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Reliability of the Langevin perturbative solution in stochastic inflation

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A method to estimate the reliability of a perturbative expansion of the stochastic inflationary Langevin equation is presented and discussed. The method is applied to various inflationary scenarios, as large field, small field, and running mass models. It is demonstrated that the perturbative approach is more reliable than could be naïvely suspected and, in general, only breaks down at the very end of inflation.

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I. INTRODUCTION

The stochastic approach to inflation [1–5] is an efficient method to study how quantum effects can influence the dynamics of the scalar field driving the acceleration of the early Universe. This formalism is based on a Langevin equation which describes the evolution of a spatially averaged field (typically over a Hubble patch), the so-called "coarse-grained" field. Solving this equation, especially when the backreaction of this coarse-grained field on the background geometry is taken into account, is notoriously known as a difficult task and various methods have been proposed in the literature; see for instance Refs. [6–9].

Recently, a method based on a perturbative expansion in the noise was presented [10]. At second order in the noise, this method is powerful enough to ensure the calculation of the probability density function of the coarse-grained field for arbitrary potentials. It was demonstrated that, in order to obtain explicit analytical expressions, the calculation of only one quadrature is necessary. If, in addition, the volume effects are also determined, then only one more quadrature is required. It turns out that these quadratures are feasible for a large class of inflationary models, for instance in the cases of the chaotic [11], new [12], hybrid [13], and running mass [14] scenarios. The stochastic effects in these models were studied in Ref. [10], where the evolution of the corresponding distributions was discussed in detail.

An important question concerns the domain of validity of the perturbative expansion used in order to obtain the above results. The aim of this article is to develop a method to treat this question and to estimate when the perturbative expansion gives reliable results. It is worth noticing that so far (and this is also valid for the other approaches used in the literature to solve the Langevin equation) this issue has never been addressed elsewhere. In general, the approximate expression for the probability function is derived without worrying about its accuracy. It will be shown that the method of Ref. [10] gives, most of the time, a better approximation than it naïvely could be guessed on general grounds and only breaks down at the very end of inflation.

Our article is organized as follows: In Sec. II, we briefly recall the main results and equations obtained in Ref. [10]. Then, in Sec. III, we present our method to study the accuracy of the perturbative expansion and apply it to the inflationary models discussed in Ref. [10], to wit, chaotic, new, hybrid, and running mass scenarios. Finally, in Sec. IV, we present our conclusions.

II. SOLVING THE LANGEVIN EQUATION

In stochastic inflation, one is interested in the behavior of a coarse-grained field φ obtained after taking the spatial average of the original inflaton field over a volume the size of which is of the order of a Hubble patch. The coarse-grained field obeys a Langevin equation that can be written as

$$\dot{\varphi} + \frac{1}{3H}\frac{\mathrm{d}V}{\mathrm{d}\varphi} = \frac{H^{3/2}}{2\pi}\xi(t), \tag{1}$$

where V is the inflaton potential and ξ a white noise defined such that its correlation function simply reads $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$, $\delta(z)$ being the Dirac distribution. The backreaction can be seen in the fact that the Hubble parameter H in Eq. (1) depends on the coarse-grained field φ via the slow-roll Friedmann equation $H^2 \sim \kappa V(\varphi)/3$, where $\kappa = 8\pi/m_{\mathrm{Pl}}^2$.

The method proposed in Ref. [10] (see also Ref. [15] for earlier attempts) consists in expanding the coarse-grained field in powers of the noise according to

$$\varphi(t) = \varphi_{\mathrm{cl}}(t) + \delta\varphi_1(t) + \delta\varphi_2(t) + \cdots, \tag{2}$$

where φ_{cl} is the classical solution, i.e. the one obtained when the noise is "switched off" in the Langevin equation. The quantities $\delta\varphi_1$ and $\delta\varphi_2$ are, respectively, first and second order in the noise. They obey the equations

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$$\frac{\mathrm{d}\delta\varphi_1}{\mathrm{d}t} + \frac{2}{\kappa}H''(\varphi_{\mathrm{cl}})\delta\varphi_1 = \frac{H^{3/2}(\varphi_{\mathrm{cl}})}{2\pi}\xi(t), \tag{3}$$

and

$$\frac{\mathrm{d}\delta\varphi_2}{\mathrm{d}t} + \frac{2}{\kappa}H''(\varphi_{\mathrm{cl}})\delta\varphi_2 = -\frac{H'''(\varphi_{\mathrm{cl}})}{\kappa}\delta\varphi_1^2 + \frac{3}{4\pi}H^{1/2}(\varphi_{\mathrm{cl}})H'(\varphi_{\mathrm{cl}})\delta\varphi_1\xi(t), \tag{4}$$

where a prime denotes a derivative with respect to the field. These equations can be solved easily since they are (by definition) linear. The expansion could, of course, be pushed to higher orders if necessary.

In Ref. [10], it was demonstrated that this formalism allows us to calculate the probability density function P_{c} of the coarse-grained field in a single Hubble patch. At second order in the noise, it is given by

$$P_{\mathrm{c}}(\varphi, t) = \frac{1}{\sqrt{2\pi\langle\delta\varphi_1^2\rangle}}\exp\left[-\frac{(\varphi - \varphi_{\mathrm{cl}} - \langle\delta\varphi_2\rangle)^2}{2\langle\delta\varphi_1^2\rangle}\right], \tag{5}$$

where the variance $\langle\delta\varphi_1^2\rangle$ and the mean $\langle\delta\varphi_2\rangle$ appearing in the above expression can, respectively, be written as

$$\langle\delta\varphi_1^2\rangle = \frac{\kappa}{2}\left(\frac{H'}{2\pi}\right)^2\int_{\varphi_{\mathrm{cl}}}^{\varphi_{\mathrm{in}}}\mathrm{d}\psi\left(\frac{H}{H'}\right)^3, \tag{6}$$

$$\langle\delta\varphi_2\rangle = \frac{H''}{2H'}\langle\delta\varphi_1^2\rangle + \frac{H'}{4\pi m_{\mathrm{Pl}}^2}\left[\frac{H_{\mathrm{in}}^3}{(H'_{\mathrm{in}})^2} - \frac{H^3}{(H')^2}\right]. \tag{7}$$

This formalism also permits the calculation of volume effects. If, instead of considering the distribution of the field in a single domain, we want to have access to its spatial distribution, one must weigh the single-domain distribution by the volume of each Hubble patch. This leads to the definition

$$P_{\mathrm{v}}(\varphi, t) = \frac{\left\langle\delta(\varphi - \varphi[\xi])\mathrm{e}^{3\int\mathrm{d}\tau H(\varphi[\xi])}\right\rangle}{\left\langle\mathrm{e}^{3\int\mathrm{d}\tau H(\varphi[\xi])}\right\rangle}. \tag{8}$$

Then, it was shown in Ref. [10] that, at second order in the noise, P_{v} takes the form

$$P_{\mathrm{v}}(\varphi, t) = \frac{1}{\sqrt{2\pi\langle\delta\varphi_1^2\rangle}}\exp\left[-\frac{(\varphi - \langle\varphi\rangle - 3\mathcal{V})^2}{2\langle\delta\varphi_1^2\rangle}\right], \tag{9}$$

where $\langle\delta\varphi_1^2\rangle$ and $\langle\varphi\rangle = \varphi_{\mathrm{cl}} + \langle\delta\varphi_2\rangle$ are still given by Eqs. (6) and (7). The term $\mathcal{V}$ describing the correction to the mean value due to volume effects can be written as

$$3\mathcal{V} = \frac{12H'}{m_{\mathrm{Pl}}^4}\int_{\varphi_{\mathrm{cl}}(t)}^{\varphi_{\mathrm{in}}}\mathrm{d}\psi\frac{H^4}{(H')^3} - 12\pi\frac{H}{H'}\frac{\langle\delta\varphi_1^2(t)\rangle}{m_{\mathrm{Pl}}^2}. \tag{10}$$

Therefore, as already mentioned, estimating the volume effects merely requires the calculation of one additional quadrature.

In Ref. [10], the results briefly described above have been applied to various concrete inflationary models. In particular, the potential

$$V(\varphi) = M^4\left[a + b\left(\frac{\varphi}{\mu}\right)^n\right], \tag{11}$$

where $a = 0, 1$ and $b = \pm 1$ has been considered. The case $a = 0$, $b = 1$ corresponds to large field (LF) models (or "chaotic inflation") [11], $a = 1$, $b = -1$ to small field (SF) models (as "new inflation") [12], and $a = 1$, $b = 1$ to hybrid inflation [13]. The scale M is fixed by the Wilkinson Microwave Anisotropy Probe normalization. The case of running mass inflation [14], namely,

$$V(\varphi) = M^4\left[1 - \frac{c}{2}\left(-\frac{1}{2} + \ln\frac{\varphi}{\varphi_0}\right)\frac{\varphi^2}{M_{\mathrm{Pl}}^2}\right], \tag{12}$$

was treated also. In the expression of the potential, $M_{\mathrm{Pl}} \equiv m_{\mathrm{Pl}}/\sqrt{8\pi}$ and the quantities c (which can be positive or negative) and φ_0 are free parameters. Running mass inflation can be realized in four classical versions and stochastic effects have been studied in Ref. [10] for the first ($c > 0$, $\varphi_{\mathrm{cl}} < \varphi_0$) and the second ($c > 0$, $\varphi_{\mathrm{cl}} > \varphi_0$) scenarios (RM1 and RM2) .

For the models described above, the behavior of P_{c} and P_{v} have been investigated in details in Ref. [10]; see, in particular, Figs. 2 and 3. As mentioned in the introduction, the issue that we now address is the reliability of the method of approximation used in order to establish these results.

III. RELIABILITY OF THE EXPANSION

An important question is the determination of the interval in which the perturbed solution of the Langevin equation that we have obtained, $\varphi_{\mathrm{cl}} + \delta\varphi_1 + \delta\varphi_2$, remains a good approximation of the exact one. Indeed, initially, the perturbed solution is "by definition" a good approximation since we have $\delta\varphi_1(\varphi_{\mathrm{in}}) = \delta\varphi_2(\varphi_{\mathrm{in}}) = 0$. Then, as the field evolves from φ_{in}, we expect $\delta\varphi_1$ and $\delta\varphi_2$ to grow and the approximation to break down at some value of $\varphi_{\mathrm{cl}} \neq \varphi_{\mathrm{in}}$. *A priori*, the criterion of validity is simply $\delta\varphi_2 < \delta\varphi_1 < \varphi_{\mathrm{cl}}$. But things can be more complicated. For instance, let us assume that the classical field is initially very small, as is the case for new inflation. Then, $\delta\varphi_1/\varphi_{\mathrm{cl}}$ becomes large very quickly (because φ_{cl} is very small), apparently signaling a breakdown of the approximation. However, it is clear that this could just be an artifact of the criterion used which, somehow, would be too naïve. To illustrate this last point, let us consider the following simple example. Suppose that we want to calculate $f(\varphi_{\mathrm{cl}} + \Delta\varphi)$, where f is a given function that we do not need to specify explicitly. Taylor expanding this expression leads to $f(\varphi_{\mathrm{cl}} + \Delta\varphi) \sim f(\varphi_{\mathrm{cl}}) + f'(\varphi_{\mathrm{cl}})\Delta\varphi$ and, in general, this expression gives a good approximation provided that $\Delta\varphi \ll \varphi_{\rm cl}$. However, if the derivatives of f are very small around $\varphi_{\rm cl}$, then the approximation can be good even if $\Delta\varphi$ is much larger than naïvely could be expected. We will see that, in the case of the perturbative expansion of the Langevin equation, we are exactly in this situation. The deep reason for that is the fact that the role of the derivatives of f is now played by the derivatives of the Hubble parameter. Since these are necessarily small as long as the slow-roll approximation is satisfied, one can expect the previous phenomenon to happen. Therefore, it is important not to underestimate the reliability of the perturbative expansion and to study this issue carefully.

In the following, we address this question from a slightly different point of view, focusing on the Langevin equation itself rather than on its exact solution which is, of course, unknown. Our goal is to find a criterion that controls when the perturbed equation that we are able to solve is a good approximation of the exact one. This is a simpler task since we now compare known "objects." At this point, one can even dare an analogy. The situation under consideration is indeed similar to what is done with the slow-roll approximation, for instance for the Klein-Gordon equation. In this case, one does not compare the exact solution (which is, most of the time, unknown as well) to the slow-roll one. One rather studies how small the term that we neglect in the exact equation ($\ddot{\varphi}$) is in comparison with the term that we keep ($H\dot{\varphi}$), i.e. we study the magnitude of the slow-roll parameter $\ddot{\varphi}/(H\dot{\varphi})$. The spirit of the method that we use below is along the same line. Finally, before embarking on the discussion of the reliability of the approximation used here, let us stress again that, so far and despite its importance, this question has not been given a satisfactory answer in the literature on the subject.

In order to determine the accuracy of the expansion, we will make use of the Lagrange remainder theorem [16] for the error in a Taylor expansion. This theorem states that any function $f(\varphi)$ around some value $\varphi_{\rm cl}$ can be written as

$$f(\varphi_{\rm cl} + \Delta\varphi) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\varphi_{\rm cl})}{k!}(\Delta\varphi)^k + \frac{f^{(n)}(\varphi_{\rm cl} + \theta\Delta\varphi)}{n!} \times (\Delta\varphi)^n, \tag{13}$$

for some value of the parameter θ between 0 and 1. Let us emphasize that this expression is exact and does not assume anything on $\Delta\varphi$, in particular, does not assume $\Delta\varphi \ll 1$.

The next step is to apply this theorem to the Langevin equation, $\dot{\varphi} + 2H'/\kappa = H^{3/2}\xi/(2\pi)$, more precisely to the function H' and $H^{3/2}$ in the left- and right-hand sides, respectively. In our case, we take $n = 3$ since we have considered the perturbative expansion of the Langevin equation up to second order in the noise. This gives

$$\frac{{\rm d}\Delta\varphi}{{\rm d}t} + \frac{2}{\kappa}H''_{\rm cl}\Delta\varphi + \frac{H'''_{\rm cl}}{\kappa}\Delta\varphi^2 + \frac{2L_2}{\kappa} = \frac{H^{3/2}_{\rm cl}}{2\pi}\xi + \frac{3}{4\pi}H'_{\rm cl}H^{1/2}_{\rm cl}\Delta\varphi\xi + \frac{R_2}{2\pi}\xi, \tag{14}$$

where we have used the classical equation of motion and where, according to Eq. (13), we have

$$L_2 \equiv \frac{H^{(4)}(\varphi_{\rm cl} + \theta_{\rm L}\Delta\varphi)}{6}(\Delta\varphi)^3, \tag{15}$$

$$R_2 \equiv \frac{(H^{3/2})''(\varphi_{\rm cl} + \theta_{\rm R}\Delta\varphi)}{2}(\Delta\varphi)^2. \tag{16}$$

We stress again that, despite its resemblance with Eqs. (3) and (4), Eq. (14) is an exact equation determining $\Delta\varphi$ (hence the exact stochastic field $\varphi_{\rm cl} + \Delta\varphi$), as long as some values of the two parameters $\theta_{\rm L}$ and $\theta_{\rm R}$ are suitably chosen between 0 and 1. At this stage, this is just a complicated way to rewrite the exact Langevin Eq. (1).

The main idea is now to assume that the truncated expansion is reliable for values of $\Delta\varphi \equiv \delta\varphi_1 + \delta\varphi_2$ such that L_2 and R_2 are small in comparison with the other terms appearing in Eq. (14). Indeed, if this is the case, then the approximated Eqs. (3) and (4) become indistinguishable from the exact one (14). More precisely, for each value of $\varphi_{\rm cl}$, we have to find the limiting values $\Delta\varphi_{\rm min}(\varphi_{\rm cl}) < 0$ and $\Delta\varphi_{\rm max}(\varphi_{\rm cl}) > 0$ such that L_2 and R_2 are small in comparison with the other terms in the equation of motion. Then, the validity of the perturbative treatment will be guaranteed as long as

$$-|\Delta\varphi_{\rm min}(\varphi_{\rm cl})| < \delta\varphi_1 + \delta\varphi_2 < \Delta\varphi_{\rm max}(\varphi_{\rm cl}), \tag{17}$$

or, in other words, as long as we have $\varphi \in [\varphi_{\rm cl} - |\Delta\varphi_{\rm min}(\varphi_{\rm cl})|, \varphi_{\rm cl} + \Delta\varphi_{\rm max}(\varphi_{\rm cl})]$. In practice, since we are dealing with stochastic quantities, instead of $\delta\varphi_1 + \delta\varphi_2$, we will apply our criterion to the quantity $\sqrt{\langle\delta\varphi_1^2\rangle} + \langle\delta\varphi_2\rangle$, $\langle\delta\varphi_2\rangle$ being evaluated with or without the volume effects.

However, to explicitly derive $\Delta\varphi_{\rm max}(\varphi_{\rm cl})$ and $\Delta\varphi_{\rm min}(\varphi_{\rm cl})$, Eq. (14) cannot be used directly because we do not know the values of $\theta_{\rm L}$ and $\theta_{\rm R}$. In fact, it is sufficient to take the maximum of the absolute value of the Lagrange remainders (for $\theta_{\rm L,R} \in [0, 1]$) in order to get an upper bound on the error. Therefore, the approximation is reliable, i.e. L_2 and R_2 are negligible, when the two following conditions

$$\max_{x\in[\varphi_{\rm cl},\varphi_{\rm cl}+\Delta\varphi]}\left|\frac{H^{(4)}(x)}{6}\Delta\varphi^3\right| \ll \left|\frac{H'''_{\rm cl}}{2}\right|\Delta\varphi^2 \tag{18}$$

and

$$\max_{x\in[\varphi_{\rm cl},\varphi_{\rm cl}+\Delta\varphi]}\left|\frac{(H^{3/2})''(x)}{2}\right|\Delta\varphi^2 \ll \left|(H^{3/2}_{\rm cl})'\Delta\varphi\right| \tag{19}$$

hold, while it breaks down when (for fixed values of $\varphi_{\rm cl}$) at least one of the two is violated. The limiting values $\Delta\varphi_{\rm max}(\varphi_{\rm cl})$ and $\Delta\varphi_{\rm min}(\varphi_{\rm cl})$ are then determined by requiring that the two above inequalities become equalities. Since we have two equations and each of them involves absolute values, this gives two positive and two negative solutions, the actual value of $\Delta\varphi_{\rm max}(\varphi_{\rm cl})$ and $\Delta\varphi_{\rm min}(\varphi_{\rm cl})$ clearly being the one leading to the tightest constraint.

Having determined $\Delta\varphi_{\rm max}(\varphi_{\rm cl})$ and $\Delta\varphi_{\rm min}(\varphi_{\rm cl})$ with the above procedure, one must also take into account the fact that we are dealing with stochastic quantities. In this respect, the validity of the perturbative treatment will be guaranteed as long as the probability of finding $-|\Delta\varphi_{\rm min}(\varphi_{\rm cl})| < \delta\varphi_1 + \delta\varphi_2 < \Delta\varphi_{\rm max}(\varphi_{\rm cl})$ is sufficiently close to 1. In terms of probability, this means that one requires

$$\frac{1}{\sqrt{2\pi\langle\delta\varphi_1^2\rangle}}\int_{\Delta\varphi_{\rm min}}^{\Delta\varphi_{\rm max}} {\rm d}\varphi\, \exp\left[-\frac{(\varphi-\langle\delta\varphi_2\rangle)^2}{2\langle\delta\varphi_1^2\rangle}\right] \simeq 1, \quad (20)$$

where we have considered $P_{\rm c}$ as the probability density function. In the case where the volume effects are taken into account, $P_{\rm v}$ should be used instead.

Finally, there is yet another constraint coming from the fact that, in general, $\delta\varphi_1 + \delta\varphi_2$, is a good approximation only if $\delta\varphi_2 \ll \delta\varphi_1$. This is necessary if we want to "separate" Eq. (14) into two equations, one for $\delta\varphi_1$ and one for $\delta\varphi_2$.

Let us now see how the previous considerations work in practice for the chaotic inflation potential $V(\varphi) = m^2\varphi^2/2$. In this particular case, $\delta\varphi_1 + \delta\varphi_2$ is an exact solution of the approximated second order equation since $H''' = 0$ and the constraint $\delta\varphi_2 \ll \delta\varphi_1$ does not apply. In addition, we also have $L_2 = 0$ and, as a consequence, the limiting values $\Delta\varphi_{\rm min}$ and $\Delta\varphi_{\rm max}$ are found only from the constraint (19) involving R_2. Using the slow-roll equations of motion, one has $(H^{3/2})'(x) = 3(\kappa/6)^{3/4}m^{3/2}x^{1/2}/2$ and $(H^{3/2})''(x) = 3(\kappa/6)^{3/4}m^{3/2}x^{-1/2}/4$. The next step is to evaluate the maximum of this last function in the interval $x \in [\varphi_{\rm cl}, \varphi_{\rm cl} + \Delta\varphi]$. Let us start with the upper bound. Since $\Delta\varphi_{\rm max} > 0$ one has $\max|(H^{3/2})''| \propto \varphi_{\rm cl}^{-1/2}$, i.e. $\theta_{\rm R} = 0$. Then one can solve for $\Delta\varphi_{\rm max}$. Applying Eq. (19), one arrives at

$$\frac{1}{2}\times\frac{3}{4}\left(\frac{\kappa}{6}\right)^{3/4} m^{3/2}\varphi_{\rm cl}^{-1/2}\Delta\varphi_{\rm max}^2 = \frac{3}{2}\left(\frac{\kappa}{6}\right)^{3/4} m^{3/2}\varphi_{\rm cl}^{1/2}\Delta\varphi_{\rm max}, \quad (21)$$

from which one obtains $\Delta\varphi_{\rm max} = 4\varphi_{\rm cl}$.

Let us now consider the lower bound $\Delta\varphi_{\rm min} < 0$. The maximum of the function $(H^{3/2})''$ is now given by $\max|(H^{3/2})''| \propto (\varphi_{\rm cl} - |\Delta\varphi_{\rm min}|)^{-1/2}$, i.e. $\theta_{\rm R} = 1$. Therefore, in this case, solving the corresponding equality (19) requires to solve a second order algebraic equation in $|\Delta\varphi_{\rm min}|$, namely,

$$\frac{1}{2}\times\frac{3}{4}\left(\frac{\kappa}{6}\right)^{3/4} m^{3/2}(\varphi_{\rm cl} - |\Delta\varphi_{\rm min}|)^{-1/2}|\Delta\varphi_{\rm min}|^2 = \frac{3}{2}\left(\frac{\kappa}{6}\right)^{3/4} m^{3/2}\varphi_{\rm cl}^{1/2}|\Delta\varphi_{\rm min}|, \quad (22)$$

and the result reads $|\Delta\varphi_{\rm min}| = (-8 \pm 4\sqrt{5})\varphi_{\rm cl}$. Gathering the two limits obtained before, one finds that the reliability interval is given by

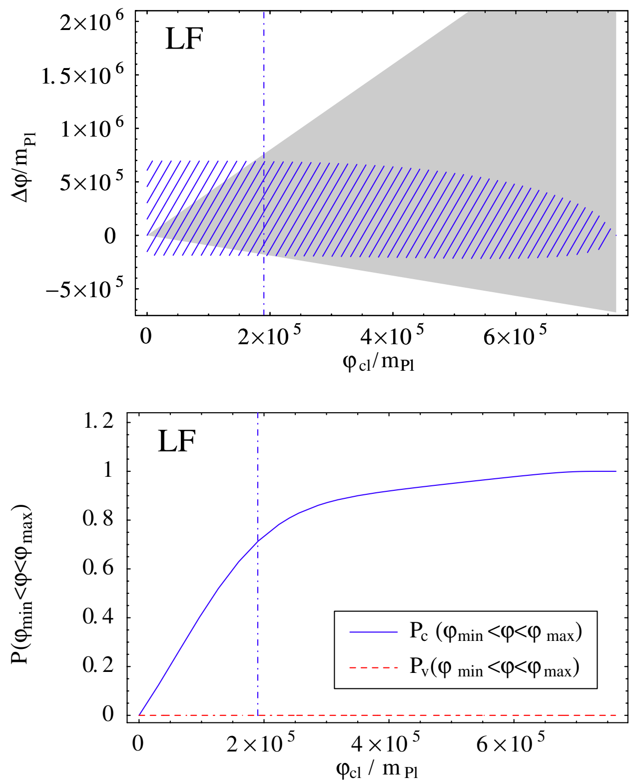

FIG. 1 (color online). Accuracy of the second order approximation for the large field (chaotic) model $V(\varphi) \propto m^2\varphi^2$ with an initial condition corresponding to $V(\varphi_{\rm in}) = m_{\rm Pl}^4/2$. The mass m is chosen so that the Wilkinson Microwave Anisotropy Probe normalization is reproduced (its value is related to the value of M used before). All quantities are plotted versus $\varphi_{\rm cl}$ and, therefore, inflation proceeds from the right (large field values) to the left (small field values). On the left panel, the allowed interval is represented by the uniformly colored region which is delimited by $\Delta\varphi_{\rm min}$ and $\Delta\varphi_{\rm max}$ obtained with Eq. (23). The hatched region represents the region delimited by the two lines $\langle\delta\varphi_2\rangle \pm \sqrt{\langle\delta\varphi_1^2\rangle}$, $\langle\delta\varphi_2\rangle$ being evaluated without the volume effects. The vertical dotted-dashed line signals the value of $\varphi_{\rm cl}$ at which the approximation breaks down. On the right panel, the probability of finding φ in the reliability range computed with $P_{\rm c}$ (solid line) and with $P_{\rm v}$ (dashed line) is displayed. Clearly, the single-domain probability starts decreasing approximately at the value of $\varphi_{\rm cl}$ where $\delta\varphi_1 + \delta\varphi_2$ is no longer in the reliability interval. On the other hand, the volume effects corrections are very large and the volume-weighted distribution is not trustable.

$$-4(\sqrt{5}-2)\varphi_{\rm cl} \ll \delta\varphi_1 + \delta\varphi_2 \ll 4\varphi_{\rm cl}. \qquad (23)$$

This interval is represented in Fig. 1 (left panel) by the uniformly colored region. One clearly sees that this region is limited by two straight lines as calculated above. As inflation proceeds, the allowed region shrinks. This has to be compared with $\langle\delta\varphi_2\rangle \pm \sqrt{\langle\delta\varphi_1^2\rangle}$ represented by the hatched area. The lower border line of the hatched region is $\langle\delta\varphi_2\rangle - \sqrt{\langle\delta\varphi_1^2\rangle}$ while the upper border line is $\langle\delta\varphi_2\rangle + \sqrt{\langle\delta\varphi_1^2\rangle}$. The two lines meet at the beginning of inflation where they vanish since $\langle\delta\varphi_1^2(\varphi_{\rm in})\rangle = \langle\delta\varphi_2(\varphi_{\rm in})\rangle = 0$. As long as the hatched region lies within the uniformly colored one, the approximation is reliable. When this is no longer the case, the approximation breaks down. In Fig. 1, this is signaled by the vertical dotted-dashed line and occurs for $\varphi_{\rm cl} \sim 2\times 10^5 m_{\rm Pl}$. It is clear that the second order approximation is good until one approaches the end of slow-roll inflation. The right panel of the same figure shows the probability of finding $\delta\varphi_1 + \delta\varphi_2$ between $\Delta\varphi_{\rm min}$ and $\Delta\varphi_{\rm max}$, computed according to Eqs. (20) and (23), and confirms the previous conclusion.

When volume effects are considered, the situation becomes conceptually more complicated but the same ideas can be utilized to check the accuracy of the volume-weighted distribution. In particular, one should now compare the region limited by $\langle\delta\varphi_2\rangle + 3\mathcal{V} \pm \sqrt{\langle\delta\varphi_1^2\rangle}$ with the reliability region. In the case of large field models, however, we do not plot this region because the volume effects are so important that the corresponding region would be outside the figure. This will be done for the other models; see below.

One also can use the criterion of Eq. (20) but this time, as already mentioned, the probability should be evaluated with the distribution $P_{\rm v}(\varphi)$ rather than with $P_{\rm c}(\varphi)$. The corresponding results can strongly differ since the field realizations having higher potential energy (i.e. with a faster expansion rate) will be favored. In particular, if their expansion rate is sufficiently large, this can give a high statistical significance to realizations outside the reliability range having a very low significance according to the original distribution. In this situation, when the difference between the two distributions is very important, the form of $P_{\rm v}(\varphi)$ obtained from the perturbed solution cannot be trusted although $P_{\rm c}(\varphi)$ is reliable. This means that, in most domains, the statistical properties of the field are correctly described by $P_{\rm c}(\varphi)$ but that the Universe is mainly made up of very big domains where $P_{\rm c}(\varphi)$ cannot be trusted. This is exactly what happens for LF models where the volume-weighted probability of finding the field in the confidence range (dashed line, right panel in Fig. 1) is basically vanishing while the single-domain one (solid line) is large. Therefore, in this case, the perturbative method does not allow us to reliably compute the volume-weighted distribution.

We have also performed the same study for new inflation and the results are displayed in Fig. 2. Now, the reliability interval, still given by the uniformly colored region, is no longer determined only from R_2 but also from L_2 because,

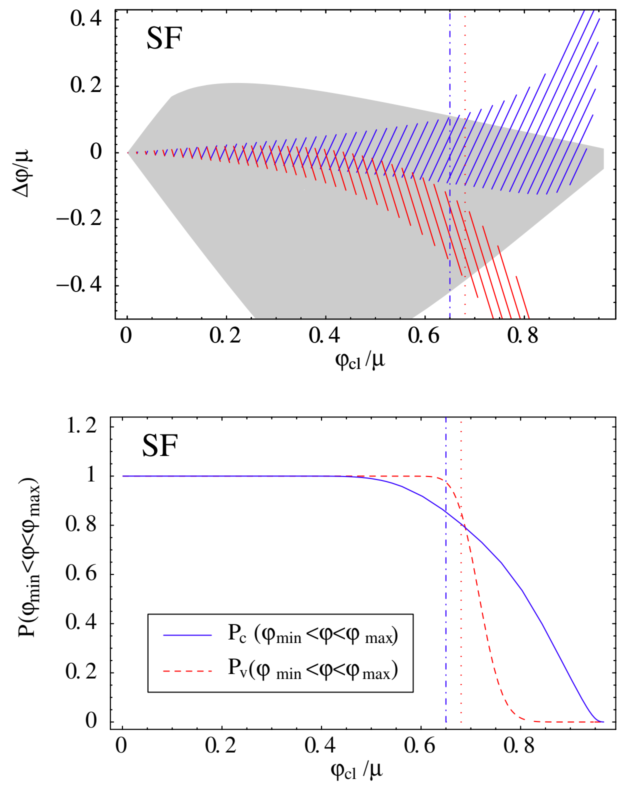

FIG. 2 (color online). Accuracy of the second order approximation for the small field (new inflation) potential $V(\varphi) \propto 1 - (\varphi/\mu)^2$ with the initial condition $\varphi_{\rm in} \simeq 10^{-5}\mu$. In this case, inflation proceeds from the left (small field values) to the right ("large" field values). On the left panel, the uniformly colored region represents the interval where the approximation is trustable. The limiting values are now obtained from the condition on the remainder R_2 but also from the one coming from the remainder L_2. The actual $\Delta\varphi_{\rm min}(\varphi_{\rm cl})$ and $\Delta\varphi_{\rm max}(\varphi_{\rm cl})$, which delimit the confidence region, must be the smallest ones (in absolute value). The hatched (top) region with positive slope lines is delimited by $\langle\delta\varphi_2\rangle \pm \sqrt{\langle\delta\varphi_1^2\rangle}$ where $\langle\delta\varphi_2\rangle$ is computed without the volume effect. On the other hand, the hatched (bottom) region with negative slope lines is also delimited by $\langle\delta\varphi_2\rangle \pm \sqrt{\langle\delta\varphi_1^2\rangle}$ but, this time, with $\langle\delta\varphi_2\rangle$ computed with the volume effects. The vertical dotted-dashed line indicates when the approximation without the volume effects breaks down while the vertical dotted line signals when the approximation with the volume effects becomes untrustworthy. On the right panel, the probability of finding $\delta\varphi_1 + \delta\varphi_2$ in the reliability range is displayed. It is clear that both $P_{\rm c}$ (solid line) and $P_{\rm v}$ (dashed line) yield a probability close to 1 for a large part of the inflationary phase.

contrary to the LF case, L_2 does not vanish for SF. Therefore, in this situation, one has to determine $\Delta\varphi_{\max}$ and $\Delta\varphi_{\min}$ from Eqs. (18) and (19) and not only from Eq. (19), as was the case for the LF models. The form of $\Delta\varphi_{\max}$ and $\Delta\varphi_{\min}$ as a function of $\varphi_{\rm cl}$ is also more complicated and is no longer given by straight lines. It must be computed numerically. As can be seen in Fig. 2, the allowed region increases at the beginning of inflation, reaches a maximum extension and shrinks as the end of inflation is approached. The region $\langle\delta\varphi_2\rangle \pm \sqrt{\langle\delta\varphi_1^2\rangle}$ without the volume effects is given by the hatched (top) region, the lines having a positive slope. The region $\langle\delta\varphi_2\rangle \pm \sqrt{\langle\delta\varphi_1^2\rangle}$ with the volume effects taken into account is represented by the hatched (bottom) region, the lines having a negative slope. When these regions are within the uniformly colored region, the approximation is reliable. As before, the border lines $\langle\delta\varphi_2\rangle \pm \sqrt{\langle\delta\varphi_1^2\rangle}$ meet at the beginning of inflation where they vanish. The break down of the approximation, without the volume effect, is signaled by the vertical dotted-dashed line and, with the volume effects, by the dotted line. One notices that the results for SF are basically similar to those that have been obtained in the chaotic model case, namely, the approximation remains reliable until the very end of the inflationary phase. However, in the case of new inflation, one clearly sees the importance of using a carefully defined criterion to estimate the reliability of the approximation. As already mentioned, the naïve criterion $\delta\varphi_1 \ll \varphi_{\rm cl}$ would have indicated that the approximation becomes untrustworthy very quickly after the beginning of inflation since we have initially $\varphi_{\rm cl}/\mu \ll 1$. We see in Fig. 2 that, on the contrary, the approximation is good during a large part of inflation. Finally, the probability of being in the reliability region, computed with $P_{\rm c}$ (solid line) or $P_{\rm v}$ (dashed line) is also displayed in Fig. 2 (right panel). In the case of SF models, the volume effects are less important and, as a consequence, the two probabilities are similar. The fact that the breakdown of the approximation occurs at the end of inflation only, at $\varphi_{\rm cl}/\mu \sim 0.65$ for the particular example studied here, is confirmed.

A similar analysis also can be done for the running mass potential. The results for the two models under consideration are displayed in Fig. 3. All the conventions concerning the allowed regions, volume effects etc. are the same as

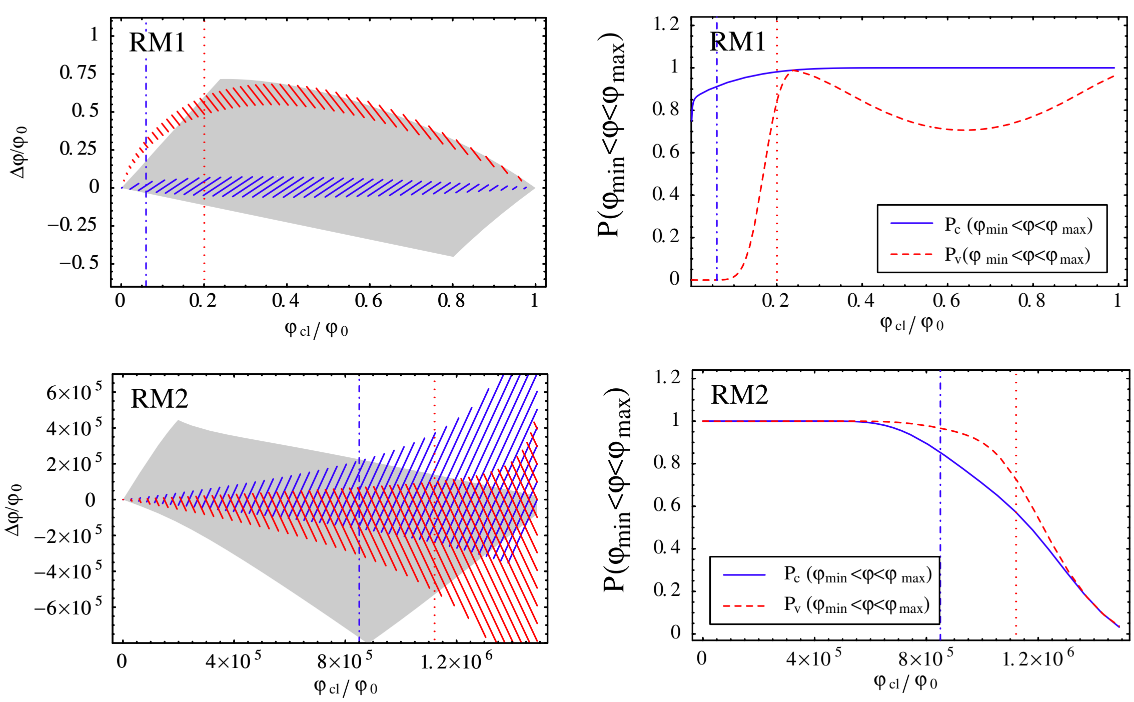

FIG. 3 (color online). Accuracy of the second order approximation for the two running mass models RM1 and RM2 with initial conditions $\varphi_{\rm in}/\varphi_0 \simeq 1 - 1.5 \times 10^{-5}$ and $\varphi_{\rm in}/\varphi_0 \simeq 1 + 10^{-3}$, respectively. Inflation proceeds from the right to the left in the RM1 model (upper panels) and from the left to the right in the RM2 model (lower panels). On the left panels, the limiting values signaling the break down of the conditions given by Eqs. (15) and (16) are represented and compared with $\langle\delta\varphi_2\rangle \pm \sqrt{\langle\delta\varphi_1^2\rangle}$. Conventions are the same as the ones used in Figs. 1 and 2. As usual, the actual values of $\Delta\varphi_{\min}$ and $\Delta\varphi_{\max}$ must be the ones with the smallest absolute values. Cusps in the curves are a consequence of taking the maximum absolute value of the Lagrange remainders and appear when one discontinuously changes the value of θ from 0 to 1 (or the opposite). On the right panels, the probability of finding φ in the reliability range is displayed. For both models the reliability of the solution does not dramatically change (and for the RM2 model is even slightly enhanced) when volume weighting is considered.

before. In the case of the RM1 model (upper panels), Eq. (18) giving the constraint on L_2 can be analytically solved to the lowest order in $c(\varphi_0/M_{\rm Pl})^2$. This gives

$$-\frac{7-\sqrt{13}}{6}\varphi_{\rm cl} \ll \Delta\varphi \ll 3\varphi_{\rm cl}. \qquad (24)$$

It turns out that this constraint is the dominating one at late times as can be checked in Fig. 3 (left panel): the shape of the allowed region near the end of inflation is delimited by two straight lines. The constraint on R_2 has been solved numerically. The figure demonstrates, and this is also confirmed in the right panel, that the perturbative solution for RM1 is very good during almost all the inflationary phase and breaks down only at the very end of inflation. Similar conclusions hold for the RM2 model, see the two bottom panels.

Finally, one notices that the two probabilities [i.e. the ones obtained with $P_{\rm c}(\varphi)$ and $P_{\rm v}(\varphi)$], and contrary to the LF case, do not dramatically differ from each other. In the case of the RM2 model, the reliability of the volume-weighted description can even be larger than the single-point one.

IV. CONCLUSIONS

In this section, we briefly summarize the new results obtained in this article. The main goal of the paper was to present a new method aimed at estimating the precision of the perturbative expansion studied in Ref. [10]. This method is based on the use of the Lagrange remainder. After having discussed the general features of this new approach, we have applied it to the inflationary models studied in Ref. [10]. We have proven that the approximate probability density functions derived in this reference are, in general, a very good approximation to the actual ones except, as expected, at the end of inflation. This conclusion holds even if the volume effects are taken into account except in the case of the large field models. We conclude that the perturbative expansion of the inflationary Langevin equation together with the method presented here, besides being the only available method with a built-in measure of its domain of validity, form a robust formalism to efficiently compute the stochastic effects during inflation.

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