Complete integrability of higher-dimensional Einstein equations with additional symmetry and rotating black holes

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A new derivation of the five-dimensional Myers-Perry black-hole metric as a 2-soliton solution on a nonflat background is presented. It is intended to be an illustration of how the well-known Belinski-Zakharov method can be applied to find solutions of the Einstein equations in D-dimensional space-time with $D - 2$ commuting Killing vectors using the complete integrability of this system. The method appears also to be promising for the analysis of the uniqueness questions for higher-dimensional black holes.

I. INTRODUCTION

There is a number of reasons to be interested in finding exact black-hole solutions of higher-dimensional general relativity. In string theory our space-time has compactified additional dimensions. As was recently noted, the radii of compactification can be large, and one can check this possibility experimentally (see [1] for a review). Another reason is the recent discovery of a duality between the quantum gauge theory in usual space-time and the classical gravitation in five-dimensional anti−de Sitter space-time [2]. At last, the recent discovery of black rings [3], the rotating black-hole solutions with the event horizon of unusual topology $S^1 \times S^2$, showed that the no-hair theorems should be nontrivially generalized in the five-dimensional case, and this generalization can give us a better understanding of the reason why the no-hair theorems exist in the case of usual four-dimensional space-time. Motivated by the growing interest in the higher-dimensional gravitation, we present here a general method to find the solutions of Einstein equations in the presence of a sufficient degree of symmetry in all dimensions using their complete integrability.

The rotating black holes in usual four-dimensional space-time [4] are described by stationary axisymmetric metrics. They possess two Killing vectors, corresponding to the time t and the azimuthal angle ϕ, and the metric can be presented in the form, independent of t and ϕ and dependent only on two spacial coordinates ρ and z. These metrics satisfy the Einstein equations which in this case have the form of a nonlinear system for two-dimensional fields. The complete integrability of this system was shown by Belinski and Zakharov [5,6], who found the explicit formula for N-soliton solutions. The particular cases of these solutions cover many systems of high physical significance. Notably, the rotating Kerr black hole corresponds to a 2-soliton solution [6]. A natural generalization for an arbitrary number of space-time dimensions is

to consider D-dimensional space-times with $D - 2$ Killing vectors. The Einstein equations in this case have the same form for all values of D [7], and all that is known for $D = 4$ (in particular, the complete integrability and N-soliton solutions) can be easily generalized to $D > 4$. While several other approaches exist for the complete integrability in 4D general relativity (for a recent review and a list of references see [8]), it is the Belinski-Zakharov method which can be most easily, in the obvious way, generalized to higher dimensions. The idea to use the complete integrability in the higher-dimensional case is not new. Previously, it was applied most often in the context of Kaluza-Klein theory [9,10]. It was applied also to some particular cases of higher-dimensional theory. These particular cases include static (Weyl) solutions [11] and a class of five-dimensional solutions, which can be reduced to the four-dimensional case [12]. Closely related models of gravity with matter fields, arising as low energy limits of superstring theory, were considered in [13] (see also references therein).

Among the solutions with the required degree of symmetry there are rotating black holes in various dimensions. In $(D - 1)$-dimensional space a body (or a black hole) can rotate in $\left[\frac{D-1}{2}\right]$ mutually orthogonal planes along the same number of angular coordinates (where square brackets [ ] stand for the integer part). If the black hole is stationary, the rotation should not change the metric with time. The metric is thus independent of these $\left[\frac{D-1}{2}\right]$ angles and each angle corresponds to a Killing vector. Together with the Killing vector along the time coordinate this gives $\left[\frac{D+1}{2}\right]$ Killing vectors for a general rotating black hole in D-dimensional space-time: two for $D = 4$, three for $D = 5$, and $D = 6$, and so on. Therefore, the Einstein equations for black holes in 4 and 5 space-time dimensions have enough symmetries to be completely integrable, while for $D \geq 6$ the number of Killing vectors is not sufficient. The integrable equations with $D \geq 6$ describe Kaluza-Klein black holes, when $D - 5$ or $D - 4$ of the spacial dimensions are compactified on circles.

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Black holes in five-dimensional space-time are especially interesting because of the recently discovered black ring solutions [3]. These solutions, with an unusual topology of the event horizon, exist in addition to previously known 5D analog of the Kerr black hole, the Myers-Perry solution [14]. One may hope that the complete integrability by the inverse scattering method can help to answer the arising questions about the uniqueness of regular 5D black-hole solutions. The structure of both the black ring and the Myers-Perry solutions suggests that they can be interpreted as soliton solutions. As an illustration of the potential of the method we give here a new derivation of the $D = 5$ Myers-Perry metric as a 2-soliton solution on a simple static background. While originally Myers and Perry found their solution in a certain degree by guessing it, and then directly verifying its validity, our derivation is based on a regular method. We were not able to do the same for the general black ring solution, so this is left for future work.

The plan of the rest of the paper is as follows. In Sec. II we remind the reader following the work [5,6] how the inverse scattering method can be applied to general relativity in a space-time with a sufficient degree of symmetry, stressing that everything works equally well for all D. In Sec. III we apply this method to give a new derivation of the 5D Myers-Perry metric as a 2-soliton solution. We conclude in Sec. IV with a short summary of the results and a discussion of future perspectives.

II. INVERSE SCATTERING METHOD AND SOLITONS

In a space-time with n commuting Killing vectors one can always introduce a coordinate system, in which the metric is independent of n coordinates. We shall consider a D-dimensional space-time with $D - 2$ Killing vectors. In this case we can write down the metric in the form in which it depends on 2 coordinates only. We shall denote these coordinates ρ and z. If furthermore the Einstein equations are satisfied, the metric can be written down in the following simple form [7]:

$$-ds^2 = g_{ab}dx^a dx^b + f(d\rho^2 + dz^2), \qquad \det g = -\rho^2, \tag{1}$$

where, as usual, the summation over the repeating indices $a, b = 0, 1, \ldots, (D - 3)$ is assumed. The Einstein equations for this metric are equivalent to the following equations for the $(D - 2) \times (D - 2)$ matrix g_{ab}:

$$\partial_i(\sqrt{-g}g^{ab}\partial_i g_{bc}) = 0, \qquad i = \rho, z, \tag{2}$$

and for the conformal factor $f(\rho, z)$:

$$\partial_\rho \ln f = -\rho^{-1} + \frac{\rho}{4}(g_{ab,\rho}g_{cd,\rho} - g_{ab,z}g_{cd,z})g^{ac}g^{bd}$$
$$= -\rho^{-1} + (4\rho)^{-1}\operatorname{Tr}(U^2 - V^2), \tag{3}$$

$$\partial_z \ln f = \frac{\rho}{2}g_{ab,\rho}g_{cd,z}g^{ac}g^{bd} = (2\rho)^{-1}\operatorname{Tr}(UV).$$

Here the following notations were introduced for matrices:

$$U = \rho\partial_\rho gg^{-1}, \qquad V = \rho\partial_z gg^{-1}. \tag{4}$$

This system [Eqs. (2) and (3)] is well known for the usual space-time, $D = 4$. For higher dimensions these equations were derived recently by Harmark [7]. Evidently, the equations are the same for all dimensions, only the dimensionality of the matrix g_{ab} depends on D.

Equations (2) for g_{ab} do not contain f. If one solves them first, then one can substitute g_{ab} in the right-hand side of Eq. (3) and find f solving arising linear differential equations of first order. Equations (3) for the conformal factor f are mutually compatible when Eqs. (2) are satisfied for the matrix g_{ab}.

It is well known that system (2) is completely integrable. It follows from the fact that Eqs. (2) can be viewed as the compatibility condition for the following overdetermined system of linear differential equations [5,6]:

$$D_\rho \psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\psi, \qquad D_z \psi = \frac{\rho V + \lambda U}{\lambda^2 + \rho^2}\psi;$$

$$D_\rho = \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2}\partial_\lambda, \qquad D_z = \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2}\partial_\lambda. \tag{5}$$

Here $\psi(\rho, z, \lambda)$ is a complex square matrix, which is non-degenerate almost everywhere, $U(\rho, z)$ and $V(\rho, z)$ are real square matrices independent of λ. The complex parameter λ is called a "spectral parameter." It is easy to check that the above system is compatible if and only if there exists a matrix field $g(\rho, z)$ (which is identified with the metric) such that U and V can be derived from g via Eq. (4), and g satisfies Eq. (2). Note that the metric g can be easily extracted from ψ as $g = \psi(\rho, z, 0)$.

One can construct new solutions from known solutions by the following "dressing" procedure. One starts from a known solution g_0 and finds the corresponding ψ_0 by solving the linear equation (5). Then one looks for the new solution ψ in the form $\psi = \chi\psi_0$. Making this substitution into Eq. (5) results in the equations for χ:

$$D_\rho \chi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho U_0 + \lambda V_0}{\lambda^2 + \rho^2},$$
$$D_z \chi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho V_0 - \lambda U_0}{\lambda^2 + \rho^2}. \tag{6}$$

These equations have the following involution symmetry: if χ is a solution, then

$$\chi' = g\tilde{\chi}^{-1}(-\rho^2/\lambda)g_0^{-1}$$

is also a solution (the tilde denotes the matrix transposition). In general, any pair of solutions is related as $\chi'\psi_0 = \chi\psi_0 K(w)$, where $w = (\rho^2/\lambda - \lambda)/2 + z$, $D_{\rho,z}w = 0$, but

Belinski and Zakharov demanded an additional condition to be satisfied: $\chi' = \chi$. This condition can be rewritten as $g = \chi(\lambda)g_0\tilde{\chi}(-\rho^2/\lambda)$, and it is easy to see that it guarantees that the matrix g is symmetric.

To find the solitonic solutions of Eq. (6) one looks for χ that are rational functions of the spectral parameter, making the following ansatz [5,6]:

$$\chi = 1 + \sum_k \frac{R_k}{\lambda - \mu_k}, \tag{7}$$

where the positions of poles μ_k depend on the coordinates ρ and z. Each pole corresponds to a soliton, and the number of poles is the number of solitons. The coordinate dependence $\mu_k(\rho, z)$ can be extracted from Eq. (6) by substituting there Eq. (7), noting that the left-hand side must have only simple poles in λ as the right-hand side has, and thus the conditions $D_\rho\mu_k = D_z\mu_k = 0$ must be satisfied. Integrating these differential equations one finds $\mu_k = w_k - z \pm \sqrt{(w_k - z)^2 + \rho^2}$, where the constant w_k is the soliton position on the z axes. We shall refer to the cases of signs plus and minus before the square root in this expression as soliton and antisoliton, respectively.

Evidently, χ^{-1} must be also a rational function of λ. For the identity $\chi^{-1}(\lambda)\chi(\lambda) = 1$ to be satisfied at the points $\lambda = \mu_k$, the matrices $\chi^{-1}(\mu_k)$ and R_k must be degenerate: $\chi^{-1}(\mu_k)R_k = 0$. This means that R_k factorizes as $R_{ka}{}^b = n_a^{(k)}m^{(k)b}$. From Eq. (6) and from the identity $\chi^{-1}(\lambda)\chi(\lambda) = 1$ one can find the vectors $m^{(k)a}$ and $n_a^{(k)}$. The result is (for more details we refer the reader to [5,6])

$$m^{(k)a} = m_{0b}^{(k)}[\psi_0^{-1}(\mu_k, \rho, z)]^{ba}, \qquad n_a^{(k)} = \sum_l \mu_l^{-1}D^{kl}N_a^{(l)}, \tag{8}$$

where the notations $\Gamma_{kl} = m^{(k)a}g_{0ab}m^{(l)b}(\rho^2 + \mu_k\mu_l)^{-1}$, $N_a^{(l)} = m^{(l)c}g_{0ca}$ were introduced, vectors $m_{0a}^{(k)}$ consist of arbitrary constants, and D^{kl} is the inverse of the matrix Γ_{kl}: $D^{km}\Gamma_{ml} = \delta_l^k$. The final expression for the metric is [5,6]

$$g_{ab} = g_{0ab} - \sum_{k,l} D^{kl}\mu_k^{-1}\mu_l^{-1}N_a^{(k)}N_b^{(l)}. \tag{9}$$

It is useful to write down the expression for the inverse metric as well:

$$(g^{-1})^{ab} = (g_0^{-1})^{ab} - \rho^{-2}\sum_{k,l} m^{(k)a}D^{kl}m^{(l)b}. \tag{10}$$

An explicit formula can be written for the conformal factor f as well [5,6]. It can be shown that the ratio f/f_0 is proportional to the determinant $\det(\Gamma_{kl})$ and it depends on the arbitrary constants $m_{0a}^{(k)}$ only through this determinant.

In an important particular case, the background metric g_0 is static (diagonal), and each vector $m_{0a}^{(k)}$ has only one nonzero component. Then, the resulting metric g is also

diagonal, and it is obtained from g_0 simply by multiplying its diagonal elements g_{0aa} corresponding to nonzero elements of $m_{0a}^{(k)}$ by $-\rho^2/\mu_k^2$.

III. MYERS-PERRY BLACK HOLE IN FIVE DIMENSIONS

In the usual four-dimensional space-time the Kerr solution describing rotating black holes was rederived in [6] as 2 solitons on the flat Minkowski background. The Kerr solution has 2 parameters: mass m and rotation parameter a (the angular momentum is ma). The nonrotating Schwarzschild black hole is a particular case, the static solution with zero angular momentum $a = 0$. While it is very fortunate that for the Kerr solution the background metric is flat, there is no known reason for this to be true *a priori*, so this seems to be a mere accident. Indeed, it is easy to see that in five-dimensional space-time the analog of the Schwarzschild solution, the Tangherlini solution [15], cannot be obtained as a 2-soliton solution on flat Minkowski background.

The metric of the Schwarzschild-Tangherlini black hole in five-dimensional space-time has the form [7,15]:

$$g_{ab}^{\text{Sch}} = \text{diag}\left(-\frac{\mu_+}{\mu_-}, \mu_-, \frac{\rho^2}{\mu_+}\right),$$

$$f^{\text{Sch}} = \frac{\mu_-(\rho^2 + \mu_+\mu_-)}{(\rho^2 + \mu_+^2)(\rho^2 + \mu_-^2)}, \tag{11}$$

where $\mu_\pm = \sqrt{\rho^2 + (z \pm \alpha)^2} - z \mp \alpha$. It can be obtained as a 2-soliton solution on the following background metric:

$$g'_{0ab} = \text{diag}\left(-\frac{\mu_+}{\mu_-}, -\frac{\rho^2}{\mu_-}, -\mu_+\right). \tag{12}$$

This metric is obtained from (11) by dividing the $\phi\phi$ component by $(-\mu_-^2/\rho^2)$ and the $\psi\psi$ component by $(-\rho^2/\mu_+^2)$. In this way we effectively remove a soliton at $z = -\alpha$ and an antisoliton at $z = \alpha$ (cf. the end of the previous section). It is convenient to use as the background a simpler metric $g_{0ab} = -(\mu_-/\mu_+)g'_{0ab}$:

$$g_{0ab} = \text{diag}\left(1, \frac{\rho^2}{\mu_+}, \mu_-\right), \tag{13}$$

using the fact that the multiplication of a background metric by a function commutes with the operation of putting solitons on this background. The corresponding solution of Eqs. (5) is

$$\psi_{0ab} = \text{diag}\left(1, \lambda + \frac{\rho^2}{\mu_+}, \mu_- - \lambda\right). \tag{14}$$

On this background we place a soliton at $z = -\alpha$ and an antisoliton at $z = \alpha$. The corresponding vectors $m^{(1,2)a}$ defined in Eq. (8) are

$$m^{(1)a} = \left(T_+, 0, \frac{\Psi_+}{\mu_- - \mu_+}\right),$$

$$m^{(2)a} = \left(T_-, \frac{\Phi_- \mu_- \mu_+}{\rho^2(\mu_- - \mu_+)}, 0\right), \tag{15}$$

where $T_{\pm}$, Φ_-, and Ψ_+ are arbitrary constants. The particular choice $T_+ = T_- = 0$ corresponds to a Schwarzschild solution. In Eq. (15) we have set $m^{(1)}_{0\phi} = m^{(2)}_{0\psi} = 0$. Nonzero values of $m^{(1)}_{0\phi}$ and $m^{(2)}_{0\psi}$ give a family of singular solutions with two additional parameters.

It is useful to introduce the prolate spherical coordinates x and y that allows one to express $\mu_{\pm}$, z, and ρ as *rational* functions of these coordinates [7]:

$$\sqrt{\rho^2 + (z \pm \alpha)^2} = \alpha(x \pm y), \qquad z = \alpha xy,$$

$$\mu_{\pm} = \alpha(x \mp 1)(1 - y), \qquad \rho^2 = \alpha^2(x^2 - 1)(1 - y^2). \tag{16}$$

It is easy to see that the obtained solution can have singularities only on the $\rho = 0$ axis. This axis is naturally divided into three parts by the positions of solitons at the points $z = \pm \alpha$. In the coordinates (x, y) these three parts correspond to the values $x = 1$, $y = \pm 1$ [7]. These parts can be identified with *rods*, introduced in [7,16], and the analysis of possible singularities on the $\rho = 0$ axis can be reduced to the analysis of the *rod structure* of the solution. The rod structure was defined in [7] as follows. Because of the condition $\det g = -\rho^2$, the metric determinant vanishes on the $\rho = 0$ axis, and the metric has a zero eigenvalue there. If there are more than one zero eigenvalues there would be a curvature singularity. It was argued also in [7] that the corresponding eigenvector can change its direction only in a discrete set of points, because otherwise there would be singular intervals on the $\rho = 0$ axis. These points divide the $\rho = 0$ axis in parts called rods. The directions of the metric eigenvectors with zero eigenvalues are called the *directions of the rods*. The rods get a natural interpretation in terms of solitons. The rod end points (that were not already present in the background solution) coincide with the positions of solitons. At the same time, the existence of an explicit relation between the rod directions and the constant vectors $m^{(k)}_{0a}$ remains an open question.

Another question, related to no-hair theorems in five dimensions, arises naturally here. Evidently, the general solution of (2) is nonsolitonic: it corresponds to a continuous density of solitons on the $\rho = 0$ axis. However, such solutions have singular intervals on the axis of symmetry, so they describe not a black hole, but the metric outside a matter distribution (a rotating "star"). The background solution, in its turn, has to be built only from a *finite* number of rods, otherwise in the generic case the resulting metric would be singular. Thus, it is natural to suppose that the black holes correspond to *solitonic* solutions on some simple backgrounds. If this turns out to be true, this proba-

bly could help one to find the most general five-dimensional black-hole metric and to prove its uniqueness.

Let us return now to the analysis of the obtained solution. Using the evident freedom in rescaling of the arbitrary constants we chose the normalization condition

$$\Psi_+^2 \Phi_-^2 - 16\alpha^2 T_+^2 T_-^2 = 1.$$

It is also convenient to introduce the following 3 parameters:

$$\rho_0^2 = 4\alpha(4\alpha T_+^2 + \Psi_+^2)(4\alpha T_-^2 + \Phi_-^2),$$

$$a_1 = 4\alpha T_- \Psi_+, \qquad a_2 = 4\alpha T_+ \Phi_-. \tag{17}$$

These parameters are not independent:

$$\alpha = \tfrac{1}{4}\sqrt{(\rho_0^2 - a_1^2 - a_2^2)^2 - 4a_1^2 a_2^2}.$$

One still has the freedom to make linear transformations in the space of coordinates t, ϕ, and ψ. This transformation can be chosen in such a way that the rod structure of the resulting metric matches the rod structure of the flat space-time. Namely, we require that the rods at $y = 1$ and $y = -1$ have directions along the ϕ coordinate and along the ψ coordinate, respectively. The resulting linear transformation of the time and angle coordinates has the form:

$$t = t_{\text{new}} + 4\alpha T_- \Psi_+ \phi_{\text{new}} + 4\alpha T_+ \Phi_- \psi_{\text{new}},$$

$$\phi = \Psi_+ \Phi_+ \phi_{\text{new}} - 4\alpha T_- T_+ \psi_{\text{new}},$$

$$\psi = \Psi_+ \Phi_- \psi_{\text{new}} - 4\alpha T_- T_+ \phi_{\text{new}}. \tag{18}$$

In this way we obtain the metric of the Myers-Perry black hole in 5D space-time [7,14]:

$$g_{00} = -(4\alpha x + (a_1^2 - a_2^2)y - \rho_0^2)/\omega,$$

$$g_{\varphi\psi} = \tfrac{1}{2}a_1 a_2 \rho_0^2(1 - y^2)/\omega,$$

$$g_{0\varphi} = -a_1 \rho_0^2(1 - y)/\omega,$$

$$g_{0\psi} = -a_2 \rho_0^2(1 + y)/\omega, \tag{19}$$

$$g_{\varphi\varphi} = \frac{1 - y}{4}(4\alpha x + \rho_0^2 + a_1^2 - a_2^2 + 2a_1^2\rho_0^2(1 - y)/\omega),$$

$$g_{\psi\psi} = \frac{1 + y}{4}(4\alpha x + \rho_0^2 - a_1^2 + a_2^2 + 2a_2^2\rho_0^2(1 + y)/\omega),$$

where the subscript "new" for the coordinates was dropped and the notation $\omega = 4\alpha x + (a_1^2 - a_2^2)y + \rho_0^2$ was introduced. As explained in the previous section, the conformal factor can be obtained as

$$f = f^{\text{Sch}} \det\Gamma_{kl} / \det\Gamma_{kl}^{\text{Sch}} = \frac{\omega}{8\alpha^2(x^2 - y^2)}, \tag{20}$$

where the Schwarzschild solution corresponds to $T_{\pm} = 0$. This is in agreement with [7,14].

The Myers-Perry solution with a single nonzero angular momentum ($a_2 = 0$) was rederived recently in [12] using the complete integrability of the system. In this particular case the matrix g_{ab} has the block-diagonal form, and the

equations reduce effectively to the four-dimensional case. Then the authors of [12] applied the results of [17] to obtain the Myers-Perry metric with a single angular momentum parameter. In contrast to this previous work, the present paper considers genuinely the five-dimensional case of a black hole with two nonzero angular momenta, which cannot be reduced to four dimensions.

IV. CONCLUSIONS

The aim of the present paper was to attract attention to the potential applications of the complete integrability of Einstein equations in D-dimensional space-time with $D - 2$ commuting Killing vectors. The integrability can be seen by an obvious generalization of the well-known Belinski-Zakharov construction for the usual four-dimensional case. In particular, in this way an explicit formula for N-soliton solutions can be written down. As an illustration of the practical usefulness of this method for finding solutions to the Einstein equations we have derived the five-dimensional Myers-Perry black-hole metric as a 2-soliton solution on a static background. The method appears promising for the analysis of the uniqueness of five-dimensional black-hole solutions. It gives a new point of view on the important notion of the rod structure, identifying the rod end points with solitons.

The next step would be to find the general black ring metric as a soliton solution on a simpler background. This is especially interesting, because the currently known black ring metric, having only one nonzero angular momentum parameter, appears to be not the most general one [3]. To this end, one can try the same approach as the one that was used in this paper to derive the Myers-Perry solution. One can start from the static black ring solution and remove a few solitons and antisolitons from it. One obtains a new static solution that has to be used as the background in the dressing procedure. Then one can put the solitons back on this background at their initial positions, but this time with generic values of the arbitrary constants $m_{0a}^{(k)}$. All our attempts in this direction so far did not result in finding regular black ring solutions, giving only various families of singular ones. Further investigations are needed to see whether the regular black ring solutions can be obtained with a different combination of solitons and background metric or one has to modify the method.

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