Process $e^+e^- \to 3\pi(\gamma)$ with final state radiative corrections

S. Bakmaev,* Yu. M. Bystritskiy,† and E. A. Kuraev‡

Joint Institute for Nuclear Research, 141980 Dubna, Russia

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The process of annihilation of e^+e^- to the three pion final state in the case of moderately high energies is considered. The final state emission of virtual and real photons is considered explicitly. The calculations are performed in QED and the pion sector of the effective chiral Lagrangian. The initial state emission is considered in terms of the lepton structure function. Some numerical estimates are given.

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I. INTRODUCTION

The problem of taking into account radiative corrections (RC) of the lowest order of perturbation theory to the process of three pion production in the annihilation channel of colliding e^+e^- at moderately high energies becomes necessary for precision measurements of a hadronic contribution to the muon anomalous magnetic moment $(g-2)_\mu$ [1]. This is the motivation of this paper. A similar calculation for production of 2π and $\mu^+\mu^-$ in the annihilation channel was performed recently [2].

We use the pion sector of chiral perturbation theory (ChPT) to carry out calculations of interaction of charged mesons with the electromagnetic field. The relevant piece of this Lagrangian including anomaly [3,4] is

$$\begin{aligned}\mathcal{L} = & \frac{f_\pi^2}{4} Sp[D_\mu U (D^\mu U)^+ + \chi U^+ + U\chi^+] \\ & - \frac{e}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta} A_\mu Sp[Q\{(\partial_\nu U)(\partial_\alpha U^+)(\partial_\beta U)U^+ \\ & - (\partial_\nu U^+)(\partial_\alpha U)(\partial_\beta U^+)U\}] - \frac{ie^2}{8\pi^2}\varepsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)A_\alpha Sp \\ & \times \Big[Q^2(\partial_\beta U)U^+ + Q^2 U^+(\partial_\beta U) \\ & + \frac{1}{2} QUQU^+(\partial_\beta U)U^+ - \frac{1}{2} QU^+QU(\partial_\beta U^+)U \Big], \end{aligned} \tag{1}$$

where $f_\pi = 94$ MeV is the pion decay constant, $U = \exp(i\frac{\sqrt{2}\Phi}{f_\pi})$, $D_\mu U = \partial_\mu U + ieA_\mu[Q, U]$, $Q = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ is the quark charge matrix, and the terms with $\chi = B\mathrm{diag}(m_u, m_d, m_s)$ introduce explicit chiral symmetry breaking due to nonzero quark masses. The constant B has dimension of mass and is determined by the equation $Bm_q = M^2$ where $m_q = m_u \approx m_d$ and M is the pion mass.

*Electronic address: bakmaev@theor.jinr.ru
†Electronic address: bystr@theor.jinr.ru
‡Electronic address: kuraev@theor.jinr.ru

The pseudoscalar meson matrix Φ has its standard form

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}. \tag{2}$$

We consider the reaction of e^+e^- annihilation into three pions

$$e^-(p_-) + e^+(p_+) \to \pi^-(q_-) + \pi^+(q_+) + \pi^0(q_0). \tag{3}$$

The matrix element for this process in the Born approximation has the form (see Fig. 1)

$$M^{(0)} = \frac{i\alpha}{\pi f_\pi^{(0)3}} \cdot \frac{1}{q^2} \cdot \bar{v}(p_+)\gamma_\mu u(p_-) \cdot (\mu q_+ q_- q_0), \tag{4}$$

where $f_\pi^{(0)}$ is the unrenormalized pion decay constant, and $s = q^2 = (p_+ + p_-)^2$ is the invariant mass of the initial state, $(\mu q_+ q_- q_0) \equiv \varepsilon_{\mu\nu\alpha\beta} q_+^\nu q_-^\alpha q_0^\beta$.

Squaring this matrix element and performing summation over initial lepton spin states we have the total cross section for this process

$$\sigma_B^{(0)} = \frac{\alpha^2 s^2}{2^8 \cdot 3 \cdot \pi^5 f_\pi^{(0)6}} \int_{x_+^{\min}}^{x_+^{\max}} dx_+ \int_{x_-^{\min}}^{x_-^{\max}} dx_- G(x_+, x_-), \tag{5}$$

where

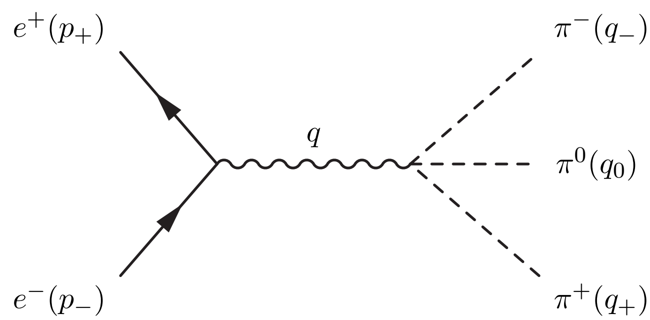

FIG. 1. Feynman diagram contributing to the process probability in the Born approximation.

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$$G(x_+, x_-) = 4(x_+^2 - \mu^2)(x_-^2 - \mu^2) - (1 - 2x_+ - 2x_- + 2x_+x_- + \mu^2)^2, \quad (6)$$

where $\mu^2 = M^2/s$, M is the pion mass, $x_\pm = \varepsilon_\pm/\sqrt{s}$, $x_0 = \varepsilon_0/\sqrt{s}$ are fractions of final pion energies, while $x_+ + x_- + x_0 = 1$. The limits of integration in (5) follow from the kinematic constrains

$$x_+^{\min} = \mu, \qquad x_+^{\max} = \frac{1}{2}\left(1 - \frac{3M^2}{s}\right) = \frac{1}{2}(1 - 3\mu^2),$$

$$x_-^{\max,\min} = \frac{1}{2(1 - 2x_+ + \mu^2)}((1 - x_+)(1 - 2x_+ + \mu^2) \pm R(x_+)),$$

where $R^2(x_+) = (x_+^2 - \mu^2)(1 - 2x_+ + \mu^2) \times (1 - 2x_+ - 3\mu^2)$.

II. VIRTUAL PHOTON EMISSION

Radiative corrections from the virtual photon emission can be represented by the Feynman diagrams (FD) of types drawn in Fig. 2.

First, we notice that FDs of types 3, 4, 6 give a zero contribution ($\delta_3 = \delta_4 = \delta_6 = 0$) due to

$$\int \frac{d^4k}{i\pi^2} \frac{(\mu k q(2q_- - k))}{k^2(k^2 - 2kq_-)(k^2 - 2kq + q^2 - M^2)} \equiv 0. \quad (7)$$

FDs of type 2 are the contributions from pion wave function renormalization and they are equal to [5]

$$\delta_c = \delta_2 = \frac{\alpha}{\pi}\left(L_\Lambda + \ln\frac{M^2}{\lambda^2} - \frac{3}{4}\right), \quad (8)$$

where $L_\Lambda = \ln(\Lambda^2/M^2)$ and Λ is the ultraviolet cutoff parameter, and λ is the fictitious photon mass. Considering the contributions of FDs of types 1, 5 we have

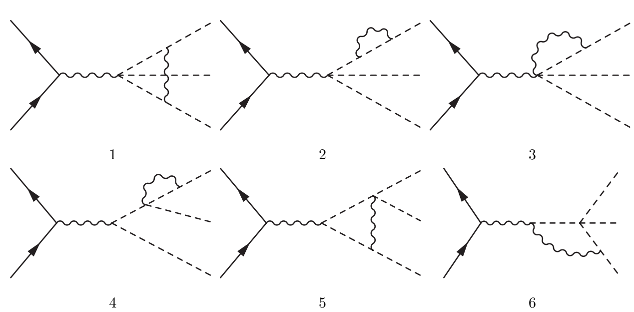

FIG. 2. Types of Feynman diagrams of emission of an additional virtual photon.

$$\delta_v = \delta_1 + \delta_5 = \frac{\alpha}{\pi}\left[1 + \frac{1}{2}L_\Lambda - \frac{1 + \beta^2}{2\beta}\ln\left(\frac{1 + \beta}{1 - \beta}\right)\ln\frac{M^2}{\lambda^2} + \frac{1 + \beta^2}{4}s_1 \mathrm{Re}\int_0^1 \frac{dx}{q_x^2}\left(\ln\frac{q_x^2}{M^2} - 2\right) + W\right], \quad (9)$$

$$W = \int_0^1 dx\left[\frac{a_+}{2b_+}\left(1 - \frac{a_+}{b_+}\ln\frac{A}{a_+}\right) - \frac{a_-}{2b_-}\left(1 - \frac{a_-}{b_-}\ln\frac{A}{a_-}\right)\right], \quad (10)$$

where $s_1 = (q_+ + q_-)^2$ is the invariant mass of a charged pion pair, $\beta^2 = 1 - 4M^2/s_1$, $q_x^2 = M^2 - s_1 x(1 - x) - i0$, $a_\pm = -x(1 - x_\pm)$, $b_\pm = x(x - x_\pm) + \mu^2$, and $A = -x(1 - x) + \mu^2$. The integrals in (9) can be calculated explicitly

$$\mathrm{Re}\int_0^1 \frac{dx s_1}{q_x^2} = -\frac{2}{\beta}L, \quad (11)$$

$$\mathrm{Re}\int_0^1 \frac{dx s_1}{q_x^2}\ln\frac{q_x^2}{M^2} = \frac{4}{\beta}\left[L\ln\left(\frac{1 + \beta}{2\beta}\right) - \frac{1}{4}L^2 + \mathrm{Li}_2\left(\frac{1 - \beta}{1 + \beta}\right) + 2\xi_2\right], \quad (12)$$

where $L = \ln\frac{1+\beta}{1-\beta}$, $\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t}\ln(1 - t)$ and $\xi_2 = \frac{\pi^2}{6} = \mathrm{Li}_2(1)$.

III. SOFT REAL PHOTON EMISSION

The standard calculation of the contribution of real soft photon emission by final pions

$$\delta_s = \frac{\sigma_{\mathrm{soft}}}{\sigma_B} = -\frac{\alpha}{4\pi^2}\int \frac{d^3k}{\omega}\left(\frac{q_-}{kq_-} - \frac{q_+}{kq_+}\right)^2\Big|_{\omega<\Delta\varepsilon}, \quad (13)$$

where $\Delta\varepsilon$ is the maximum energy of soft photon (i.e. $\omega < \Delta\varepsilon$), leads to

$$\delta_s = \frac{2\alpha}{\pi}\left\{\left(\ln\Delta - \frac{1}{2}\ln(x_+x_-) + \ln\frac{M}{\lambda}\right)\left(-1 + \frac{1 + \beta^2}{2\beta}L\right) + \frac{1 + \beta^2}{4\beta}\left[-g - \frac{1}{2}L^2 + L\ln\left(\frac{4}{1 - \beta^2}\right) - \xi_2 - 2\mathrm{Li}_2\left(-\frac{1 - \beta}{1 + \beta}\right)\right]\right\}, \quad (14)$$

where $\Delta = \Delta\varepsilon/\sqrt{s}$ and the quantity g is defined by

$$g = 2\beta\int_0^1 \frac{dt}{1 - \beta^2 t^2}\ln\left(1 + \frac{1 - t^2}{4}\frac{(x_+ - x_-)^2}{x_+x_-}\right). \quad (15)$$

Its expression in terms of the Spence function is

$$g = \ln\frac{1+\beta}{1-\beta}\ln\left|\frac{\beta^2a^2-b^2}{\beta^2}\right| + \mathrm{Li}_2\left(\frac{b(1-\beta)}{b-\beta a}\right) + \mathrm{Li}_2\left(\frac{b(1-\beta)}{b+\beta a}\right) - \mathrm{Li}_2\left(\frac{b(1+\beta)}{b-\beta a}\right) - \mathrm{Li}_2\left(\frac{b(1+\beta)}{b+\beta a}\right), \tag{16}$$

where

$$a = \frac{x_+ + x_-}{2\sqrt{x_+x_-}}, \qquad b = \frac{|x_+ - x_-|}{2\sqrt{x_+x_-}}. \tag{17}$$

IV. HARD REAL PHOTON EMISSION

Let us consider the contribution of radiative corrections which arises from the emission of additional hard photon by final particles, i.e., the process

$$e^-(p_-) + e^+(p_+) \to \pi^-(q_-) + \pi^+(q_+) + \pi^0(q_0) + \gamma(k). \tag{18}$$

The amplitude for this process can be written in the form

$$M = (4\pi\alpha)^2\frac{1}{q^2}\cdot\bar{v}(p_+)\gamma_\mu u(p_-)\cdot\frac{1}{4\pi^2f_\pi^3}\cdot T_{\mu\nu}e^\nu(k), \tag{19}$$

where k, $e_\mu(k)$ are the momenta and the polarization vector of the final real photon; $T_{\mu\nu}$ is the tensor corresponding to the $\gamma^*(\mu, Q) \to \pi^+(q_+)\pi^-(q_-)\pi^0(q_0)\gamma(\nu, k)$ vertex which follows from (1):

$$T^{\mu\nu} = (\mu\nu Qk)A + (\mu\nu(Q+k)q_0) + (\mu\lambda Qq_0)\left(\frac{q_-^\nu q_+^\lambda}{(q_-k)} + \frac{q_+^\nu q_-^\lambda}{(q_+k)}\right) - (\nu\lambda kq_0)\left(\frac{(2q_- - Q)^\mu q_+^\lambda}{Q^2 - 2(q_-Q)} + \frac{(2q_+ - Q)^\mu q_-^\lambda}{Q^2 - 2(q_+Q)}\right), \tag{20}$$

where $Q = q_+ + q_- + q_0 + k$, (thus $Q_0 = \sqrt{s}$), $A = 1 - (s_1 - M^2)/((Q-k)^2 - M^2)$. This tensor satisfies the gauge-invariance condition for both photon legs

$$Q_\mu T^{\mu\nu} = k_\nu T^{\mu\nu} = 0. \tag{21}$$

Thus, we may write the cross section in the form

$$\sigma^{\mathrm{hard}} = \frac{(4\pi\alpha)^3}{8s}\cdot\frac{1}{s^2}Sp[\hat{p}_+\gamma^\mu\hat{p}_-\gamma^\nu]\left(\frac{1}{4\pi^2f_\pi^3}\right)^2 \times\frac{1}{3}\left(g_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2}\right)\int d\Gamma_4\sum_\lambda|T^{\alpha\beta}e_\beta^\lambda|^2. \tag{22}$$

The phase volume for the final state has the form

$$d\Gamma_4 = (2\pi)^4\delta^4(p_+ + p_- - q_+ - q_- - q_0 - k) \times\frac{d^3q_+}{(2\pi)^32\varepsilon_+}\frac{d^3q_-}{(2\pi)^32\varepsilon_-}\frac{d^3q_0}{(2\pi)^32\varepsilon_0}\frac{d^3k}{(2\pi)^32\omega} = (2\pi)^{-8}\frac{s^2\pi^2}{16}xdxdx_+dx_-dO_\gamma, \tag{23}$$

where $x = \omega/\sqrt{s}$, $x + x_+ + x_- + x_0 = 1$. Now the cross section acquires the form

$$\sigma^{\mathrm{hard}} = \frac{\alpha^3}{2^8\cdot 3\cdot\pi^7f_\pi^6}\int dxxdx_+dx_-dO_\gamma(-|T^{\mu\nu}|^2)|_{x>\Delta}. \tag{24}$$

We should like to note that the sum of hard photon emission $\delta_h = \sigma^{\mathrm{hard}}/\sigma_B$ and soft real photon emission δ_s contributions ($\delta_h + \delta_s$) do not contain the auxiliary parameter Δ. In order to see this explicitly, let us consider the small $x = \omega/\sqrt{s}$ limit of σ^{hard}. Really, if we consider the case $\omega \ll \Delta\sqrt{s}/2$, then [see (20)]

$$T^{\mu\nu}e_\nu(k)|_{\omega\ll\Delta\sqrt{s}/2} \approx (\mu q_+q_-q_0)\left(\frac{(q_-e)}{(q_-k)} - \frac{(q_+e)}{(q_+k)}\right). \tag{25}$$

We can calculate the hard photon emission contribution in this limit

$$\int\frac{d^3k}{2\pi\omega}(-|T^{\mu\nu}e_\nu(k)|^2)|_{\omega\ll\Delta\sqrt{s}/2} = 4\ln\frac{1}{\Delta}\left(\frac{1+\beta^2}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right), \tag{26}$$

and we get for δ_h

$$\delta_h|_{\omega\ll\Delta\sqrt{s}/2} \approx \frac{2\alpha}{\pi}\left[\ln\frac{1}{\Delta}\left(\frac{1+\beta^2}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right) + O(\Delta)\right]. \tag{27}$$

We redefine the contributions in the following way:

$$\delta_s + \delta_h \to \bar{\delta}_s + \bar{\delta}_h, \tag{28}$$

$$\bar{\delta}_s = \delta_s + \frac{2\alpha}{\pi}\ln\frac{1}{\Delta}\left(\frac{1+\beta^2}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right), \tag{29}$$

$$\bar{\delta}_h = \delta_h - \frac{2\alpha}{\pi}\ln\frac{1}{\Delta}\left(\frac{1+\beta^2}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right), \tag{30}$$

where both $\bar{\delta}_s$ and $\bar{\delta}_h$ do not depend on Δ anymore.

V. CONCLUSION

The final result is

$$\sigma^{ee\to3\pi(\gamma)} = \sigma_B^{(0)}(1 + \delta_{sw})(1 + \delta) = \sigma_B(1 + \delta), \quad \sigma_B = \sigma_B^{(0)}(f_\pi^{(0)} \to f_\pi) \tag{31}$$

$$\delta = (\delta_c + \delta_v + \bar{\delta}_s + \bar{\delta}_h)|_{L_\Lambda=0}, \qquad \delta_{sw} = \frac{3\alpha}{2\pi} L_\Lambda, \tag{32}$$

where we extracted the short-distance contributions in the form $(1+\delta_{sw})$ and used this factor to renormalize the $f_\pi^{(0)}$ pion decay constant in the form $f_\pi^{(0)-6}(1+\delta_{sw}) = f_\pi^{-6}$ [6].

The explicit form of $\delta - \bar{\delta}_h$ is

$$\delta - \bar{\delta}_h = \frac{\alpha}{\pi}\Big\{-\frac{1}{2}\ln(x_+x_-)\Big(-1+\frac{1+\beta^2}{2\beta}L\Big)+\frac{1}{4}+W + \frac{1+\beta^2}{4\beta}\Big[-g-\frac{1}{2}L^2+L\ln\frac{4}{1-\beta^2}-\xi_2 - 2\mathrm{Li}_2\Big(-\frac{1-\beta}{1+\beta}\Big)\Big]+\frac{1+\beta^2}{\beta}\Big[L+L\ln\frac{1+\beta}{2\beta} - \frac{1}{4}L^2+\mathrm{Li}_2\Big(-\frac{1-\beta}{1+\beta}\Big)+2\xi_2\Big]\Big\}, \tag{33}$$

with $L = \ln(\frac{1+\beta}{1-\beta})$, g defined in (15) and W defined in (10). We should like to note that $g(x_+ = x_-) = W(x_+ = x_-) = 0$. The form of $\bar{\delta}_h$ depends on the experimental conditions of the final state particle registration and is not considered here.

In Fig. 3, we present the value $\delta - \bar{\delta}_h$ for a typical experimental situation.

It is known that the main contribution comes from the channels with rho-mesons in the intermediate state. To take this into account in the presented above Born approximation (4), we should incorporate an additional factor $|F|^2$ into the right-hand side of (5) (see [7])

$$|F(x_+,x_-)|^2 = \Big(\frac{3}{4\pi^2}\Big)^2\frac{1}{f_\pi^6}\Big(\frac{M_\rho}{f_\pi}\Big)^2\alpha_K|1-3\alpha_K - \alpha_K H(x_+,x_-)|^2, \tag{34}$$

with $\alpha_K = (f_\pi g_{\rho\pi\pi}/M_\rho)^2$, $H(x_+,x_-) = R_\rho(Q_0^2) + R_\rho(Q_+^2) + R_\rho(Q_-^2)$, where $R_\rho(Q^2)$ is the ρ-meson Breit-Wigner propagator

$$R_\rho(Q^2) = \Big[\frac{Q^2}{M_\rho^2} - 1 + i\frac{\Gamma_\rho(Q^2)}{M_\rho}\Big]^{-1}, \tag{35}$$

and $Q_0^2 = s(2x_+ + 2x_- - 1) + M^2$, $Q_+^2 = s(1-2x_-) + M^2$, $Q_-^2 = s(1-2x_+) + M^2$.

RC calculation within the effective Lagrangian framework with pions and rho-mesons is considerably complicated (we have about 50 Feynman diagrams at the one-loop level). Nevertheless, the value of RC can be evaluated within the effective Lagrangian approach with pions only (considering rho-meson being heavy enough). This results in the given above expressions (31). The accuracy of this calculation is defined by our approximation of heavy rho-meson

$$\frac{g_{\rho\pi\pi}^2}{M_\rho^2 - s_1} = \frac{1}{f_\pi^2}\Big(1 + O\Big(\frac{s_1}{M_\rho^2}\Big)\Big), \tag{36}$$

and for intermediate invariant masses of the $\pi^+\pi^-$ pair ($\sqrt{s_1} \le 400$ MeV) it is about $\approx 20\%$.

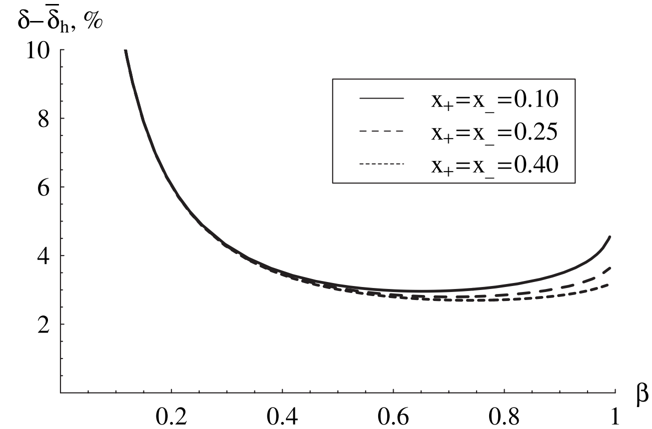

FIG. 3. The value $\delta - \bar{\delta}_h$ [see (33)] in percents for typical experimental conditions as a function of β.

Finally, we present the cross section of the process $ee \to 3\pi(\gamma)$ with the radiation from particles of the initial state in terms of structure functions [8]:

$$d\sigma^{ee\to 3\pi(\gamma)} = \int dx F(x)\sigma_B^{ee\to 3\pi}(sx)\Big(1 + \frac{\alpha}{\pi}K + \delta\Big), \tag{37}$$

with $F(x) = \beta(1-x)^{\beta-1}[1+\frac{3}{4}\beta] - \frac{\beta}{2}(1+x)$ being the known initial state radiation function and $K = \frac{\pi^2}{3} - \frac{1}{2}$ with δ taken from (33) with the replacement $x_\pm \to \tilde{x}_\pm = x_\pm/\sqrt{x}$.

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[1] G. W. Bennett *et al.* (Muon g-2 Collaboration), Phys. Rev. Lett. **92**, 161802 (2004).
[2] Y. M. Bystritskiy, E. A. Kuraev, G. V. Fedotovich, and F. V. Ignatov, hep-ph/0505236.
[3] E. Witten, Nucl. Phys. **B223**, 433 (1983).
[4] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[5] Ngee-Pong Chang, Phys. Rev. **131**, 1272 (1963).
[6] B. R. Holstein, Phys. Lett. B **244**, 83 (1990).
[7] E. A. Kuraev and Z. K. Silagadze, Yad. Fiz. **58N9**, 1687 (1995) [Phys. At. Nucl. **58**, 1589 (1995)].
[8] E. A. Kuraev and V. S. Fadin, Yad. Fiz. **41**, 733 (1985) [Sov. J. Nucl. Phys. **41**, 466 (1985)].