Regularization ambiguities in loop quantum gravity

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One of the main achievements of loop quantum gravity is the consistent quantization of the analog of the Wheeler-DeWitt equation which is free of ultraviolet divergences. However, ambiguities associated to the intermediate regularization procedure lead to an apparently infinite set of possible theories. The absence of an UV problem—the existence of well-behaved regularization of the constraints—is intimately linked with the ambiguities arising in the quantum theory. Among these ambiguities is the one associated to the $SU(2)$ unitary representation used in the diffeomorphism covariant "point-splitting" regularization of the nonlinear functionals of the connection. This ambiguity is labeled by a half-integer m and, here, it is referred to as the *m ambiguity*. The aim of this paper is to investigate the important implications of this ambiguity. We first study $2+1$ gravity (and more generally BF theory) quantized in the canonical formulation of loop quantum gravity. Only when the regularization of the quantum constraints is performed in terms of the fundamental representation of the gauge group does one obtain the usual topological quantum field theory as a result. In all other cases unphysical local degrees of freedom arise at the level of the regulated theory that conspire against the existence of the continuum limit. This shows that there is a clear-cut choice in the quantization of the constraints in $2+1$ loop quantum gravity. We then analyze the effects of the ambiguity in $3+1$ gravity exhibiting the existence of spurious solutions for higher representation quantizations of the Hamiltonian constraint. Although the analysis is not complete in $3+1$ dimensions—due to the difficulties associated to the definition of the physical inner product—it provides evidence supporting the definitions quantum dynamics of loop quantum gravity in terms of the fundamental representation of the gauge group as the only consistent possibilities. If the gauge group is $SO(3)$ we find physical solutions associated to spin-two local excitations.

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I. INTRODUCTION

The discovery of connection variables for general relativity led to the definition of a new approach for the nonperturbative quantization of gravity known as loop quantum gravity (LQG) [1–3]. The introduction of $SU(2)$ connection variables for classical canonical general relativity [4,5], and the corresponding use of Wilson loop variables in the quantum theory [6,7], allowed the resolution of the long-standing technical problems that had stopped the development of the *quantum geometrodynamics* of Dirac, Wheeler, and DeWitt, among others [8]. Among these new achievements are the rigorous definition of the kinematical Hilbert space of quantum gravity, the rigorous quantization of geometric operators such as area and volume (with the associated prediction of discreteness of quantum geometry), and the quantization of the highly nonlinear Hamiltonian constraint—analog of the Wheeler-DeWitt equation—governing the dynamics of quantum gravity. The latter is an important technical achievement of the approach where background independence and diffeomorphism invariance play a central role in the elimination of the UV divergences that plague standard quantum field theories.

Polymerlike excitations known as *spin-network states* form a basis of the kinematical Hilbert space $\mathcal{H}_{\rm kin}$. Quantum Einstein's equations are given by the quantum counterpart of the classical constraints of canonical general relativity. A subset of the constraints—characterized by the vector and Gauss constraints—requires the physical states of quantum gravity to be $SU(2)$ gauge invariant and space diffeomorphism invariant.[1] Since the action of the $SU(2)$ gauge group and space diffeomorphism can be unitarily represented in the kinematical Hilbert space, it is easy to characterize the set of invariant states and hence the solutions of this subset of quantum constraint equations by group averaging. Gauge invariant states are given by equivalence classes of spin-network states under diffeomorphisms, i.e., two polymerlike excitations are regarded

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[1] In the Dirac program one starts by defining the so-called kinematical Hilbert space $\mathcal{H}_{\rm kin}$. One proceeds by representing the set of classical constraints—here simply denoted by $C \approx 0$—as quantum operators in $\mathcal{H}_{\rm kin}$. In the classical theory the constraints generate through the Poisson bracket infinitesimal gauge transformations; therefore, in the quantum theory $\hat{C}$ become the generators of gauge transformations. The Hilbert space of solutions of the constraint equations $\hat{C}\Psi = 0$ is hence given by the gauge invariant states and is called the *physical Hilbert space*, denoted $\mathcal{H}_{\rm phys}$.

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as the same if they can be deformed into each other by the action of a diffeomorphism.

Dynamics is governed by the so-called Hamiltonian constraint, whose classical form is

$$H(E^a_j, A^i_b) = \frac{E^a_i E^b_j}{\sqrt{\det(E)}} F^k_{ab}(A)\epsilon^{ij}_k \approx 0, \quad (1)$$

where A^i_a is an $SU(2)$ connection, $F^k_{ab}(A)$ is its curvature tensor, E^a_i is its conjugate momentum with the geometric interpretation of a (densitized) triad field, and we have considered the constraint in the Riemannian theory (this simplifying assumption will be made throughout this article). In the quantum theory the Hamiltonian constraint must be promoted to a quantum operator whose kernel defines the so-called physical Hilbert space $\mathcal{H}_{\rm phys}$ of quantum gravity. The quantization of the Hamiltonian constraint was introduced by Thiemann in [9,10]. Shortly thereafter it was pointed out [11] that in addition to (potential) factor order ambiguities, Thiemann's prescription had an intrinsic ambiguity labeled by a half-integer $m \in \mathbb{Z}/2$ associated to the $SU(2)$ unitary representation used to regularize the curvature tensor $F^k_{ab}(A)$ appearing in the classical expression of the Hamiltonian constraint. In this paper we refer to this problem as the *m ambiguity.*

For every $m \in \mathbb{Z}/2$ one obtains a different quantum Hamiltonian constraint $\hat{H}_m$. As argued below, linear combinations of different regularizations are also good regularizations; therefore, one obtains an infinite-dimensional set of possibly different theories. In this respect one viewpoint is that the understanding of the dynamics in each theory would allow the pinpointing of the correct one by confronting its prediction with observations. For instance, the analog of the m ambiguity appears also in the coupling of quantum gravity with matter. This ambiguity is known to lead to important physical consequences in the context of cosmological models inspired by loop quantum gravity known as loop quantum cosmology [12], in particular in the evolution of the universe near the classical big bang singularity [13]. These effects are potentially observable so that comparison with observations is expected to put constraints on the set of viable theories. Although, this viewpoint might be argued in the phenomenological framework of loop quantum cosmology it is not tenable for a fundamental theory as we will discuss in what follows.

The existence of the m ambiguity is intimately related to the mechanism leading to the absence of UV problems in loop quantum gravity. More precisely, in order to regularize quantum operators corresponding to nonlinear functionals of the fundamental fields (e.g. the Hamiltonian constraint) one uses a diffeomorphism covariant prescription of "point splitting" consisting of replacing the connection by holonomies along infinitesimal paths. The origin of the ambiguity resides in the choice of the $SU(2)$ representation in which these holonomies are taken. Because of diffeomorphism invariance it turns out that the regulator can be removed without ever encountering UV divergences. In this way one ends up with a well-defined quantum Hamiltonian constraint, but only at the price of having an infinite number of consistent but (in principle) different quantum theories.

The situation is reminiscent of the problem of renormalization in standard background dependent quantum field theories. There, in order to make sense of products of operator valued distributions (representing interactions) one has to provide a regularization prescription (e.g. an UV cutoff, dimensional regularization, point splitting, etc.). Removing the regulator is a subtle task involving the tuning of certain terms in the Lagrangian (counterterms) that ensure finite results when the regulator is removed. In fact by taking special care in the mathematical definition of the "products of distributions at the same point" one can provide a definition of the quantum theory which is completely free of UV divergences [14] (see also [15–17]). However, any of these regularization procedures is intrinsically ambiguous. The dimension of the parameter space of ambiguities depends on the structure of the theory. The right theory must be fixed by comparing predictions with observations (by the so-called renormalization conditions). According to this, in loop quantum gravity one has only achieved the first step: a rigorous regularization provided by the mathematical framework of the theory. It remains to settle the crucial issue of how to fix the associated ambiguities.

According to the previous discussion, ambiguities associated to the UV regularization allow for the classification of theories in two important types: *renormalizable* and *nonrenormalizable* quantum field theories. In a renormalizable theory such as QED there are finitely many ambiguities which can be fixed by a finite number of renormalization conditions, i.e., one selects the suitable theory by appropriate tuning of the ambiguity parameters in order to match observations. In a nonrenormalizable theory (e.g. perturbative quantum gravity) the situation is similar except for the fact that there are infinitely many parameters to be fixed by renormalization conditions. As the latter must be specified by observations, a nonrenormalizable theory has little predictive power.

Removing UV divergences by a regularization procedure is intimately related to the appearance of ambiguities in the quantum theory. Although this can happen in different ways, in particular, formulations, this problem is intrinsic to the formalism of quantum field theory (QFT). In this respect, it is illustrative to analyze the nonperturbative treatment of gauge theories in the context of lattice gauge theory (where the true theory is studied by means of a regulated theory defined on a space-time discretization or lattice). It is well known that here too the regulating procedure leads to ambiguities; the relevance of the example resides in the fact that these ambiguities resemble in nature those appearing in loop quantum gravity. More precisely, let us take for concreteness $SU(2)$ Yang-Mills theory which can be analyzed nonperturbatively using the standard (lattice) Wilson action

$$S_{\rm LYM} = \frac{1}{g^2}\sum_p \left(1 - \frac{1}{4}\,{\rm Tr}[U_p + U_p^\dagger]\right). \tag{2}$$

In the previous equation $U_p \in SU(2)$ is the holonomy around plaquettes p, and the sum is over all plaquettes of a regulating (hypercubic) lattice. It is easy to check that the previous action approximates the Yang-Mills action when the lattice is shrunk to zero for a fixed smooth field configuration. This property is referred to as the *naive continuum limit*. Moreover, the quantum theory associated to the previous action is free of any UV problem due to the UV cutoff provided by the underlying lattice.

Is this procedure unique? As it is well known the answer is no. Among the many ambiguities let us mention the one that, as it will become clear later, is the closest in spirit to the m ambiguity in loop quantum gravity. More precisely one can regulate Yang-Mills theory equally well using the following action instead of (2):

$$S^{(m)}_{\rm LYM} \propto \frac{1}{g^2}\sum_p \left(1 - \frac{1}{2(2m+1)}\,{\rm Tr}^{(m)}[\Pi^{(m)}(U_p) + \Pi^{(m)}(U_p^\dagger)]\right), \tag{3}$$

where $\Pi^{(m)}(U_p)$ denotes to the $SU(2)$ unitary irreducible representation matrix (of spin m) evaluated on the plaquette holonomy U_p. Or more generally one can consider suitable linear combinations

$$S_{\rm LYM} = \sum_m a_m S^{(m)}_{\rm LYM}. \tag{4}$$

From the viewpoint of the classical continuum theory all these actions are equally good as they all satisfy the naive continuum limit. Do these theories approximate in a suitable sense the continuum quantum field theory as well and are these ambiguities unimportant in describing the physics of quantum Yang-Mills theory? The answer to both of these questions is yes and the crucial property that leads to this is the renormalizability of Yang-Mills theory. Different choices of actions lead indeed to different discrete theories. However, in the low energy effective action the differences appear only in local operators of dimension five or higher. A simple dimensional argument shows that in the continuum limit (i.e. when the regulating lattice dependence is removed by shrinking it to zero) all the above theories lead to the same predictions in the sense that can safely ignore nonrenormalizable contributions. Therefore, the ambiguities at the "microscopic level" do not have any effect at low energies where we recover quantum Yang-Mills theory.

The situation in LQG looks at first quite similar. In order to quantize the Hamiltonian constraint one also needs to make mathematical sense of the highly nonlinear (not even polynomial) form of the Hamiltonian constraint (1). The Hamiltonian constraint is quantized by means of a regularization procedure that, due to the manifest background independence of the approach, does not lead to any UV divergencies when removed (no hidden infinities are ever encountered). However, as in standard QFT ambiguities arise as a consequence of the regularization. Here we are concerned with what we have called the m ambiguity which appears when nonlinear functions of the connection A are replaced by holonomies in the regularization of the Hamiltonian constraint (1). The m ambiguity is associated (in analogy to the previous example in the context of lattice gauge theory) with the $SU(2)$ representation chosen in the regularization. As a consequence one obtains an m worth ($m \in \mathbb{Z}/2$) of (smeared)[2] quantum Hamiltonians, $\hat{H}_m[N]$, that are consistent in the sense of Thiemann. More generally any linear combination

$$\hat{H}[N] = \sum_m a_m \hat{H}_m[N] \quad \text{with} \sum_m a_m = 1 \tag{5}$$

is also a consistent quantization. The nature of this ambiguity is very similar to the example considered in the context of lattice gauge theory above but the naive implications seem rather dangerous in the case of gravity.

If one would argue in analogy to the lattice gauge theory case one immediately runs into trouble because of the nonrenormalizability of gravity. Indeed for gravity the nontrivial information about the quantum theory is encoded in the dimension five and higher local operators in the effective action (i.e. the infamous higher curvature quantum corrections to the Einstein-Hilbert action). Consequently, and according to our previous argument, these are precisely the terms that would be affected by the ambiguities of the microscopic theory, and one would need to perform an infinite set of independent measurements in order to fix the ambiguities of the fundamental theory. Such a scenario would place the nonperturbative approach of LQG at the same footing as the standard perturbative approach in the sense of predictive power.

However, one should doubt the validity of the previous argument on the basis that it is constructed from a notion of "continuum limit" which is only applicable to background dependent theories. For example in lattice gauge theories it is relatively easy to define the notion of a continuum limit by simply studying the dependence of the observables of the theory as a function of the lattice constant. Because of background independence there is no analog of the lattice constant in loop quantum gravity. Geometry is dynamical

[2] The smeared Hamiltonian constraint is defined as

$$H[N] = \int \frac{N(x)E^a_i(x)E^b_j(x)}{\sqrt{\det(E(x))}} F^k_{ab}(A(x))\epsilon^{ij}_k d^3x,$$

and $N(x)$ is a scalar test function called the lapse.

and the only scale entering the theory is the fixed Planck length that modulates the spectrum of geometric operators. Because of both technical and conceptual difficulties associated to the definition of the continuum in LQG, an explicit treatment of the question of the effects of the ambiguities at low energies is not possible at this stage. There are indeed indications that the low energy limit in a background independent theory is very different from what one would naively hope from the experience in standard QFT [18]. However, even though it may be wrong to use the heuristics of standard QFT, this perspective poses a genuine question that requires an answer. The goal of this paper is to shed some light onto this important issue.

It is interesting to notice that in the simplified context of loop quantum cosmology one can study the effects of the m ambiguity, and arrive at conclusions that are in agreement with the previous motivation. Even though, in this framework, one deals with finitely many degrees of freedom, the ambiguities of the full theory are inherited by the model due to the particular way in which the model is derived from the full theory. In this simplified setting one can compute quantum corrections to the classical theory in the sense of an effective theory. These appear in fact as higher curvature corrections to the Hamiltonian constraint [19]. The precise form of these corrections depends indeed on the value of the parameter m [20]. As in the previous case one should interpret these results with due care. In particular loop quantum cosmology is not a fundamental description of quantum gravity, and it is not even diffeomorphism invariant. Nevertheless, it provides a new perspective to arrive at the key question that motivates this work.

Finally, it is also possible that some set of the ambiguities found in the quantization of the Hamiltonian constraint are of no physical relevance due to consistency conditions that can already be found by studying in more detail the dynamics of the theory. If that is the case then there is a chance that we can shed some light on the issue before completely resolving the problem of the low energy limit of LQG. This is in fact the avenue that will be explored in this work.

These considerations confront loop quantum gravity with two obvious alternatives:

(i) From the infinite-dimensional set of quantum Hamiltonians (5) only a finite dimensional subset leads to mathematically consistent and physically different theories.

(ii) The infinite-dimensional set of quantum Hamiltonians (5) leads to an infinite-dimensional space of mathematically consistent and physically different theories.

The possibility (i) is desirable while possibility (ii) is equivalent to the status of perturbative quantum gravity in the sense of predictive power. Despite its central role in understanding the theory of LQG, this question has been only marginally posed [21]. We will explicitly show that in the case of $2 + 1$ gravity the first possibility holds. In fact there is an infinite-dimensional set of mathematically consistent regularizations of the constraints of $2 + 1$ gravity, yet they all lead to the same physical theory. In this case we are free to choose the simplest one which corresponds to using the fundamental representation in the regularization of the curvature constraint.

The fact that $2 + 1$ gravity is a topological quantum field theory might indicate that there cannot be any UV related renormalization problems because the theory has no local degrees of freedom. However, we must emphasize that before implementing the curvature constraint the kinematical Hilbert space of the theory corresponds to the kinematical Hilbert space of LQG, and thus that of a full fleshed field theory. Hence, ambiguities arise in the definition of the dynamics in a way that mimics the four dimensional case. If these ambiguities should also disappear in $3 + 1$ gravity we might expect to learn something about the underlying mechanism by studying how it happens in $2 + 1$ dimensions.

Indeed, using the insight from the $2 + 1$ theory we provide evidence to support a similar conclusion in $3 + 1$ quantum gravity. Our result in $3 + 1$ gravity is weaker due to the present lack of a suitably defined notion of the physical inner product. Our analysis will be performed in the Riemannian theory in the framework of Thiemann's quantization; however, the general ideas presented here are expected to be relevant for other prescriptions for the definition of the quantum dynamics—such as the *master constraint program* [22] or that of *consistent discretizations* [23–26]—where the same regularization ambiguity arises. This work is meant to provide a direction that could lead to a possible resolution of the ambiguity issue. A stronger result (as the one in $2 + 1$ gravity) would require explicit knowledge of the (yet not available) notion of physical probability derived from the theory.

Are there other ambiguities? Perhaps the most obvious ambiguity in the quantization of the classical expression (1) concerns the ordering of the densitized triad fields and the connection. However, background independence and consistency with the (recently shown to be unique [27]) kinematical structure of loop quantum gravity appears to drastically reduce factor ordering ambiguities. The only known mechanism for the quantization of the nonlinear E-dependent part of the Hamiltonian constraint is due to Thiemann and based on the observation that one can write (1) as

$$H \propto \epsilon^{abc}\epsilon_{ijk}e^i_a F^{jk}_{bc}$$
$$\text{with } e^i_a(x) = \left\{A^i_a(x), \int dy^3\sqrt{|\det(E)|}\right\}. \qquad (6)$$

The previous expression is used in the quantization where the Poisson bracket is promoted to a commutator and the integral corresponds to the well-understood quantum volume operator (there are in fact two proposed versions of the latter [28,29]; however, resent results indicate that only one appears to be consistent [30,31]). Because Thiemann's prescription requires the E dependent part of the Hamiltonian to be treated in this way we end up with only two factor ordering possibilities: $\hat{e}^i_a$ on the left or on the right of $\hat{F}(A)$. The first of the previous possibilities does not lead to a well-defined operator [32], technically it is ruled out by cylindrical consistency [33]. So we conclude that factor ordering seems not to be a source of infinitely many ambiguities. Another ambiguity noticed in the literature is associated to, what we can call, the "combinatorial" possibilities in the regularization of the curvature part of (1). As we mentioned above one regularizes the connection dependence in the Hamiltonian by using holonomies. There at least two natural choices: one where the action of the Hamiltonian constraint on a spin network creates new nodes, and the other where only the valence of nodes is altered by the action of the constraint but no new nodes are created. A new manifold of ambiguities appears if one considers the coupling of gravity to matter [34,35]. The effects of these ambiguities will not be studied here. We concentrate on the m ambiguity which gives rise to infinitely many *a priori* consistent theories and is most clearly related to the regularization procedure.

II. THE m AMBIGUITY IN QUANTUM CANONICAL 2 + 1 RIEMANNIAN GRAVITY

A complete account of the canonical quantization of 2 + 1 gravity using LQG techniques is provided in [36]; we will follow the notation therein. If one starts from the kinematical Hilbert space $\mathcal{H}_{\rm kin}$ spanned by spin-network states the only remaining constraint in 2 + 1 gravity is the quantum curvature constraint

$$\hat{F}(A)|\psi\rangle = 0.$$

The physical inner product and the physical Hilbert space, $\mathcal{H}_{\rm phys}$, of 2 + 1 gravity can be defined by introducing a regularization of the formal expression defining the generalized projection operator into the kernel of F, namely,

$$P = \text{``}\prod_{x\in\Sigma}\delta(\hat{F}(A))\text{''} = \int D[N]\exp\Big(i\int_\Sigma \mathrm{Tr}[N\hat{F}(A)]\Big), \tag{7}$$

where $N\in su(2)$, and Σ denotes the 2-dimensional Riemann surface representing space. In [36] it is shown how the previous object can be given a precise definition leading to a rigorous expression for the physical inner product of the theory. However, in order to give a precise meaning to the previous formal expression it is necessary to introduce a regularization as an intermediate step for the quantization due to the nonlinear dependence of the constraint on the fundamental variables. In this section we observe that the analog of the m ambiguity in 3 + 1 gravity appears when the regulator is introduced. Therefore we first generalize the construction of [36] to this case.

In order to motivate the regularization consider a local patch $U\subset\Sigma$ where we choose the cellular decomposition to be square with cells of coordinate length ϵ. In that patch, the integral in the exponential in (7) can be written as a Riemann sum

$$F[N] = \int_U \mathrm{Tr}[NF(A)] = \lim_{\epsilon\to 0}\sum_p \epsilon^2\,\mathrm{Tr}[N_p F_p], \tag{8}$$

where p labels plaquettes, $N_p\in su(2)$, and $F_p\in su(2)$ are values of $N^i\tau_i$, $\tau_i\epsilon^{ab}F^i_{ab}[A]$ at some interior point of the plaquette p, and τ_i are the generators of $su(2)$. The tensor ϵ^{ab} is the 2-dimensional Levi-Cività tensor. The quantity $F[N]$ corresponds to the smeared curvature constraint.

The basic observation is that given the holonomy $U_p\in SU(2)$ around the plaquette p and a unitary irreducible representation of $SU(2)$, $\Pi^{(m)}$, one can write

$$\Pi^{(m)}[U_p] = 1^{(m)} + \epsilon^2 F^i_p(A)\tau^{(m)}_i + \mathcal{O}(\epsilon^2),$$

where $1^{(m)}$ is the identity in the representation m and $\tau^{(m)}_i$ is the ith generator in the corresponding representation, which implies

$$F[N] = \int_U \mathrm{Tr}[NF(A)] = \lim_{\epsilon\to 0}\sum_p \frac{\mathrm{Tr}^{(m)}[N_p\Pi^{(m)}[U_p]]}{C^{(m)}}, \tag{9}$$

where the $\mathrm{Tr}^{(m)}$ in the right-hand side (rhs) is taken in the representation m, $N_p = N^k_p\tau^{(m)}_k$ and $C^{(m)} = \mathrm{Tr}^{(m)}[\tau^{(m)}_3\tau^{(m)}_3]$. Notice that the explicit dependence on the regulator ϵ has dropped out of the sum on the rhs, a sign that we should be able to remove the regulator upon quantization. The rhs can be easily promoted to a sum of self-adjoint operators acting in the kinematical Hilbert space, so the previous prescription provides a half-integer worth of quantizations of $F[N]$ in the sense of Gaul-Rovelli [11] (the operator $\hat{\Pi}^{(m)}[U_p]$ acts simply by multiplication in $\mathcal{H}_{\rm kin}$ [27]). The use of holonomies in the quantization of $F[N]$ (which is the natural point-split-like regularization adapted to the kinematical structure of the theory) is responsible for the occurrence of the m ambiguity.

Following [36] one introduces $P^{(m)}_\epsilon$—a regularization of the generalized projection operator in terms of the representation m—in terms of a definition of its matrix elements between elements of the spin-network basis denoted $\{|s\rangle\}$, namely

$$\begin{aligned}\langle P^{(m)}s, s'\rangle &= \lim_{\epsilon\to 0}\Big\langle\prod_p\int dN_p \exp\Big(i\frac{\mathrm{Tr}^{(m)}[N_p\hat{\Pi}^{(m)}[U_p]]}{C^{(m)}}\Big)s, s'\Big\rangle\\ &= \lim_{\epsilon\to 0}\Big\langle\prod_p d^{(m)}(U_p)s, s'\Big\rangle,\end{aligned} \tag{10}$$

where in the last equation we have introduced the distribution $d^{(m)}(U)$ that we formally write as

$$\begin{aligned} d^{(m)}(U) &= \int dN \exp\left(i\frac{\mathrm{Tr}^{(m)}[N\hat{\Pi}^{(m)}[U]]}{C^{(m)}}\right) \\ &= C^{(m)3}\delta(\mathrm{Tr}^{(m)}[\tau_1^{(m)}\Pi^{(m)}[U]]) \\ &\quad\times \delta(\mathrm{Tr}^{(m)}[\tau_2^{(m)}\Pi^{(m)}[U]]) \\ &\quad\times \delta(\mathrm{Tr}^{(m)}[\tau_3^{(m)}\Pi^{(m)}[U]]). \end{aligned} \tag{11}$$

It is easy to check that $d^{(1/2)}(U) = \delta(U)$, i.e., the delta distribution on $SO(3)$ directly from the N integration. Therefore, $d^{(1/2)}(U)$ projects into the identity in the sense that $\int dU f(U) d^{(1/2)}(U) = f(1)$. However, for $m \neq 1/2$ the group averaging is more subtle and the rhs of the previous equation is not well defined as a distribution. We will give a precise definition of $d^{(m)}(U)$ for $m \neq 1/2$ below. The properties of $d^{(m)}(U)$—as shown in [36]—completely determine the physical scalar product of the theory. In fact the above property of $d^{(1/2)}(U)$ implies that $P^{(1/2)}$ defines a projection operator into flat-connection configurations and therefore yields a physical Hilbert space corresponding to finitely many topological degrees of freedom.

Before studying the case $m \geq 1$ we will illustrate the main idea in a simpler case: three dimensional BF theory with internal gauge group $G = U(1)$. This example illustrates the main idea that will be applied in the rest of the paper. The analog of Eq. (11) is given by the expression[3]

$$d^{(m)}(\phi) = \int dN \exp\left(iN\left[\frac{\sin(m\phi)}{m}\right]\right) = \delta\left[\frac{\sin(m\phi)}{m}\right], \tag{12}$$

which we can expand in terms of $U(1)$ unitary irreducible representations as

$$d^{(m)}(\phi) = \sum_k c_k^{(m)} e^{ik\phi}, \tag{13}$$

where

$$c_k^{(m)} = \frac{1}{2\pi}\sum_{\alpha=1}^{2m-1} e^{-ik\phi_\alpha}, \tag{14}$$

where $\phi_\alpha = \alpha\delta$ with $\delta = \pi/m$ are the roots of the argument of the delta function above. These roots are the solutions to the regularized constraint $F = \sin(m\phi)/m$. We see that as a consequence of our regularization the constraint admits extra solutions in addition to the flat one $\phi = 0$. The sum corresponds to a geometric sum, namely

$$\begin{aligned} c_k^{(m)} &= \frac{1}{2\pi}\sum_{\alpha=1}^{2m-1}[e^{-ik\delta}]^\alpha \\ &= \begin{cases} m/\pi & \forall\ k = 2pm, \quad p \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{15}$$

If we proceed as in [36] we would find that unless $m = 1$ (i.e. the fundamental representation) we would obtain a theory with infinitely many degrees of freedom. This is because the vanishing of infinitely many Fourier components of $d^{(m)}(\phi)$ for $m \neq 1/2$ implies a reduction of the space of zero-norm states with respect to the $m = 1$ quantization. Hence, the physical Hilbert space becomes larger. The argument presented here is rather formal. This is because the $U(1)$ case presents some extra subtleties at the time of defining the physical inner product which are not present in the non-Abelian case which will be treated in more detail in the following section.[4] The choice of $m > 1$ introduces *spurious* solutions to the regularized constraints.

Now we essentially repeat the previous derivation for $SU(2)$, but we go further removing the regulator and constructing in this way the physical inner product. We shall see that the spurious solutions appearing in the previous example are also present in $2 + 1$ gravity for certain bad regularizations. We will show that for these choices the regulator cannot indeed be removed and such regularizations must be ruled out as inconsistent. This will lead to a unique theory in the case of three dimensional gravity.

Let us analyze $d^{(m)}(U)$ defined in Eq. (11) in more detail. The simplest way is to use the isomorphism between $SU(2)$ and S^3. Any element $U \in SU(2)$ can be written as

$$U = x^\mu \tau_\mu, \quad \text{where } x^\mu x^\nu \delta_{\mu\nu} = 1, \tag{16}$$

$\mu, \nu = 1, \ldots, 4$, and $\tau_0 = 1$ and $\tau_\alpha = i\sigma_\alpha$ for σ_α the Pauli matrices for $\alpha = 1, 2, 3$. In terms of this parametrization of $SU(2)$ one can write the unitary irreducible representations of spin m as

$$\Pi^{(m)}[U]^{A_1\cdots A_{2m}}_{B_1\cdots B_{2m}} = x^{\mu_1}\cdots x^{\mu_{2m}}\tau^{(A_1}_{\mu_1 B_1}\cdots\tau^{A_{2m})}_{\mu_{2m}B_{2m}}, \tag{17}$$

from where it follows that

$$\mathrm{Tr}^{(m)}[\tau_i^{(m)}\Pi^{(m)}[U]] = x^i Q^{(2m-1)}(x^\mu), \tag{18}$$

where $Q^{(2m-1)}(x^\mu)$ is a polynomial of degree $2m - 1$. The fact that $d^{(m)}(U) = d^{(m)}(gUg^{-1})$ implies that $Q^{(2m-1)}(x^\mu)$ is rotational invariant as a function of $\vec{x}$ or, equivalently,

[3]The analogy is self-evident observing that for $SU(2)$ one has $\mathrm{Tr}^{(m)}[N\hat{\Pi}^{(m)}[U]] = -\mathrm{Tr}^{(m)}[N\hat{\Pi}^{(m)}[U^{-1}]]$.

[4]When the fundamental representation is used in the regularization, the vacuum-to-vacuum physical transition amplitude if the theory is defined on $M = \Sigma \times \mathbb{R}$ with Σ given by a Riemann surface of genus g is given by $\langle P, 1\rangle = \sum_k \Delta_k^{2-2g}$—where Δ_k is the dimension of the irreducible unitary representation k—which is convergent for the $SU(2)$ case and $g > 1$ but always ill defined for $U(1)$. This is an example of the kind of technical difficulties we would encounter if we would like to completely analyze the $U(1)$ case.

only dependent on x^0 [using (16)]. Therefore, only for $m = 1/2$ the only solution to the three traces in (11) equal to zero is $\vec{x} = 0$ which implies $U = 1$. However, for $m \geq 1$ in addition to $\vec{x} = 0$ we have the roots of $Q^{(2m-1)}(x^\mu) = 0$. For example for $m = 1$ one has

$$Q^{(1)}(x^\mu) = 2x^0 = 2\sqrt{1 - \vec{x} \cdot \vec{x}}. \tag{19}$$

In this case Eq. (18) vanishes for the point $\vec{x} = 0$ and the 2-sphere $\vec{x} \cdot \vec{x} = 1$. The fact that the configurations that solve the constraint is the union of submanifolds of $SU(2)$ with different dimensions (the point $x^0 = 1$ and the sphere $|\vec{x}| = 1$) implies that (11) is ill defined as a distribution. In order to carry on one has to introduce a regularization having in mind that $d^{(m)}[U]$ must project onto the identity $x^0 = 1$ and each of the 2-spheres of S^3 that are solutions of $Q^{(2m-1)}(x^\mu) = 0$. The regularization procedure is ambiguous. The ambiguity can be parametrized by two parameters, namely

$$d^{(m)}(U[x^\mu]) = \lambda_1 \prod_{i=1}^{3} \delta(x^i) + \lambda_2 \delta(Q^{(2m-1)}(x^\mu)). \tag{20}$$

Notice that if we would choose $\lambda_2 = 0$ we would immediately reproduce the standard quantization based on the fundamental representation. Since our aim is to explore the possibility of constructing a theory which is both well defined but different from the one obtained for $m = 1/2$ we proceed by assuming that $\lambda_2 \neq 0$.

As in Ref. [36] it will be convenient to expand the distribution $d^{(m)}(U)$ in terms of unitary irreducible representations. This allows us to write the plaquette contributions (10) in terms of sums over Wilson loops which can be easily represented by self-adjoint operators in $\mathcal{H}_{\text{kin}}$. More precisely we want

$$d^{(m)}(U) = \sum_j c_j^{(m)} \chi_j[U], \tag{21}$$

where $\chi_j(U)$ is the character or trace of the j-representation matrix of $U \in SU(2)$ and the coefficients $c_j^{(m)}$ are given by the Peter-Weyl theorem, namely

$$\begin{aligned} c_j^{(m)} &= \int dU d^{(m)}(U) \chi_j[U^{-1}] \\ &= \frac{1}{\pi^2} \int dx^\mu \delta(x^\mu x^\nu \delta_{\mu\nu} - 1) \\ &\quad \times d^{(m)}(U[x^\mu]) \chi_j[U^{-1}[x^\mu]], \end{aligned} \tag{22}$$

where the integration is performed with the Haar measure of $SU(2)$ that, in the coordinates we are using, takes the simple form

$$d\mu_H = \pi^{-2} dx^\mu \delta(x^\mu x^\nu \delta_{\mu\nu} - 1).$$

For $m = 1/2$ we obtain the familiar result $c_j^{(1/2)} = 2j + 1$ for $j \in \mathbb{Z}$ and zero otherwise, i.e. $d^{1/2}[U]$ is the $SO(3)$ delta distribution.[5] For $m = 1$, $Q^{(1)}(x^\mu) = 2x_0$; therefore using (20) and (22) we obtain

$$c_j^{(1)} = \lambda_1 (2j + 1) + \lambda_2 \chi_j[U_0], \tag{23}$$

where U_0 is in the conjugacy class of the element labeled by coordinates $x^0 = 0$, $x^1 = x^2 = 0$, $x^3 = 1$. Since in the expression of the physical inner product we can absorb an overall factor, we will define

$$\begin{aligned} c_j^{(1)} &= (2j + 1) + \lambda \chi_j[U_0] \\ &= (2j + 1)\left[1 + \lambda \frac{\sin[(2j + 1)\frac{\pi}{2}]}{(2j + 1)}\right], \end{aligned} \tag{24}$$

where λ parametrizes the remaining ambiguity.

The previous equation allows us to write the distribution $d^{(1)}[U]$ as a sum of holonomy operators in the corresponding irreducible representations. We can represent the regulated projector as a sum of the product of such fundamental Wilson loops based on the plaquettes of the regulating lattice. In order to complete the definition of the theory we must take the limit $\epsilon \to 0$ in the definition of the physical inner product. This amounts to shrinking to zero the cellular decomposition of Σ used as a regulator of P. To make our point it will be sufficient to consider the vacuum-to-vacuum transition amplitude defined by Eq. (10) when the states $|s\rangle = |s'\rangle = |1\rangle \in \mathcal{H}_{\text{kin}}$.

In the case $m = 1/2$ the limit $\epsilon \to 0$ is straightforward because the integration of the connection on the boundary of neighboring plaquettes is, in that case, simply equivalent to a fusion of plaquettes with no change in the amplitude (see Fig. 1 with $c_j^{(1/2)} = 2j + 1$). In this sense we have a trivial scaling or renormalization of the amplitudes for $m = 1/2$ so that the continuum limit produces a topological quantum field theory (for details see [36]). In fact the vacuum-to-vacuum amplitude is

$$\langle P^{(1/2)}, 1 \rangle = \sum_j (2j + 1)^{2-2g}, \tag{25}$$

where g is the genus of Σ.

For an arbitrary m the situation changes radically. This can be illustrated in our present example $m = 1$. Because of the additional solution $U_0 \neq 1$, the fusion move no longer implies a trivial renormalization of the face amplitude (see Fig. 1). Integrating over all the internal connections we obtain

$$\langle P_\epsilon^{(1)}, 1 \rangle = \sum_j (2j + 1)^{2-2g} \left[1 + \lambda \frac{\sin[(2j + 1)\frac{\pi}{2}]}{(2j + 1)}\right]^{A/\epsilon^2}, \tag{26}$$

[5] In fact one cannot get the mode expansion of the $SU(2)$ delta function coming from the integral definition of $d^{1/2}[U]$. In order to have the half-integer representations in 2 + 1 gravity one must define $c_j^{(1/2)} = 2j + 1$ for all half-integers [37].

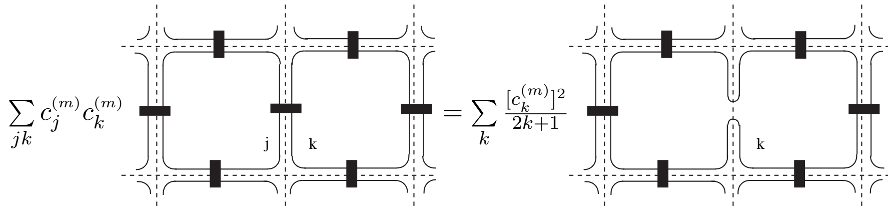


FIG. 1. Infinitesimal plaquette delta distributions can be integrated and fusioned with the corresponding modification of the face amplitude. The lines in the previous figure represent the holonomy around plaquettes of the regulator in the representation denoted by the Latin index k and j in this case. The dark boxes denote integration of the generalized connection associated to the corresponding edge. The equation represented by the figure is a trivial consequence of the orthogonality of unitary irreducible representations of $SU(2)$. The plaquettes in the picture are square as a matter of simplicity.

where $\langle P_\epsilon^{(1)}, 1\rangle$ denotes the vacuum-to-vacuum amplitude before the regulator has been removed, A is the coordinate area of Σ, and A/ϵ is the number of plaquettes in the cellular decomposition.

For $\lambda \neq 0$ the limit $\epsilon \longrightarrow 0$ of the previous face amplitude is ill defined. For constant λ the face amplitude will either diverge or converge to zero depending on the value of the representation j. In order to avoid this problem we could renormalize λ as we shrink the lattice. For instance the limit would in fact be well defined if we chose $\lambda = \epsilon^{-2}\lambda_0$. In this case we get

$$\langle P^{(1)}, 1\rangle = \lim_{\epsilon\longrightarrow 0}\langle P_\epsilon^{(1)}, 1\rangle = \sum_j (2j+1)^{2-2g} \exp\left[\lambda_0 A \frac{\sin[(2j+1)\frac{\pi}{2}]}{(2j+1)}\right]. \tag{27}$$

The previous amplitude explicitly depends on the coordinate area of Σ. We can insist upon defining the limit by renormalizing the ambiguity parameter but at the cost of losing background independence. It is clear that the theory obtained for $m = 1$ has nothing to do with $2 + 1$ quantum gravity. In other words we have taken the limit $\epsilon \longrightarrow 0$ but the result is not even diffeomorphism invariant: it remains in the dependence of the amplitude on the coordinate area. We have run into an anomaly of the kind described in [38]. We can avoid the previous problem if we choose $\lambda(\epsilon) = \mathcal{O}(\epsilon^2)$. In that case we would recover the topological amplitude (25) as in the case $m = 1/2$. This is not surprising as we are simply suppressing any contribution of the spurious solutions in the continuum limit.

For $m = 3/2$ the situation is simply the same, which illustrates the generic case. In this case $Q^{(2)}(x^\mu) = 8x^i x_i - 20(x^0)^2$. A similar analysis gives

$$c_j^{(3/2)} = \lambda_1(2j+1) + \lambda_2 \chi_j[U_0] \tag{28}$$

where U_0 is in the conjugacy class of the element labeled by the point $x^0 = \sqrt{2/7}$, $x^1 = x^2 = 0$, $x^3 = \sqrt{5/7}$. As before (and for any $m \geq 1$) the presence of spurious solutions would spoil the existence of a diffeomorphism invariant continuum limit unless the physical inner product is defined in such a way that the extra solutions have zero physical norm. In that case the theory obtained coincides with the one constructed in terms of the fundamental representation $m = 1/2$.

A. Linear combinations

The problem with the quantization of the curvature constraint in terms of a single representation m that is different from the fundamental one can be traced back to the existence of nontrivial configurations that solve the regulated constraint. These extra solutions do not correspond, in classical terms, to $F = 0$. In the limit $\epsilon \longrightarrow 0$ the spurious solutions define wild oscillatory configurations at the coordinate scale set by ϵ. These solutions conspire to make the elimination of the regulator ill defined. We have seen in the previous section that unless the spurious solutions are appropriately suppressed (which leads to the quantum theory obtained for $m = 1/2$) the continuum limit does not exist or is anomalous.

One can avoid the previous undesired effect by considering those *good* regularizations that do not introduce spurious solutions. In fact this can be easily characterized as follows: Instead of using a regularization consisting of a single irreducible representation one can study the general case where the curvature constraint is quantized by an arbitrary linear combination of Wilson lines in any representation. Namely we replace (9) by

$$\hat{F}[N] = \sum_m a_m \hat{F}^{(m)}[N] = \lim_{\epsilon\longrightarrow 0} C^{-1} \sum_p \sum_m a_m \,\mathrm{Tr}^{(m)}[N_p \hat{\Pi}^{(m)}[U_p]], \tag{29}$$

where C^{-1} is the appropriate normalization factor for $\sum_m a_m = 1$.

There exists an infinite-dimensional space of such regularizations, parametrized by the coefficients $\{a_m\}$. From this infinite-dimensional set of theories only those which satisfy

$$\sum_m a_m \,\mathrm{Tr}^{(m)}[N_p \Pi^{(m)}[U] = 0 \quad \text{iff } U = 1 \tag{30}$$

lead to theories where the continuum limit is well defined. In fact the individual values of the coefficients $\{a_m\}$ play no physical role, and as long as the previous equation holds the corresponding physical inner product is unique.

In the $U(1)$ example this corresponds to any periodic function $F[\phi]$ on the interval $[0, 2\pi]$ vanishing at 0. It is obvious that there is an infinite-dimensional space of such functions. The analog of Eq. (12) becomes

$$d(\phi) = \int dN \exp(iNF[\phi]) = F'[0]\delta[\phi]. \tag{31}$$

Except for a trivial overall factor renormalization we obtain the result that follows from the quantization based on the fundamental representation. The result is exactly the same in the non-Abelian case. So we conclude that considering arbitrary linear combinations of representations we can obtain well-defined quantizations of 2 + 1 gravity. However, the resulting theory is completely equivalent to the $m = 1/2$ quantization. We are in fact in the situation (i) described in the Introduction.

B. Covariant spin foams

At this stage it should be clear that the analysis presented above can be extended with mild modifications to the covariant picture. More precisely in the lattice definitions of the path integral for 2 + 1 quantum gravity that leads to the Ponzano-Regge model one can also study the effect of the modification of the simplicial action by replacing the customary regularization of the curvature tensor in terms of the Wilson line in the fundamental representation by an arbitrary function of the holonomy around plaquettes satisfying the naive continuum limit property.

In the case of a regularization based on a single unitary representation, for $m \neq 1/2$ discretization independence of the partition function is lost and the path integral is no longer well defined. The continuum limit is lost. The good regularizations are characterized as in the previous section and are equivalent to that defined in terms of the fundamental representation, in terms of which we recover a unique result: the standard Ponzano-Regge model. Notice that this can also be interpreted from the point of view developed in [38], if the spin foam face amplitude is not equal to the dimension of the representation labeling the face, the spin foam amplitudes are not well defined in the equivalence classes of spin foams and hence are regarded as anomalous.

III. THE *m* AMBIGUITY IN 3 + 1 GRAVITY

In 3 + 1 gravity our strategy is similar to that of 2 + 1 gravity. We will show that unless $m = 1/2$ [for $SU(2)$]—or $m = 1$ [for $SO(3)$]—is used in the regularization of the Hamiltonian constraint, the resulting theory contains spurious local degrees of freedom. These are the analog of the new solutions found above which interfere with the existence of the continuum limit in 2 + 1 gravity. We will explicitly demonstrate the existence of such solutions in 3 + 1 gravity by constructing explicit examples when $m > 1$. Their existence is due exactly to the same mechanism as in our previous lower dimensional example. These solutions also correspond to wildly Planck-scale-oscillatory configurations. In view of the result of the previous section these regularizations correspond to bad suited quantizations of the curvature part of the Hamiltonian constraint.

Unfortunately the construction of the physical inner product of the theory is not yet well understood and it is in this respect that our argument cannot be as strong as the one made for 2 + 1 gravity in the first part of this paper. Nevertheless, the fact that quantizations of the theory in terms of $m > 1$ produce these extra local excitations, i.e. new degrees of freedom, strongly discourages the choice of such theories. One should expect these spurious solutions to be zero norm in the physical inner product of loop quantum gravity.

A. Quantization of the Hamiltonian constraint

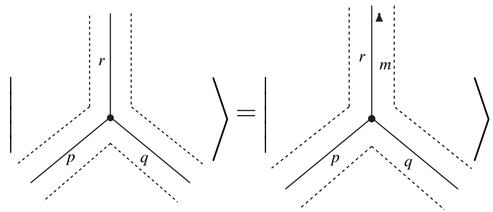
As explained in the Introduction Thiemann's prescription leads to $\hat{H} = \hat{F}(A)\widehat{EE/\det(E)}$ as the only consistent factor ordering in the quantization of the Hamiltonian constraint (1). We use the notation of Ref. [11]. With all this in mind the action of the (regulated) quantum Hamiltonian constraint on a spin-network vertex $|v\rangle$ is given by

$$\hat{\mathcal{H}}^{m}_{\Delta}|v\rangle = \frac{N_v i}{3l_0^2 C(m)}\epsilon^{ijk}\,\mathrm{Tr}[(\hat{h}^{(m)}[\alpha_{ij}] - \hat{h}^{(m)}[\alpha_{ji}])\hat{h}^{(m)}[s_k]\hat{V}\hat{h}^{(m)}[s_k^{-1}]]|v\rangle, \tag{32}$$

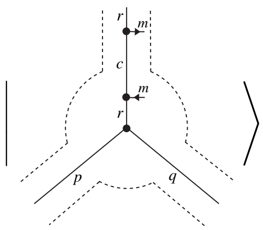
where the subindex Δ in $\hat{\mathcal{H}}^m_{\Delta}$ denotes the triangulation used for the regularization of the action of the constraint, N_v is the value of the lapse function at the vertex, and the supraindex m denotes the fact that we are using the unitary representation of spin m to regularize the curvature term in terms of the holonomies $\hat{h}^{(m)}[\alpha_{ij}]$ around to certain loops α_{ij} and $\hat{h}^{(m)}[s_k]$ along segments s_k respectively. The latter are defined in detail in [11] and will be graphically introduced in what follows. The m-dependent factor $C(m)$ is a normalization factor needed to satisfy the naive continuum limit.

Now we will briefly remind the reader of the basic technicalities associated to the quantization of the Hamiltonian constraint. In this part of the paper we are following [11] almost literally. For simplicity we use 3-valent nodes in our pictures; however, our argument is completely general and applies to arbitrary n-valent nodes. We describe the regularization of the Hamiltonian constraint by analyzing the action of the different terms in (32) separately. We start with the action of the holonomy $\hat{h}^{(m)}[s_k^{-1}]$ operator on the right which after a simple exercise of recoupling theory gives

$$\hat{h}^{(m)}[s_k^{-1}]$$



(33)

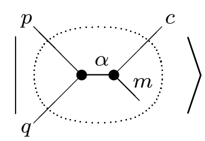


(34)

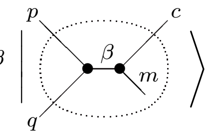
where the two new 3-intertwiners are normalized. The dotted line denotes a region of zero size introduced for illustrative purposes. For instance, the vertical lines labeled by representations r and m in the second diagram above are to be thought of as overlapping.

The next operator appearing in (32) from right to left is the volume operator. Following the notation of [11] the action of the volume operator on the vertex is given by

$$\hat{V}\hat{h}^{(m)}[s_k^{-1}]|v\rangle = \hat{V}$$



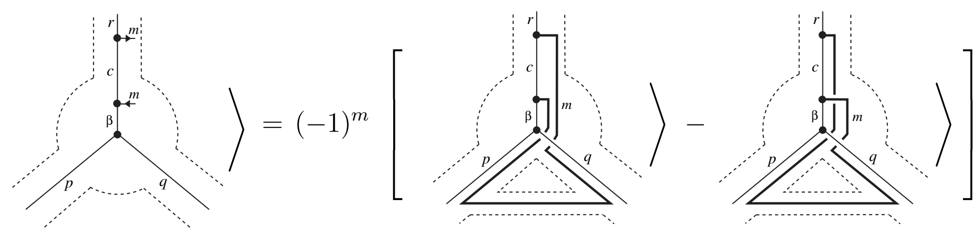
$$= \sum_{\beta} V(p,q,m,c)_{\alpha}{}^{\beta}$$



(35)

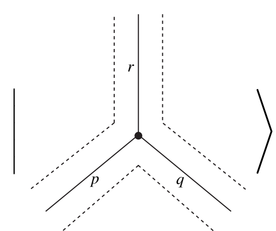
where $V(p, q, m, c)_{\alpha}{}^{\beta}$ denotes the matrix elements of the volume operator, and the dotted region corresponds to a single point. Inside this dotted region we graphically represent the elements of the finite dimensional vector space $\mathrm{Inv}[p \otimes q \otimes m \otimes c]$ in terms of normalized 3-intertwiners (labeled by α and β in the previous expression) in the standard fashion. We recall that 3-valent nodes are used here as a matter of convenience. In general the previous equation remains true with the obvious modifications. As we will see below our argument is completely independent of the volume part of the quantum Hamiltonian, and hence valid for any node valence. Next one acts with the operators that represents the action of the curvature tensor—the last term on the left of (32)—obtaining

$$(\hat{h}^{(m)}[\alpha_{ij}] - \hat{h}^{(m)}[\alpha_{ji}])\,\hat{h}^{(m)}[s_k]$$

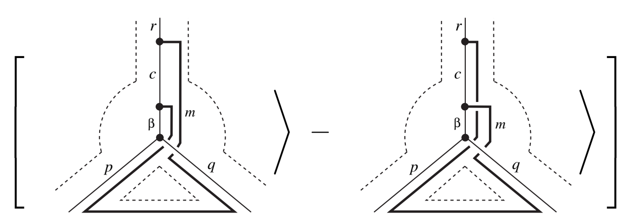


Putting all together and ignoring the prefactor in (32) the action of the regulated Hamiltonian becomes

$$\mathrm{Tr}\left((\hat{h}^{(m)}[\alpha_{ij}] - \hat{h}^{(m)}[\alpha_{ji}])\,\hat{h}^{(m)}[s_k]\,\hat{V}\,\hat{h}^{(m)}[s_k^{-1}]\right)$$



$$= (-1)^m \sum_{c\beta} V(p, q, m, c)_r^{\beta}$$



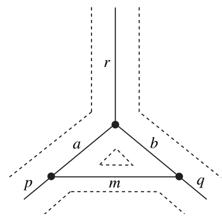
(36)

We call the new edge created by the action of the curvature *exceptional edge*. This edge has special properties that grant the absence of anomalies in the quantum theory (for details see [39]).

Expanding the result in the spin-network basis and projecting on the connection representation we can write

$$<A|\hat{\mathcal{H}}^m_\Delta|v(p,q,r)\rangle = \sum_{a,b} H^{(m)}(p,q,r;a,b)$$

$$\times \Big\langle A \Big|$$



$$\Big\rangle + \dots$$

$$= \sum_{a,b} H^{(m)}(p,q,r;a,b)$$

$$\times\, \Psi^{p,q,r;a,b}(A_{out};A_{exc}) + \dots, \quad (37)$$

where we have only explicitly written the term where the exceptional edge is created on the bottom (there are two more terms in this case but they are not important for the rest of the argument), and $H^{(m)}(p, q, r; a, b)$ are the corresponding matrix elements of the quantum Hamiltonian constraint. The functional $\Psi^{p,q,r;a,b}(A_{\text{out}}; A_{\text{exc}})$ is the spin-network function of the generalized connection along the edges of the underlying graph. The variable A_{exc} denotes the value of the holonomy along the exceptional edge created by the action of the regulated Hamiltonian constraint, which by appropriate gauge fixing at the original vertex can be taken as the value of the holonomy around the triangular loop created by the action of the constraint. On the other hand A_{out} denotes the generalized connection along the edges of the spin-network graph which are different from the three edges mentioned above.

It is important to notice that if we write $A_{\text{exc}} = x^\mu \tau_\mu$, using the parametrization of $SU(2)$ of the previous section, the action of the Hamiltonian constraint implies that

$$\sum_{a,b} H^{(m)}(p,q,r;a,b)\Psi^{p,q,r;a,b}(A_{\text{out}};x^0,\vec{x}) = -\sum_{a,b} H^{(m)}(p,q,r;a,b)\Psi^{p,q,r;a,b}(A_{\text{out}};x^0,-\vec{x}). \quad (38)$$

In other words the resulting state has a definite "parity" under inversion of the generalized connection along the exceptional loop as a consequence of Eq. (36). This property will be important in the following sections.

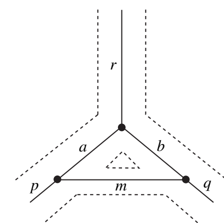
B. Constructing solutions

We assume in this subsection that the corresponding vertex is 3-valent. This will simplify the discussion of the action of the quantum Hamiltonian constraint. This restriction is however a simple matter of convenience as in the case that the matrix elements of the quantum constraint can be evaluated in a simpler way. In principle one could generalize the argument presented here to arbitrary valence. Notice however that such generalization is not necessary for the validity of our conclusions as our objective is to show the presence of spurious local degrees of freedom and not to fully characterize them. In particular we will exhibit explicit spurious solutions in the next subsection by means of a general argument valid for arbitrary vertices.

We come back to Eq. (37) and the notation defined there. Now we define a diffeomorphism invariant state $(\Psi_{A_{\text{out}},x^\mu}|$ by

$$(\Psi_{A_{out},x^\mu}| = \sum_{\phi\in \mathrm{Diff}(\Sigma)} \sum_{ab} \Psi^{p,q,r;a,b}(A_{out},x^\mu)$$

$$\times \Big\langle$$



$$\Big| U_\phi\ , \quad (39)$$

where U_ϕ is the unitary operator that represents the diffeomorphism ϕ. The previous states are labeled by the parameters A_{out} and x^μ (or simply $A_{\text{exc}} = x^\mu \tau_\mu$). The coefficients $\Psi^{p,q,r;a,b}(A_{\text{out}}, x^\mu)$ are given by the evaluation of the corresponding spin-network function defined in (37) for a definite choice of configuration, i.e., the generalized connection (holonomies) along the edges of the corresponding graph.

We also assume that the rest of the spin-network state is annihilated by the quantum Hamiltonian constraint acting on the other vertices. This assumption is realized, for example, by a spin-network state that has no exceptional edges apart from the one on the vertex of interest. More precisely, because the action of the quantum Hamiltonian constraint creates exceptional links on spin-network states we have that $(\Psi|\hat{H}[N]s\rangle = 0$ if the diffeomorphism invariant state $(\Psi|$ does not have any exceptional edge. From this basic solution one can obtain infinitely many solutions by adding local excitations—solutions to the local conditions imposed by the Hamiltonian constraint at a vertex—at different vertices. This is precisely what we do in order to construct the new solution.

It is direct to check that for any spin-network state $|\phi\rangle$ we have

$$(\Psi_{A_{\text{out}},x^\mu}|\hat{H}[N]\phi\rangle = \begin{cases} 0, & \text{if the state } \phi \notin [\Psi\text{–exceptional edge}], \\ N_v P^{(2m)}_{A_{\text{out}}}(x^\mu), & \text{otherwise}, \end{cases} \quad (40)$$

where $[\Psi - \text{exceptional edge}]$ denotes the equivalence class under diffeomorphisms of Σ of the spin-network state obtained from any element in the sum (39) by setting $m = 0$, $a = p$ and $b = q$ respectively, and N_v is the value of the lapse function at the corresponding vertex. The quantity $P^{(2m)}_{A_{\rm out}}(x^\mu)$ is an order $2m$ polynomial of the variable x^μ explicitly given by

$$P^{(2m)}_{A_{\rm out}}(x^\mu) = \sum_{a,b} H^{(m)}(p, q, r; a, b)\Psi^{p,q,r;a,b}(A_{\rm out}; x^\mu). \tag{41}$$

The coefficients of the previous polynomial can be shown to be real: the reality of $\Psi^{p,q,r;a,b}(A_{\rm out}; x^\mu)$ follows from the fact that spin-network functions can be normalized to be real functions of the generalized connection. Spin networks can be taken as real because they can be expressed as real linear combinations of products of traces of Wilson loops in the fundamental representation and hence real. The matrix elements of the Hamiltonian constraint are also real in this basis. This might seem strange as the Hamiltonian constraint is not self-adjoint. This is perhaps the reason why this property of the Hamiltonian constraint has not been previously noticed in the literature. It is a simple matter to prove the reality of the matrix elements of the Hamiltonian for 3-valent vertices.[6] It is not obvious whether the reality holds for general matrix elements. This would be interesting to explore.

The state $(\Psi_{A_{\rm out}, x^\mu}|$ would be in fact a physical state for every solution x^μ of the equation $P^{(2m)}_{A_{\rm out}}(x^\mu) = 0$ with $x^\mu x_\mu = 1$. As the order of the polynomial increases with m, it is natural to expect that the number of solutions of $P^{(2m)}_{A_{\rm out}}(x^\mu) = 0$ will do so as well. However, it could happen that for some reason none of the nontrivial solutions of the polynomial equation satisfy $x^\mu x_\mu = 1$. Notice however that the reality of the coefficients of $P^{(2m)}_{A_{\rm out}}(x^\mu)$ plus the fact its coefficient depends on the external (continuum) parameters $A_{\rm out}$ suggests that it should be possible to tune the polynomial equation so that its solutions lay on the unit sphere. Nevertheless, in order to show this explicitly one would need the explicit evaluation of the matrix elements of the Hamiltonian constraint. This is not a serious obstacle as such an analysis for 3-valent vertices would require a simple generalization of the results of [42]. However, such a strategy will take us for a considerable technical detour in the paper; so we will instead demonstrate the existence of spurious solutions by a different method.

Assume for the moment that these solutions exist for $m > 1/2$. The existence of these solutions is directly linked to our choice of regularization indicating that the physically correct quantizations must be those for which the curvature tensor is regularized in terms of the fundamental representation. If on the contrary one wants to insist on using a higher m representation in the definition of the theory one must provide a strong justification for the inclusion of the extra local degrees of freedom. The understanding of the construction of the physical inner product from the quantum constraints would certainly make the result more robust. Our results in 2 + 1 gravity suggest in this respect that the spurious solutions appearing for higher m regularizations would be of zero norm and hence would disappear from $\mathcal{H}_{\rm phys}$.

C. Solutions from an algebraic argument

Instead of explicitly computing the matrix elements of the quantum Hamiltonian—which would present a quite formidable task—we construct solutions in this section by a simple algebraic argument. The idea is to make use of Eq. (33). The argument presented here is valid for any vertex valence.

1. Example in quantum mechanics

As an example we consider a quantum mechanical particle on the unit sphere. An orthonormal basis of the Hilbert space can be taken to be the angular momentum basis whose elements we label $|\ell m\rangle$(s.t. $L^2|\ell m\rangle = \ell(\ell + 1)|\ell m\rangle$ and $L_z|\ell m\rangle = m|\ell m\rangle$). We have that the wave function $\langle \vec{x}, \ell m\rangle = Y_{\ell m}(\vec{x})$ transforms under parity as

$$Y_{\ell m}(-\vec{x}) = (-1)^\ell Y_{\ell m}(\vec{x}).$$

Because of the previous property the action of the parity operator $\hat{p}$ on basis elements is simply

$$\hat{p}|\ell m\rangle = (-1)^\ell|\ell m\rangle.$$

The action of our (toy) Hamiltonian constraint $\hat{H}$ is defined by

$$\langle \vec{x}|\hat{H}|\ell m\rangle = \sum_{n,q} h_{\ell m;nq} Y_{nq}(\vec{x}), \tag{43}$$

which is the simplified analog of Eq. (37). To complete the

[6] Let us briefly support the statement of reality. In fact the result is a simple consequence of properties of the reality properties of the spin-network basis and the volume operator. From Eq. (36) one concludes that the matrix elements of the quantum Hamiltonian constraint are real if the matrix elements of the volume operator are real (the reality of the combination of spin networks on the right follows directly from the reality of spin-network basis elements). Therefore it remains to show that the matrix elements of the volume operator appearing in (36) are real. Recall that the finite dimensional matrix $V(p, q, m, c)_\alpha{}^\beta$ is defined as $V = \sqrt{|W|}$ where W essentially corresponds to the quantization of $\epsilon_{abc}E^a_i E^b_j E^c_j \epsilon^{ijk}$. Acting on finite valence nodes, and because its action does not change the valence, W can be represented by a finite dimensional Hermitian matrix. In order to define the square root one must go to the basis that diagonalizes W, namely

$$V = \sqrt{|W|} = U\sqrt{|W_D|}U^{-1} \tag{42}$$

where W_D is the diagonal form of W [11]. An important property of W is that it is purely imaginary and skew symmetric [40,41]. Hence W^2 is real and symmetric U is orthogonal from where it follows the reality of V. This completes the proof of the reality of the matrix elements of the quantum Hamiltonian constraint.

analogy we require $\hat{H}$ to be such that $\langle\vec{x}|\hat{H}|\ell m\rangle = -\langle-\vec{x}|\hat{H}|\ell m\rangle$ which can be achieved if $\hat{H} = (1-\hat{p})\hat{H}_0$. In this analogy we associate

$$(1-\hat{p}) \rightarrow \frac{\hat{h}^{(m)}[\alpha_{ij}] - \hat{h}^{(m)}[\alpha_{ji}]}{2}$$

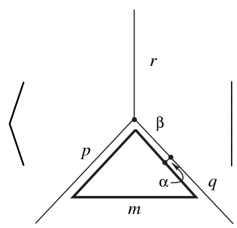
and

$$\hat{H}_0 \rightarrow \hat{h}^{(m)}[s_k]\hat{V}\hat{h}^{(m)}[s_k^{-1}].$$

For an operator like this it is very easy to find solutions. In fact any dual state of even parity will be obviously annihilated by $\hat{H}$. A basis of solutions will be given by the states $\langle 2n, m|$ for any positive integer n.

2. Solutions of Thiemann's Hamiltonian

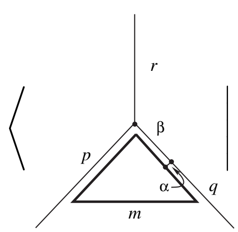
In order to find solutions of the quantum Hamiltonian we must first construct states with a definite parity under the "reflection" $A_{\mathrm{exc}} \rightarrow A_{\mathrm{exc}}^{-1}$. A family of candidate states is given by the following spin-network states



(44)

which under the transformation $A_{\mathrm{exc}} \rightarrow A_{\mathrm{exc}}^{-1}$ transform by a factor $(-1)^\alpha$ [42]. The next step is to find a diffeomorphism invariant state starting from the previous spin network by means of summing over the action of diffeomorphisms. The corresponding state is an element of the set of distributions or linear functionals $Cyl^\star$ and can be written as

$$(\Psi| = \sum_{\phi\in\mathrm{Diff}[M]} \sum_{\alpha\beta} c^{\Psi}_{\alpha\beta}$$



$\mathrm{U}[\phi]$ (45)

where $U[\phi]$ is the unitary operator that generates diffeomorphism and the only condition on the coefficients is that $c^{\Psi}_{\alpha\beta} = 0$ if α is odd.

Direct calculation shows that the previous is a solution of the m-quantum Hamiltonian constraint, namely, that $(\Psi, H^{(m)}[N]s\rangle = 0$ for any arbitrary $|s\rangle \in \mathcal{H}_{\mathrm{kin}}$. The previous statement is nontrivial only in the case when $|s\rangle$ is in the diff-equivalent class of the spin-network state we started with. In that case the answer is zero because we are computing the superposition between an even parity with an odd parity state which must vanish.

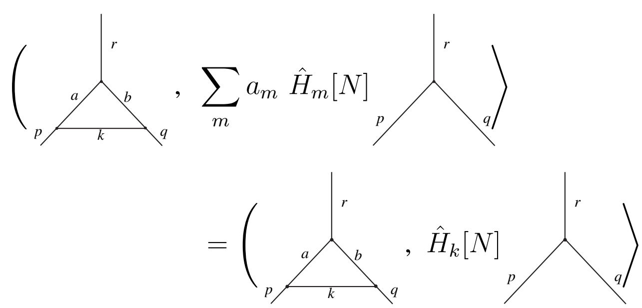
The solutions found in the previous subsection are labeled by two quantum numbers α and β. The set of possible values for these two quantum numbers grows with the value of the ambiguity parameter m. There are in fact $2m+1$ allowed values for α which lead to IntegerPart$(m+1)$ even values. If $m = 1/2$ we have only the possibility $\alpha = 0$. However, if we use the fundamental representation of $SO(3)$, i.e., $m = 1$ we have two possibilities: $\alpha = 0$, already present in the previous case and $\alpha = 2$. This solution corresponds to a spin-two local excitation. For higher values of m there are more solutions as an artifact of a bad choice of regularization. According to the results in $2+1$ gravity these solutions should be regarded as spurious.

D. Linear combinations

One should also consider the possibility of arbitrarily combining different m regularizations to produce an infinite-dimensional family of quantum Hamiltonian constraints

$$\hat{H}[N] = \sum_m a_m \hat{H}_m[N] \quad \text{with} \sum_m a_m = 1. \tag{46}$$

Now the previous solutions will continue to exist since the action of the quantum constraint on them is governed by a single term in the sum. The key equation is



where the $(s|$ denotes a diff-invariant state associated to the spin network $\langle s|$. The validity of the previous equation allows for the construction of spurious solutions by simply using the spurious solutions found in the previous section for individual terms in (46). This seems quite different from what we found in Sec. II A, where some linear combinations would lead to quantizations that were equivalent to the one based on the fundamental representation.

Even though this might be interpreted as a positive result one should keep in mind that this happens because of a property of Thiemann's quantum constraint that is also seen as a problem. More precisely, the fact that among the solutions of Thiemann Hamiltonian constraint there is a vast set of solutions of a rather trivial nature. For example a diffeomorphism invariant state labeled by a spin network with no exceptional edge is a trivial solution of the constraints. This is related to the special character of the exceptional edges that are added by the action of H required by the conditions that imply the absence of an anomaly [33]. The triviality of these solutions is puzzling and seems to indicate that the restrictions imposed by quantum constraint quantized *à la Thiemann* are too weak to lead to a theory with propagating degrees of freedom [43]. This problem is one of the main motivation for the exploration of alternative definitions of the dynamics such as the one proposed in the master constraint program, the consistent discretization approach and the covariant spin foams approach.

IV. DISCUSSION

The absence of divergences in the quantization of the Hamiltonian constraint is a remarkable feature of loop quantum gravity. In this work we point out that this important characteristic of the theory does not, by itself, resolve the issue of renormalization in quantum gravity as having a sound mathematical framework (free of infinities) as intimately related to the existence of ambiguities. In the case of loop quantum gravity there is an infinite-dimensional space of possible theories. Until the problem of the ambiguities has settled the situation, regarding the predictive power of the theory, it is not much different from that of standard perturbative approaches. In this paper we investigated the so-called m ambiguity associated to the unitary representation used in the quantization of the configuration variables. In the case of $2 + 1$ gravity the problem is completely resolved. In $3 + 1$ gravity we provide evidence pointing to a possible resolution of the question. In what follows we discuss these results in more detail.

A. 2 + 1 loop quantum gravity

We have showed that consistency of the quantum theory eliminates the ambiguities related to the quantization of the curvature constraint in $2 + 1$ loop quantum gravity. If the regularization is not performed using the holonomy in the fundamental representation of the gauge group the appearance of extra (spurious) solutions conspires against the possibility of removing the regulator in the definition of the physical scalar product. There are other prescriptions that lead to a well-defined theory but they are fully equivalent to the quantum theory defined in terms of the fundamental representation. Pure gravity in three dimensions is an example of theory belonging to the first class mentioned in the Introduction.

The spurious solutions to the quantum constraint regulated with the representation m (with $m > 1/2$) correspond to wildly oscillatory curvature configurations down to the Planck scale. These solutions are annihilated by the regulated constraint but because of the latter feature they are not well defined in the ''continuum'' (i.e., independently of the regulator). Nevertheless, if one defines the physical inner product in terms of the good regularizations (e.g., $m = 1/2$) then the spurious solutions of the regulated constraint for the bad quantizations (e.g., $m > 1/2$) have zero physical norm.

Because $2 + 1$ gravity is a topological theory, the fact that the issue of ambiguities can be completely settled in this case is not entirely surprising—the renormalizability of $2 + 1$ gravity is advocated since Witten's seminal work [44]. Gravity in $2 + 1$ dimensions has finitely many degrees of freedom and from this perspective one would not expect serious difficulties dealing with the UV problem. Our results make the previous statement precise in the framework of loop quantum gravity and provide the starting point for the analysis of the issue in $3 + 1$ dimensions. The results of the first part of this work extend trivially to the case of spinning particles coupled to $2 + 1$ gravity studied in [45].

B. 3 + 1 loop quantum gravity

The effects of the m ambiguity in $3 + 1$ loop quantum gravity are similar. Regularizing the holonomies used in the quantization of the Hamiltonian with unitary representations of spin $m > 1$ introduces new local degrees of freedom. These solutions correspond, as in the lower dimensional case, to highly oscillatory excitations at the Planck scale. The mechanism leading to the existence of such solutions is the analog of the $2 + 1$ case: higher representation regularizations of the curvature tensor appearing in the Hamiltonian constraint correspond to functions on the groups with additional roots.

The direct computation of the spurious solutions of Sec. III B would require the explicit computation of the matrix elements of the Hamiltonian constraint for arbitrary regularizations. In Sec. III C 2 we used a symmetry argument to explicitly exhibit the existence of new local degrees of freedom associated with the choice of higher m quantizations. These local degrees of freedom correspond to higher spin local excitations—for example the quantum number α in Eq. (45) takes values $\alpha = 4, \dots, 2m$ for $m =$ integer.

At this stage one cannot construct a complete argument as in $2 + 1$ gravity due to the lack of an explicit definition of the physical inner product in $3 + 1$ gravity. More precisely we cannot prove that the spurious solutions would spoil the existence of a well-defined continuum limit unless they are zero norm in $\mathcal{H}_{\text{phys}}$. Nonetheless the existence of spurious solutions of the quantum constraints associated to $m > 1$ quantizations provides an argument against such theories that changes our perspective regarding the ambiguity problem: if one would like to use values of $m > 1$ in the quantization one would need to provide a clear justification for the inclusion of the associated extra degrees of freedom. This is evidence pointing in the right direction; we hope that future studies will shed more light on this important issue.

An interesting possibility is to study the ambiguity problem in $3 + 1$ LQG by first analyzing a formulation of $2 + 1$ quantum gravity that mimics the structure of the $3 + 1$ theory. A classical formulation of $2 + 1$ can be obtained from $3 + 1$ general relativity assuming for instance that there is a translation Killing vector field (i.e. by symmetry reduction). If one does this at the Hamiltonian level the structure of the constraints of the theory remains the same as in $3 + 1$. In particular there is a Hamiltonian constraint of the form (1) (where now space indices $a, b = 1, 2$) in addition to the appropriate diffeomorphism constraint. The quantization of this formulation has been studied by Thiemann [46]. It is easy to see that the ambiguity studied here will subsist in this case. It would be interesting to study whether dynamical considerations in this simplified case lead to a reduction of the ambiguity similar to the one found in Sec. II. This toy model might represent an instructive exercise in trying to resolve this open problem of $3 + 1$ gravity.

Finally, let us mention that a study of the effects of the m ambiguity in the quantum mechanical context of loop quantum cosmology has been performed in [20]. The results are consistent with the ones presented here for the field theory. In fact there are new solutions associated to a higher m quantization of the Hamiltonian constraint. Most of these solutions are unphysical or spurious in view of certain semiclassicality criteria [47] applied in the context of loop quantum cosmology. It is interesting to notice that in the simple model studied in [48] one can also study the effects of the ambiguity with the advantage of knowing the physical inner product. In this case one can explicitly show that spurious solutions are indeed zero-norm states. At first sight this looks promising at it shows that the physical Hilbert space for different m values are all isomorphic. However, the physical interpretation of the physical states in standard terms (e.g. interpreting them as representing the wave function of the universe evolving in terms of the universe scale factor) does indeed depend on the ambiguity parameter. This shows that little quantitative predictive power is to be associated to these models at least until the ambiguity issue is fixed in the fundamental theory.

C. Physical Hamiltonian and other approaches to dynamics

Our analysis in $3 + 1$ gravity has been performed entirely in the context of the framework of Thiemann's quantization of the Hamiltonian constraint. Even when Thiemann's prescription provides a mathematically consistent quantum operator, concerns have been raised about its physical viability. Problems related to the so-called ultralocal character of the quantum dynamics—which are rooted in the way the constraint algebra of gravity is represented (for a review see [21])—have been pointed out as a serious obstacle for the theory to reproduce general relativity in the classical limit [43] (for a different perspective of the same problem see [1]).

This has motivated the search for an alternative definition of dynamics such as the covariant definition given by the so-called *spin foam models* [49], alternative quantizations proposed by Thiemann in his master constraint program [22], and the program of Gambini and Pullin of consistent discretizations [23]. In the latter two alternative formulations similar regularization problems give rise to ambiguities which are the analog of the m ambiguity studied here. Therefore, the questions raised by this article must also be addressed in these cases.

Since our argument is based on the existence of multiple solutions of the quantized constraints we expect its conclusions to be sufficiently general to provide a nontrivial insight in cases in which the details of the dynamics are different. In fact, in the first part of the paper we showed how the analysis of the ambiguity in the canonical formulation of $2 + 1$ gravity had a precise parallel in the covariant formulation (or spin foam representation) of the theory. For this reason we think that our results obtained in the context of Thiemann's constraint should apply in suitable form to any definition of the quantum dynamics where the connection is represented by holonomies.

D. Spin foam models from constrained BF theory

In Sec. II B we showed how the potential ambiguities arising in the definition of the path integral of BF theory can be eliminated. Our results in three dimensions can be easily generalized to arbitrary dimensions. Therefore, there are no ambiguities of the type analyzed here in the quantization of BF theory in four dimensions. This provides an extra incentive for the search of a covariant formulation (or spin foam representation) based on the idea of viewing gravity as a constrained BF theory. Many of the spin foam models studied in the literature are of this kind [50,51]. Particularly attractive in this respect is the treatment proposed by Freidel and Starodubtsev [52].

E. General considerations about first order gravity

In the Introduction we advocated the similarities between the renormalization problem in perturbative and loop quantum gravity with the purpose of stressing the importance of a clear understanding of the ambiguity issue in the latter. Now we would like to point out an important difference which provides an independent (heuristic) argument supporting the idea that the background independent quantum field theory of gravity pursued by loop quantum gravity should be rather restrictive instead of infinitely ambiguous.

Loop quantum gravity—or spin foam models as their covariant formulation—is a general framework for the nonperturbative quantization of gravity in the first order formulation. By the first order formulation we mean here the most general diffeomorphism invariant theory that one can write in terms of a tetrad of 1-forms and a Lorentz connection A.[7] The most general form of such action in three dimensions is

$$S[e, A] = \int \mathrm{Tr}[e \wedge F(A)] + \Lambda \int \mathrm{Tr}[e \wedge e \wedge e\epsilon], \quad (47)$$

which was first quantized and argued to be renormalizable by Witten [44]. In four dimensions the most general action becomes

$$\begin{aligned} S[e,A] = &\frac{1}{2\kappa}\int \mathrm{Tr}[e\wedge e\wedge F^{\star}(A)] + \frac{1}{\kappa\gamma}\int \mathrm{Tr}[e\wedge e\wedge F(A)] \\ &+ \Lambda \int \mathrm{Tr}[e\wedge e\wedge (e\wedge e)^{\star}] \\ &+ \alpha \int \mathrm{Tr}[F(A)\wedge F^{\star}(A)] + \beta \int \mathrm{Tr}[F(A)\wedge F(A)], \end{aligned} \quad (48)$$

where γ is the Immirzi parameter, and α and β are coupling constants. Notice that from this perspective it is natural to introduce a nontrivial Immirzi parameter which is essential for the definition of loop quantum gravity.

Heuristically, in standard renormalization framework, the simplicity of the previous action is reminiscent of a "renormalizable" theory: all the possible terms compatible with the postulated fundamental symmetries are finitely many.[8] However this argument cannot be made in the standard way because the previous action is not quadratic around the diffeomorphism invariant vacuum $e = 0$ and $A = 0$ and one cannot make use of the usual perturbative treatment [44]. Moreover, if one instead defines the perturbative theory around an invertible configuration, say $e_a^I = \delta_a^I$, then the perturbation theory generates the infinitely many terms in the effective action that can be written in terms of the inverse e^{-1}. Hence we arrive in this case to the standard *no renormalizability* of gravity. If the striking simplicity of the general action (48) is of an indication in some sense of the uniqueness of the associated quantum theory the question must be explored nonperturbatively.

Loop quantum gravity and spin foam models are nonperturbative approaches based on this action. The fundamental excitations, spin-network states, represent in fact quantum geometries that are degenerate almost everywhere. Indeed, strictly speaking states corresponding to nondegenerate geometries do not exist. Only complicated superpositions of polymerlike excitations approximate metric configurations such as $e_a^I = \delta_a^I$ in the weak sense given by *coarse graining* [53,54]: probing the state at low energies yields a metric manifold while the geometry is almost-everywhere degenerate ($e = 0$) at the Plank scale. The simple form of the action (48) in terms of these variables suggests that the resulting quantum theory could be rather restrictive.

If there are no ambiguities at the fundamental level, then how is one to recover the infinite series of higher dimensional operators in the effective action of gravity? Coarse graining would be the mechanism. In the semiclassical limit the quantum geometry states approximate a spacetime geometry when probed at sufficiently low energies. Deviations from the classical behavior due to quantum fluctuations will appear as higher powers of the curvature tensor corrections in the effective action because e^{-1} now exists in the coarse-grained sense. In this process only coarse graining would generate the higher curvature corrections in the effective action description. These terms should be calculable from the fundamental theory and the properties of the semiclassical states considered. In other words, the nonperturbative formulation of first order gravity (where ambiguities are controlled by a finite number of parameters) could play the role of renormalizable theory underlying the nonrenormalizable metric gravity. From this perspective we could expect that, as in 2 + 1 gravity, the (infinite-dimensional set of) regularization ambiguities in the quantization would have to be drastically reduced in the definition of $\mathcal{H}_{\mathrm{phys}}$. This question will have to be explored further in future work; our present results provide some supporting evidence in this direction.

F. Gravitons?

Let us conclude our discussion with a speculative interpretation of an intriguing type of solution to the quantum Hamiltonian constraint found in Sec. III C. We constructed an argument to rule out higher spin regularizations of the quantum Hamiltonian. The case $m = 1/2$ and $m = 1$ are special as they correspond to the fundamental representation of $SU(2)$ and $SO(3)$ respectively. Therefore, we might expect the quantization based on $m = 1$ to be of interest. In this case the solutions found in Sec. III C have a clear-cut interpretation as spin-two excitations. It would be interesting to further investigate the possibility of interpreting the solutions presented in (45) as the fundamental degrees of freedom leading, in the low energy limit, to the notion of graviton. Notice that if we assume that the continuum limit is dominated by four valent vertices (i.e., *quantum tetrahedra:* the simplest excitations of 3-volume), these solutions are labeled by two local quantum numbers as illustrated in Fig. 2. In this speculative interpretation we see the infinitesimal loop that is attached to the geometry by a link labeled with $\alpha = 2$ as a spin-two particle. Notice that this is fully analogous to the way in which spin $1/2$ fermions are coupled to the geometry [55,56].

[7] The canonical quantization of these theories directly leads to the fundamental variables of LQG: fluxes of non-Abelian electric field and generalized connections.

[8] Here we are assuming that there are no matter couplings. In order to couple the theory to standard matter one needs to use the inverse tetrad e^{-1} which is not a fundamental variable. Notice that fermions can be brought into the game without introducing the inverse tetrad.

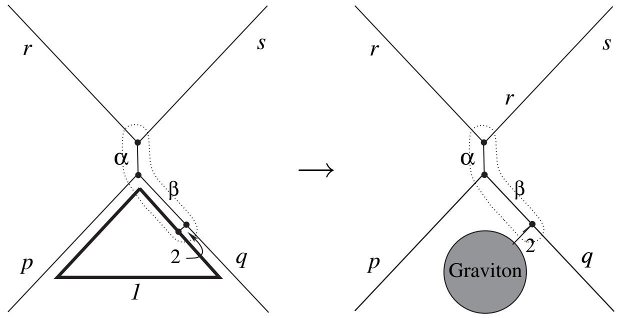


FIG. 2. Interpretation of the solutions (45) for $m = 1$ as graviton excitations. Starting from a solution to the constraints given by a diff-invariant spin network with a vertex with no exceptional edge we can construct a new solution as explained in Sec. III C and illustrated here. The solution space is parametrized by the quantum numbers α and β in this figure. The dotted region corresponds to a single point in the spin-network graph.

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