Color SU(3) symmetry, confinement, stability, clustering and quark mass dependence in the $q^4\bar{q}$ system

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We examine the color SU(3) dynamics of $q^4\bar{q}$ system, i.e. of the *pentaquark*. First we study this system in the model with two-body interaction proportional to the color charges. We construct the potential matrix and show (1) Confinement: the color singlet $q\bar{q}$ potential energy rises infinitely with the separation distance, (2) Stability: All colorless states' energies are bounded from below, (3) Color singlet clustering: the pentaquark color-singlet state Hamiltonian turns into a sum of a three-quark (baryon) and a quark-antiquark (meson) cluster Hamiltonian, in the limit of asymptotically large separations. We evaluate the four excitation eigenfrequencies of pentaquarks in the harmonic oscillator two-body confining potential and the ground states' dependence on the quark-antiquark mass ratio. We show that the pentaquark is unlikely to bind even with a top antiquark, in contrast to tetraquarks.

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I. INTRODUCTION

Irrespective of the exotic $\Theta^+(1540)$ resonance's existence or nonexistence, multiquarks deserve a serious theoretical study. The pentaquark ($q^4\bar{q}$), i.e., the system that is made up of four quarks and one antiquark is the second simplest example of a multiquark, right after the well-studied tetraquark [1–3]. It can be in one of three linearly independent, mutually orthogonal color singlet states: one that is the direct product of two ordinary color singlets (one baryon and one meson $q\bar{q}$), which we shall call the "asymptotic baryon-meson" state, and two others that are overall singlet combinations of two (distinct) color octets, which we call "hidden color" confined pentaquark states. The pentaquark interaction potential in color space is generally unknown: even so-called "dynamical" pentaquark calculations [4] have used some oversimplified (essentially Abelian Lorentz scalar interaction) *Ansätze* for the color dependence.

The color SU(3) Yang-Mills ("gauge") field dynamics, also known as quantum chromodynamics (QCD), has been proposed as the solution to all of the quark dynamical problems. The QCD equations of motion are nonlinear and strongly coupled, so no exact solution has been found to date. In the following we shall use only QCD's exact ("unbroken") color SU(3) symmetry, which ought to be beyond doubt, to constrain and/or predict the properties of the mathematically allowed dynamical pentaquark states. Thus all of our conclusions must also hold in QCD, though we shall not attempt to derive them explictly.

Rather than try to solve QCD, we shall use the so-called $F_i \cdot F_j$ two-quark potential model [1,5,6], which is unique in that it has the following beneficial properties: (0) It is globally color SU(3) symmetric; (1) Confinement: the color singlet $q\bar{q}$ potential energy rises infinitely with the separation distance, (2) Stability: All color-singlets' energies are bounded from below, as assured by the positivity (Orsay) theorem [2],[1] (3) Color singlet clustering: In the limit of asymptotically large separations, the pentaquark "asymptotic color-singlet state" Hamiltonian turns into the sum of a three-quark (baryon) and a quark-antiquark (meson) cluster Hamiltonian [5,8].

This is important because even so-called dynamical calculations of pentaquark spectra have relied on very simple *Ansätze* for the color dependence of the quark-quark interaction, viz. equal strength of all quark-quark and quark-antiquark interactions. This assumption is equivalent to a Lorentz scalar, color independent (Abelian) interaction, which we shall explicitly rule out in our analysis below.

Importance of these deliberations lies in their consequences for the spectra of pentaquarks: they determine (a) the energy of the first orbital angular momentum excited state; and (b) the effective strength of the hyperfine interaction, which determine the detailed flavor-spin splittings of pentaquarks.

We shall not concern ourselves much with these HF interactions here other than to point out that they are invariably of short range (they are usually proportional to a Dirac delta function) and hence their expectation values are sensitive to the wave function at the origin: The HFI-induced mass/energy shift in a harmonic oscillator confined nucleon is proportional to $I = \langle\Psi|\delta(\mathbf{r}_1 - \mathbf{r}_2)|\Psi\rangle = (m_q\omega/\sqrt{2\pi})^{3/2}$, where m_q is the constituent quark mass and $\omega = \sqrt{k/m_q}$ is the oscillator frequency of the pair of quarks. As the average color factor of the two-body force

[1]We were (mis)led to believe that M. Rosina was the first (and only) to have published the conjecture and partial proof of this result; consequently we referred to it as Rosina's conjecture in the second paper of Ref. [7]. It turns out that the Orsay group published the complete proof [2] some five years before Rosina and many other authors had partial proofs much earlier, see [8]; we shall henceforth call it the Orsay positivity theorem.

in pentaquarks equals only one quarter of that in mesons, one should therefore expect an average decrease in the HFI matrix elements in pentaquarks, as compared to that in mesons, inversely proportional to the ratio $(I_{\rm meson}/I_{\rm pentaq}) = (\omega_{\rm meson}/\omega_{\rm pentaq})^{3/2} = 4^{3/4} \simeq 2.83$, thus reducing the naive schematic approximation predictions of the HFI mass shifts up to 3 times on the average, and perhaps even more in special cases. Thus we see that the color structure of the pentaquark and the two-quark confining interaction play a vital role in moving towards a reliable prediction of pentaquark spectra.

There is another potentially important application of these ideas, *viz.* to the unusually small decay width of the Θ^+: If the Θ^+ is created in one of the "nonasymptotic" color-singlet states, as is allowed by color SU(3) conservation, then its decay into a nucleon and a kaon is strongly supressed/forbidden and may proceed only through two-gluon/glueball intermediate states.

This paper falls into four sections. After the Introduction, in Sec. II we give a reminder of the basic facts regarding the $\mathrm{F}_i \cdot \mathrm{F}_j$ model and the $q^4\bar{q}$ system's color SU(3) wave functions. Then in Sec. III we examine the predictions of the (standard) $\mathrm{F}_i \cdot \mathrm{F}_j$ model of quark interactions for pentaquarks. Finally, in Sec. IV we draw our conclusions.

II. PRELIMINARIES

A. The color exchange, or $\mathrm{F}_i \cdot \mathrm{F}_j$ interaction

Now, the so-called $\mathrm{F}_i \cdot \mathrm{F}_j$ color dependent two-quark interaction

$$V_{ij} = (\mathrm{F}_i \cdot \mathrm{F}_j) v_{ij}, \tag{1}$$

leads to a mixing of the color singlet states without breaking of the color SU(3) symmetry. In applications of this model there were basically two schools: (a) the MIT bag model [6], which dealt (mostly schematically) with consequences of the Breit interaction, Eq. (4), between relativistic quarks confined in a spherical bag; and (b) the nonrelativistic constituent quark model , which assumes a confining two-body potential $v_{12}^{\rm conf.}$, usually a harmonic oscillator, or a linearly rising potential

$$-v_{12}^{\rm conf.} = \begin{cases} \frac{k}{2}(\mathbf{r}_1 - \mathbf{r}_2)^2 \\ \lambda|\mathbf{r}_1 - \mathbf{r}_2| \end{cases}, \tag{2}$$

where $k = m\omega^2$. The "realistic" potential consists of the linear + Coulomb + constant + Breit (see below) terms [9]

$$-v_{12}^{\rm real.} = -\frac{\alpha_C}{r_{12}} + \lambda r_{12} + \Lambda - v_{12}^{\rm Breit}, \tag{3}$$

where α_C is the strong fine structure constant and

$$v_{12}^{\rm Breit} = -\frac{2\alpha_S \pi}{3m_1 m_2} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{4}$$

Note the overall minus sign that cancels the negative sign of the color factor in the color singlet $q\bar{q}$, and the $\bar{\mathbf{3}}$ diquark state.[2]

The (quantum mechanical) mixing of distinct color singlets (see Sec. II C below) is due to the fact that the color factor $\mathrm{F}_i \cdot \mathrm{F}_j$ is a part of the $i \leftrightarrow j$ two-quark color *exchange* operator P_{ij}^C.[3] That does not mean that color $SU(3)$ symmetry is broken, however.

The color exchange nature of the confining interaction, however, is not enough for a full description of hadron spectra: one must also have simultaneously color- (or flavor-) and spin-dependent forces, such as the Fermi-Breit force Eq. (4).

Now remember that the color-dependent two-body confining potential has its strength reduced by a factor of 2 as one goes from the $q\bar{q}$ system to the color $\bar{\mathbf{3}}$ q^2 substate of the q^3 nucleon. This ratio changes to 1:4, on the average, in pentaquarks. The precise value depends on the color state the pentaquark is in.

As $\omega \simeq k^{1/2}$, where k is the oscillator spring constant, we would expect $(\omega_{\rm meson}/\omega_{\rm nucleon}) \simeq (k_{\rm meson}/k_{\rm nucleon})^{1/2} \simeq \sqrt{2} = 1.41$ for mesons and ordinary three-quark baryons, but the three-body nature (reduced mass effects for the three-body Jacobi coordinates) of the baryon changes this to $(\omega_{\rm meson}/\omega_{\rm nucleon}) \simeq (4/3)^{1/2} = \sqrt{1.333} \simeq 1.155$, in good agreement with experiment (the fitted values yield $(\omega_{\rm meson}/\omega_{\rm nucleon}) = 550/450 \simeq 1.22$ for this ratio), see Ref. [11]. Similarly, for pentaquarks we expect $(\omega_{\rm meson}/\omega_{\rm pentaq}) \simeq (k_{\rm meson}/k_{\rm pentaq})^{1/2} \simeq \sqrt{4} = 2$, and/or $(\omega_{\rm nucleon}/\omega_{\rm pentaq}) \simeq (k_{\rm nucleon}/k_{\rm pentaq})^{1/2} \simeq \sqrt{2} = 1.41$.

B. C conjugation and the Lorentz scalar vs vector potentials

As both the Lorentz scalar and the Lorentz vector two-body interactions reduce to the same form in the (lowest order) nonrelativistic limit, the distinction between them may seem an academic point. That is indeed so for interactions solely between quarks, or solely between antiquarks, but when it comes to quark-antiquark interactions, the vector and scalar interactions differ by an overall sign, i.e., if one is attractive, the other is repulsive. That is a consequence of the opposite C conjugation properties of Lorentz scalars and Lorentz vectors. This leads to opposite signs in $q\bar{q}$ potentials: For scalar vertices

[2]The constant term Λ has an interesting role: it effectively changes the total mass of the hadron (or the gravitational mass of the constituent quark, but not its inertial mass!) in different color states. For example a negative Λ lowers the color singlet $q\bar{q}$ mass and increases the color octet one; and similarly for q^3 states.

[3]One can also construct three-body color exchange operators from SU(3) invariant products of three-quark color charge matrices [10].

$$\bar{C}_{12} = \begin{cases} -\mathrm{F}_1 \cdot \bar{\mathrm{F}}_2 \\ \bar{\mathrm{F}}_1 \cdot \bar{\mathrm{F}}_2 \end{cases} ; \tag{5}$$

whereas, for the vector ones,

$$\bar{C}_{12} = \begin{cases} \mathrm{F}_1 \cdot \bar{\mathrm{F}}_2 \\ \bar{\mathrm{F}}_1 \cdot \bar{\mathrm{F}}_2 \end{cases} , \tag{6}$$

where the antiquark color factor is defined by

$$\bar{\mathrm{F}}^a = -\tfrac{1}{2}\boldsymbol{\lambda}^{aT} = -\tfrac{1}{2}\boldsymbol{\lambda}^{a*}. \tag{7}$$

Therefore, of course, the difference cannot be seen in systems made up entirely of (constituent) quarks, such as baryons. Nor can it be seen in the $q\bar{q}$ system alone, because the sign of this interaction can always be changed to agree with experiment. It is only in the tetra- and pentaquark systems that the distinction between Lorentz scalar and Lorentz vector interactions leads to dramatic differences.

We shall show that Lorentz scalar confining potentials are not allowed in this scenario: they lead to unstable color singlet states. Note that this is more than an academic point: The Breit interaction is a standard part of the (higher order in v/c) nonrelativistic reduction of the Lorentz vector two-body potential, i.e. of the one-gluon exchange potential, but not of the Lorentz scalar one. Another standard part of the (higher order in v/c) nonrelativistic reduction of the Lorentz vector two-body potential is the spin-orbit potential, which, however does not feature prominently in hadron spectra. The idea of a purely Lorentz scalar confining two-body potential as a remedy for the phenomenological spin-orbit problem has been tossed around for well over two decades [3] and is still being revived on occasion, see Ref. [12], despite this glaring defect.

C. $q^4\bar{q}$ color singlet states, their mixing and crossing (quark exchange) channels

In the $q^4\bar{q}$ system, there are three linearly independent and mutually orthogonal color singlets. One can label them by their symmetry properties under the interchange of the four quark indices; for example: one state ($|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle$ is antisymmetric in the first two indices and symmetric in the third and fourth indices, another state $|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$) is symmetric in the first two indices and antisymmetric in the third and fourth indices, and the third one is antisymmetric in the first two and the third and fourth indices ($|\bar{\mathbf{3}}_{12}\mathbf{3}_{34}\bar{\mathbf{3}}_5\rangle$). This basis is unsuitable, however, for the description of asymptotic states—the baryon and the meson.

The asymptotic "baryon-meson" color singlet state is a linear combination of two of the three states:

$$|\mathbf{1}_{123}\mathbf{1}_{45}\rangle = \sqrt{\frac{2}{3}}|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle + \frac{1}{\sqrt{3}}|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle. \tag{8}$$

The indices 1, 2, 3, 4 and 5 denote all the other quantum numbers, such as flavor and spin, of the four quarks and the antiquark, respectively. Clearly there are two other linearly independent color singlet states, the

$$|\mathbf{8}^{\rho}_{123}\mathbf{8}_{45}\rangle = \frac{1}{\sqrt{3}}|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle - \sqrt{\frac{2}{3}}|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle, \tag{9}$$

that is orthogonal to the first one, as well as the

$$|\mathbf{8}^{\lambda}_{123}\mathbf{8}_{45}\rangle = |\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle, \tag{10}$$

that is orthogonal to the first two. Hence it should be clear that there is nothing special about the state $|\mathbf{1}_{123}\mathbf{1}_{45}\rangle$, in the subsequent developments; one may equally well use the states

$$|\mathbf{1}_{124}\mathbf{1}_{35}\rangle = -\sqrt{\frac{2}{3}}|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle + \frac{1}{\sqrt{3}}|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle \tag{11}$$

$$|\mathbf{8}^{\rho}_{124}\mathbf{8}_{35}\rangle = \frac{1}{\sqrt{3}}|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle + \sqrt{\frac{2}{3}}|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle \tag{12}$$

$$|\mathbf{8}^{\lambda}_{124}\mathbf{8}_{35}\rangle = |\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle. \tag{13}$$

Moreover, it should be clear that the whole procedure can be repeated with the first pair of quark indices replaced by the second pair and with the Pauli basis states suitably replaced:

$$|\mathbf{1}_{134}\mathbf{1}_{25}\rangle = \sqrt{\frac{2}{3}}|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle + \frac{1}{\sqrt{3}}|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle \tag{14}$$

$$|\mathbf{8}^{\rho}_{134}\mathbf{8}_{25}\rangle = \frac{1}{\sqrt{3}}|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle - \sqrt{\frac{2}{3}}|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle, \tag{15}$$

$$|\mathbf{8}^{\lambda}_{134}\mathbf{8}_{25}\rangle = |\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle. \tag{16}$$

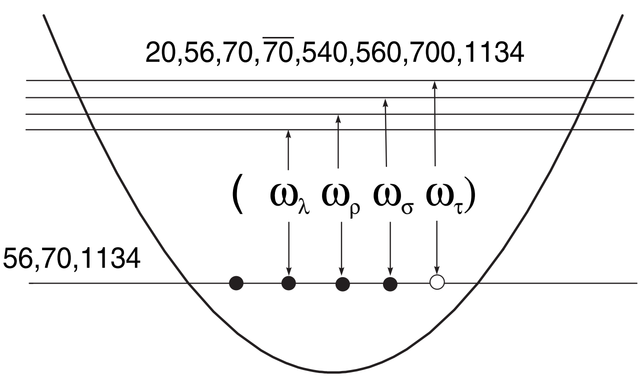

FIG. 1. Pauli allowed color singlet pentaquark states in the harmonic oscillator potential. The excitation energies $\hbar\omega_\alpha$, where $\alpha = \lambda, \rho, \sigma, \tau$ denotes the eigenmodes of the pentaquark (not necessarily identical to the Jacobi coordinates of five particles) depends on the color singlet state the pentaquark is in. The numbers denote the dimensions of the $\mathrm{SU}_{FS}(6)$ irreducible representations allowed by the Pauli principle in the corresponding state, see Ref. [13].

The Pauli principle holds only for identical particles, i.e., it antisymmetrizes only the four quarks, but not the $q\bar{q}$ pairs. For this reason, the (''unphysical'') basis spanned by states $|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle$, $|\mathbf{6}_{12}\bar{\mathbf{6}}_{34}\bar{\mathbf{3}}_5\rangle$, and $|\bar{\mathbf{3}}_{12}\mathbf{3}_{34}\bar{\mathbf{3}}_5\rangle$ is better suited to the application of the Pauli principle than the (''physical'') asymptotic basis. The linear independence and orthogonality of the three color singlet states, however, provides an additional constraint, even on the $q\bar{q}$ pairs to which the Pauli principle does not apply. Thus, the combination of the color singlet requirement and the Pauli principle leads to flavor-spin selection rules determining the pentaquark spectra, Fig. 1, for an explanation see Ref. [13].

Proper asymptotic behavior of the $q^4\bar{q}$ system imposes an additional ''clustering'' condition on its Hamiltonian.

D. Color singlet clustering in the $q^4\bar{q}$ system

Technically, in the pentaquark case color singlet clustering[4] means that one color singlet state (the ''baryon-meson'' one) should be able to reach asymptotically large separations, or equivalently that the corresponding potential should approach the sum of the potentials in the two separate color singlets, i.e., in the baryon and the meson, in the limit of asymptotically large separations of the two clusters' center-of-masses $\Delta R = |\Delta\mathbf{R}| = |\frac{1}{3}\times(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3) - \frac{1}{2}(\mathbf{r}_4+\mathbf{r}_5)| \to \infty$:

$$\lim_{\Delta R\to\infty}\langle V\rangle_{\mathbf{11}} \equiv \lim_{\Delta R\to\infty}\langle \mathbf{1}_{123}\mathbf{1}_{45}|V|\mathbf{1}_{123}\mathbf{1}_{45}\rangle = \langle\mathbf{1}_{123}|V|\mathbf{1}_{123}\rangle + \langle\mathbf{1}_{45}|V|\mathbf{1}_{45}\rangle. \quad (17)$$

In order to verify color singlet clustering Eq. (17) in QCD, one needs to know the precise form of the 2-, 3-, 4- and 5-body potentials. That is (still) impossible at this stage, both empirically and theoretically.[5] Below we look only at the $\mathrm{F}_i\cdot\mathrm{F}_j$ model, which obeys color singlet clustering/saturation exactly.

III. THE COLOR EXCHANGE, OR $\mathrm{F}_i\cdot\mathrm{F}_j$ INTERACTION IN THE $q^4\bar{q}$ SYSTEM

A. The $q^4\bar{q}$ Hamiltonian in the Pauli basis

We define the color space interaction matrix in the ''Pauli basis'' as follows:

$$V_{\mathrm{Pauli}} = \begin{pmatrix} \langle\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle & \langle\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle & \langle\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle \\ \langle\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle & \langle\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle & \langle\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle \\ \langle\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle & \langle\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle & \langle\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle \end{pmatrix}. \quad (18)$$

We find the following color singlet diagonal and off-diagonal potentials in the $q^4\bar{q}$ system with Lorentz vector two-body interactions

$$\langle\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle = -\tfrac{5}{12}(v_{13}+v_{14}+v_{23}+v_{24}) + \tfrac{1}{3}(-2v_{12}+v_{34}) + \tfrac{1}{6}(v_{15}+v_{25}-5(v_{35}+v_{45})) \quad (19)$$

$$\langle\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \frac{v_{13}-v_{14}+v_{23}-v_{24}}{2\sqrt{2}} + \frac{-v_{35}+v_{45}}{\sqrt{2}} \quad (20)$$

$$\langle\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \tfrac{1}{4}[-v_{13}+v_{14}+v_{23}-v_{24} + v_{15}+v_{25}-v_{35}-v_{45}] \quad (21)$$

$$\langle\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = -\tfrac{1}{6}(v_{13}+v_{14}+v_{23}+v_{24}) - \tfrac{2}{3}(v_{12}+v_{34}) - \tfrac{1}{3}(v_{15}+v_{25}+v_{35}+v_{45}) \quad (22)$$

$$\langle\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \frac{-v_{13}-v_{14}+v_{23}+v_{24}}{2\sqrt{2}} + \frac{-v_{15}+v_{25}}{\sqrt{2}} \quad (23)$$

$$\langle\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = -\tfrac{5}{12}(v_{13}+v_{14}+v_{23}+v_{24}) + \tfrac{1}{3}(v_{12}-2v_{34}) + \tfrac{1}{6}[-5(v_{15}+v_{25})+v_{35}+v_{45}]. \quad (24)$$

Equations (19)–(24) are direct analogons of the tetraquark equations of motion first derived in Ref. [1]. Only Eq. (22) has also been derived by Okiharu *et al.* [14].

To obtain the Lorentz scalar interaction potentials, merely flip the sign of all the $q\bar{q}$ terms in the above results, i.e., $v_{i5}\to -v_{i5}$.

B. Asymptotic bases

Identities (8)–(10) are summarized by the (basis) transformation matrix

[4] This property also goes by the name of '' color saturation'', for historical reasons, named after similarities with the saturation of the nucleon interactions.

[5] In Ref. [7] we made some *Ansätze* for the two- and three-quark potentials, and constrained them by the requirements of confinement, stability and proper color ordering in the $q\bar{q}$ and q^3 systems. The general case will not be discussed here.

$$L_{123} = L_{124}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

and similarly for Eqs. (11)–(13)

$$L_{124} = L_{123}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{26}$$

and

$$L_{234} = L_{134}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \tag{27}$$

and

$$L_{134} = L_{234}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \tag{28}$$

which describe the remaining two asymptotic channels. The color singlet potential matrices in the asymptotic bases read

$$\begin{aligned} V_{(123)(45)} &= L_{123} V_{\rm Pauli} L_{123}^{-1} \qquad V_{(124)(35)} = L_{124} V_{\rm Pauli} L_{124}^{-1} \\ V_{(234)(15)} &= L_{234} V_{\rm Pauli} L_{234}^{-1} \qquad V_{(134)(25)} = L_{134} V_{\rm Pauli} L_{134}^{-1}. \end{aligned} \tag{29}$$

C. Color singlet clustering in the $q^4\bar{q}$ system

Now use Eq. (25) to find

$$\begin{aligned} \langle V \rangle_{\mathbf{11}} &= \langle \mathbf{1}_{123} \mathbf{1}_{45} | V | \mathbf{1}_{123} \mathbf{1}_{45} \rangle = -\left(\tfrac{2}{3} \textstyle\sum_{i<j}^{3} v_{ij} + \tfrac{4}{3} v_{45} \right) \\ &= \langle \mathbf{1}_{123} | V_{2-b} | \mathbf{1}_{123} \rangle + \langle \mathbf{1}_{45} | V_{2-b} | \mathbf{1}_{45} \rangle, \end{aligned} \tag{30}$$

which proves that in this channel the two color singlet clusters move freely (‘‘cluster’’) at all distances, not just asymptotically, in this model. The factors $-2/3$ and $-4/3$ are just the values of the color factor $\mathrm{F}_i \cdot \bar{\mathrm{F}}_j$, for $(i, j = 1, 2, 3)$ and $(i = 4, j = 5)$, in their respective color singlet states. Together with the overall minus sign in the confining potential Eq. (2) this yields positive confining two-body potentials for the q^3 and $q\bar{q}$ cluster, with no restoring force in the two cluster relative coordinate.

Similarly, the two ‘‘hidden color’’ states

$$\begin{aligned} \langle V \rangle_{\mathbf{8}_\varrho \mathbf{8}} &= \langle \mathbf{8}_{123}^{\varrho} \mathbf{8}_{45} | V | \mathbf{8}_{123}^{\varrho} \mathbf{8}_{24} \rangle \\ &= -\tfrac{1}{3}(2v_{12} + v_{34}) + \tfrac{1}{12}(v_{13} + v_{23}) - \tfrac{7}{12}(v_{14} + v_{24}) \\ &\quad - \tfrac{1}{6}(v_{15} + v_{25} - v_{45} + 7v_{35}) \end{aligned} \tag{31}$$

$$\begin{aligned} \langle V \rangle_{\mathbf{8}_\lambda \mathbf{8}} &= \langle \mathbf{8}_{123}^{\lambda} \mathbf{8}_{45} | V | \mathbf{8}_{123}^{\lambda} \mathbf{8}_{24} \rangle = \langle \mathbf{6}_{12} \bar{\mathbf{3}}_{34} \bar{\mathbf{3}}_5 | V | \mathbf{6}_{12} \bar{\mathbf{3}}_{34} \bar{\mathbf{3}}_5 \rangle \\ &= -\tfrac{5}{12}(v_{13} + v_{14} + v_{23} + v_{24}) + \tfrac{1}{3}(v_{12} - 2v_{34}) \\ &\quad - \tfrac{1}{6}[5(v_{15} + v_{25}) - v_{35} - v_{45}], \end{aligned} \tag{32}$$

are permanently confined, i.e., they cannot decay into the baryon and meson color singlet state (without simultaneous emission of at least two gluons) even if their energy is higher than that of the asymptotic state. This fact (selection rule) directly affects the observability of these new states. We shall not discuss these states here, but hope to return to their study elsewhere.

D. Pentaquark chromo-harmonic Hamiltonian

For this purpose it is best to use the center-of-mass and the relative (Jacobi) coordinates, defined by

$$\mathbf{r}_\varrho = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2) \tag{33}$$

$$\mathbf{r}_\lambda = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3) \tag{34}$$

$$\mathbf{r}_\tau = \frac{1}{2\sqrt{3}}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 - 3\mathbf{r}_4) \tag{35}$$

$$\mathbf{r}_\sigma = \frac{1}{2\sqrt{5}}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 - 4\mathbf{r}_5) \tag{36}$$

$$\mathbf{R} = \frac{1}{\sqrt{5}}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_5). \tag{37}$$

Because of translational invariance, the complete potential matrix is independent of the center-of-mass coordinate $\mathbf{R}$.

1. Pauli basis

Thus we find the following (vector) potentials (we set $k = 1$ for simplicity)

$$\begin{aligned} \langle \bar{\mathbf{3}}_{12} \mathbf{6}_{34} \bar{\mathbf{3}}_5 | V | \bar{\mathbf{3}}_{12} \mathbf{6}_{34} \bar{\mathbf{3}}_5 \rangle &= \frac{5}{6}\mathbf{r}_\lambda{}^2 + \mathbf{r}_\rho{}^2 + \frac{5}{6}\mathbf{r}_\sigma{}^2 - \frac{1}{2}\sqrt{\frac{5}{3}}\mathbf{r}_\sigma \mathbf{r}_\tau \\ &\quad + \frac{2}{3}\mathbf{r}_\tau{}^2 - \sqrt{\frac{5}{6}}\mathbf{r}_\lambda \cdot \mathbf{r}_\sigma + \frac{\sqrt{2}}{3}\mathbf{r}_\lambda \cdot \mathbf{r}_\tau \end{aligned} \tag{38}$$

$$\begin{aligned} \langle \bar{\mathbf{3}}_{12} \bar{\mathbf{3}}_{34} \bar{\mathbf{3}}_5 | V | \bar{\mathbf{3}}_{12} \bar{\mathbf{3}}_{34} \bar{\mathbf{3}}_5 \rangle &= \frac{2}{3}\mathbf{r}_\lambda{}^2 + \mathbf{r}_\rho{}^2 - \frac{\sqrt{2}}{3}\mathbf{r}_\lambda \cdot \mathbf{r}_\tau \\ &\quad + \frac{5}{6}(\mathbf{r}_\sigma{}^2 + \mathbf{r}_\tau{}^2) \end{aligned} \tag{39}$$

$$\langle \mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \mathbf{r}_\lambda{}^2 + \frac{\mathbf{r}_\rho{}^2}{2} + \sqrt{\frac{5}{6}}\mathbf{r}_\lambda\cdot\mathbf{r}_\sigma + \frac{5}{6}\mathbf{r}_\sigma{}^2 + \frac{1}{2}\sqrt{\frac{5}{3}}\mathbf{r}_\sigma\cdot\mathbf{r}_\tau + \mathbf{r}_\tau{}^2 \tag{40}$$

$$\langle \bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \frac{-1}{3\sqrt{2}}(\mathbf{r}_\lambda{}^2 - \mathbf{r}_\tau{}^2) + \mathbf{r}_\lambda \cdot\left(\frac{-1}{2}\sqrt{\frac{5}{3}}\mathbf{r}_\sigma + \frac{\mathbf{r}_\tau}{6}\right) + \sqrt{\frac{5}{6}}\mathbf{r}_\sigma\mathbf{r}_\tau \tag{41}$$

$$\langle \bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \frac{\mathbf{r}_\lambda{}^2}{24} - \frac{\mathbf{r}_\rho{}^2}{8} - \sqrt{\frac{1}{6}}\mathbf{r}_\rho\cdot\mathbf{r}_\tau - \frac{1}{4}\sqrt{\frac{5}{3}}\mathbf{r}_\sigma \cdot\mathbf{r}_\tau + \frac{\mathbf{r}_\tau{}^2}{12} + \frac{1}{2\sqrt{3}}\mathbf{r}_\lambda \cdot\left(\mathbf{r}_\rho - \sqrt{\frac{5}{2}}\mathbf{r}_\sigma - \frac{\mathbf{r}_\tau}{\sqrt{6}}\right) \tag{42}$$

$$\langle \bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \frac{1}{\sqrt{6}}\mathbf{r}_\lambda\cdot\mathbf{r}_\rho - \frac{1}{2}\mathbf{r}_\rho \cdot\left(\sqrt{5}\mathbf{r}_\sigma - \frac{\mathbf{r}_\tau}{\sqrt{3}}\right). \tag{43}$$

2. Eigenfrequencies and eigenmodes of pentaquarks

(1) $|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle$

$$\langle \bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle = \mathbf{r}_\rho{}^2 + \frac{1}{2}\left(-\frac{1}{\sqrt{3}}\mathbf{r}_\lambda + \sqrt{\frac{2}{3}}\mathbf{r}_\tau\right)^2 + \frac{11-\sqrt{46}}{12}\left(\sqrt{1-\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\lambda}{\sqrt{3}} + \sqrt{1+\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\sigma}{\sqrt{2}} + \sqrt{1-\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\tau}{\sqrt{6}}\right)^2 + \frac{11+\sqrt{46}}{12}\left(\sqrt{1+\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\lambda}{\sqrt{3}} - \sqrt{1-\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\sigma}{\sqrt{2}} + \sqrt{1+\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\tau}{\sqrt{6}}\right)^2 \tag{44}$$

Note the presence of a large ($(11+\sqrt{46})/12$) and a small eigenvalue ($(11-\sqrt{46})/12$). This state is confined and orthogonal to the asymptotic state. It cannot decay into a meson and baryon (in this model and NR approximation). Experimental production and observation of such states is an open problem.

(2) $|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$ Similarly for the second state:

$$\langle \bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \mathbf{r}_\rho{}^2 + \frac{5}{6}\mathbf{r}_\sigma{}^2 + \left(-\frac{1}{\sqrt{3}}\mathbf{r}_\lambda + \sqrt{\frac{2}{3}}\mathbf{r}_\tau\right)^2 + \frac{1}{2}\left(\sqrt{\frac{2}{3}}\mathbf{r}_\lambda + \frac{1}{\sqrt{3}}\mathbf{r}_\tau\right)^2 > 0 \tag{45}$$

(3) $|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$ Finally the third state is analogous to the first one

$$\langle \mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5|V|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle = \frac{1}{2}\mathbf{r}_\rho{}^2 + \left(-\frac{1}{\sqrt{3}}\mathbf{r}_\lambda + \sqrt{\frac{2}{3}}\mathbf{r}_\tau\right)^2 + \frac{11-\sqrt{46}}{12}\left(\sqrt{1-\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\lambda}{\sqrt{3}} - \sqrt{1+\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\sigma}{\sqrt{2}} + \sqrt{1-\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\tau}{\sqrt{6}}\right)^2 + \frac{11+\sqrt{46}}{12}\left(\sqrt{1+\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\lambda}{\sqrt{3}} + \sqrt{1-\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\sigma}{\sqrt{2}} + \sqrt{1+\frac{1}{\sqrt{46}}}\frac{\mathbf{r}_\tau}{\sqrt{6}}\right)^2. \tag{46}$$

The eigenvalues of this potential are equivalent to those of Eq. (44), but two of the three eigenmodes are different.

From Eqs. (44)–(46), one can convince oneself that all three diagonal color singlet potentials are positive semidefinite (remember that the Orsay positivity theorem holds only for diagonal terms) in all four Jacobi coordinates, i.e. they confine in all four relative motions.

In the asymptotic bases, some of the diagonal potential matrix elements may be independent of one or more linear combinations of Jacobi coordinates, however. These "zero-modes" indicate free motion, i.e., possibility of reaching asymptotically large separations in the respective relative coordinate; they are a consequence of the color singlet clustering, or of the color saturation property.

3. Physical consequences

The oscillator eigenfrequencies $\omega_{1,2,3,4}$ in the three color singlet states are not determined merely by the eigenvalues of the potential matrix, but also by the reduced masses of the corresponding eigenmodes. The standard normalized Jacobi coordinates Eqs. (33)–(36) are rather convenient in this regard, as all four have the same reduced mass m equal to the constituent quark mass m_q, in the limit of good flavor

SU(3) symmetry. The only proviso is that the eigenmodes must be normalized as well, which is fulfilled here. The frequencies are tabulated in Table I and the corresponding ground state energies in

$$E_{|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle} = \frac{3}{2}(\omega_\sigma + \omega_\rho + \omega_\lambda + \omega_\tau) = \frac{3}{2}\left(1 + \sqrt{\frac{5}{3}} + 2\sqrt{2}\right)\sqrt{\frac{k}{m}} \tag{47}$$

$$E_{|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle} = E_{|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle} = \frac{3}{2}(\omega_\sigma + \omega_\rho + \omega_\lambda + \omega_\tau) = \frac{3}{2}\left(1 + \sqrt{2} + \sqrt{\frac{11 - \sqrt{46}}{6}} + \sqrt{\frac{11 + \sqrt{46}}{6}}\right)\sqrt{\frac{k}{m}} \tag{48}$$

$$E_{|\mathbf{1}_{123}\mathbf{1}_{45}\rangle} = \frac{3}{2}(2\omega_{\rm nucleon} + \omega_{\rm meson}) = \frac{3}{2}\left(\sqrt{\frac{8}{3}} + 2\sqrt{2}\right)\sqrt{\frac{k}{m}}. \tag{49}$$

Compare these results with $\omega^2_{\rm meson} = 8k/3m \simeq 2.67k/m$, $\omega^2_{\rm nucleon} = 2k/m$ and note that the largest and the lowest pentaquark oscillator frequencies/excitation energies bracket the meson and nucleon ones $\omega^2_{\rm meson} < [(11 + \sqrt{46})/6](k/m) = 2.96k/m$ and $\omega^2_{\rm nucleon} > [(11 - \sqrt{46})/6](k/m) = 0.70k/m$. Note that the degeneracy is completely lifted in states $|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle$ and $|\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$, whereas there is one doubly degenerate level in the $|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$ state.

Importance of these deliberations lies in their consequences for the spectra of pentaquarks: they determine the energy of the first orbital angular momentum excited state. Many models of the Θ^+ assume even parity, i.e. a P-wave state, in spite of its low mass. This could be explained by the low excitation energies of the λ mode in the first two, and the σ modes in the first and third states. Now use $\omega_{\rm meson} = 550$ MeV, or $\omega_{\rm nucleon} = 450$ MeV to find $\omega_\sigma = 282$ MeV, or $\omega_\sigma = 266$ MeV, respectively. Note, moreover, that (in the equal mass limit and with HO potential) both pentaquark ground states' energies lie above the asymptotic baryon-meson system's: $E_{|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle} > E_{|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle} > E_{|\mathbf{1}_{123}\mathbf{1}_{45}\rangle}$. This is a consequence of the concavity of the energy as a function of the (reduced) mass in the harmonic oscillator potential and may change in a more realistic confining potential, such as the "Coulomb + linear" one, if the tetraquark system can be used for guidance Ref. [3]. The ordering of states in the tetraquarks, Ref. [3] is function of the (reduced) quark masses; we have tried to find level crossings, i.e., changes of the level ordering $E_{|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle} > E_{|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle}$, but we could not find one for quark-antiquark mass ratios ranging from unity, see Fig. 2, up to and exceeding 10^6.

Another important consequence of a smaller/larger ω value in pentaquarks is in the spatial matrix elements of the HFI, which scale like $\omega^{3/2}$ with the HO frequency ω, so the pentaquark HFI splittings are inversely proportional to the ratio $(I_{\rm meson}/I_{\rm pentaq}) = (\omega_{\rm meson}/\omega_{\rm pentaq})^{3/2}$, whose maximum value equals $(8/3 \times 0.70)^{3/4} = 3.81^{3/4} \simeq 2.73$ and the minimum value $(8/3 \times 2.96)^{3/4} = 0.901^{3/4} = 0.95$; and/or $(I_{\rm nucleon}/I_{\rm pentaq}) = (\omega_{\rm nucleon}/\omega_{\rm pentaq})^{3/2}$, whose maximum value reads $(2/0.70)^{3/4} = 2.86^{3/4} \simeq 2.20$ and the minimum value $(2/2.96)^{3/4} = 0.676^{3/4} = 0.746$.

These numerical values effectively establish upper bounds on the size of the *pentaquark* HFI matrix elements (or, physically speaking, of the HF *pentaquark* mass splittings) in terms of *baryon* HFI matrix elements, i.e., *baryon* mass splittings. The former have been assumed in the

TABLE I. The pentaquark (excitation) eigen-energy spectrum in the chromo-harmonic oscillator model.

State	ω_1^2	ω_2^2	ω_3^2	ω_4^2
$\lvert\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle$	$\frac{k}{m}$	$\frac{2k}{m}$	$\frac{11-\sqrt{46}}{6}\frac{k}{m}$	$\frac{11+\sqrt{46}}{6}\frac{k}{m}$
$\lvert\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$	$\frac{k}{m}$	$\frac{2k}{m}$	$\frac{5k}{3m}$	$\frac{2k}{m}$
$\lvert\mathbf{6}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$	$\frac{2k}{m}$	$\frac{k}{m}$	$\frac{11-\sqrt{46}}{6}\frac{k}{m}$	$\frac{11+\sqrt{46}}{6}\frac{k}{m}$

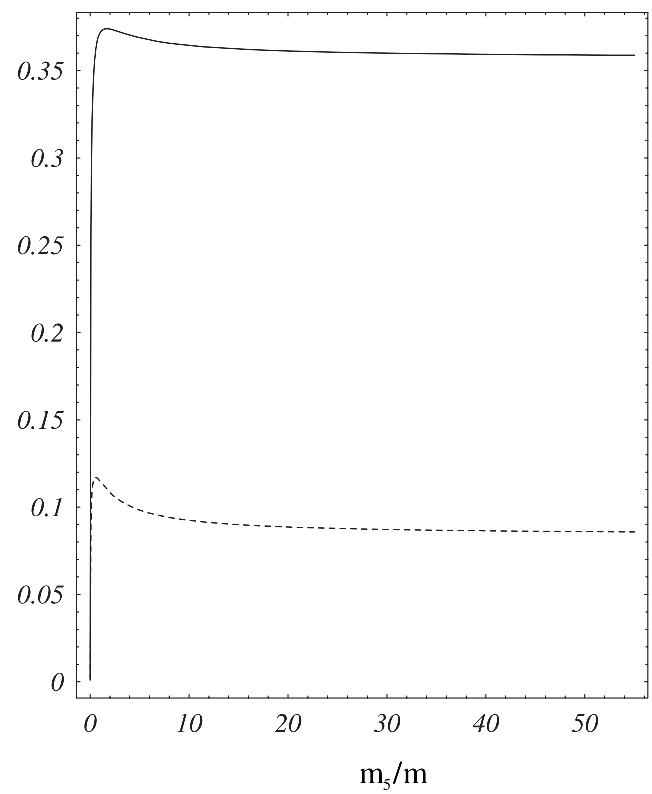

FIG. 2. Ratios of pentaquark energies $E_{|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle}/E_{|\mathbf{1}_{123}\mathbf{1}_{45}\rangle} - 1$ (solid), and $E_{|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle}/E_{|\mathbf{1}_{123}\mathbf{1}_{45}\rangle} - 1$ (dashed) in the nonrelativistic chromo-harmonic quark model as a function of the antiquark-quark mass ratio m_5/m.

literature as being independent of the latter, and as a rule larger than the here provided upper bounds.

E. Color singlet mixing in the chromo-harmonic model in the $q^4\bar{q}$ system

It is not obvious, however, how the asymptotic pentaquark state $|\mathbf{1}_{123}\mathbf{1}_{45}\rangle$ "evolves" as the separation of the two hadrons is reduced to zero $\Delta R \to 0$. In other words: which linear combination of the Pauli basis states does the $|\mathbf{1}_{123}\mathbf{1}_{45}\rangle$ state turn into in this limit? At first sight one might guess that it is that state in Eq. (8) whose expansion coefficient is closest to unity, i.e., that it is the $|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle$, but that conjecture turns out to be incorrect. The correct answer is the $|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$.

That much can be seen from the spatial dependence of the mixing angle $\Theta(\Delta R)$, Fig. 3, which diagonalizes either diagonal 2×2 submatrix of the color potential Eq. (18): We write the spatial "evolution" of the asymptotic "-baryon-meson" color singlet state as a function of two of the three Pauli states and a spatially varying mixing angle:

$$|\mathbf{1}_{123}\mathbf{1}_{45}\rangle = \sin\Theta(\Delta R)|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle + \cos\Theta(\Delta R)|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle. \tag{50}$$

One can see that $\Theta(\Delta R \to \infty) \to 54.74^0$, i.e., precisely the angle such that $\cos\Theta(\Delta R \to \infty) = 1/\sqrt{3}$, and $\cos\Theta(\Delta R \to 0) = 1$, which proves our claim. The other, orthogonal linear combination represents one of two hidden- color states

$$|\mathbf{8}_{123}\mathbf{8}^{\rho}_{45}\rangle = \cos\Theta(\Delta R)|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle - \sin\Theta(\Delta R)|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle, \tag{51}$$

turns into $|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle$ in the zero-separation limit.

Note that the Pauli basis states $|\bar{\mathbf{3}}_{12}\bar{\mathbf{3}}_{34}\bar{\mathbf{3}}_5\rangle$, $|\bar{\mathbf{3}}_{12}\mathbf{6}_{34}\bar{\mathbf{3}}_5\rangle$ correspond precisely to the phenomenological *Ansätze* promoted by Jaffe and Wilczek (JW) and by Karliner and Lipkin (KL), respectively. Moreover, note that the KL *Ansatz* state cannot be reached from the asymptotic KN state, by using the time evolution operator in the nonrelativistic quark model without gluons.

IV. DISCUSSION AND SUMMARY

Note that both hidden- color singlet states $|\mathbf{8}^{\rho,\lambda}_{123}\mathbf{8}_{45}\rangle$ are confined and orthogonal to the asymptotic meson-nucleon one. That means that they cannot decay into a meson and a nucleon in the nonrelativistic quark model, which does not contain a color-dependent operator that could turn two color octets into a color singlet state (such as the two-gluon annihilation/scattering operator). In a realistic model with gluon creation/annihilation such a decay would exist, but it would be suppressed as compared to the "fall-apart" decay mode.

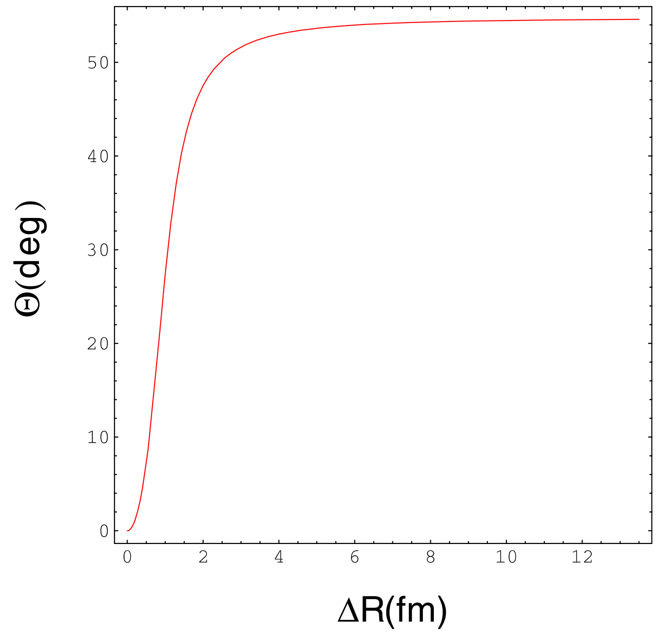

FIG. 3 (color online). Mixing angle Θ (degrees) as a function of the cluster separation ΔR (in units of meson mean square radii $\sqrt{\langle r^2\rangle}_{\text{meson}} = \omega^{-1}_{\text{meson}} \sim 1$ fm) in the nonrelativistic chromo-harmonic quark model.

There is a potentially important application of these ideas, viz. to the unusually small decay width of the Θ^+: If the Θ^+ is created in one of the "nonasymptotic" color-singlet states $|\mathbf{8}^{\rho,\lambda}_{123}\mathbf{8}_{45}\rangle$, as it may, due to color SU(3) conservation, then its decay into a nucleon and a kaon is forbidden in the NR approximation and may proceed only through two-gluon/glueball intermediate states, which is supressed in the real world.

In summary we have considered the stability, confinement and color singlet clustering/decays of pentaquarks in the simplest admissible color-exchange ($\mathrm{F}_i \cdot \mathrm{F}_j$) two-body potential model. Most of the results in Sec. II and all of the results in Sec. III are new, so far as we know. We do not repeat here the (partial) summaries given above. We have evaluated the heavy antiquark mass dependence of the pentaquark ground state's energy and showed that the system is not bound even with top antiquark.

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