PHYSICAL REVIEW D **73,** 034017 (2006)

Dispersive approach to the axial anomaly and nonrenormalization theorem

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(Received 23 November 2005; revised manuscript received 11 January 2006; published 10 February 2006)

Anomalous triangle graphs for the divergence of the axial-vector current are studied using the dispersive approach generalized for the case of higher orders of perturbation theory. The validity of this procedure is proved up to the two-loop level. By direct calculation in the framework of dispersive approach we have obtained that the two-loop axial-vector-vector (AVV) amplitude is equal to zero. According to the Vainshtein's theorem, the transversal part of the anomalous triangle is not renormalized in the chiral limit. We generalize this theorem for the case of finite fermion mass in the triangle loop.

DOI: 10.1103/PhysRevD.73.034017 PACS numbers: 12.38.Bx, 11.55.Fv

I. INTRODUCTION

There is a class of electroweak contributions to the muon $g - 2$ containing a fermion triangle along with a virtual photon and Z boson as shown at Fig. 1 that is discussed in [1–6]. For the determination of the muon anomalous electromagnetic moment we are interested in the $Z^* \to \gamma^*$ transition in the presence of the external magnetic field to first order in this field. In this approximation one may consider the current j_μ as a source of a soft photon with polarization vector $e^\mu(k)$ and momentum $k \to 0$. In such kinematics, projection of the amplitude on $e^\mu(k)$ contains only two Lorentz-invariant structures

$$\begin{aligned} T_{\alpha\nu}(p^2, m^2) &= T_{\alpha\mu\nu}(k, p)e^\mu(k)|_{k\to 0} \\ &= w_T(p^2, m^2)(-p^2\tilde{f}_{\nu\alpha} + p_\nu p^\rho \tilde{f}_{\rho\alpha} \\ &\quad - p_\alpha p^\rho \tilde{f}_{\rho\nu}) + w_L(p^2, m^2)p_\alpha p^\rho \tilde{f}_{\rho\nu}, \\ \tilde{f}_{\mu\nu} &= \tfrac{1}{2}\varepsilon_{\mu\nu\gamma\delta}f^{\gamma\delta}, \qquad f_{\mu\nu} = k_\mu e_\nu - k_\nu e_\mu, \end{aligned} \tag{1.1}$$

and can be viewed as a correlator of the axial and vector currents in the external electromagnetic field with strength tensor $f_{\mu\nu}$. The same expression appears while analyzing the dominant contribution of light-by-light scattering to $g - 2$.

Both structures w_T and w_L are transversal with respect to vector current, but only the first structure is transversal with respect to axial current while the second is longitudinal. According to the classical papers by Rosenberg [7], Adler [8], Bell and Jackiw [9] at the one-loop level the invariant functions $w_{L,T}$ satisfy the relation

$$w_L^{(1-\mathrm{loop})}(p^2, m^2) = 2w_T^{(1-\mathrm{loop})}(p^2, m^2). \tag{1.2}$$

It was shown by A. I. Vainshtein [10] that there is the symmetry of the triangle amplitude $T_{\alpha\mu\nu}$ under permutation $\nu \leftrightarrow \alpha$ in the chiral limit. This symmetry preserves the relation between w_L and w_T (1.2) for the case $m = 0$ in any order of perturbation theory.

Moreover, according to the Adler-Bardeen theorem [11] the anomalous longitudinal part of the triangle is not renormalized in the chiral limit. It is worthy to note that this statement implies an operator relation, while the matrix elements get the corrections [12] due to anomalous dimension of axial current. At the same time, the validity of Adler-Bardeen theorem at the operator level allows to express [13] these three-loop corrections [12] in terms of earlier two-loop and even one-loop calculations of anomalous divergencies.

To apply the Adler-Bardeen theorem to the problem in question, one should recall that the axial anomaly is expressed only through the longitudinal part w_L [8,9]

$$p^\alpha T_{\alpha\nu}(p^2, m^2 = 0) = p^2 w_L(p^2, m^2 = 0)p^\sigma \tilde{f}_{\sigma\nu} \sim p^\sigma \tilde{f}_{\sigma\nu}, \tag{1.3}$$

and its nonrenormalization leads to the fact that the one-loop result $w_L^{(1-\mathrm{loop})} \sim 1/p^2$ does not get the perturbative

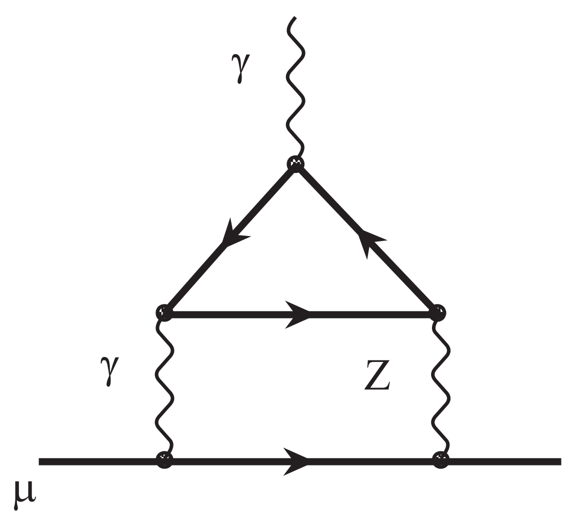

FIG. 1. Effective $Z\gamma\gamma^*$ coupling contributing to a_μ^{EW}.

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corrections from gluon exchanges in the higher orders [14]. Nonrenormalization of w_L implies the same for w_T:

$$w_L(p^2, m^2 = 0) = 2w_T(p^2, m^2 = 0),\qquad w_{L,T}^{(>1-\text{loop})}(p^2, m^2 = 0) = 0. \tag{1.4}$$

This nonrenormalization, in contrast with longitudinal part, holds on only perturbatively and, seemingly, is not directly related to the phenomenon of anomaly. More general nonrenormalization theorems in perturbative QCD were proved in [15].

It is very interesting to look at the phenomenon of anomaly and perturbative nonrenormalizability of $w_{L,T}$ using the dispersive approach [16]. In framework of this approach anomaly becomes quite simple and represents itself just as an obvious subtraction constant. As a result, the two-photon matrix element of the axial-vector current acquires a pole in the chiral limit so the anomaly appears as a pure infrared effect. Detailed investigation of the one-loop AVV triangle graph within such approach has been performed earlier [17].

The language of dispersion relations allows us to extract some new specific properties of the higher order corrections to fermion triangles. Relation (1.2). in the context of dispersive approach. emerges due to the universality of anomaly when it appears in the dispersion relation in the axial and vector channels. This resembles the mentioned symmetry with respect to the permutation $\nu \leftrightarrow \alpha$, observed by A. I. Vainshtein. At the same time, here this relation is entirely related to the anomaly phenomenon.

By direct analytical calculations of relevant form factors in two-loop approximation taking into account symmetry properties of the amplitude we obtained that the full two-loop AVV amplitude is equal to zero for arbitrary fermion masses. Nonanomalous Ward identity for imaginary parts of form factors has been proven, which provides the correctness of dispersive approach usage at least in two-loop approximation. We also make a suggestion that the dispersive approach is applicable in any order of perturbation theory. Together with Adler-Bardeen theorem, immediate consequence of such suggestion is that the Vainshtein's theorem is correct for nonzero fermion mass.

The paper is organized as follows. In Sec. II the dispersive approach to the anomaly is briefly described for particular configurations of the external momenta in the standard one-loop triangle graph following to [17]. The Born approximation to Vainshtein theorem in the framework of dispersive approach is interpreted as equality of two expressions for axial anomaly and valid in the case of finite fermion mass. In Sec. III the generalization of dispersion approach to the axial anomaly for higher orders of perturbation is suggested. The two-loop radiative corrections to anomalous triangle with arbitrary fermion masses for the same kinematical configurations are considered in the context of such an approach. We found that all two-loop form factors are zero, justifying the postulated generalization of dispersive approach and Vainshtein's theorem. Section IV contains some concluding remarks and discussion of the higher orders of perturbation theory and nonperturbative effects. The appendix contains some details of the two-loop calculations.

II. AXIAL ANOMALY AND NONRENORMALIZATION THEOREM IN THE FRAMEWORK OF DISPERSIVE APPROACH

We use the standard tensor representation of the VVA triangle graph amplitude (Fig. 2) due originally to Rosenberg [7]

$$\begin{aligned} T_{\alpha\mu\nu}(k,p) &= \varepsilon_{\alpha\mu\nu\rho}k^\rho F_1 + \varepsilon_{\alpha\mu\nu\rho}p^\rho F_2 \\ &\quad + k_\nu \varepsilon_{\alpha\mu\rho\sigma}k^\rho p^\sigma F_3 + p_\nu \varepsilon_{\alpha\mu\rho\sigma}k^\rho p^\sigma F_4 \\ &\quad + k_\mu \varepsilon_{\alpha\nu\rho\sigma}k^\rho p^\sigma F_5 + p_\mu \varepsilon_{\alpha\nu\rho\sigma}k^\rho p^\sigma F_6. \end{aligned} \tag{2.1}$$

Here $F_j = F_j(q^2; k^2, p^2, m^2)$, $j = 1, \ldots, 6$ are the Lorentz-invariant form factors. The Bose symmetry of the amplitude $T_{\alpha\mu\nu}(k,p) = T_{\alpha\nu\mu}(p,k)$ is equivalent to

$$F_1(k,p) = -F_2(p,k),\qquad F_3(k,p) = -F_6(p,k),\qquad F_4(k,p) = -F_5(p,k). \tag{2.2}$$

The gauge invariance leads to the vector Ward identities $k^\mu T_{\alpha\mu\nu} = 0$, $p^\nu T_{\alpha\mu\nu} = 0$ which in terms of form factors

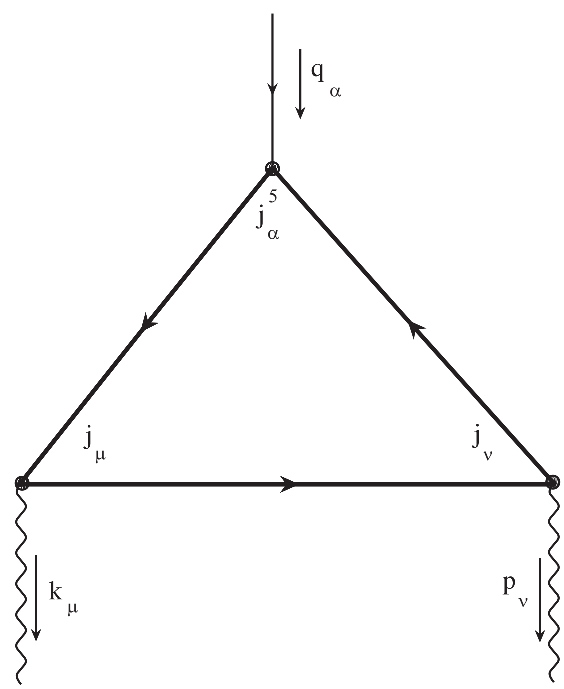

FIG. 2. Anomalous triangle diagram.

gives

$$F_1 = (kp)F_3 + p^2F_4, \qquad F_2 = k^2F_5 + (kp)F_6. \quad (2.3)$$

In one-loop approximation all form factors F_j may be expressed in Feynman-parametric form. For relevant form factors we have

$$F_3(k, p) = -\frac{1}{\pi^2}I_{11}(k, p),$$
$$F_4(k, p) = \frac{1}{\pi^2}[I_{20}(k, p) - I_{10}(k, p)], \quad (2.4)$$

where

$$I_{mn}(k, p) = \int_0^1 dx \int_0^{1-x} dy \frac{x^m y^n}{y(1-y)k^2 + x(1-x)p^2 + 2xy(kp) - m^2}. \quad (2.5)$$

Here $N_c = 1$ for simplicity. These integrals have the following symmetry properties

$$I_{mn}(k, p) = I_{nm}(p, k). \quad (2.6)$$

It is useful to observe that (2.2) together with (2.3) and (2.6) implies

$$F_6(k, p) = -F_3(k, p). \quad (2.7)$$

It is well-known from the classic Adler paper [8] that the accurate calculation of loop-momentum integrals at the one-loop level leads to the anomalous axial-vector Ward identity

$$q^\alpha T_{\alpha\mu\nu}(k, p) = 2mT_{\mu\nu}(k, p) + \frac{1}{2\pi^2}\varepsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma. \quad (2.8)$$

The pseudotensor $T_{\mu\nu}$ may be written as

$$T_{\mu\nu}(k, p) = G(k, p)\varepsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma. \quad (2.9)$$

In terms of form factors (2.8) reads

$$F_2 - F_1 = 2mG + \frac{1}{2\pi^2}. \quad (2.10)$$

According to the Adler-Bardeen theorem the axial anomaly occurs only at one-loop level. So, relation (2.10) for full form factors remains the same

$$F_2^{(tot)} - F_1^{(tot)} = 2mG^{(tot)} + \frac{1}{2\pi^2}. \quad (2.11)$$

In the framework of dispersive approach we deal with the imaginary parts of relevant form factors. We start with the kinematical configuration of external momenta $k^2 = 0$, $p^2 \neq 0$ for particular case where $q^2 > 4m^2$ and $p^2 < 4m^2$. Then there is a cut for $q^2 \in (4m^2, \infty)$ while there is no such singularity with respect to variable p^2. For the form factors F_j, $j = 1, \ldots, 6$ and G, in any order of perturbation theory, one may write unsubtracted dispersion relations with respect to q^2

$$F_j(q^2; p^2, m^2) = \frac{1}{\pi}\int_{4m^2}^\infty \frac{A_j^A(t; p^2, m^2)}{t - q^2}dt,$$
$$G(q^2; p^2, m^2) = \frac{1}{\pi}\int_{4m^2}^\infty \frac{B^A(t; p^2, m^2)}{t - q^2}dt. \quad (2.12)$$

We use the notations $F_j(q^2; p^2, m^2) \equiv F_j(q^2; k^2 = 0, p^2, m^2)$ and $G(q^2; p^2, m^2) \equiv G(q^2; k^2 = 0, p^2, m^2)$ below in the current section. The A_j^A and B^A are the corresponding imaginary parts, implying the cut with respect to variable $q^2 = t$, for example

$$A_j^A(q^2; p^2, m^2) = \frac{F_j(q^2 + i\varepsilon; p^2, m^2) - F_j(q^2 - i\varepsilon; p^2, m^2)}{2i}.$$

The imaginary parts of the relevant form factors satisfy nonanomalous Ward identities, because they do not contain the linear divergences in the momentum integrals

$$(p^2 - t)A_3^A(t; p^2, m^2) - p^2A_4^A(t; p^2, m^2) = 2mB^A(t; p^2, m^2). \quad (2.13)$$

Using (2.12) and (2.13) one gets finally

$$F_2(q^2; p^2, m^2) - F_1(q^2; p^2, m^2) - 2mG(q^2; p^2, m^2) = \frac{1}{\pi}\int_{4m^2}^\infty A_3^A(t; p^2, m^2)dt. \quad (2.14)$$

Comparing with (2.10) and taking into account (2.7), we find that the occurrence of the axial anomaly at one-loop level is equivalent to a "sum rule" [17]

$$\int_{4m^2}^\infty A_3^A(t; p^2, m^2)dt = \frac{1}{2\pi}. \quad (2.15)$$

It is easy to evaluate the A_3^A by taking the imaginary part of the corresponding integral in (2.4) [17]

$$A_3^A(q^2; p^2, m^2) = \frac{1}{2\pi}\frac{1}{(q^2 - p^2)^2}\left(-p^2R + 2m^2\ln\frac{1+R}{1-R}\right),$$
$$R = \left(1 - \frac{4m^2}{q^2}\right)^{1/2}. \quad (2.16)$$

By integration of this expression over q^2 one can check the relation (2.15).

In the preceding discussion we have employed dispersion relations with respect to variable q^2 as these were appropriate for considered kinematical region. Let us now discuss another version of dispersive calculation of the anomaly using the cuts with respect to p^2. For this purpose we will consider another kinematical region $p^2 > 4m^2$, $q^2 < 4m^2$. Writing now instead of (2.12) unsubtracted dispersion relations with respect to variable $p^2 = t$ for F_j and G, we obtain

$$F_2(q^2; p^2, m^2) - F_1(q^2; p^2, m^2) - 2mG(q^2; p^2, m^2) = \frac{1}{\pi}\int_{4m^2}^{\infty}[A_4^V(q^2; t, m^2) - A_3^V(q^2; t, m^2)]dt.$$

The A_j^V and B^V are the corresponding imaginary parts, implying the cut with respect to variable $p^2 = t$

$$A_j^V(q^2; p^2, m^2) = \frac{F_j(q^2; p^2 + i\varepsilon, m^2) - F_j(q^2; p^2 - i\varepsilon, m^2)}{2i}.$$

Thus, to recover the standard one-loop anomaly (2.10) taking into account (2.7) one has to show that

$$\int_{4m^2}^{\infty}[A_4^V(q^2; t, m^2) - A_3^V(q^2; t, m^2)]dt = \frac{1}{2\pi}, \tag{2.17}$$

for an arbitrary m and for any considered value of q^2.

A straightforward calculation at one-loop level using (2.4) gives a result [17]

$$A_3^V(q^2; p^2, m^2) = \frac{1}{2\pi}\frac{1}{(p^2 - q^2)^2}\left(p^2 S - 2m^2 \ln\frac{1+S}{1-S}\right), \tag{2.18}$$

$$A_4^V(q^2; p^2, m^2) = \frac{1}{2\pi}\frac{1}{p^2 - q^2}S, \qquad S = \left(1 - \frac{4m^2}{p^2}\right)^{1/2}. \tag{2.19}$$

It was observed [17] that the integrands $A_4^V(q^2; p^2, m^2) - A_3^V(q^2; p^2, m^2)$ occurring in the sum rule (2.17) at one-loop level are equal to the expression for $A_3^A(q^2; p^2, m^2)$ from sum rule (2.15) with q^2 and p^2 being interchanged. As a result we have

$$A_4^V(q^2; p^2, m^2) = A_3^A(p^2; q^2, m^2) + A_3^V(q^2; p^2, m^2). \tag{2.20}$$

Let us write the unsubtracted dispersion relations with respect to $q^2 = t$ of the both sides of (2.20):

$$\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{A_4^V(q^2; t, m^2)}{t - p^2}dt = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{A_3^A(t; q^2, m^2)}{t - p^2}dt + \frac{1}{\pi} \times \int_{4m^2}^{\infty}\frac{A_3^V(q^2; t, m^2)}{t - p^2}dt.$$

One can immediately get from this expression that

$$F_4(q^2; p^2, m^2) = F_3(p^2; q^2; m^2) + F_3(q^2; p^2; m^2). \tag{2.21}$$

In the case with one soft $k \to 0$ and one virtual photons $p^2 \neq 0$ we have $q^2 = (k + p)^2 \simeq p^2$. For this kinematics one can obtain the expressions for the longitudinal and transversal parts of amplitude $T_{\alpha\nu}$ (1.1) in terms of Rosenberg's form factors

$$\begin{aligned} w_L(p^2, m^2) &= F_4(p^2; p^2, m^2), \\ w_T(p^2, m^2) &= F_4(p^2; p^2, m^2) - F_3(p^2; p^2, m^2). \end{aligned} \tag{2.22}$$

The relation (1.2) between w_T and w_L in terms of form factors

$$F_4(p^2; p^2, m^2) = 2F_3(p^2; p^2; m^2) \tag{2.23}$$

immediately follows from (2.21) with $q^2 = p^2$.

In the framework of Vainshtein's approach, the axial anomaly is expressed only through the longitudinal part of triangle w_L in the chiral limit (1.3). Within the dispersive approach, we have two dispersive relations for axial anomaly (2.15) and (2.17), including imaginary parts of both structures w_L and w_T for arbitrary mass.

III. CALCULATION OF TWO-LOOP AXIAL ANOMALY AND CHECK OF DISPERSIVE APPROACH AND VAINSHTEIN'S THEOREM

We propose the generalization of dispersion approach to the axial anomaly for any order of perturbation theory. We suggest that the imaginary parts of the relevant form factors satisfy nonanomalous Ward identities for higher order of perturbation theory as well. This implies that anomaly will be also given by corresponding finite subtractions.

To check that, we will calculate the triangle diagram in two-loop approximation. The results for QED and QCD corrections differ only by the obvious color factor. We consider the full amplitude of anomalous triangles $T_{\alpha\mu\nu}^{(2-\text{loop})}$ with all possible types of radiative corrections shown at Fig. 3.

At first we construct four possible scalars from amplitude $T_{\alpha\mu\nu}$ (2.1) in the particular kinematics $k^2 = 0$, $p^2 \neq 0$:

$$\begin{aligned} S_1 &\equiv T_{\alpha\mu\nu}k^\alpha \varepsilon^{\xi\mu\nu\eta}k_\xi p_\eta = -2(kp)^2 F_2, \\ S_2 &\equiv T_{\alpha\mu\nu}p^\alpha \varepsilon^{\xi\mu\nu\eta}k_\xi p_\eta = 2(kp)^2 F_1, \\ S_3 &\equiv T_{\alpha\mu\nu}\varepsilon^{\alpha\mu\nu\eta}k_\eta = 6(kp)F_2 + 2(kp)^2 F_4, \\ S_4 &\equiv T_{\alpha\mu\nu}\varepsilon^{\alpha\mu\nu\eta}p_\eta \\ &= 6(kp)F_1 + 2(kp)^2 F_5 - 2(kp)^2 F_3 + 6p^2 F_2. \end{aligned} \tag{3.1}$$

Together with vector Ward identities for form factors in the

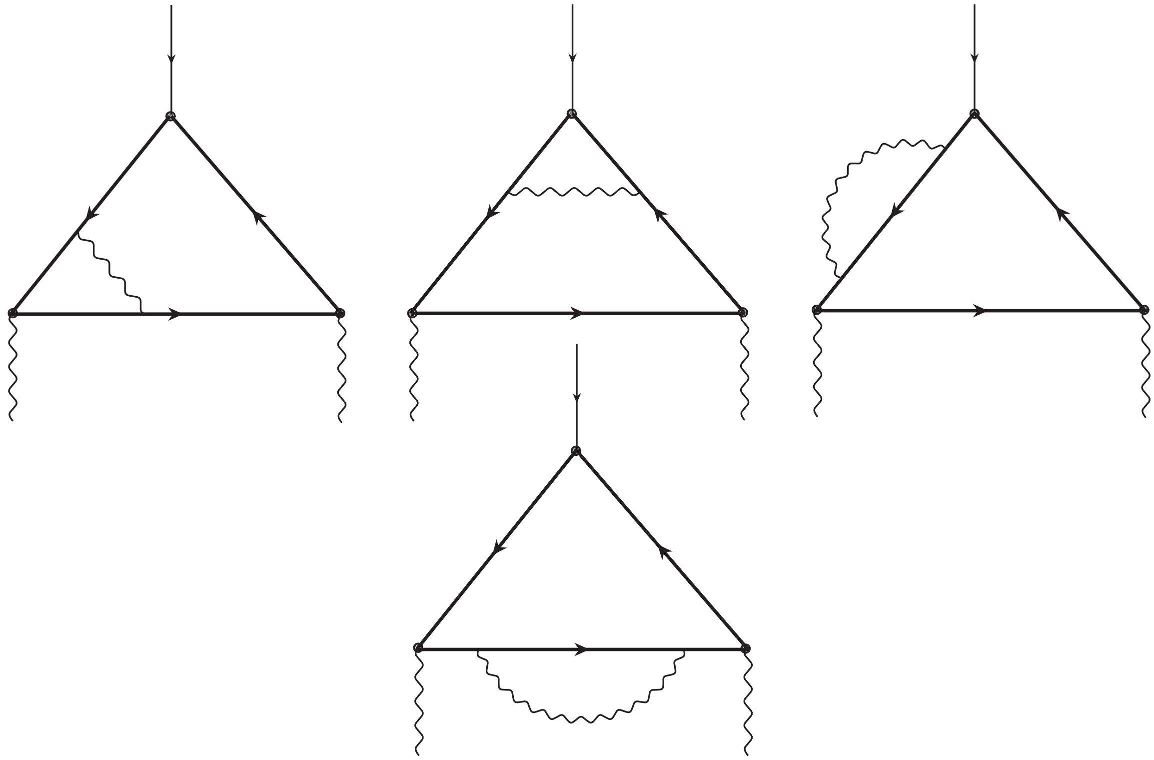

FIG. 3. Two-loop triangle diagrams.

same kinematics

$$F_1 = (kp)F_3 + p^2F_4, \qquad F_2 = (kp)F_6. \tag{3.2}$$

We have the closed system of equations for all form factors $F_j = F_j(q^2; k^2 = 0, p^2, m^2)$, $j = 1, \ldots, 6$. Its solution is

$$\begin{aligned} F_1 &= \frac{S_2}{2(kp)^2}, \\ F_2 &= -\frac{S_1}{2(kp)^2}, \\ F_3 &= \frac{(kp)S_2 - p^2(kp)S_3 - 3p^2S_1}{2(kp)^4}, \\ F_4 &= \frac{(kp)S_3 + 3S_1}{2(kp)^3}, \\ F_5 &= \frac{S_4(kp) - 2S_2 - p^2S_3}{2(kp)^3}, \\ F_6 &= -\frac{S_1}{2(kp)^3}. \end{aligned} \tag{3.3}$$

In the framework of dispersive approach we are interested in calculation of imaginary parts of corresponding form factors that is $A_j^{A(2-\text{loop})}(q^2; k^2 = 0, p^2, m^2)$, so let us calculate the imaginary parts of scalars $S_n = S_n(q^2; k^2 = 0, p^2, m^2)$, $n = 1, \ldots, 4$, for example, with respect to q^2

$$\Delta_n^A(q^2; k^2 = 0, p^2, m^2) = \frac{1}{2i}[S_n(q^2 + i\varepsilon; k^2 = 0, p^2, m^2) - S_n(q^2 - i\varepsilon; k^2 = 0, p^2, m^2)].$$

We use the Pauli-Villars regularization with the parameter M. After integration over loop momenta we have obtained the expressions for $\Delta_n^{A(2-\text{loop})}(q^2; k^2 = 0, p^2, m^2)$ in the form of Feynman-parametric integrals (see Appendix). We drop all the terms with the imaginary parts with respect to q^2 which do not survive in the limit $M \to \infty$.

The next step is to calculate the imaginary part of Feynman-parametric integrals and to get the expressions for $\Delta_n^{A(2-\text{loop})}(q^2; k^2 = 0, p^2, m^2)$ in explicit form before taking off the regularization. We get the imaginary part of each integrand, and then we integrate the δ-functions with complicated arguments analytically step by step. After the second or third integration all Feynman-parametric integrals in $\Delta_n^{A(2-\text{loop})}(q^2; k^2 = 0, p^2, m^2)$ turned to zero, so

$$\Delta_n^{A(2-\text{loop})}(q^2; k^2 = 0, p^2, m^2) \equiv 0, \qquad n = 1, \ldots, 4. \tag{3.4}$$

According to (3.3) we see that the imaginary parts of all form factors with respect to q^2 turned to zero in the considering kinematics

$$A_j^{A(2-\text{loop})}(q^2; k^2 = 0, p^2, m^2) \equiv 0, \qquad j = 1, \ldots, 6. \tag{3.5}$$

As a result the anomaly sum rule (2.15) preserves one-loop form also at two-loop level.

The investigation of the Vainshtein's theorem requires the discontinuities of form factors with the respect to variable p^2 (or k^2). Unsubtracted dispersion relations (2.12) guarantee that all two-loop form factors $F_j^{(2-\text{loop})}$ are equal to zero

$$F_j^{(2-loop)}(q^2; k^2 = 0, p^2, m^2) \equiv 0, \quad j = 1, ..., 6.$$

So, the full two-loop AVV amplitude in the considering kinematics is equal to zero. Consequently, the discontinuities with respect to p^2 (or k^2) are also zero

$$A_j^{V(2-loop)}(q^2; k^2 = 0, p^2, m^2) \equiv 0, \quad j = 1, ..., 6,$$

and the anomaly sum rule (2.17) also preserves one-loop form. This immediately leads to validity of Vainshtein's theorem with finite mass at two loops

$$w_L(p^2, m^2) = 2w_T(p^2, m^2), \qquad w_{L,T}^{(2-\text{loop})}(p^2, m^2) = 0. \tag{3.6}$$

For completeness we also explicitly checked the correctness of nonanomalous Ward identities for imaginary parts. To do so we have considered the imaginary part of pseudoscalar form factor $B^{A(2-\text{loop})}(q^2; k^2, p^2, m^2)$ in the particular kinematics. By direct diagrammatic calculations of the two-loop amplitude $T_{\mu\nu}^{(2-\text{loop})}$ we found that

$$B^{A(2-\text{loop})}(q^2; k^2 = 0, p^2, m^2) \equiv 0.$$

So, taking into account (3.5), the relation (2.13) does not obtain the perturbative corrections in two-loop approximation.

IV. DISCUSSION

In our work the axial anomaly and Vainshtein's nonrenormalization theorem are considered in the framework of dispersive approach. We found that all two-loop contributions to form factors $F_j(q^2; k^2 = 0, p^2, m^2)$ and $G(q^2; k^2 = 0, p^2, m^2)$ are equal to zero for arbitrary fermion mass. It allows us to prove the suggested generalization of dispersive approach to axial anomaly and to expand the Vainshtein's nonrenormalization theorem for arbitrary fermion masses in the triangle loop.

Although we thus proved these properties only at two-loop level, they are likely to be valid at any order of perturbation theory. Indeed, the validity of nonanomalous Ward identities for imaginary parts is due to the absence of the linear divergencies and should hold at all loops. This, together with Adler-Bardeen theorem, would result in validity of anomaly sum rules. As soon as the corrections to their integrands are zero in chiral limit due to Vainshtein's theorem, it is rather hard to imagine the function which is nonzero for finite mass case, while its integral is still zero. In turn, all other functions should be also zero due to gauge invariance.

At the same time, the further studies of dispersive approach at higher orders and (especially) beyond perturbative theory are of most interest. One should note here the recent calculations in the framework of instanton model [18] leading to the exponential, rather than power corrections to Vainshtein's theorem.

It seems that it is the nonlocality of this model, rather than instanton specifics, that provides this exponential behavior. In fact, it is analogous to the exponential falloff of transverse momentum dependent parton distributions whose coordinate description [19] bears similarity to the vacuum nonlocal condensates. One should also recall another observation and suggestion of [17], namely, that the local vacuum condensates do not match the dispersive description of axial anomaly and that the nonlocal condensates may improve the situation.

ACKNOWLEDGMENTS

We are grateful to A. E. Dorokhov, J. Hořejší, M. Knecht, J. Stern, O. V. Tarasov, and A. I. Vainshtein for useful discussions and comments. This work is partially supported by Grant No. RFBR 03-02-16816.

Note added.—After finishing our work we learned about the preprint of F. Jegerlehner and O. V. Tarasov [20] whose conclusions are similar to ours in the chiral limit for arbitrary external momenta.

APPENDIX

The imaginary parts of invariant scalars (3.1) are given by the following Feynman-parametric integrals

$$\begin{aligned}\Delta_1^{A(2-loop)} \sim \mathrm{Disc}_{q^2}\Big[&M^4 \int_0^1 da \int_0^{1-a} db \int_0^1 dx \int_0^{1-x} dy[4(2(kp)X_1 - p^2X_2)(x+2y-1)] + M^2 \int_0^1 da \int_0^{1-a} db \\ &\times \int_0^1 dx \int_0^{1-x} dy[(32y(kp)^2 + 16xa(kp)p^2 + 16b(kp)^2 - 32m^2y(kp) + 8(kp)p^2 - 8m^2yp^2 - 16(kp)^2 \\ &+ 16x(kp)^2 + 32ya(kp)p^2 - 8x(kp)p^2 - 32yb(kp)^2 - 16xb(kp)^2 - 16m^2x(kp) - 16y(kp)p^2 \\ &- 16a(kp)p^2)X_1 + (4yp^4 - 8(kp)p^2 - 16yap^4 - 8xap^4 - 8m^2x(kp) + 8ap^4 - 16m^2p^2 + 24m^2yp^2 \\ &+ 16m^2xp^2 + 8y(kp)p^2)X_2 + (4p^4 - 8x(kp)p^2 - 4xp^4 - 8m^2p^2 - 4yp^4 + 8m^2yp^2 + 8m^2xp^2 \\ &+ 8(kp)p^2 - 8y(kp)p^2)X_3] + \int_0^1 da \int_0^{1-a} db \int_0^{1-a-b} dc \int_0^1 dx \int_0^{1-x} dy[-16(kp)^2m^2(2m^2y - 2m^2 \\ &- p^2y + p^2 + 2(kp))X_4^2 + 32m^4y(kp)^2X_5^2] + \int_0^1 da \int_0^{1-a} db \int_0^1 dx \int_0^{1-x} dy[4m^2(-8y(kp)^2 - 4xa(kp)p^2 \\ &- 4b(kp)^2 + 4m^2y(kp) - 2(kp)p^2 + 2m^2yp^2 + 4(kp)^2 - 4x(kp)^2 - 8ya(kp)p^2 + 2x(kp)p^2 + 8yb(kp)^2 \\ &+ 4xb(kp)^2 + 2m^2x(kp) + 4y(kp)p^2 + 4a(kp)p^2 + 2m^2(kp))X_1 + 4m^2(-yp^4 + 2(kp)p^2 + 4yap^4 \\ &+ 2xap^4 + 2m^2x(kp) - 2ap^4 + 3m^2p^2 - 4m^2yp^2 - 3m^2xp^2 - 2y(kp)p^2)X_2 + 4m^2(-p^4 + 2x(kp)p^2 \\ &+ xp^4 + 2m^2p^2 + yp^4 - 2m^2yp^2 - 2m^2xp^2 - 2(kp)p^2 + 2y(kp)p^2)X_3]\Big];\end{aligned} \tag{A1}$$

$$\begin{aligned}\Delta_2^{A(2-\mathrm{loop})} \sim \mathrm{Disc}_{q^2}\Big[&M^4 \int_0^1 da \int_0^{1-a} db \int_0^1 dx \int_0^{1-x} dy[-4(2(kp)X_1 - p^2X_2)(x+2y-1)] + M^2 \int_0^1 da \int_0^{1-a} db \\ &\times \int_0^1 dx \int_0^{1-x} dy[(-32y(kp)^2 + 16(kp)^2 - 16xa(kp)p^2 - 8(kp)p^2 + 16m^2y(kp) + 16m^2x(kp) \\ &+ 8x(kp)p^2 - 16x(kp)^2 + 16xb(kp)^2 - 16b(kp)^2 + 16y(kp)p^2 + 32yb(kp)^2 - 32ya(kp)p^2 \\ &- 8m^2p^2 - 16m^2(kp) + 16a(kp)p^2)X_1 + (-24y(kp)p^2 + 16yap^4 + 8(kp)p^2 - 8m^2xp^2 - 8ap^4 + 4xp^4 \\ &- 8m^2yp^2 - 12yp^4 + 8xap^4)X_2 + (-4p^4 + 4xp^4 + 8y(kp)p^2 - 8m^2yp^2 + 4yp^4 - 8m^2xp^2 + 8m^2p^2 \\ &+ 8x(kp)p^2 - 8(kp)p^2)X_3 - 8m^2x(kp)X_6] + \int_0^1 da \int_0^{1-a} db \int_0^{1-a-b} dc \int_0^1 dx \\ &\times \int_0^{1-x} dy[-16(kp)^2m^2(-2(kp)y - 2m^2 + 2(kp) + 2m^2y + p^2 + xp^2)X_4^2 - 16(kp)^2m^2(2m^2 - 4m^2y)X_5^2] \\ &+ \int_0^1 da \int_0^{1-a} db \int_0^1 dx \int_0^{1-x} dy[-4m^2(4(kp)^2 - 4x(kp)^2 - 8y(kp)^2 + 4xb(kp)^2 - 2(kp)p^2 + 2x(kp)p^2 \\ &+ 8yb(kp)^2 - 4xa(kp)p^2 - 8ya(kp)p^2 + 4y(kp)p^2 + 4a(kp)p^2 + 2m^2x(kp) - 2m^2p^2 - 2m^2(kp) \\ &- 4b(kp)^2)X_1 - 4m^2(-2ap^4 + 4yap^4 + 2(kp)p^2 - 6y(kp)p^2 - m^2xp^2 - m^2p^2 + xp^4 - 3yp^4 \\ &+ 2xap^4)X_2 - 4m^2(-p^4 + 2x(kp)p^2 + xp^4 + 2m^2p^2 + yp^4 - 2m^2yp^2 - 2m^2xp^2 - 2(kp)p^2 \\ &+ 2y(kp)p^2)X_3 + 8m^4x(kp)X_6]\Big];\end{aligned} \tag{A2}$$

$$\Delta_3^{A(2-\mathrm{loop})} \sim \mathrm{Disc}_{q^2}\bigg[M^4\int_0^1 da\int_0^{1-a} db\int_0^1 dx\int_0^{1-x} dy[-8(X_1+X_2)(x+2y-1)] + M^2\int_0^1 da\int_0^{1-a} db\int_0^1 dx \times\int_0^{1-x} dy[(-32(kp)y+32yb(kp)-16xap^2-32yap^2+16p^2y-16x(kp)-8p^2+16m^2x+8xp^2 + 16(kp)+32m^2y+16m^2-16b(kp)+16ap^2+16xb(kp))X_1+(-32(kp)+8xp^2+40p^2y+16m^2x + 16m^2y-16p^2+16x(kp)+16ap^2+48(kp)y-32yap^2-16xap^2)X_2+(-8p^2y+16m^2x+16m^2y - 16(kp)y+8p^2-16m^2+16(kp))X_3] + \int_0^1 da\int_0^{1-a} db\int_0^{1-a-b} dc\int_0^1 dx \times\int_0^{1-x} dy[16m^2(kp)(-2(kp)y-4m^2+4m^2y+4(kp)+2p^2-2p^2y+xp^2)X_4^2-64m^4y(kp)X_5^2] + \int_0^1 da\int_0^{1-a} db\int_0^1 dx\int_0^{1-x} dy[-8m^2(-2x(kp)-4(kp)y+4yb(kp)-4yap^2+2p^2y-p^2+m^2x + xp^2+2(kp)+2m^2y+3m^2-2b(kp)+2ap^2+2xb(kp)-2xap^2)X_1-8m^2(5p^2y-2p^2+m^2x + xp^2+m^2-4yap^2-4(kp)+2ap^2+6(kp)y-2xap^2+2x(kp))X_2-8m^2(2m^2x-2m^2+2m^2y - p^2y+2(kp)-2(kp)y+p^2)X_3\bigg]; \tag{A3}$$

$$\Delta_4^{A(2-\mathrm{loop})} \sim \mathrm{Disc}_{q^2}\bigg[M^4\int_0^1 da\int_0^{1-a} db\int_0^1 dx\int_0^{1-x} dy[8(-1+2y+x)X_1+8(-1+y+x)X_6] + M^2\int_0^1 da\int_0^{1-a} db \times\int_0^1 dx\int_0^{1-x} dy[(24p^2+16(kp)-16x(kp)-16m^2+16b(kp)-24xp^2-32m^2y-32yb(kp) + 32yap^2-16ap^2+16xap^2-32p^2y-16m^2x-16xb(kp))X_1-16yp^2X_2+(8p^2y-16m^2x-16(kp) + 16m^2+16(kp)y-8p^2-16m^2y)X_3+(-16b(kp)-16x(kp)+16yb(kp)+16xb(kp)-16p^2y+16p^2 + 32(kp)-32(kp)y-16m^2x)X_6] + \int_0^1 da\int_0^{1-a} db\int_0^{1-a-b} dc\int_0^1 dx \times\int_0^{1-x} dy[8(-6m^2y(kp)p^2+8m^2(kp)p^2+4y(kp)p^4-8m^4y(kp)-2x(kp)p^4-8m^2(kp)^2+4y(kp)^2p^2 - p^6+2m^2xp^4+8m^2y(kp)^2+8m^4(kp)+8m^4yp^2-4(kp)p^4-4(kp)^2p^2+yp^6-xp^6+6m^2p^4 - 4m^2x(kp)p^2--8m^4p^2-6m^2yp^4)X_4^2+8(-4m^2y(kp)^2+2m^2(kp)p^2+2m^2yp^4-8m^4(kp)-8m^4yp^2 - 4m^2x(kp)p^2-4m^2x(kp)^2+16m^4y(kp)+2m^2y(kp)p^2+4m^2(kp)^2)X_5^2] + \int_0^1 da\int_0^{1-a} db\int_0^1 dx \times\int_0^{1-x} dy[8m^2(-2(kp)+2x(kp)+3m^2-2b(kp)-3p^2+2m^2y+4yb(kp)-4yap^2+2ap^2-2xap^2 + 4p^2y+m^2x+2xb(kp)+3xp^2)X_1+16m^2yp^2X_2+8m^2(2m^2x-2m^2+2m^2y-p^2y+2(kp) - 2(kp)y+p^2)X_3+8m^2(-2xb(kp)+2b(kp)+2p^2y+m^2-m^2y+2x(kp)+m^2x-4(kp)-2yb(kp) - 2p^2+4(kp)y)X_6\bigg], \tag{A4}$$

where

$$X_1 = 1/(-m^2ya - m^2yb + m^2y - m^2 + 2ab(kp) + p^2a - p^2a^2);$$
$$X_2 = 1/(m^2xa + m^2xb - m^2x + m^2ya + m^2yb - m^2y - m^2a - m^2b - 2xya(kp) - 2xyb(kp) + 2xy(kp) - p^2xa - p^2xb + p^2x + p^2x^2a + p^2x^2b - p^2x^2 + p^2a - p^2a^2);$$
$$X_3 = 1/(m^2xa + m^2xb - m^2x + m^2ya + m^2yb - m^2y - m^2a - m^2b - 2xya(kp) - 2xyb(kp) + 2xy(kp) + 2ab(kp) - p^2xa - p^2xb + p^2x + p^2x^2a + p^2x^2b - p^2x^2 + p^2a - p^2a^2);$$
$$X_4 = 1/(m^2xa + m^2xb + m^2xc - m^2x + m^2ya + m^2yb + m^2yc - m^2y - m^2a - m^2b - m^2c - 2xya(kp) - 2xyb(kp) - 2xyc(kp) + 2xy(kp) + 2ac(kp) - p^2xa - p^2xb - p^2xc + +p^2x + p^2x^2a + p^2x^2b + p^2x^2c - p^2x^2 + p^2a - p^2a^2);$$
$$X_5 = 1/(-m^2ya - m^2yb - m^2yc + m^2y - m^2 + 2ac(kp) + p^2a - p^2a^2);$$
$$X_6 = 1/(2xy(kp) - 2xyb(kp) - 2xya(kp) - m^2b - m^2a - m^2y + m^2yb + m^2ya - m^2x + m^2xb + m^2xa + p^2x - p^2xb - p^2xa - p^2x^2 + p^2x^2b + p^2x^2a). \quad \text{(A5)}$$

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