LRS case two different behaviors close to the initial singularity was found in [15], one where the tilt v goes to zero and one where it passes the extreme value of one and grows towards infinity. In the second case, however, often singularities in some of the other quantities occur while v and the density still are finite. The numerical runs seem to show that generically the tilt grows towards one, even if it initially may be decreasing for a long density interval. In Fig. 5 v, σ_{11}, and σ_{22} for $\gamma = 4/3$ is shown. In

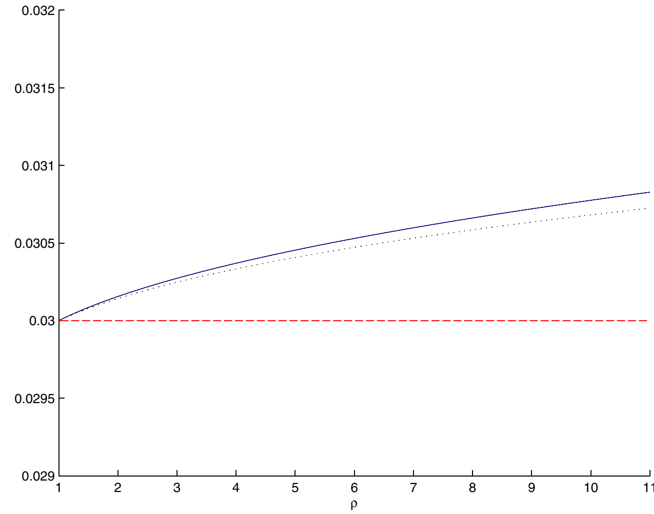

FIG. 2 (color online). Comparison between numerical calculation (solid) and perturbations to first (dashed) and second order (dotted) for v in the case of radiation and $\Lambda = 0$. Initial values: $\gamma_{131} = 1$, $v = 0.03$, $B = 0.03$, $\sigma_{12} = 0.03$, and $\sigma_{22} = 0.03$.

this particular case v was decreasing as far as the numerical calculations could be carried out (about 20 times as long as shown in the picture). Note that asymptotically $\sigma_{22} = -\sigma_{11}$ and hence this spacetime does not approach LRS.

Small changes of the initial values are sufficient to make v eventually turn and start growing as shown in Fig. 6.

Because of an apparent singularity in the equations for $v = 1$, we have not been able to follow the evolution beyond this value. For the LRS case it is possible to rewrite the only remaining differential equation to avoid this problem, but in the general case the expressions become quite complex. The equation for $\gamma = 4/3$ in the LRS case is

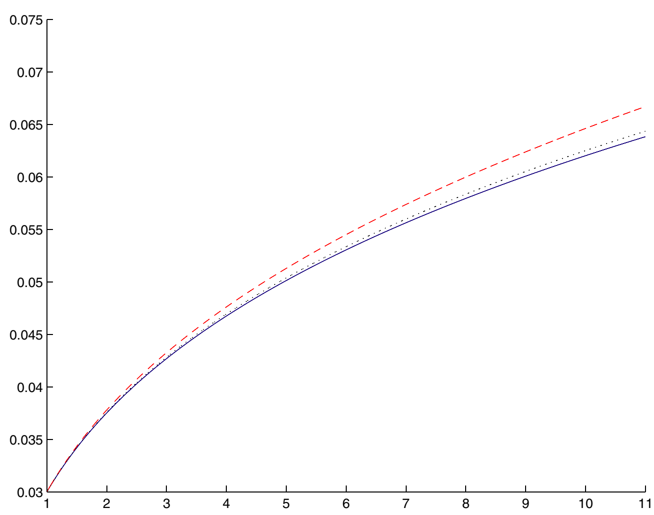

FIG. 1 (color online). Comparison between numerical calculation (solid curve) and perturbations to first (dashed) and second order (dotted) for v in the case of dust and $\Lambda = 0$. Initial values: $\gamma_{131} = 1$, $v = 0.03$, $B = 0.03$, $\sigma_{12} = 0.03$, and $\sigma_{22} = 0.03$.

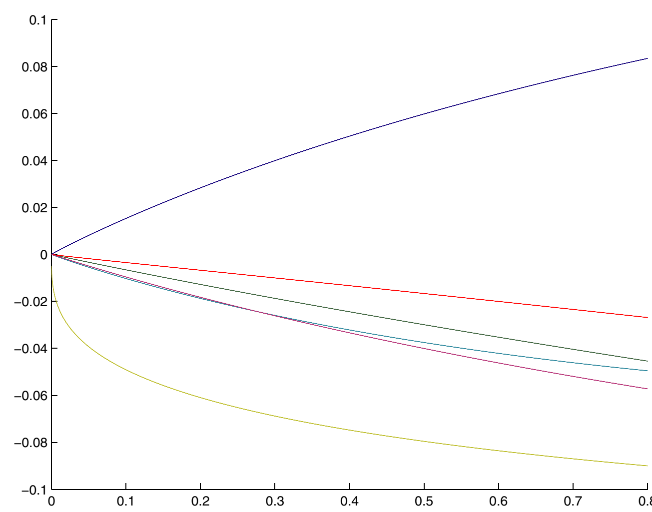

FIG. 3 (color online). $\sigma_{ij}/\theta \longrightarrow 0$ as $\rho \longrightarrow 0$ for dust and $\Lambda = 1$. Initial values: $\rho = 0.8$, $v = -0.09$, $\gamma_{131} = 0.08$, $B = 0.1$, $\sigma_{12} = 0.11$, $\sigma_{22} = 0.12$. From top to bottom (at $\rho = 0.8$): σ_{11}/θ, σ_{13}/θ, σ_{12}/θ, σ_{22}/θ, σ_{23}/θ, and v.

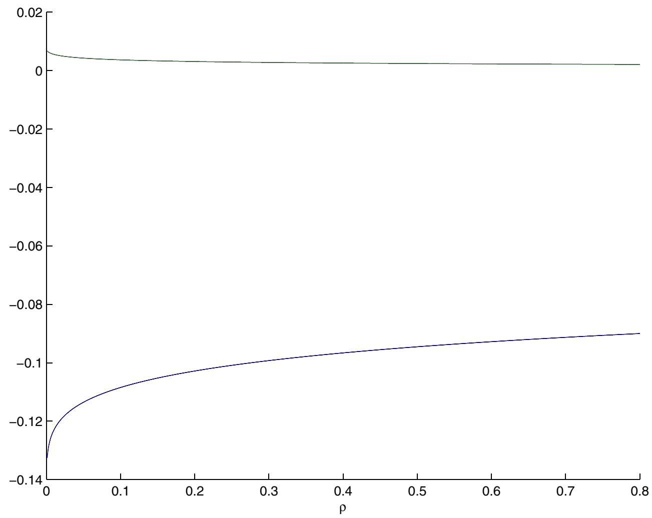

FIG. 4 (color online). ω_{13}/θ (upper curve) and v as $\rho \to 0$ for radiation and $\Lambda = 0$. Initial values: $\rho = 0.8$, $v = -0.09$, $\gamma_{131} = 0.08$, $B = 0.1$, $\sigma_{12} = 0.11$, and $\sigma_{22} = 0.12$.

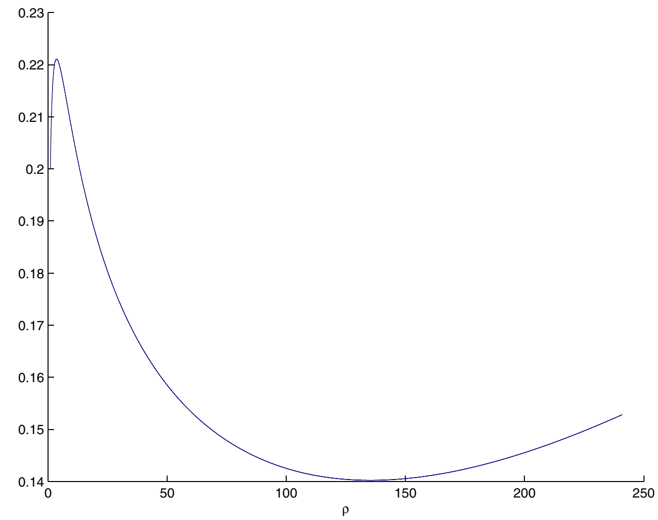

FIG. 6 (color online). v for dust and $\Lambda = 0$ when $\rho \to \infty$. Initial values: $\rho = 1$, $v = 0.2$, $\gamma_{131} = 1$, $B = -0.2$, $\sigma_{12} = 0.01$, and $\sigma_{22} = 0.7$.

given by

$$\frac{\dot{v}}{v^2} = \frac{(3B^2 + 2\rho)[2v(9B^2 + 2v^2\rho) \pm (v^2 - 3)\sqrt{81B^4 + 9B^2v^2\rho + 27B^2\rho + 4v^2\rho^2}]}{2\rho(3B^2(9 - v^2) + 4v^2\rho)(9B^2 + (v^2 + 3)\rho)} \tag{46}$$

where B is given by (30). For the negative root the tilt may grow larger than 1. For this case, however, the kinematic quantities as well as the Weyl tensor diverge for $v = \sqrt{3}$, as also found in [15].

VI. CONCLUSIONS

In this paper it was shown that the general Bianchi V cosmology with linear equation of state and cosmological constant can be reduced to an integrable system of five ordinary first order differential equations for quantities that give a complete local description of the geometry. The full line-element was found in terms of quadratures of these quantities. In general the solutions have expansion, shear, and vorticity. The system was cast in a form suitable for perturbative calculations and the first order perturbations around the open Friedmann model with vorticity, being approximations to exact solutions, were constructed. Perturbative calculations to higher orders would be straightforward up to quadratures.

A numerical study was done and the results agree well with the perturbative ones in the appropriate domains. For large times (small densities) the results agree well with previous works [15,18,19]. Numerically we found that the tilt probably falls off towards zero when the density grows for special initial values, but generically it seems as if the tilt eventually always grows unlimited for large densities. In [18,19] the late time behavior of (among others) Bianchi V solutions was studied using dynamical systems analysis. It would be of great interest to extend this analysis to early times.

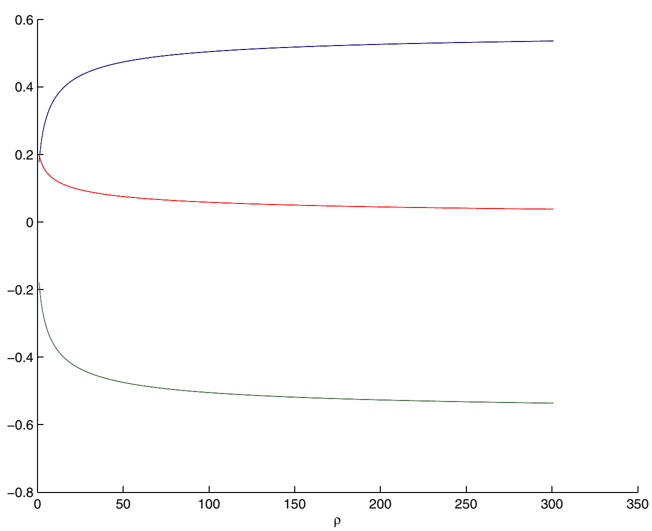

FIG. 5 (color online). v, σ_{11}/θ, and σ_{22}/θ for radiation and $\Lambda = 0$ when $\rho \to \infty$. Initial values: $\rho = 1$, $v = 0.2$, $\gamma_{131} = 1$, $B = -0.2$, $\sigma_{12} = 0$, and $\sigma_{22} = 0.7$. From top to bottom: σ_{11}/θ, v, and σ_{22}/θ.

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[1] K. Rosquist, Phys. Lett. **97A**, 145 (1983).
[2] M. Novello and M. J. Rebouças, Astrophys. J. **225**, 719 (1978).
[3] J. M. Bradley and E. Sviestins, Gen. Relativ. Gravit. **16**, 1119 (1984).
[4] A. R. King and G. F. R. Ellis, Commun. Math. Phys. **31**, 209 (1973).
[5] A. A. Peresetskii, Russ. Math. Notes **21**, 39 (1977).
[6] M. Demianski and L. P. Grishchuk, Commun. Math. Phys. **25**, 233 (1972).
[7] N. Batakis and J. M. Cohen, Phys. Rev. D **12**, 1544 (1975).
[8] S. S. Bayin and F. I. Cooperstock, Phys. Rev. D **22**, 2317 (1980); S. S. Bayin, Phys. Rev. D **32**, 2241 (1985).
[9] J. Bičák, D. Lynden-Bell, and J. Katz, Phys. Rev. D **69**, 064011 (2004).
[10] S. W. Hawking, Mon. Not. R. Astron. Soc. **142**, 129 (1969).
[11] C. B. Collins and S. W. Hawking, Mon. Not. R. Astron. Soc. **162**, 307 (1973).
[12] J. D. Barrow, R. Juszkiewicz, and D. H. Sonoda, Nature (London) **305**, 397 (1983).
[13] I. S. Shikin, Sov. Phys. JETP **41**, 794 (1976).
[14] C. B. Collins, Commun. Math. Phys. **39**, 131 (1974).
[15] C. B. Collins and G. F. R. Ellis, Phys. Rep. **56**, 65 (1979).
[16] T. V. Ruzmaikina and A. A. Ruzmaikin, Sov. Phys JETP **29**, 934 (1969).
[17] C. B. Collins and S. W. Hawking, Astrophys. J. **180**, 317 (1973).
[18] C. G. Hewitt and J. Wainwright, Phys. Rev. D **46**, 4242 (1992).
[19] A. Coley and S. Hervik, Classical Quantum Gravity **22**, 579 (2005).
[20] J. D. Barrow and S. Hervik, Classical Quantum Gravity **20**, 2841 (2003).
[21] C. H. Brans, J. Math. Phys. (N.Y.) **6**, 94 (1965).
[22] A. Karlhede and U. Lindström, Gen. Relativ. Gravit. **15**, 597 (1983).
[23] M. Bradley and A. Karlhede, Classical Quantum Gravity **7**, 449 (1990).
[24] M. Bradley and M. Marklund, Classical Quantum Gravity **13**, 3021 (1996).
[25] E. Cartan, *Leçons sur la Geometrie des Espaces de Riemann* (Gauthier-Villars, Paris, 1946).
[26] A. Karlhede, Gen. Relativ. Gravit. **12**, 693 (1980).
[27] D. L. Farnsworth, J. Math. Phys. (N.Y.) **8**, 2315 (1967).
[28] M. Marklund and M. Bradley, Classical Quantum Gravity **16**, 1577 (1999).
[29] M. Marklund, Classical Quantum Gravity **14**, 1267 (1997).
[30] A. Karlhede and M. A. H. MacCallum, Gen. Relativ. Gravit. **14**, 673 (1982).
[31] G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. **12**, 108 (1969).
[32] M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, NJ, 1975).
[33] K. Rosquist and R. T. Jantzen, Classical Quantum Gravity **2**, L129 (1985).