Alternative proposal to modified Newtonian dynamics

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From a study of conserved quantities of the so-called modified Newtonian dynamics we propose an alternative to this theory. We show that this proposal is consistent with the Tully-Fisher law, has conserved quantities whose Newtonian limit are the energy and angular momentum, and can be useful to explain cosmic acceleration. The dynamics obtained suggests that, when acceleration is very small, time depends on acceleration. This result is analogous to that of special relativity where time depends on velocity.

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Nowadays there are various observational results in astrophysics whose explanation represents a challenge for theoretical physics. One of those problems is to explain the rotation curves of the galaxies. Observations indicate a relationship $V^4 \propto M$ for the speed V of the distant stars in a galaxy of mass M. However, as the only force acting on those stars is gravity and their trajectories are circles, Newtonian dynamics indicates that the relationship to hold is $V^2 = GM/r$, where r is the distance from the star to the center of the galaxy. To account for the difference, some authors assume the existence of a sort of matter that does not radiate: the so-called dark matter. There are, however, other proposals which assume modifications to the gravitational field or to the laws of dynamics. By considering the behavior of the speed of the distant stars, Milgrom proposed a modification to Newton's second law as [1]

$$m\mu(z)\frac{d^2x^i}{dt^2} = F^i, \qquad i = 1, 2, 3; \tag{1}$$

where $z = |\ddot{x}|/a_0 = \sqrt{\ddot{x}_i\ddot{x}^i}/a_0$, $a_0 \approx 10^{-8}$ cm/s^2 and $\mu(z)$ is a function satisfying

$$\mu(z) = \begin{cases} 1 & \text{if } z \gg 1, \\ z & \text{if } z \ll 1. \end{cases} \tag{2}$$

This proposal is usually called modified Newtonian dynamics (MOND). From it one can see that, in the MOND limit ($z \ll 1$, $\mu(z) = z$), a particle describing a circular trajectory in the potential $U = -GMm/r$ satisfies

$$V^4 = a_0GM; \tag{3}$$

which is consistent with the Tully-Fisher law: $L_K \propto V^4$, where L_K is the infrared luminosity of the disk galaxy [2]. Also interesting appears the fact that the constant a_0 can be written as $a_0 = cH_0/6 \approx 10^{-8}$ cm/s^2, with H_0 the Hubble constant and c the speed of light; or alternatively by using the Eddington-Weinberg relation [3], $\hbar^2H_0 \approx G\, cm_N^3$, as $a_0 \approx m_N^3c(6m_p^3t_p)^{-1}$, where m_N is the proton mass and $m_p = (\hbar c/G)^{1/2}$ and $t_p = (\hbar G/c^5)^{1/2}$ are the Planck mass and time, respectively. This can be just a coincidence, but it could also indicate the existence of a fundamental relation between physics at very large and very small scales.

MOND is a purely phenomenological theory but it explains most of the galaxy rotation curves without introducing dark matter [4]. Its simplicity is what makes it attractive. Extensions to MOND at the level of the gravitational field can be found in [5–9]. Phenomenological implications of those can be seen in [9,10]. But despite its achievements, MOND has problems of its own. A crucial one is the lack of conserved quantities as energy. In this work we perform a study of MOND's constants of motion and, by defining an energy, propose an equation of motion alternative to (1). This proposal has several conserved quantities that in the Newtonian limit ($z \gg 1$, $\mu(z) = 1$) reduce to the usual ones: energy and angular momentum are two of them. A generalization of the virial theorem is also provided. It is shown, in addition, that this proposal can be useful to explain cosmic acceleration. Finally, we show that a possible interpretation of the dynamics is that, for accelerations of the order of a_0, time depends on acceleration. This is analogous to special relativity where time depends on velocity.

Let us start by considering modified Newton's second law (1). By using spherical polar coordinates and assuming a central force field, this equation can be written as

$$m\mu(z)(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta) = -\frac{\partial U}{\partial r}, \tag{4}$$

$$m\mu(z)(r\ddot{\theta} + 2\dot{r}\,\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) = 0, \tag{5}$$

$$m\mu(z)\frac{d}{dt}(r^2\dot{\phi}\sin^2\theta) = 0. \tag{6}$$

If $\mu(z) \neq 0$, then $\theta = \pi/2$ is a solution to (5); and Eq. (6) implies that the quantity

$$L = r^2\dot{\phi} \tag{7}$$

is conserved. By using these and $U = -GMm/r$, Eq. (4) reduces to

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$$\mu(z)\left(\ddot{r} - \frac{L^2}{r^3}\right) = -\frac{GM}{r^2}, \qquad z = \frac{1}{a_0}\left|\left(\ddot{r} - \frac{L^2}{r^3}\right)\right|. \quad (8)$$

In the MOND limit this equation becomes

$$\left|\left(\ddot{r} - \frac{L^2}{r^3}\right)\right|\left(\ddot{r} - \frac{L^2}{r^3}\right) = -\frac{a_0 GM}{r^2}; \quad (9)$$

which implies the constraint

$$\left(\ddot{r} - \frac{L^2}{r^3}\right) < 0. \quad (10)$$

By using this, Eq. (9) can be written as

$$\left(\ddot{r} - \frac{L^2}{r^3} - \frac{\sqrt{a_0 GM}}{r}\right)\left(\ddot{r} - \frac{L^2}{r^3} + \frac{\sqrt{a_0 GM}}{r}\right) = 0. \quad (11)$$

This implies that the particle's trajectory must satisfy either

$$\ddot{r} - \frac{L^2}{r^3} - \frac{\sqrt{a_0 GM}}{r} = 0, \quad (12)$$

or

$$\ddot{r} - \frac{L^2}{r^3} + \frac{\sqrt{a_0 GM}}{r} = 0, \quad (13)$$

or both, but the constraint (10) is not compatible with (12) and therefore the whole Eq. (9) is reduced to (13). Clearly, for Eq. (13) the quantity

$$\mathcal{E} = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} + \sqrt{a_0 GM}\ln r, \quad (14)$$

is conserved. This corresponds to the energy per unit mass of a particle moving in the potential $U(r) = \sqrt{a_0 GM}\ln r$. It is tempting to take $\mathcal{E}$ as the energy of the system; however, this quantity is conserved only in the MOND limit and does not reduce to the usual energy in the Newtonian limit ($\mu(z) = 1$). This makes it unsuitable.

Looking for alternatives, one can see that for a particle describing trajectories with $\dot{z} = 0$ (circles are examples), the quantity

$$E = \frac{m\mu(z)}{2}\frac{dx^i}{dt}\frac{dx_i}{dt} + U(x) \quad (15)$$

is conserved. In fact,

$$\dot{E} = \left(m\frac{\mu\prime(z)\dot{z}}{2}\frac{dx^i}{dt} + m\mu(z)\frac{d^2x^i}{dt^2} + \frac{\partial U(x)}{\partial x_i}\right)\frac{dx_i}{dt} = 0 \quad (16)$$

because of MOND equation (1). Here $\mu'(z) = d\mu(z)/dz$. Notice that this quantity is conserved for every $\mu(z)$ and $U(x)$, and reduces to the usual energy in the Newtonian limit. In this sense it can be said that E does provide a good definition of energy. Requesting conservation of this quantity, now for any trajectory, implies that the equation of motion

$$m\mu(z)\frac{d^2x^i}{dt^2} + m\frac{\mu\prime(z)\dot{z}}{2}\frac{dx^i}{dt} = F^i \quad (17)$$

must hold. Clearly, when $\dot{z} \approx 0$ this reduces to the modified Newton's second law (1) and is therefore consistent with the Tully-Fisher law.

Equations (1) and (17) coincide in the Newtonian limit, but differ in any other case for noncircular trajectories. This is not an issue as stars with the more noncircular trajectories are those close to the galaxy center; and they are outside the MOND regime. Distant stars, on the other hand, are in the MOND regime and have trajectories that can be approximated by circles. Let us then see how Eq. (17) differs from (1) for trajectories close to the circle. In general only magnitudes of velocity and acceleration of the distant stars can be measured, so it is appropriate to look at magnitude differences only. For Eq. (1), $|F| = m\mu(z)|\ddot{x}|$; but for (17),

$$|F| = m\mu(z)|\ddot{x}|\sqrt{1 + \frac{\mu'(z)\dot{z}}{\mu(z)|\ddot{x}|^2}\left(\ddot{x}\cdot\dot{x} + \dot{x}^2\frac{\dot{z}\mu'(z)}{4\mu(z)}\right)}. \quad (18)$$

Now, by assuming an elliptical trajectory: $x^i = r_0(\cos\omega t, \sqrt{1-e^2}\sin\omega t, 0)$, with e the eccentricity; in the MOND limit and to the lowest order in e, one obtains

$$|F| = m\mu(z)|\ddot{x}|\left(1 - \frac{3}{32}e^4 f(t)\right), \quad (19)$$

where $0 \le f(t) \le 1$. Thus, for the correction term to be 1% of the magnitude $|F| = m\mu(z)|\ddot{x}|$, a large eccentricity $e \approx 0.57$ is required. In this sense Eq. (17) is not so different from (1).

An advantage of (17) over (1), though, is that in addition to energy it has several conserved quantities. For instance, for potentials U depending on the distance r only, Eq. (17) implies conservation of the quantity

$$L_i = \epsilon_{ijk}x^j m\sqrt{\mu(z)}\frac{dx^k}{dt}, \quad (20)$$

which in the Newtonian limit reduces to angular momentum. If $U(r) = -GMm/r$, also the quantity

$$A_i = m\sqrt{\mu(z)}\epsilon_{ijk}\dot{x}_j L_k - \frac{GMm^2}{r}x_i, \quad (21)$$

that in the Newtonian limit reduces to the Runge-Lenz vector, is conserved. In addition, it can be seen that for $U(r) = 0$, the quantity

$$p_i = m\sqrt{\mu(z)}\frac{dx_i}{dt}, \quad (22)$$

is also conserved. This reduces to the usual momentum in the Newtonian limit.

Considering now $\langle\dot{\mathcal{G}}\rangle = \lim_{T\to 0}\frac{1}{T}\int_0^T \dot{\mathcal{G}}dt = 0$, where $\mathcal{G} = p_i x^i$ with p_i above, from Eq. (17) we obtain

$$\langle \dot{\mathcal{G}} \rangle = \left\langle \frac{F_i x^i}{\sqrt{\mu(z)}} \right\rangle + \langle \sqrt{\mu(z)} m \dot{x}_i \dot{x}^i \rangle = 0, \qquad (23)$$

which is a generalization of the virial theorem [11]. For $U = -GMm/r$, and in the MOND limit, this equation yields $\langle GMm/r \rangle \ll \langle m\dot{x}_i \dot{x}^i \rangle$; which is qualitatively consistent with observations in galaxy clusters [12].

Equation (17) is nonrelativistic but from Newtonian cosmology one can still get some implications. It is worth noticing that Newtonian cosmology is an appropriate approximation when pressure can be neglected [13]. Now, there are several ways to construct a Newtonian cosmology [3,13,14] and, as all of them yield the same equations of motion, we take the simplest one. Let us assume the cosmological principle $x_i = R(t)\hat{x}_i(t_0)$, with $\hat{x}_i(t_0)$ a unit vector. Therefore a unit-mass particle in the gravitational field has energy

$$E = \frac{1}{2}\mu(z)\dot{R}^2 - \frac{GM}{R}, \qquad z = \frac{\ddot{R}}{a_0}, \qquad (24)$$

from where

$$\mu(z)\frac{\dot{R}^2}{R^2} = -\frac{k}{R^2} + \frac{8\pi G}{3}\rho, \qquad \rho = \frac{3M}{4\pi R^3}, \qquad (25)$$

with $k = -E/2$. In the Newtonian limit ($\mu(z) = 1$) this equation is equivalent to Friedmann's for a pressureless-matter dominated universe. In fact, if $E = 0$ then $k = 0$ and if $E \neq 0$, R can always be changed to λR in such a way that k only takes values ± 1. Outside the Newtonian limit Eq. (25) is a MOND-like pressureless Friedmann equation. The $k = 0$ case is particularly interesting as recent observations indicate compatibility of the universe with this value [15]. For $k = 0$ and in the Newtonian limit, the solution to (25) is of the form $R(t) \propto t^{2/3}$. In this case the deceleration parameter $q_0 = -\ddot{R}R/\dot{R}^2 > 0$. Recent observations [16], however, provide strong evidence of an accelerated universe with $q_0 < 0$. Now, by considering the MOND limit ($\mu(z) = z$) of Eq. (25) one obtains $\ddot{R}\dot{R}^2/R^2 = 8\pi G a_0 \rho/3$. From this, $\dot{R} = \beta[\ln(R/R_0)]^{1/4}$, with $\beta^4 = 8GMa_0$ and R_0 an integration constant. Therefore, $\ddot{R} = \beta^2(4R[\ln(R/R_0)]^{1/2})^{-1}$. Notice that to be within the MOND regime, $R_0 < R$ must hold and therefore $q_0 = -(4\ln(R/R_0))^{-1} < 0$; which suggests that a relativistic generalization to the theory here presented could be useful to explain the universe acceleration without introducing dark energy.

The problem of structure formation can, in principle, also be tackled with Eq. (25). However, from the equation of motion of the usual Newtonian cosmology at the structure formation epoch (SFE) one gets $|\ddot{R}/a_0| = |4\pi G\rho R/3| \approx 10^8$, which indicates that Newton dynamics must not be replaced by MOND. Notice that if $a_0 = cH_0/6$ is changed to $a_{0,\mathrm{SFE}} = cH_{\mathrm{SFE}}/6$, with H_{SFE} being the Hubble's constant at the SFE, then $|\ddot{R}/a_{0,\mathrm{SFE}}| \approx 1$, and therefore it is necessary to consider MOND's corrections to Newton dynamics in this universe epoch. It is possible that a relativistic generalization to MOND may imply variation of a_0 with time so as to have implications in the SFE. Some of the properties a relativistic generalization to MOND must have can be found in [17].

To interpret Eq. (17) let us consider

$$m\frac{1}{\dot{\tau}^2}\frac{d^2x^i}{dt^2} - m\frac{\ddot{\tau}}{\dot{\tau}^3}\frac{dx^i}{dt} = F^i, \qquad \dot{\tau} = \frac{d\tau}{dt}. \qquad (26)$$

Notice that if $\tau = t$, this equation reduces to Newton's second law. Equation (26) is in fact a generalized Newton's second law where the time τ can depend on other variables. In particular, by taking

$$\frac{1}{\dot{\tau}^2} = \mu(z), \qquad z = \frac{1}{a_0}\sqrt{\frac{d^2x_i}{dt^2}\frac{d^2x^i}{dt^2}}, \qquad (27)$$

Equation (26) equals (17). Thus, Eq. (17) can be interpreted as a Newton's second law where time depends on acceleration.

Another dynamics where time depends on other variables is the relativistic one. Newton's second law in the relativistic case can be written as [18]

$$m\frac{d^2x^\alpha}{d\tau^2} = m\frac{1}{\dot{\tau}^2}\frac{d^2x^\alpha}{dt^2} - m\frac{\ddot{\tau}}{\dot{\tau}^3}\frac{dx^\alpha}{dt} = \frac{f^\alpha}{c}, \qquad \alpha = 0, 1, 2, 3; \qquad (28)$$

where

$$\dot{\tau} = \gamma^{-1}, \qquad \gamma^{-1} = \sqrt{1 - \frac{\dot{x}^i\dot{x}_i}{c^2}}. \qquad (29)$$

This provides analogies between the well-known relativistic dynamics and that given by Eq. (17). Similarities between conserved quantities can be seen, for instance, by remembering that in special relativity the conserved momentum is no longer $p_i = m\dot{x}_i$, but $p_i = m\gamma\dot{x}_i$ [18]; whereas for the dynamics of (17) is that from Eq. (22). Finally, it is straightforward to see that Eq. (29) can be obtained from the line element

$$ds^2 = c^2(dt)^2 - dx^i dx_i = c^2(d\tau)^2, \qquad (30)$$

whereas Eq. (27) follows from

$$dS^2 = a_0(dt)^2 - \frac{(1 - \mu^{-1}(z))}{a_0 z^2}dv^i dv_i = a_0(d\tau)^2. \qquad (31)$$

This suggests that a more general theory to the one here presented may imply that, in addition to time, some geometrical quantities as distance also depend on acceleration.

To summarize, we have presented an alternative proposal to MOND which is consistent with the Tully-Fisher law and that has several conserved quantities whose Newtonian limit is the usual one. A generalization of the virial theorem is also provided. It is shown that this proposal is useful to explain cosmic acceleration. The dynamics obtained suggests that, for accelerations of the order of a_0, time depends on acceleration. It is worth mentioning that there are already proposals to tackle the problem of MOND's constants of motion by modifying Poisson's equation for the gravitational field [5]. Those conserved quantities are, however, not for the particle but for the gravitational field.

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