Cosmological evolution of a torsion-induced quintaxion

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In an affine prolongation of general relativity, the minimal coupling of Dirac fields to gravity naturally provides an axial current interaction. We demonstrate that the cancellation of the translational curvature, i.e. torsion, in the chiral anomaly induces a dynamical axion coupled with gravitational strength. Because of a geometrical identity, our torsion-induced pseudoscalar couples to the Einstein equations with an effective energy-momentum tensor which automatically satisfies the quintessence condition $w < -1/3$ for the equation of state parameter. In a toy model of an axion-dominated Universe, this leads to an anharmonic oscillatory evolution for which the deceleration parameter is within the range of current observations.

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I. INTRODUCTION

The discovery of instanton solutions in non-Abelian gauge theories topologically classified by a Pontrjagin term of type $\vec{E}\cdot\vec{B}$ has posed a problem in quantum chromodynamics (QCD): The experimental data for the electric dipole moment of the neutron lead to the bound $\overline{\theta} := \theta_{\rm QCD} + {\rm ArgDet}\, M < 2\times 10^{-10}$ on the effective vacuum angle, after diagonalizing the quark masses. A nonzero θ would imply CP violation.

The Peccei-Quinn (PQ) solution [1] to the strong CP problem is to introduce a dynamical field, the axion a, as a pseudo-Nambu-Goldstone boson associated with a new global $U(1)_{\rm PQ}$ symmetry, spontaneously broken at a scale f_a. Nonperturbative effects of QCD induce a potential $U(\theta)$ whose minimum at $a := \overline{\theta} f_a$ cancel $\overline{\theta}$ and thus solve the strong CP problem.

It is characteristic for the axion that it couples derivatively to spinor matter, it couples nonderivatively to two gluons and it has, via the Primakoff effect, an effective coupling to two photons. Such a coupling is important from the point of view of a possible detection.

If the axion exist in its "invisible" form [2] and its energy scale f_a is not far from 10^{12} GeV, it may constitute a substantial fraction of the dark matter of the universe [3]. However, an experiment converting an axion into a single photon via the inverse Primakoff effect seems to have already excluded the $\mu{\rm eV}/c^2$ mass range as a possible constituent of the local dark matter halo [4].

The standard introduction of the axion through the spontaneous breaking of the Peccei-Quinn global chiral symmetry $U(1)_{\rm PQ}$ is, in some sense, unsatisfactory [5]; in particular, gravity may not respect such a global symmetry.

On the other hand, a dynamical pseudoscalar field arises automatically from the gravitational coupling of fermions to gravity in an affine generalization [6] of Einstein's general relativity (GR) to a Riemann-Cartan (RC) spacetime. As is well known, Dirac fields minimally coupled to gravity are only sensitive to the axial part $\mathcal{A}$ of the torsion, interacting with the standard axial fermion current, cf. Ref. [7]. In this paper, we therefore explore the possibility that the cancellation of the additional torsion pieces in the generalization of the chiral anomaly to RC spacetimes will transmute the axial torsion to a dynamical axion.

This line of reasoning has been considered before, notably by Duncan *et al.* [8]; for improvements we are using a recent reanalysis [9] of the axial anomaly with torsion and consistently work on the level of the Einstein-Cartan (EC) equations [10]. As a result of a geometrical identity, the effective Einstein equation couples to the torsion-induced axion in an unusual way, cf. Ref. [11], which implies the upper bound $w := p/\rho < -1/3$ on the equation of state parameter, as is required for a quintessential axion. Moreover, spin conservation of the Cartan equation corresponds to the usual dynamical equation for axions, coupled to the QCD and gravitational chiral anomaly. Therefore, and because the torsion potential necessarily couples with gravitational strength, we call this pseudoscalar a torsion-induced quintessential axion ("quintaxion").

Our paper is organized as follows: In Sec. II, we recall the dual reformulation of the non-Abelian Yang-Mills theory amended by the topological Pontrjagin term and the effective axion potential induced by QCD instanton effects. Already there, an equivalent coupling of the corresponding non-Abelian Chern-Simons term to the axial torsion is indicated. The minimal coupling of gravity to Dirac fields is briefly recapitulated in Sec. III. A coupling of the axial torsion $\mathcal{A} := \mathcal{A}_i dx^i$ to the axial current j_5^i is the only additional term in a prolongation to RC spacetimes. Up to normalizations, this formally runs parallel to the minimal coupling of the electromagnetic $U(1)$ potential A to the charge current j^i. In the case that $\mathcal{A}$ is a gradient, an axial-type derivative coupling to two fermions would result. In quantum field theory, however, the axial current is not conserved; the corresponding axial anomaly picks up a Pontrjagin-type term $\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}$ solely constructed from the axial torsion $\mathcal{A}$ as the only additional piece [9]. Then

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the full cancellation of torsion in the anomaly requires the axial torsion to be a gradient of a pseudoscalar, i.e., $\mathcal{A}_i \propto \partial_i\theta$, as shown in Sec. IV. This condition transmutes the quadratic axial torsion piece in the Einstein-Cartan equation into a dynamical axion, necessarily coupled with gravitational strength. Cartan's algebraic equation relates torsion to spin, i.e. to the axial current j_5^i in the case of Dirac fields. The conservation of Cartan's equation, due to the properties of the translation Chern-Simons term [12] and the axial anomaly, automatically converts into the usual dynamical axion equation coupled to the QCD and gravitational anomaly, as is shown in Sec. V. Because of a geometrical identity for the EC equation, the energy-momentum current of the torsion-induced axion contains negative pressure, which necessarily enforces the quintessence condition $w < -1/3$. In order to study this effect in some more detail, we consider in Sec. VI a toy model of an axion-dominated, spatially flat Friedman Universe. Solving numerically an exact third-order nonlinear evolution equation, we find *anharmonic* oscillations of our axion coupled to the periodic instanton potential U(θ).

Large parts of the paper employ differential forms in an anholonomic basis [6]; an appendix displays geometrical identities involving the axial torsion.

II. YANG-MILLS THEORY WITH TOPOLOGICAL TERM

Standard QCD is based on a non-Abelian gauge theory for the Yang-Mills-type *gluon* field strengths

$$G := dA + A \wedge A = \tfrac{1}{2} G_{ij}^J \lambda_J dx^i \wedge dx^j, \qquad (2.1)$$

defined in terms a Lie-algebra-valued one-form $A := A_i^J \lambda_J dx^i$ of a non-Abelian gauge group SU(N) with Lie generators λ_J. Their self- and anti-self-dual parts $G^{\pm} := \frac{1}{2}(G \pm {}^*G)$ involve the (anti)involutive Hodge dual $*$. Because of the discovery of instantons, the four-form

$$V_{\mathrm{Pontr}} := \tfrac{1}{2} dC = \tfrac{1}{2} Tr(G \wedge G) \qquad (2.2)$$

of the Pontrjagin type also needs to be considered. It is derived from the Chern-Simons term

$$C := \mathrm{Tr}(A \wedge G - \tfrac{1}{3} A \wedge A \wedge A), \qquad (2.3)$$

and violates *CP*. Instanton solutions satisfying $G^{\pm} = 0$ in a Euclidean space are classified topologically via a winding number $\int V_{\mathrm{Pontr}} \propto \mathbf{Z}$.

The Yang-Mills Lagrangian amended with such a topological term yields the self- and anti-self-dual combination

$$\tilde{L}_{\mathrm{YM}} := -\frac{1}{2}\mathrm{Tr}(G \wedge^* G) + \frac{\theta}{2} dC = \frac{\theta - 1}{8}\mathrm{Tr}(G^+ \wedge G^+) + \frac{\theta + 1}{8}\mathrm{Tr}(G^- \wedge G^-), \qquad (2.4)$$

where the coupling to the Pontrjagin term (2.2) is proportional to the so-called θ angle. Variational principles similarly to the Maxwell case provide the *Yang-Mills equations*

$$\mathcal{D}^* G = J, \qquad \mathcal{D} G \equiv 0. \qquad (2.5)$$

After integrating out the fermion fields, cf. Ref. [2], the generating functional for QCD including the term $\theta V_{\mathrm{Pontr}}$ induces an effective axion potential

$$U(\theta) = \Lambda_\theta (1 - \cos\theta). \qquad (2.6)$$

This potential displays a periodicity with a period of 2π, has a minimum at $\theta = 0$, as required, and leads to the induced axion mass of $m_a = \sqrt{\Lambda_\theta}/f_a$.

In the Abelian case, i.e. Maxwell's theory with local U(1) invariance, the four-potential is $A := A_i dx^i$ and the nonlinear term in the gauge curvature (2.1) drops out such that $F = dA$ remains, a relation valid also in RC spacetime or more general spacetime geometries, cf. Refs. [6,13]. Via the "electromagnetic anomaly," the axion has an *effective* coupling:

$$L_{a\gamma\gamma} = -\frac{\theta}{2} dC = -\frac{\theta}{2} F \wedge F = \frac{a}{f_a} \vec{E} \cdot \vec{B} \eta, \qquad (2.7)$$

to two photons which is *CPT*-invariant for a pseudoscalar θ. Accordingly, a radiation field satisfying $\vec{E} \cdot \vec{B} = 0$ will not interact with the axion. Because of the Primakoff effect (2.7), the axion is not a stable particle; its lifetime scales as $\tau(a \to 2\gamma) \sim f_a^5$.

Alternatively, in a RC spacetime one could tentatively consider a coupling of the axial torsion $\mathcal{A} = 2d\theta$ via

$$L_{a\gamma\gamma} = \tfrac{1}{4}\mathcal{A} \wedge C - \tfrac{1}{2} d(\theta C) = \tfrac{1}{4}\mathcal{A} \wedge A \wedge F - \tfrac{1}{2} d(\theta C), \qquad (2.8)$$

to the U(1) Chern-Simons term (2.3), cf. Eq. (15) of Ref. [14] for a similar four-dimensional interaction.

Although, the Chern-Simons three-form $C := A \wedge F$ is not gauge invariant, covariance[1] is typically recovered on the level of the field equations. For example, the interaction (2.8) without the boundary term provides the contribution $\delta L_{a\gamma\gamma}/\delta A = A \wedge d\mathcal{A}/4 - \mathcal{A} \wedge F/2$ in the gauge field equations which is covariant, provided the axial torsion is closed, i.e. $d\mathcal{A} = 0$. [The mixed boundary term $d\mathcal{A} \wedge F = d(\mathcal{A} \wedge dA)$ can be disregarded, since it would give only a trivial contribution to the field equations.] The equivalent form (2.8) of the axion-photon coupling will

[1]In the case that the vector $t_\alpha := e_\alpha \rfloor \mathcal{A} = m_a n_\alpha$ is constant in the direction n_α of a rotational invariant preferred frame, the corresponding amendment (2.8) of Maxwell's theory would account for a possibly anisotropy of electromagnetic propagation on a cosmological scale, cf. Jackiw [15]. However, a recent analysis [16] of cosmological data give rather stringent limits on possible violations of Lorentz invariance, accompanied by a possible *CPT* violation. From the relative mass difference in the neutral kaon system, nowadays' tightest bound on *CPT* violation is $<10^{-18}$, cf. Ref. [17] for new bounds from neutrino oscillations.

naturally arise from the minimal coupling of torsion to quantized Dirac fields, as we are going to demonstrate.

III. DIRAC FIELDS IN RIEMANN-CARTAN SPACETIME

In our notation, a Dirac field is a bispinor-valued zero-form ψ for which $\overline{\psi} := \psi^\dagger \gamma_0$ denotes the Dirac adjoint, $\mathcal{D} := D + iA\wedge$ accounts for the minimal coupling to the gauge (electromagnetic) potential, and $D\psi := d\psi + \Gamma \wedge \psi$ is the exterior covariant derivative with respect to the Riemann-Cartan connection one-form $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$, providing a minimal gravitational coupling with torsion.

The Dirac Lagrangian is given by the manifestly *Hermitian* four-form

$$L_{\mathrm{D}} = L(\gamma, \psi, \mathcal{D}\psi) = \frac{i}{2}\{\overline{\psi}{}^*\gamma \wedge \mathcal{D}\psi + \overline{\mathcal{D}\psi} \wedge^* \gamma\psi\} + m\overline{\psi}\psi\eta, \qquad (3.1)$$

where $\gamma := \gamma_\alpha \vartheta^\alpha$ is the Clifford algebra-valued coframe, cf. [18]. Since $L_{\mathrm{D}} = \overline{L}_{\mathrm{D}} = L_{\mathrm{D}}^\dagger$ even in an unholonomic frame, the minimal coupling prescribes us automatically the *Hermitian* charge current and standard axial current three-forms

$$j := \overline{\psi}{}^*\gamma\psi = j^\mu \eta_\mu \quad \text{and} \quad j_5 := \overline{\psi}{}^*\gamma\gamma_5\psi = j_5^\mu \eta_\mu, \qquad (3.2)$$

respectively, as in quantum electrodynamics (QED). In order to separate out the purely Riemannian piece from torsion terms, let us decompose the Riemann-Cartan connection $\Gamma = \Gamma^{\{\}} - K$ into the Riemannian (or Christoffel) connection $\Gamma^{\{\}}$ and the *contortion* one-form $K = \frac{i}{4}K^{\alpha\beta}\sigma_{\alpha\beta}$, obeying $D\gamma = [\gamma, K] = \gamma_\alpha T^\alpha$. Accordingly, the Dirac Lagrangian (3.1) splits [18] into a Riemannian and a spin-contortion piece:

$$\begin{aligned} L_{\mathrm{D}} &= L(\gamma, \psi, D^{\{\}}\psi) - \frac{i}{2}\overline{\psi}({}^*\gamma \wedge K - K \wedge {}^*\gamma)\psi + A \wedge j \\ &= L(\gamma, \psi, D^{\{\}}\psi) + \frac{1}{4}\mathcal{A} \wedge j_5 + A \wedge j \\ &= L(\gamma, \psi, D^{\{\}}\psi) - T^\alpha \wedge \mu_\alpha + A \wedge j. \end{aligned} \qquad (3.3)$$

The covariant derivative with respect to the Riemannian connection $\Gamma^{\{\}}$ satisfies $D^{\{\}}\gamma = 0$. Hence, in a RC spacetime a Dirac spinor only feels the *axial torsion* one-form

$$\mathcal{A} := \tfrac{1}{4}{}^*\mathrm{Tr}(\gamma \wedge D\gamma) = {}^*(\vartheta^\alpha \wedge T_\alpha) = \tfrac{1}{2}T^{[\alpha\beta\gamma]}\eta_{\alpha\beta\gamma} = \mathcal{A}_i dx^i, \qquad (3.4)$$

which is invariant under classical Weyl rescalings and *chiral transformations* $\gamma \to \gamma^\beta = e^{i\gamma^5\beta}\gamma e^{-i\gamma^5\beta}$ of the coframe, but odd under parity $P{:}\vartheta^B \to -\vartheta^B$, where $B = 1, 2, 3$. The *spin current* of the Dirac field is given by the Hermitian three-form

$$\tau_{\alpha\beta} := \frac{\partial L_{\mathrm{D}}}{\partial \Gamma^{\alpha\beta}} = \frac{1}{8}\overline{\Psi}({}^*\gamma\sigma_{\alpha\beta} + \sigma_{\alpha\beta}{}^*\gamma)\Psi = \frac{1}{4}\eta_{\alpha\beta\gamma\delta}\overline{\Psi}\gamma^\delta\gamma_5\Psi\eta^\gamma = \tau_{\alpha\beta\gamma}\eta^\gamma \qquad (3.5)$$

with *totally antisymmetric* components $\tau_{\alpha\beta\gamma} = \tau_{[\alpha\beta\gamma]}$. Equivalently, torsion merely[2] couples to the *spin-energy potential*

$$\mu_\alpha = \tfrac{1}{4}\vartheta_\alpha \wedge {}^*j_5, \qquad (3.6)$$

i.e. to a two-form that is proportional to the axial current j_5, cf. Ref. [7].

Let us tentatively assume that the dimensionless pseudoscalar θ serves as a potential [19] for the axial torsion via $\mathcal{A} = 2d\theta$. If confirmed quantum field-theoretically, there arises in (3.3) a derivative coupling of the would-be axion θ to two fermions via the *CPT*-invariant term

$$L_{a\psi\psi} = \frac{1}{2}d\theta \wedge j_5 = \frac{1}{2f_a}da \wedge \overline{\psi}{}^*\gamma\gamma_5\psi, \qquad (3.7)$$

exactly as in the usual formulation, where the axial current j_5 is the Noether current associated with a spontaneously broken Peccei-Quinn symmetry $\mathrm{U}(1)_{\mathrm{PQ}}$.

IV. AXIAL ANOMALY IN RIEMANN-CARTAN SPACETIME

In quantum field theory (QFT), however, the axial current is not conserved, rather there arises in RC spacetime the *axial anomaly*

$$\langle dj_5 \rangle = 2im\langle\overline{\psi}\gamma_5\psi\rangle\eta - \frac{1}{4\pi^2}\mathrm{Tr}(G \wedge G) - \frac{1}{96\pi^2}[2R^{\{\}}_{\alpha\beta} \wedge R^{\{\}\alpha\beta} + d\mathcal{A} \wedge d\mathcal{A}] \qquad (4.1)$$

for its vacuum expectation value. This result [9], which can easily be transferred to the chiral current $j_\pm$, is based on the Pauli-Villars regularization scheme. It deviates from the heat kernel method by terms which are not scale invariant, such as $M^2 d^* \mathcal{A}$. For the regulator mass $M \to \infty$, which corresponds to the high temperature limit [20], they would be *divergent* and therefore need to be discarded from the renormalized anomaly. The same applies to the higher order term $d\mathcal{K} = d^*\breve{D} \wedge^* \breve{D}^* \mathcal{A}$, where $\breve{D} = D^{\{\}} + i\mathcal{A}\gamma_5/4$ is a covariant derivative modified by the axial torsion.

Thus only the Weyl invariant term $d\mathcal{A} \wedge d\mathcal{A} = -2\vec{\mathcal{E}} \cdot \vec{\mathcal{B}}\eta$ for the axial torsion contributes to the axial anomaly, resembling the U(1) part $F \wedge F = dA \wedge dA$ of the Pontrjagin term (2.2).

[2]The irreducible vector or tensor pieces of the torsion do not couple to Dirac fields, but to the Rarita-Schwinger spinor-valued one-form $\Psi = \Psi_i dx^i$.

The torsion terms $d\mathcal{A}\wedge d\mathcal{A}$ and $d^*\mathcal{A}\wedge {}^*(d^*\mathcal{A}) = 4\ell^4 V_{\rm NY}\wedge {}^*V_{\rm NY}$ have been considered previously, as part of the Lagrangian (2.9) of Ref. [21], in order to make the axial torsion propagating. Because of the geometric identity (A5) for the Nieh-Yan term $d^*\mathcal{A} = 2\ell^2 dC_{\rm TT} = 2\ell^2 V_{\rm NY}$, the second term is really quartic in torsion and not scale invariant. In our approach, however, the Pontrjagin-type term $d\mathcal{A}\wedge d\mathcal{A}$ arises rather naturally from a careful analysis of the axial anomaly in RC spacetimes.

The renormalized conformal (or trace) anomaly [22]

$$\langle \vartheta^\alpha \wedge \sigma_\alpha\rangle = -\frac{1}{3\pi^2}\Big[\mathrm{Tr}(G\wedge {}^*G) + \frac{1}{24}(2R^{\alpha\beta\{\}}\wedge R^{\{\}\star}_{\alpha\beta} + d\mathcal{A}\wedge {}^*d\mathcal{A})\Big] \tag{4.2}$$

for the symmetric energy-momentum current σ_α receives, in addition to the Riemannian Euler term, a kinetic contribution of the Maxwell type from the axial torsion $\mathcal{A}$. The coefficients are similar to those in Eq. (4.1), due to the fact that chiral and trace anomalies constitute a supermultiplet [23].

A. The axion solution for a cancellation of the torsion in the anomaly

As is well known, by adding Chern-Simons-type terms to the axial current via

$$\hat{j}_5 := j_5 + \frac{1}{4\pi^2}C + \frac{1}{96\pi^2}(4C_{\rm RR} + \mathcal{A}\wedge d\mathcal{A}), \tag{4.3}$$

a conservation law can be obtained, even in the presence of the anomaly. For true gauge fields like A and $\Gamma^{\{\}}$, however, this procedure would spoil gauge invariance. Since the axial torsion $\mathcal{A}$ is *not* a gauge field, it is legitimate to absorb its contribution to the anomaly (4.1) into the redefined current. Then we find that

$$\langle d\hat{j}_5\rangle = \langle dj_5\rangle + \frac{1}{4\pi^2}\mathrm{Tr}(G\wedge G) + \frac{1}{96\pi^2}(d\mathcal{A}\wedge d\mathcal{A} + 2R^{\{\}}_{\alpha\beta}\wedge R^{\{\}\alpha\beta}) = 2im\langle\overline{\psi}\gamma_5\psi\rangle\eta, \tag{4.4}$$

i.e. that the new current is conserved for massless fermions or in the chiral limit, but would explicitly depend on the axial torsion.

A *complete cancellation* of the axial torsion in the anomaly occurs, however, not only for a closed $\mathcal{A}$, but more generally for

$$\mathcal{A} = 2e^{\varphi/f_\varphi}d\theta, \tag{4.5}$$

where the pseudoscalar θ is proportional to the would-be axion a and the scalar field φ will play the role of a dilaton. Since then $d\mathcal{A} = 2e^{\varphi/f_\varphi}d\varphi\wedge d\theta/f_\varphi$, this implies for the Pontrjagin-type term that $d\mathcal{A}\wedge d\mathcal{A} = d(\mathcal{A}\wedge d\mathcal{A}) = 0$ in (4.1), due to $d\theta\wedge d\theta = -d\theta\wedge d\theta \equiv 0$. For the same reason, the Chern-Simons-type term $\mathcal{A}\wedge d\mathcal{A}$ for the axial torsion gets completely removed in the redefined current (4.3). As expected, the kinetic term $d\mathcal{A}\wedge {}^*d\mathcal{A} = 4e^{2\varphi/f_\varphi}d\varphi\wedge d\theta\wedge {}^*(d\varphi\wedge d\theta)/f_\varphi^2$ in the trace anomaly (4.2) gets only removed for constant dilaton, i.e. $\langle d\varphi\rangle = 0$.

V. DYNAMICAL AXION FROM EINSTEIN-CARTAN THEORY

The Einstein-Cartan Lagrangian

$$V_{\rm EC} := -\frac{1}{2\kappa}R^{\alpha\beta}\wedge\eta_{\alpha\beta} = V_{\rm HE} + \frac{1}{12\kappa}\mathcal{A}\wedge {}^*\mathcal{A}, \tag{5.1}$$

where $\kappa = 8\pi G_{\rm N}$ is the gravitational constant, generalizes the metrical Hilbert-Einstein Lagrangian $V_{\rm HE}$ to a RC spacetime with torsion. Because of the geometric identities (A8) and (A9), the axial torsion $\mathcal{A}$ enters only algebraically.

The Einstein-Cartan equation [10]

$$G_\alpha := \tfrac{1}{2}R^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma} = \kappa\Sigma_\alpha, \tag{5.2}$$

coupled to the canonical energy-momentum current Σ_α of matter, is obtained by varying for the coframe ϑ^α. In RC spacetime, the Einstein current three-form G_α satisfies the first Noether identity

$$\widehat{D}G_\alpha \equiv \tfrac{1}{2}(e_\alpha\rfloor R^{\beta\gamma})\wedge\eta_{\beta\gamma\mu}\wedge T^\mu \tag{5.3}$$

with respect to the transposed connection $\widehat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha\rfloor T^\beta$, cf. Eq. (5.4.13) of Ref. [6]. Only for vanishing torsion, it reduces to the conservation law $D^{\{\}}G^{\{\}}_\alpha \equiv 0$ as a consequence of the contracted second Bianchi identity (A3).

By varying with respect to the linear connection $\Gamma^{\alpha\beta}$, we obtain the second field equation of EC theory, i.e. Cartan's algebraic relation

$$\eta_{\alpha\beta\gamma}\wedge T^\gamma = 2\kappa\tau_{\alpha\beta} \tag{5.4}$$

between torsion and the canonical spin of matter. Because of (3.5), in the case of Dirac fields this is equivalent[3] to

$$C_{\rm TT} = \frac{1}{4}j_5 \Leftrightarrow {}^*\mathcal{A} = \frac{\kappa}{2}j_5, \tag{5.5}$$

where $C_{\rm TT}$ is the translational Chern-Simons term (A7) with the fundamental length $\ell = \sqrt{\kappa}$. Classically, we then have $d^*\mathcal{A} = im\kappa\overline{\psi}\gamma_5\psi\eta$, which again vanishes for massless fermions or in the chiral limit.

Within EC theory, we will assume that spin conservation holds even at the level of QFT, i.e., that the vacuum expectation value satisfies

[3]Formally the same relation [18] arises in simple ($N = 1$) supergravity, where the three-form $j_5 := i\overline{\Psi}\wedge\gamma\wedge\Psi$ is given in terms of the Rarita-Schwinger one-form Ψ.

$$\langle dC_{\rm TT}\rangle = \tfrac{1}{4}\langle dj_5\rangle. \tag{5.6}$$

Commonly, topological invariants do not renormalize. Then the classical Nieh-Yan term $V_{\rm NY} = dC_{\rm TT}$ will surface *dynamically*, in contradistinction to the case of the scale-independent axial anomaly. Consequently, we find

$$d^*\mathcal{A} = \frac{\kappa}{2}\langle dj_5\rangle, \tag{5.7}$$

where the fermionic spin "drags some ... spin" of the axial torsion, and vice versa, similarly as in QED, cf. Ref. [24].

Let us insert here the axion solution (4.5) for the cancellation of the torsion in the axial anomaly (4.1). In the chiral limit this leads to

$$d(e^{\varphi/f_\varphi}{}^*d\theta) = -\frac{\kappa}{16\pi^2}\left[\mathrm{Tr}(G\wedge G) + \frac{1}{12}R^{\{\}}_{\alpha\beta}\wedge R^{\{\}\alpha\beta}\right], \tag{5.8}$$

i.e., our axion $a := \theta f_a$ induced by the torsion part of anomaly has become *dynamical* in the EC theory.[4]

For a constant dilaton $\langle d\varphi\rangle = 0$, we find the usual axion equation

$$d^*da = -\frac{1}{32\pi^2 f_a}\left[\mathrm{Tr}(G\wedge G) + \frac{1}{12}R^{\{\}}_{\alpha\beta}\wedge R^{\{\}\alpha\beta}\right] \tag{5.9}$$

coupled not only to the triangle anomaly of QCD, but also to the Pontrjagin four-form of Riemannian gravity. By comparing (5.8) with (5.9), the resulting PQ-type symmetry breaking scale

$$f_a = (e^{\varphi_0/2f_\varphi}/\sqrt{2\kappa}) \tag{5.10}$$

turns out to be close to the Planck energy of 10^{19} GeV, slightly rescaled by the vacuum expectation value $\varphi_0 = \langle\varphi\rangle$ of the dilaton and its coupling constant f_φ. A similar rather large scale emerges not only for the model-independent axion of superstrings, but also for axionic phantom energy [26,27].

Consequently, the axial torsion part $\mathcal{A}\wedge{}^*\mathcal{A} = 4d\theta\wedge{}^*d\theta$ in the decomposition of the EC Lagrangian (5.1) has become *dynamical* due to the anomaly. This makes quantum electrodynamics in RC spacetimes equivalent to QED on a *torsionless* spacetime geometry coupled to the axion. In contradistinction to Ref. [8], we have not imposed, via counterterms, the constraint $\langle d^*\mathcal{A}\rangle \simeq 0$ on the Nieh-Yan term (A5), which otherwise would induce Heisenberg's nonrenormalizable four-fermion contact interaction $j_5\wedge{}^*j_5$ into the action, cf. also Ref. [18]. Since the geometry couples, via the topological Pontrjagin term, back to the effective axion field equation (5.9), black holes may get restyled by "axion hair" of odd parity . Moreover, the geometrical Pontrjagin term can produce macroscopic effects for rotating mini-black holes [28].

Not only QCD instantons, but also electroweak instanton effects [29] in the supersymmetric standard model[5] can induce an effective axion potential $\mathrm{U}(\theta)$ such as (2.6). Since a Pontrjagin four-form constructed from the Riemannian curvature occurs in the axial anomaly, in Yang's quadratic generalization of gravity, gravitational instantons residing in Einstein spaces [31,32] may contribute to the effective axion potential, as well. In topological 4D self-dual models of gravity [33] including the Euler term, also torsional instantons occur, which may induce even nonminimal couplings [34] of torsion to gauge fields.

In any case, the induced potential $\mathrm{U}(\theta)$ will provide us with an instructive toy model for studying the cosmological evolution of the axion.

VI. AXION EVOLUTION IN A FRIEDMANN SPACETIME

Let us consider a homogeneous and isotropic cosmos of the Robertson-Walker type

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{6.1}$$

where $R(t)$ is the dimensionless expansion factor. An open, flat, or closed universe is characterized by $k = -1, 0, 1$, respectively. Recent observations [35] strongly favor a spatially *flat* cosmos, i.e., $k = 0$.

We are looking for solutions of the Einstein-Cartan equations (5.2). Using the decomposition (A10) for the axial torsion $\mathcal{A} = 2d\theta$ with the torsion-induced axion θ as potential, the *effective* axion-coupled Einstein equations are

$$G^{\{\}}_\alpha = \tfrac{1}{2}R^{\{\}\beta\gamma}\wedge\eta_{\alpha\beta\gamma} = \kappa[\sigma_\alpha + \sigma_\alpha(\theta)], \tag{6.2}$$

where $\sigma_\alpha = \Sigma_\alpha - D^{\{\}}\mu = \sigma_\alpha{}^\beta\eta_\beta$ is the symmetric Belinfante-Rosenfeld energy-momentum current. The three-form

$$\sigma_\alpha(\theta) = \frac{1}{3\kappa}\left(e_\alpha\rfloor d\theta\wedge{}^*d\theta - \frac{1}{3}d\theta\wedge e_\alpha\rfloor{}^*d\theta\right) \tag{6.3}$$

arises from the geometrical identity (A10) and is *not* of the Maxwell type $\Sigma^{\rm Max}_\alpha = [(e_\alpha\rfloor F)\wedge H - F\wedge(e_\alpha\rfloor H)]/2$, cf. Ref. [13]. The unusual form (6.3) accounts for the *stress-energy* current of the torsion-induced axion field, cf. Ref. [11] for generalizations to D dimensions. Thanks

[4]In effective *dual string models* [25], the theta angle θ of the topological boundary term θdC is liberated to a pseudoscalar, the Kalb-Ramond *axion*. Let us recall how it enters, besides the Chern-Simons terms, in the Kalb-Ramond three-form $H = dB - C - C^{\{\}}_{\rm RR}$. Anomaly cancellation for strings then requires the effective Bianchi identity $dH = \mathrm{Tr}(R^{\{\}}\wedge R^{\{\}} - G\wedge G)$. Since a coupling to the dilaton φ is also present, this would lead us via $H := e^{\varphi/f_\varphi}{}^*d\theta$ and compactification again to the effective axion Eq. (5.8).

[5]Superconnections [30] including the Higgs field may induce a related effect.

to the condition $\mathcal{A} = 2d\theta$ for a cancellation of torsion in the anomaly, the derivative term in the decomposition (A10) of the EC equation has dropped out due to the Poincaré lemma $dd \equiv 0$. Observe that the effective Einstein equations (6.2) are not conserved due to the torsion-dependent Mathisson-Papapetrou-type force [10] on the right-hand side of the Noether identity (5.3).

Let us consider the rather idealized scenario of an axion-dominated Universe where other matter can be disregarded, with the exception of a self-interaction potential $U(\theta)$ for the axion. Moreover, let us assume that the axion field depends only on cosmic time t, i.e. $\theta = \theta(t)$. Then, the only nonvanishing components of the energy-momentum tensor read

$$\rho_\theta = \sigma_{\hat{0}}{}^{\hat{0}} = \frac{(\dot{\theta})^2}{3\kappa} + U(\theta), \tag{6.4}$$

$$p_\theta = -\sigma_{\hat{1}}{}^{\hat{1}} = -\sigma_{\hat{2}}{}^{\hat{2}} = -\sigma_{\hat{3}}{}^{\hat{3}} = -\frac{(\dot{\theta})^2}{9\kappa} - U(\theta), \tag{6.5}$$

such that

$$p_\theta = -\tfrac{1}{3}[\rho_\theta + 2U(\theta)] \tag{6.6}$$

holds. For vanishing self-interaction $U(\theta)$, this is a "pseudorelativistic" equation of state as is characteristic for a photon gas, except for the rather "exotic" sign as a remnant of the torsional origin (A10) of our axion. A comparison with the equation of state $p_\theta = w\rho_\theta$ necessarily implies in our case the restriction

$$\rho_\theta = \frac{2(\dot{\theta})^2}{9\kappa(1+w)} \geq 0 \tag{6.7}$$

on the energy density. If the data would favor "phantom energy" [27] violating the dominant energy condition $\rho + p > 0$, an equation of state parameter $w < -1$ would require a purely imaginary axion, in order to render the energy density (6.7) positive.

A. Friedman-Raychaudhuri equations

Let us assume that $R(t) \neq 0$ in order to avoid "moving singularities," cf. Ref. [36]: Then, we can express our results in a more condensed form in terms of the Hubble expansion rate

$$H := \frac{\dot{R}(t)}{R(t)}. \tag{6.8}$$

Only the diagonal components of the effective Einstein equation (6.2) are nonvanishing. In general, the (0, 0) component is

$$3\left(H^2 + \frac{k}{R^2}\right) = \kappa\rho_\theta. \tag{6.9}$$

This equation is not dynamical, but describes the condition of zero total energy, as required by Einstein's equations. The spatial components yield the *Friedman equation*

$$2\dot{H} + 3H^2 + \frac{k}{R^2} = -\kappa p_\theta. \tag{6.10}$$

In view of $\mathcal{A} = 2d\theta$, Cartan's algebraic torsion equation (5.4) also has become dynamical: For vanishing dilaton and inducing the axion potential $U(\theta)$ via the topological terms, Eq. (5.8) converts into the nonlinear Klein-Gordon equation

$$\ddot{\theta} = -3H\dot{\theta} - \frac{dU}{d\theta} \tag{6.11}$$

for the axion, cf. Eq. (172) of Ref. [1] for a related evolution of the vacuum expectation value $\langle\theta\rangle$ in an expanding Universe. By eliminating the k/R^2 terms in Eqs. (6.9) and (6.10), we alternatively obtain the *Raychaudhuri equation*

$$\dot{H} + H^2 = \frac{\ddot{R}}{R} = -\frac{\kappa}{6}(\rho_\theta + 3p_\theta). \tag{6.12}$$

Together, they form a coupled system of second order differential equations.

In the special case of constant axion, i.e., $\dot{\theta} = 0$, Eq. (6.11) requires a constant axion potential $U(\theta) = \Lambda_\theta$ and, for a spatially flat Universe with $k = 0$, the (anti)de Sitter expansion $R = R_0 \exp(\mp\sqrt{\kappa\Lambda_\theta/3}t)$ emerges, familiar from inflation [37]. Generically, for $\dot{\theta} \neq 0$ we may eliminate from (6.11) the Hubble parameter and insert $H = -(\ddot{\theta} + dU/d\theta)/(3\dot{\theta})$ into the Raychaudhuri equation (6.12), in order to obtain the nonlinear third-order differential equation

$$\dot{\theta}\theta^{(3)} - \frac{1}{3}\left(4\ddot{\theta} + \frac{dU}{d\theta}\right)\left(\ddot{\theta} + \frac{dU}{d\theta}\right) + (\dot{\theta})^2\frac{d^2U}{d\theta^2} = \frac{\kappa}{2}(\rho_\theta + 3p_\theta)(\dot{\theta})^2. \tag{6.13}$$

B. Anharmonic oscillatory evolution

In the following, let us consider the periodic axion potential (2.6) normalized to $\Lambda_\theta = 1$.

For the initial conditions $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$, and $\ddot{\theta}(0) = -1$ of a maximal "misaligned" axion field at the onset of the big bang, we have solved numerically the resulting nonlinear differential equation (6.13) with MATHEMATICA 5.1 and plotted the result in Fig. 1. Shortly after the big bang, the axion starts oscillating around $\langle\theta\rangle = 0$. The corresponding axion density ρ_θ is drawn in Fig. 2, whereas Fig. 3 monitors the equation of state parameter $w := p_\theta/\rho_\theta$. In accordance with (6.6), it oscillates between -1 corresponding to a constant self-interaction $U = \Lambda_\theta$ at the origin and $-1/3$ for momentarily vanishing self-interaction.

With all this information at hand, we can easily infer from (6.9) together with the definition (6.8) the expansion factor

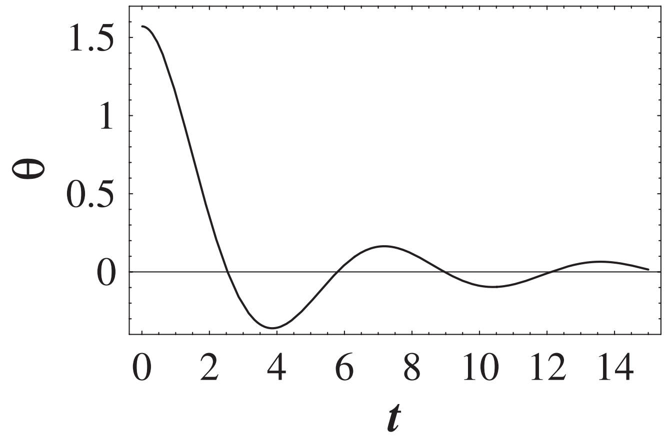

FIG. 1. Evolution of the dimensionless axion field θ after the big bang, initially "misaligned" by $\theta_0 = \pi/2$. Its oscillatory behavior reminds us of the Bessel function $J_0(t)$ of the first kind ($\kappa = \Lambda_\theta = 1$).

$$R = R_0 \exp\left\{\int_0^t \sqrt{\rho_\theta/3\kappa}dx\right\} \tag{6.14}$$

of our axion-dominated Universe. The drawing in Fig. 4 indicates a steadily increasing expansion in our toy model after the onset of the big bang. Small wiggles in the expansion are visible, but seem to be smoothed out due to the integration.

However, more precise information can be gained by considering the dimensionless *deceleration parameter*

$$q := -\frac{\ddot{R}R}{(\dot{R})^2} = -\frac{\ddot{R}}{RH^2} = \frac{\rho_\theta + 3p_\theta}{2(\rho_\theta - 3k/\kappa R^2)} \tag{6.15}$$

which, for $k = 0$, is related to the equation of state parameter w via $q = (1 + 3w)/2$. Accordingly, our axion-driven Universe oscillates between a maximal acceleration with $q = -1$, and a "momentarily" constant expansion rate with $q = 0$, cf. Fig. 5. This is a rather oversimplified

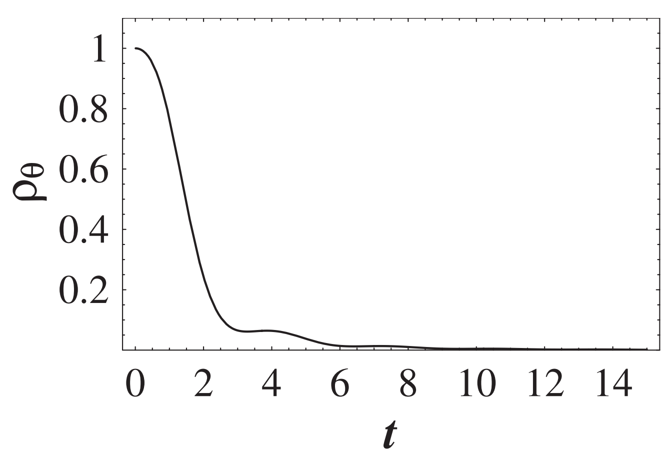

FIG. 2. Decay of the axion density ρ_θ during the evolution of the cosmos ($\kappa = \Lambda_\theta = 1$).

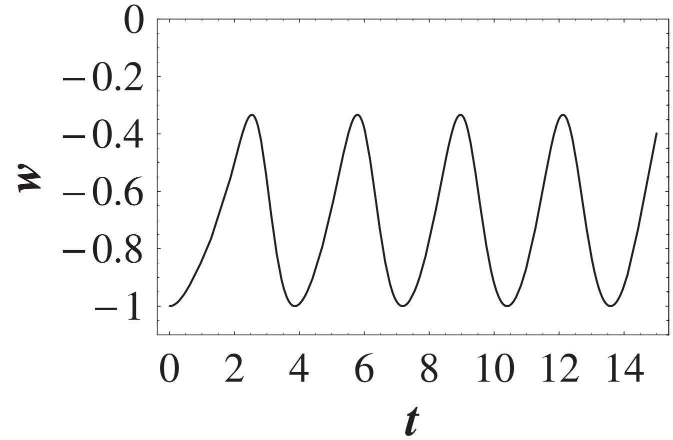

FIG. 3. Anharmonic oscillatory evolution of the equation of state parameter $w := p_\theta/\rho_\theta$ with ever decreasing periods.

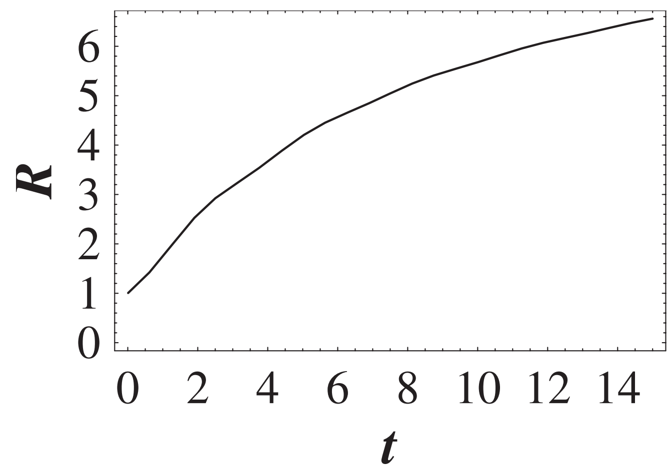

FIG. 4. Expansion of the axion-driven flat Universe with regions of accelerated and constant expansion, normalized to $R_0 = \kappa = \Lambda_\theta = 1$.

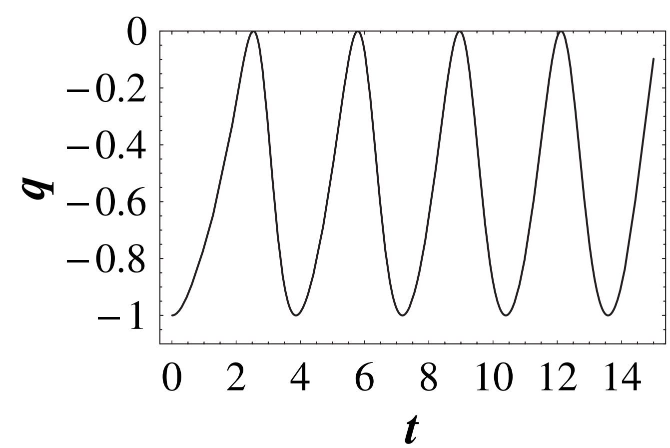

FIG. 5. The deceleration parameter q oscillates between constant expansion rate and acceleration.

model of the real Universe. However, also in the radiation-dominated era with the predescribed Hubble parameter $H(t) = 1/2t$ as "background," similar *anharmonic* oscillations of the axion following a modified Bessel function have been found [38]. It would be more realistic to start from a later-time Universe in which the axion is initially frozen close to the maximum of $\mathrm{U}(\theta)$ due to the "friction" term $H\dot{\theta}$ in Eq. (6.11) caused by matter. Such a more phenomenological analysis is left for the future.

Recent type Ia supernova observations [39] indeed reveal that our Universe has undergone a transition from deceleration to acceleration in the past, at redshift $z \simeq 0.5$. The accelerated expansion of the present epoch of our Universe corresponds to an equation of state parameter $w_{\rm DE} = -0.98 \pm 0.12$ for "dark energy," in concordance with all recent combined data [35]. This is rather close to a cosmological constant Λ of Einstein with $w_\Lambda = -1$, nowadays not at all regarded as his "biggest blunder." In our toy model, it surfaces as a "momentary" phase due to the axion potential induced by instanton effects in QCD or from the nontrivial topology of quantum gravity, as well.

VII. DISCUSSION

Why can we interpret the potential θ of an irreducible piece $\mathcal{A} = 2d\theta$ of the enigmatic torsion as a torsion-induced axion? First, by construction it is a pseudoscalar, since $\mathcal{A}$ is an axial covector, odd under space reflections P. Second, due to the axial anomaly, it has the same interaction (2.7) with two photons as the PQ or invisible axion, and therefore would be produced in the Sun via the Primakoff process. Only the interaction strength $g_a = 1/f_a = \sqrt{2\kappa}e^{-\varphi_0/2f_\varphi}$ of its coupling to two photons is, due to (5.10), much less than the upper limit of 10^{-10} GeV^{-1} recently reported by the group [40] running the CERN Axion Solar Telescope (CAST) for the mass range $m_a \leq 0.02$ eV$/c^2$. Third, it couples in the usual manner (3.7) to two fermions, albeit a similarly reduced coupling strength.

The underlying Einstein-Cartan framework is macroscopically indistinguishable from GR, since torsion is not propagating there and, via the Cartan equation (5.4), *very weakly* coupled to the spin density. However, in our model the axial torsion becomes dynamically due to the anomaly. Then, we can infer from (3.3) that the torsion-fermion coupling is rather weak, whereas the effective torsion mass occurring in (5.1) is almost of the Planck scale. This is well within the recent collider bounds [41] on the coupling parameters of a dynamical axial torsion. After its transmutation to a torsion-induced axion, a decay constant (5.10) close to the reduced Planck scale is inherited.

In this respect, it shares some features in common with the model-independent axion of superstrings, including its problems. However, there are scenarios [42,43] in which an ultralight axion or pseudo-Nambu-Goldstone boson with $m_a = \sqrt{\Lambda_\theta}/f_a < 3H_0$ can avoid the usual cosmological constraints [4,44]. In the supersymmetric standard model [29], the potential (2.6) is scaled to $\Lambda_\theta \simeq \Lambda = (3 \times 10^{-3}\ \mathrm{eV})^4$ typical for a *quintessence axion*. In the context of string cosmology, massless axions are able to seed the observed anisotropy of the cosmic microwave background. In the pre-big bang model of Gasperini and Veneziano [45,46], there exists a branch of a less efficient relaxation of an ultralight axion with a mass given by $m_a = 7.4 \times (10^7\ \mathrm{GeV}/f_a)\ \mathrm{eV}/c^2 > 10^{-11}\ \mathrm{eV}/c^2$ for which the axion energy density remains well undercritical, i.e. $\Omega_a < 0.1$. Such low mass constituents could possibly be "weighted" by the maximal total mass $M \simeq 16/(\kappa m_a)$ of a stable Bose-Einstein condensation of astrophysical scale, i.e., via the so-called axidilaton stars [47]. More recently, the embedded "hidden" sector [2] has been interpreted [48] as containing a (hidden sector) axion a_h which mixes with the model-independent axion such that an ultralight quintessential axion with similar properties emerges.

For the interpretation of its cosmological implications, it is important to stress that our torsion-induced axion θ automatically satisfies the *quintessence condition* $w < -1/3$ as a result of the unusual axial torsion coupling (6.3) in the EC equations. More precisely, the equation of state parameter oscillates anharmonically in the range $-1 \leq w < -1/3$, marginally within the bounds of recent data [39] on type SnIa supernovae. As a consequence, our toy model of the Universe encounters phases of constant expansion and "momentary" acceleration, the latter apparently the case in our present epoch. Marginally, such an oscillatory evolution may fit better to the combined data than the predictions of the standard ΛCDM model. Moreover, our cosmological oscillations are rather conventional, since they do not cross the "demarcation" line $w < -1$ of phantom energy [26]. Although such a remote possibility is still permitted [49] by observations, it seems to be already disfavored by the first year data of the Supernova Legacy survey [50].

Summarizing, our "quintaxion" resembles in many aspects the model-independent axion of superstrings or M-theory. However, in our straightforward four-dimensional approach, no supersymmetry nor higher dimensions are needed; only a torsional prolongation of Einstein's GR as envisioned by Cartan [51]. Nevertheless, our exact results for the evolution of an axion-dominated Friedman cosmos may, as well, provide further clues for the pre-big bang cosmology [46], or need to be embedded into the supersymmetric standard model with a hidden sector [48], in order to avoid cosmological constraints [44].

ACKNOWLEDGMENTS

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APPENDIX: AXIAL TORSION IDENTITIES IN RIEMANN-CARTAN SPACETIME

The coframe $\vartheta^\alpha = e_j{}^\alpha dx^j$ of dimension (length) and the dimensionless connection one-form $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta dx^i$ are the gauge potentials of nonlinearly realized *local translations* [52] and *local linear transformations*, respectively. The dual basis $\{\eta_\alpha, \eta_{\alpha\beta}, \eta_{\alpha\beta\gamma}, \eta_{\alpha\beta\gamma\delta}\}$ of exterior forms can be generated from the volume four-form $\eta = \eta_{\alpha\beta\gamma\delta}\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta/4!$ by consecutive interior products: $\eta_\alpha := e_\alpha \rfloor \eta = {}^*\vartheta_\alpha$, etc.

The translational field strength is the *torsion* two-form

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \tfrac{1}{2}T_{ij}{}^\alpha dx^i \wedge dx^j, \tag{A1}$$

of dimension (length), whereas the Riemann-Cartan *curvature* two-form

$$R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta = \tfrac{1}{2}R_{ij\alpha}{}^\beta dx^i \wedge dx^j \tag{A2}$$

is dimensionless. These field strengths obey the *first* and *second Bianchi identities*

$$DT^\alpha \equiv R_\gamma{}^\alpha \wedge \vartheta^\gamma, \qquad \text{and} \qquad DR^{\alpha\beta} \equiv 0. \tag{A3}$$

The RC connection $\Gamma^{\alpha\beta} = \Gamma^{\{\}\alpha\beta} - K^{\alpha\beta}$ can be split into the unique Levi-Civita connection $\Gamma^{\{\}\alpha\beta}$ of Riemannian geometry and a *contortion* one-form $K_{\alpha\beta} = -K_{\beta\alpha}$ implicitly related to torsion via $T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta$. In turn, the RC curvature two-form

$$R^{\alpha\beta} = R^{\{\}\alpha\beta} - D^{\{\}}K^{\alpha\beta} - K^\alpha{}_\mu \wedge K^{\mu\beta} \tag{A4}$$

decomposes into the Riemannian curvature $R^{\{\}\alpha\beta}$ plus contortion pieces.

The Lagrangians corresponding to the Bianchi identities (A3) are the boundary terms

$$dC_{\rm TT} = \frac{1}{2\ell^2}(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta) =: V_{\rm NY}, \tag{A5}$$

$$\begin{aligned} dC_{\rm RR} &= \frac{1}{2}R_{\alpha\beta} \wedge R^{\alpha\beta} =: V_{\rm Pontr} \\ &= \frac{1}{2}R^{\{\}}_{\alpha\beta} \wedge R^{\{\}\alpha\beta} + \frac{1}{12}d\Big[{}^*\mathcal{A} \wedge R^{\{\}} - \frac{1}{3}\mathcal{A} \wedge d\mathcal{A} \\ &\quad + \frac{1}{9}{}^*\mathcal{A} \wedge {}^*(\mathcal{A} \wedge {}^*\mathcal{A})\Big]. \end{aligned} \tag{A6}$$

The latter contains, among others, a term proportional to the curvature scalar $R := {}^*(R^{\alpha\beta} \wedge \eta_{\beta\alpha})$ and the axial torsion piece $d\mathcal{A} \wedge d\mathcal{A}$ of the axial anomaly with a relative factor 9 as required by the supersymmetric path integral [53], cf. also Ref. [54].

The translational Chern-Simons term

$$C_{\rm TT} := \frac{1}{2\ell^2}(\vartheta^\alpha \wedge T_\alpha) = -\frac{(-1)^s}{2\ell^2}{}^*\mathcal{A} \tag{A7}$$

in (A5) is not Weyl invariant, cf. (3.14.9) of Ref. [6], due to the occurrence of a fundamental length ℓ. Up to normalizations, the four-forms (A5) and (A6) are known as Nieh-Yan [12] and gravitational Pontrjagin term, respectively.

The *geometric identity*

$$\begin{aligned} R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} &\equiv R^{\alpha\beta} \wedge \eta_{\alpha\beta} - K^{\alpha\mu} \wedge K_\mu{}^\beta \wedge \eta_{\alpha\beta} \\ &\quad + K^{\alpha\beta} \wedge T^\gamma \wedge \eta_{\alpha\beta\gamma} + d(K^{\alpha\beta} \wedge \eta_{\alpha\beta}) \\ &= R^{\alpha\beta} \wedge \eta_{\alpha\beta} + T^\alpha \wedge {}^*(-{}^{(1)}T_\alpha + 2{}^{(2)}T_\alpha \\ &\quad + \tfrac{1}{2}{}^{(3)}T_\alpha) + 2d(\vartheta^\alpha \wedge {}^*T_\alpha), \end{aligned} \tag{A8}$$

relating the Hilbert-Einstein and the EC Lagrangian to proper teleparallelism, is a consequence of (A4). Here ${}^{(i)}T_\alpha$ are the three irreducible pieces of the torsion, cf. Ref. [6]. In particular, ${}^{(3)}T_\alpha = [(-1)^s/3]^*(\vartheta_\alpha \wedge \mathcal{A})$ is the irreducible axial torsion two-form algebraically related to the *axial torsion* one-form (3.4).

In RC space(times) where only axial torsion is present, the boundary term drops out due to $\vartheta^\alpha \wedge \vartheta_\alpha = 0$ and ${}^{(3)}K_{\alpha\beta} = -[(-1)^s/6]^*(\vartheta_\alpha \wedge \vartheta_\beta \wedge \mathcal{A})$. Then the identity (A8) reduces to

$$R^{\alpha\beta} \wedge \eta_{\alpha\beta} = R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{(-1)^s}{6}\mathcal{A} \wedge {}^*\mathcal{A}. \tag{A9}$$

This has been corroborated by EXCALC/REDUCE [55]. Likewise, the Einstein-Cartan three-form

$$\begin{aligned} G_\alpha &:= \frac{1}{2}R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} \\ &= G^{\{\}}_\alpha + \frac{(-1)^s}{12}\Big(e_\alpha \rfloor \mathcal{A} \wedge {}^*\mathcal{A} - \frac{1}{3}\mathcal{A} \wedge e_\alpha \rfloor^* \mathcal{A}\Big) \\ &\quad + \frac{(-1)^s}{6}\vartheta_\alpha \wedge d\mathcal{A} \end{aligned} \tag{A10}$$

decomposes into the Einstein three-form $G^{\{\}}_\alpha = G_\alpha{}^\beta \eta_\beta$ with respect to the Riemannian connection and axial torsion pieces, cf. Eq. (21) of Ref. [56].

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