PHYSICAL REVIEW D **73,** 025020 (2006)

S-matrix renormalization in effective theories

K. Semenov-Tian-Shansky,[1,3,*] A. Vereshagin,[2,3,†] and V. Vereshagin[3,‡]

[1]*Département de Physique, Université de Liège au Sart Tilman, B5a, B4000 Liège 1, Belgium*

[2]*Institute of Physics and Technology, Allegaten 55, N5007 Bergen, Norway*

[3]*Theoretical Physics Department, Institute of Physics, St. Petersburg State University, St. Petersburg, Petrodvoretz, 198504, Russia*

(Received 31 October 2005; published 31 January 2006)

This is the fifth paper in the series devoted to explicit formulation of the rules needed to manage an effective field theory of strong interactions in *S*-matrix sector. We discuss the principles of constructing the meaningful perturbation series and formulate two basic ones: uniformity and summability. Relying on these principles, one obtains the bootstrap conditions which restrict the allowed values of the physical (observable) parameters appearing in the extended perturbation scheme built for a given localizable effective theory. The renormalization prescriptions needed to fix the finite parts of counterterms in such a scheme can be divided into two subsets: minimal, needed to fix the *S*-matrix, and nonminimal, for eventual calculation of Green functions; in this paper we consider only the minimal one. In particular, it is shown that, in theories with the asymptotic behavior governed by known Regge intercepts, the system of independent renormalization conditions only contains those fixing the counterterm vertices with $n \leq 3$ lines, while other prescriptions are determined by self-consistency requirements. Moreover, the prescriptions for $n \leq 3$ cannot be taken arbitrarily: an infinite number of bootstrap conditions should be respected. The concept of localizability, introduced and explained in this article, is closely connected with the notion of resonance in the framework of perturbative quantum field theory. We discuss this point and, finally, compare the cornerstones of our approach with the philosophy known as ''analytic *S*-matrix.''

DOI: 10.1103/PhysRevD.73.025020 PACS numbers: 02.30.Lt, 11.10.Gh, 11.10.Lm, 11.15.Bt

I. INTRODUCTION

This paper, together with the previous one [1], is aimed at forming the philosophical and theoretical base for calculations proposed in [2–4] and to outline the ways for further analysis; therefore it is natural to review first the material presented in those publications.

We are interested in constructing a self-consistent perturbation technique for the infinite component effective field theories[1] of strong interactions. We work with Dyson's scheme, because it is the only known way to combine Lorentz invariance, unitarity, and the cluster decomposition principle with postulates of quantum mechanics. Thus, the problems we have to deal with include an infinite number of graphs to be summed up at each loop order and the problem of ordering the required renormalization conditions, since in such a theory one needs to fix an infinite number of parameters to be able to calculate amplitudes. In [3] it has been shown that already the requirement of summability of tree graphs—the existence of well-defined tree-level amplitudes—leads to strong limitations on the possible values of coupling constants of a theory. However, in those articles some theoretical statements, like *meromorphy* and *polynomial boundedness* of the tree-level amplitude, were taken as postulates and only some general arguments in their favor were given. With these assumptions, it became possible to obtain the system of *bootstrap equations* for masses and coupling constants of $\pi\pi$ and πK resonances in nice agreement with experimental data. The main tool used to derive them was the *Cauchy expansion*, based on the celebrated Cauchy integral formula, which represents the tree-level amplitude as a well-defined series in a given domain of the space of kinematical variables.

In subsequent publications we fill some gaps left in the previous analysis and discuss new concepts. Thus, in [4] (see also [7]), we suggest the notion of *minimal parametrization* and in [1] the corresponding reduction theorem is proven. This theorem explains why it is sufficient to consider only the minimal (''on-shell-surviving'') vertices at each loop order of the perturbation theory—the fact implicitly used in [3] to parametrize amplitudes. Besides, in [4] we briefly discuss what we call the *localizability* requirement—the philosophy which, in particular, serves as a background for the requirements of polynomial boundedness and meromorphy of tree-level amplitudes. Since the last point was not explained clear enough, we address it in this paper.

We start with summing up the main results of a previous publication [1] in Sec. II. Continuing the logic line of that article we consider the *extended perturbation scheme* based on the interaction Hamiltonian[2] which, along with

*Electronic address: K.Semenov@ulg.ac.be

†Electronic address: Alexander.Vereshagin@ift.uib.no

‡Electronic address: vvv@av2467.spb.edu

[1]We use this term in its original meaning (see [5,6]) with small modifications suggested in [1,4]. Our definition is formulated in the beginning of Sec. II, the details are discussed in Sec. VIII.

[2]Throughout the paper when saying Hamiltonian we mean the Hamiltonian density. Besides, we always imply that this density (in the interaction picture) is written in the Lorentz-covariant form thus using Wick's *T*-product in Dyson's series; see Sec. VIII below.

 © 2006 The American Physical Society

the fields that describe true asymptotic states, contains also auxiliary fields of fictitious unstable particles—resonances. In Sec. III we formulate two mathematical principles: *asymptotic uniformity* and *summability*, which create a base for constructing the meaningful perturbation series in such an effective theory. Further, in Sec. IV we briefly discuss the mathematical tool which allows us to present the finite loop order amplitudes as convergent functional series. This technique is exploited in Secs. V and VI, where we demonstrate how the requirement of crossing symmetry gives rise to the system of bootstrap conditions and analyze the renormalization procedure. The results of these two sections are generalized in Sec. VII, where phenomenological constraints are imposed and the renormalization prescriptions are explicitly written; besides, it is shown there that the bootstrap conditions obtained at any loop order can be treated as the relations between *physical observables*, which justify the legitimacy of our preceding analysis of experimental data [3,7,8].

The connection between the extended perturbation scheme and the *localizable initial theory* (based on the Hamiltonian constructed solely from the fields of true asymptotic states) is outlined in Sec. VIII which, along with Sec. III, is the central one in this article. In particular, we discuss here a (rather hypothetic) step-by-step process of localization and explain our usage of the term ''*strong interaction*.'' The localization process requires introducing auxiliary free fields. The physical interpretation of those fields is given in Sec. IX where we discuss the meaning of the terms ''mass'' and ''width'' as the resonance classification parameters.

At last, Sec. X is devoted to comparative analysis of the effective scattering theory philosophy with respect to that of the analytic S-matrix.

II. CLASSIFICATION OF THE PARAMETERS: A BRIEF REVIEW

For the following discussion it is essential to recall the results obtained in [1]. Referring to the analysis presented there, we attempt to be not too rigorous trying, instead, to make a picture clear.

We say that the field theory is effective if the interaction Hamiltonian in the interaction picture contains all the monomials consistent with a given algebraic (linear) symmetry.[3] *Effective scattering* theory is just the effective field theory only used to compute the S-matrix, while the calculation of Green functions is not implied. In particular, only the S-matrix elements should be renormalized.

The fields of stable particles and resonances of arbitrary high spin are present in effective Hamiltonian and every vertex is dotted by the corresponding coupling constant. Therefore we are forced to deal with an infinite set of parameters. By construction, the effective theory is renormalizable but, to make use of this property, one needs an infinite number of renormalization prescriptions. This looks impractical until certain regularity reducing the number of independent prescriptions, possibly up to some basic set, is found. As shown in [1], the S-matrix in effective theory only depends upon a certain set of parameters, which we called the *resultant parameters* of various levels (or loop orders) l. The value $l = 0$ labels the tree-level parameters, $l = 1$—one-loop level ones, and so on.

The parametrization implies the use of *renormalized* perturbation theory (see, e.g. [9–11]), so the interaction Hamiltonian is written as a sum of basic one plus counterterms:

$$\mathcal{H}_{\text{int}} = \mathcal{H}_{\text{b}} + \mathcal{H}_{\text{ct}}.$$

The coupling constants in $\mathcal{H}_{\text{b}}$ are the physical ones, while each counterterm contributes starting from appropriate loop order. The resultant parameters of order $l = 0$ include mass parameters[4] and certain infinite polylinear combinations of the Hamiltonian coupling constants with coefficients depending on masses. The lth level resultant parameters $\boldsymbol{v}^{(l)}_{\ldots}$ with $l \geq 1$ contain also the items depending on counterterm couplings of orders $l' \leq l$. In fact, it is the new counterterms arising at each new loop order that make this classification convenient in perturbative calculations.

The construction of resultant parameters implies transition to the *minimal* parametrization, in which every S-matrix graph is built of minimal propagators and minimal vertices. The numerator of the minimal propagator is just a covariant spin sum considered as a function of four independent components of momentum.[5] The essence of the term ''minimal'' when it relates to a scalar function is that the latter looks similar on and off the mass shell. When it relates to a tensor structure, this term means that the

[3]The original definition given in [5] employs Lagrangian. The reasons for this difference were explained in [1,4], see also Sec. VIII below. In general we do not imply any other symmetry but Lorentz invariance, inclusion of any linear internal symmetry being trivial. The dynamical (nonlinear) symmetries are briefly discussed in [1].

[4]Real masses, appearing in Feynman propagators. For stable particles these are the physical (observable) masses—see Secs. VII, VIII, and IX.

[5]The conventional way to extend the spin sum—numerator of the propagator—out of the mass shell is explained, e.g., in [6], Chap. 6.2. In principle, it can be done in many ways, so that additional regular terms may arise in propagator. In [1] we allowed those terms just for the sake of generality, calling the resulting structure a ''nonminimal'' propagator. However, as one can deduce from the discussion in the chapter cited above, these nonminimal items can always be canceled by adding certain local terms in the Hamiltonian. That is why from now on we shall use the minimal (sometimes called transverse) propagators only. Besides, due to peculiar features of spin sums for massless particles, we imply that the Hamiltonian does not contain the massless fields of spin $J \geq 1$. The latter is quite sufficient for work with the hadron spectrum.

structure does not vanish when dotted by a relevant (on-shell) wave function. The central object is the minimal effective vertex.

To explain what it is, we start from the basic Hamiltonian (without counterterms). Let us single out all its items constructed from a given set of, say, n normally ordered field operators with quantum numbers collectively referred to as $i_1, \ldots, i_n$. These items differ from each other by the Hamiltonian coupling constants, by the number of derivatives and/or, possibly, by their matrix structure due to fermions or a linear symmetry group. Now, consider a momentum space matrix element of the (formal) infinite sum of all these terms. This matrix element should be calculated on the mass shell, presented in a Lorentz-covariant form and considered as a function of $4(n-1)$ *independent* components of particle momenta p_k^μ (four-momentum conservation δ-function is retained, but on-shell restriction is relaxed). The wave functions should be crossed out. The resulting structure we call the n-leg minimal effective vertex of the Hamiltonian level. Every such vertex $V^{(\mathcal{H})}$ presents a finite sum[6]

$$V^{(\mathcal{H})}_{\ldots}(i_1, \ldots, i_n; p_1, \ldots, p_n) = \delta(\Sigma p_k) \sum_a T^a_{\ldots} V^{(\mathcal{H})i_1 \ldots i_n}_a(\nu_1, \ldots, \nu_{3n-10}), \quad (2.1)$$

of tensor/matrix structures $T^a_{\ldots}$ dotted by scalar form factors $V^{(\mathcal{H})}_a$ linear in Hamiltonian coupling constants, each form factor being a formal power series in relevant scalar kinematical variables $\nu_1, \ldots, \nu_{3n-10}$ (the amount of independent scalars formed of p_k^μ that can survive on shell is $3n-10$):

$$V^{(\mathcal{H})i_1 \ldots i_n}_a(\ldots) = \sum_{r_1, \ldots, r_d = 0}^{\infty} g^{(a,\mathcal{H})i_1 \ldots i_n}_{r_1 \ldots r_d} \nu_1^{r_1} \ldots \nu_d^{r_d},$$

$$d = 3n - 10;$$

here $g^{(a,\mathcal{H})i_1 \ldots i_n}_{r_1 \ldots r_d}$ stand for linear combinations of the Hamiltonian couplings.

We use the term "minimal vertex of the Hamiltonian level" (not effective) to denote any separate contribution to the above series—a momentum space vertex produced by the basic Hamiltonian which does not alter on shell when dotted by the wave functions of external particles. Except the trivial cases like ϕ^4, a Hamiltonian term (like, e.g. $\phi^2 \partial_\mu \phi \partial^\mu \phi$) gives rise to both minimal and nonminimal vertices; that is why the notion of minimality makes sense in momentum space only. It is also sensible to the choice of variables ν_k, thereby in actual calculations one should fix this choice which, however, is not essential here.

[6]Compare [1], Eqs. (4)–(7) and Eqs. (12)–(13).

[7]These are the graphs computed on the mass shell of all external particles and dotted by the relevant wave functions.

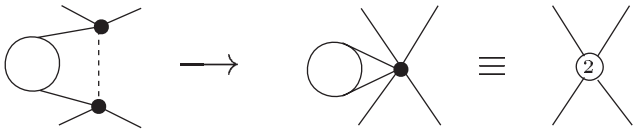


FIG. 1. Example of graph reduction: the dashed propagator is canceled by the nonminimal (vanishing on-shell, or when dotted by on-shell numerator of the propagator) structure coming from one of the vertices. Two initial vertices merge together into a new—secondary—vertex, and the 2-loop bubblelike structure is absorbed by the new coupling constant. As a result, a new second level vertex arises.

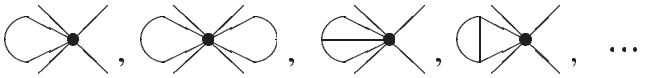


FIG. 2. Vertices of the levels $l = 1, 2$ pictured via Hamiltonian vertices: bubbles just rescale couplings and increase the level index.

Minimal vertices of tree and higher levels can be constructed after all the amplitude graphs[7] are subjected to the reduction procedure [1]. In this process some propagators disappear, being canceled, e.g., by $(p^2 - M^2)$ factors from nonminimal vertices. The two vertices connected by such a propagator flow together to form a single secondary vertex, like e.g. in Fig. 1. To preserve loop counting, we assign to this new vertex the level index equal to the loop order of the initial structure reduced (contracted) to form the vertex, plus the loop order of bubblelike structure[8] got attached to this vertex after the reduction of all graphs is completed (Fig. 2). For example, if two Hamiltonian vertices were connected with one another by one $(l-k)$-loop self-energy subgraph and, in addition, by the k simple propagators, the reduction of the latter ones leads to the appearance of a new, secondary, vertex of the lth level: see e.g. Fig. 1, where $k = 1$ and $l = 2$. Further, if one of the initial vertices was, say, 1-loop counterterm, then the level assigned to the secondary vertex is $l + 1$, and so on, so that the initial loop order is kept. The idea, of course, is that self-closed lines do not alter the tensor/matrix structure of the vertex, only rescaling the vertex coupling (the regularization is implied). Thus, it is natural to treat the vertex together with bubbles as a new single vertex, where a new coupling is given by the product of the two (or more) initial ones, times whatever bubbles give. However, since there are hidden loops, this new vertex should not appear in calculations until the needed loop order is reached.

[8]This does not relate to *tadpoles*. Note that in [1] we considered tadpoles (1-leg graphs) attached to a given vertex on the same footing as self-closed lines. Here, however, we consider tadpoles as independent elements of Feynman rules for constructing graphs in terms of resultant parameters. This allows us to avoid (rather formal) problems with the definition of one-particle irreducibility. Anyway, the tadpoles can always be removed by relevant renormalization prescription (see Sec. VII).

In [1] we have shown that, after full reduction is done, all the amplitude graphs are expressed via the minimal propagators and minimal vertices, some of them being the secondary vertices of various levels. All of them are minimal in a sense that they do not change their Lorentz-covariant form when put on mass shell and multiplied by the relevant wave functions. All bubblelike structures disappear forming minimal vertices of higher levels.

Now, we define the zeroth (tree) level minimal effective vertex as that of the Hamiltonian level plus the sum of all zeroth level secondary vertices with the same external legs. Clearly, the most general tensor structure of the vertex is defined by the external legs only, therefore the tensor structure of the tree-level minimal effective vertex is the same as that of the Hamiltonian level vertex (2.1). However, due to secondary vertices, the form factors $V_a^{(0)i_1\ldots i_n}$ are not anymore linear in Hamiltonian couplings.

The lth level minimal effective vertex with the same set of external legs is just a sum of all lth level minimal vertices with those legs, without adding the lth level minimal effective counterterm vertex with the same legs.[9] As we just mentioned, the presence of bubbles does not change the tensor structure of the effective vertex, it only affects the coefficients of scalar form factors. These latter coefficients (eventually supplied with the index l) are called the lth level minimal parameters.

Consider now a process involving a given set of external particles. Along with other graphs, the renormalized lth loop order amplitude of this process acquires contributions from both lth level minimal effective vertex and lth level minimal effective counterterm with the same set of external lines. Since both vertices have the same tensor structure, we can finally combine them into a single effective vertex, which we call the resultant vertex of the lth level. Simply speaking, awkward vertices with bubbles (see Fig. 2), including those that came from the reduction of nonminimal elements of graphs produced by the initial Feynman rules, are absorbed by the counterterms of the corresponding loop order and we treat the resulting combination as a single item. Analogous to minimal parameters, the lth level resultant parameters (couplings) are the coefficients in formal power series representing relevant form factors or, in general, any other set of independent parameters describing the resultant vertex. They are, of course, the functions of initial Hamiltonian couplings. However, the latter functional dependence is not of interest anymore: we are not going to fix any of couplings in the initial Hamiltonian. Rather, we will prefer to operate with minimal or resultant parameters directly. The simplest case is the 3-, 2-, and eventual 1-leg resultant vertices. One can easily check that, when put on shell and multiplied by relevant wave functions, the 1- and 2-leg vertices do not depend on external momenta, while those with 3 legs can only depend on it through the tensor structures like p^μ or γ^μ. Hence, the form factors in corresponding minimal effective (and resultant) vertices are reduced to constants.

The reduction technique introduced in [1] allows one to show that any amplitude graph of loop order L can always be presented as a sum of graphs built of minimal propagators and the minimal vertices of the levels $l \leq L$. In turn, the full (renormalized) sum of such Lth order graphs describing certain scattering process can be reexpressed solely in terms of minimal propagators and the resultant vertices $V_{res}^{(l)\cdots}$ with level indices $l \leq L$. Therefore, as long as the S-matrix is considered, the only building blocks we need in the Feynman rules are the resultant vertices and the minimal propagators.

The special convenience of dealing with the set of resultant parameters is that it is full and its members are independent. It is full in the sense that no other constants are needed to describe the renormalized S-matrix elements of the Lth loop order but the resultant parameters with $l \leq L$. They are independent in the sense that taking account of the higher loop order $l > L$ graphs leaves the structure of the lower level $l \leq L$ parameters unchanged. The reason for this is of course a freedom in the counterterm couplings which we consider independent at this stage. Thus, two resultant vertices $V^{(l_1)}$ and $V^{(l_2)}$ with the same external lines but of different levels l_1 and l_2 are described by precisely the same tensor structures, but the coefficients in power series (in the same set of variables)—the resultant parameters—do not depend on each other, which is indicated by different level indices.[10] Besides, by the very construction, the resultant parameters of the same level are independent, as far as we consider independent all the coupling constants in the effective Hamiltonian.

However, there is one thing, unpleasant from the technical point of view, that happens during the reduction.[11] As above, suppose that one works with regularized expressions. Before the reduction, one could think about all the amplitude graphs at any loop order as being finite: the counterterms were adjusted in a way that all subdivergencies for each given graph are canceled when regularization is removed. As it is clearly seen, the reduction is nothing but rearrangement of parameters within the graphs of a given loop order—it does not change the values of S-matrix elements. Imagine, however, that some graph had a subgraph with the divergency proportional to $p^2 - M^2$, where p and M are the momenta and mass of a particle

[9]Being considered separately, an lth level minimal effective counterterm vertex is built of counterterm vertices of the lth loop order in the same way as the Hamiltonian level minimal effective vertex is built of the Hamiltonian vertices. Of course, it has the same tensor/matrix structure.

[10]An analogous statement was made in [1], p. 9, with respect to minimal parameters of different levels. This is not quite correct, until all the counterterms are taken into account and, hence, the resultant parameters are formed.

[11]It does not affect the tree-level calculations of [2–4], neither the results of [1], thereby it was not mentioned there.

on corresponding external [with respect to (w.r.t.) the subgraph] leg. Before the reduction this subdivergency had been removed by the relevant explicitly drawn counterterm vertex of the form

$$-C(p^2 - M^2) + \cdots,$$

where C possesses exactly the same singular behavior w.r.t. regulator as the relevant part of the subgraph. But during the reduction the situation changes. Because of the nonminimal structure, $(p^2 - M^2)$, the corresponding propagator disappears from the graph and the nonminimal counterterm gets absorbed by the new (secondary) vertex. As a result, we may have a subgraph with divergency proportional to $(p^2 - M^2)$ and no explicit counterterm to kill it. Instead, one of the new couplings acquires singular behavior so that the S-matrix remains finite. This is technically inconvenient, because one is then forced to keep working with regularization until all the amplitude graphs of a given loop order are calculated.

Looking for a remedy, one may find it convenient to reintroduce some nonminimal counterterms after the reduction is done. This, in turn, may require renormalization prescriptions fixing the relevant nonminimal parameters. Since, as stressed above, the S-matrix does not depend on the latter, the only thing one needs to take care of is that the chosen values of nonminimal quantities do not fall in contradiction with various self-consistency relations. We shall treat this technical problem in a forthcoming publication. In this paper we just assume that it is solved in one way or another (see also the discussion in Sec. VI and Appendix C).

It is now clear that renormalization prescriptions (RP's) required to calculate the finite S-matrix are of two types. The first type RP's restricts the off-shell behavior of subgraphs. These prescriptions play no role in fixing the on-shell value of the graph itself; they are only needed to make convenient the intermediate steps of amplitude calculations and, in principle, would be required to get finite Green functions. We do not consider them in this paper. In contrast, the second set of RP's (called below as minimal) fixes the finite parts of counterterm constants contained in resultant parameters, which determine the value of each S-matrix element. Therefore the first step towards reducing the required number of independent RP's is to study the structure of this latter set.

There are certain subtleties in the usage of resultant vertices for constructing the amplitude. First, the only possible tadpoles are the 1-leg resultant vertices which, as explained in Sec. VII, can be safely dropped. Next, as mentioned above, the self-closed lines are also not present anymore being absorbed in resultant vertices. It makes no sense to picture explicitly those bubbles, because the resultant couplings are independent parameters of a theory. Therefore, due to ''hidden'' loops present in vertices with levels $l \geq 1$, the true loop order of a graph L_{true} may differ from the number of explicitly drawn loops L_{expl}. To keep the right loop order, we have to take account of the level index l_i of each resultant vertex $V_i^{(l_i)}$. The true loop order is then given by the number of explicitly drawn loops plus the sum of levels of the vertices used to construct the graph under consideration:

$$L_{\text{true}} = L_{\text{expl}} + \sum_{\text{vertices}} l_i. \tag{2.2}$$

We can now sum up the results of [1] discussed in this section in a form of instruction. To construct the L-loop contribution to the amplitude of a given process in the framework of effective scattering theory, one needs to:

(i) Use the system of Feynman rules only containing the minimal propagators and minimal effective (resultant) vertices of the levels $l \leq L$.

(ii) Construct all the graphs with $L_{\text{expl}} \leq L$ explicit loops with no bubbles involved and take account of the relevant symmetry coefficients. Below (Sec. VIII), we argue that there is no need in calculating amplitudes of the processes with external lines corresponding to resonances.

(iii) Pick up and sum all the graphs respecting relation (2.2) with $L_{\text{true}} = L$.

We will need these results in Secs. V, VI, and VII to explore the structure of minimal RP's needed to fix the physical content of effective scattering theory. But first we shall formulate two principles which we use as the basis for constructing the well-defined finite loop order amplitudes.

III. BASIC PRINCIPLES OF CONSTRUCTING THE PERTURBATION SERIES

In a sense, the argumentation in this section and in Sec. VIII below is inspired by the philosophy originally developed by Krylov and Bogoliubov [12] for nonlinear oscillation theory. It is concerned with perturbation series with singular behavior and allows one to group the items in a way that the summation procedure acquires meaning. In this spirit we specify certain requirements for the Dyson's type series arising in the strong interaction effective theories.

First of all, one needs a parameter to put the terms of perturbation series in certain order. Since the effective Hamiltonian involves an infinite number of coupling constants, the conventional logic (weak coupling or, the same, small perturbation) does not work, especially in strong interaction physics. That is why it is commonly accepted to classify the terms in perturbation series according to the (true) number of loops in Feynman graphs (see, e.g., [13]).

However, in effective theories the problem of the meaning of the loop series expansion (is it convergent? asymptotic? ...) is even more intricate than in conventional renormalizable theories. Indeed, in this case each item of the loop expansion, in turn, presents an infinite unordered sum of graphs. This is because the interaction Hamiltonian

is a formal sum of all possible monomials constructed from the field operators and their derivatives of arbitrary high degree and order. The problem of strong convergence of such operator series is not simple, if ever meaningful in the framework of perturbation theory. Instead, below we formulate two conditions of weak convergence for the functional series for S-matrix elements of a given loop order.

One of the most important requirements which we make use of when constructing the meaningful items of the Dyson perturbation series is that of polynomial boundedness. Namely, the full sum of S-matrix graphs with given set of external lines and fixed number L of loops must be polynomially bounded in every pair energy at fixed values of the other kinematical variables. There are two basic reasons for imposing this limitation. First, from general postulates of quantum field theory (see, e.g., [14]) it follows that the full (nonperturbative) amplitude must be a polynomially bounded function of its variables. Second, the experiment shows that this is quite a reasonable requirement. Since we never fit data with nonperturbative expressions for the amplitude, it is natural to impose the polynomial boundedness requirement on a sum of terms up to any fixed loop order and, hence, on the sum of terms of each order. A similar argument also works with respect to the bounding polynomial degrees. To avoid unnecessary mutual contractions between different terms of the loop series, we attract the following *asymptotic uniformity* requirement: the degree of the bounding polynomial which specifies the asymptotics of a given loop order amplitude must be equal to that specifying the asymptotics of the full (nonperturbative) amplitude of the process under consideration. Surely, this latter degree may depend on the type of the process as well as on the values of the variables kept fixed.[12]

The condition of asymptotic uniformity (or, simply, uniformity) is concerned with the asymptotic behavior of the total contribution at some fixed loop order, but does not tell us how the unordered infinite sum of graphs with the same number of loops (and, of course, describing the same process) can be converted into the well-defined summable[13] functional series. To solve the latter problem, we rely upon another general principle which we call *summability* requirement.[14] It is formulated as follows: in every sufficiently small domain of the complex space of kinematical variables there must exist an appropriate order of summation of the formal series of contributions coming from the graphs with given number of loops, such that the reorganized series converges. Altogether, these series must define a unique analytic function with only those singularities that are present in individual graphs.

At first glance, the summability requirement may seem somewhat artificial. This is not true. There are certain mathematical and field-theoretical reasons for taking it as the guiding principle that provides a possibility to manage infinite formal sums of graphs in a way allowing to avoid inconsistencies. It is, actually, both the summability and uniformity principles that allow us to use the Cauchy formula to obtain a well-defined expression for the amplitude of a given loop order. This will be demonstrated many times in the rest of the article.

We would like to stress that the requirements of uniformity and summability are nothing but independent subsidiary conditions fixing the type of perturbation scheme which we only work with. Surely, there is no guarantee that in this way one can construct the most general expressions for the S-matrix elements in effective theory. Nevertheless, there is a hope to construct at least meaningful ones presented by the Dyson's type perturbation series only containing the well-defined items.

IV. THE CAUCHY FORMULA IN HYPERLAYERS

Applying the famous Cauchy integral formula to the scattering amplitude is a basic tool of the analytic S-matrix approach and it is very well treated in the literature. We also use this tool but in a way essentially different from the conventional one.

First, we apply the Cauchy formula to the finite loop order amplitudes. Second, being armed with the polynomial boundedness principle discussed in the previous section, we pay special attention to the convergency of the resulting series of integrals. Basically, we treat the Cauchy integral as the tool to put in order the so far unordered scope of Feynman graphs of a given loop order in a way that the resulting series converge and, therefore, make sense. This turns out especially useful when we need to express the amplitude of a given loop order in terms of resultant vertices and for deriving bootstrap equations for the physical parameters. For use in the rest of the article and for future references, we shall thus outline the main steps of the Cauchy integral formula application.

Consider a function $f(z, \mathbf{x})$ analytic in the complex variable z and smoothly depending on a set of parameters $\mathbf{x} \equiv \{x_i\}$. Suppose further that when $\mathbf{x} \in D$, where D is a small domain in the parameter space, this function has only a finite number of singular points in every finite domain of the complex-z plane. In Fig. 3 it is shown the geography of singular points $s_k \equiv s_k(\mathbf{x})$, $k = \pm 1, \pm 2, \ldots$, typical for the finite loop order scattering amplitudes in quantum field theory. Both left and right singular points are enumerated in order of increasing modulo. Note that the cuts are drawn in an unconventional way—just to simplify the figure. If the point s_k corresponds to the pole-type singularity there

[12]This is a generalization of the requirement first suggested in [15], see also [16].

[13]This is a loan term widely used in modern theory of divergent series; see, e.g., [17].

[14]By analogy with the maximal analyticity principle used in the analytic theory of the S-matrix (see, e.g. [18]), sometimes we call it the analyticity principle.

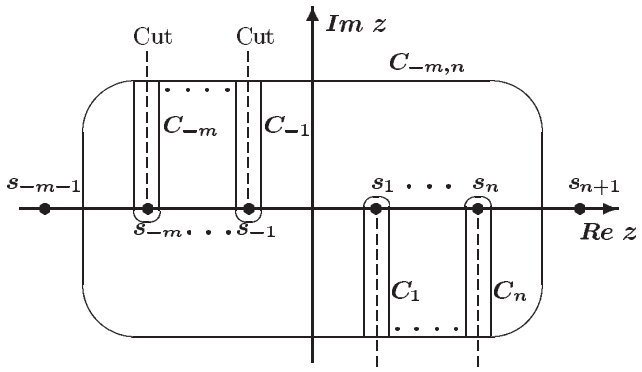


FIG. 3. System of embedded contours on the complex-z plane.

is no need in a cut, but its presence makes no influence on the following discussion.

Let us recall the definition of the polynomial boundedness property adjusted for the case of many variables [3]. Consider the system of closed embedded contours $C(i) \equiv C_{-m_i,n_i}$ (Fig. 3) such that every $C(i)$ surrounds $C(i-1)$ and does not cross the singular points. We say that the function $f(z, \mathbf{x})$ is N-bounded in the hyperlayer $B_x\{z \in \mathbf{C}, \mathbf{x} \in D\}$ if there is an infinite system of contours $C(i)$, $i = 1, 2, \ldots$, and an integer N such that

$$\max_{\mathbf{x}\in D; z\in C(i)} \left| \frac{f(z, \mathbf{x})}{z^{N+1}} \right| \to 0 \tag{4.1}$$

when $i \to \infty$. The minimal N (possibly, negative), which provides the correctness of the uniform (in x) estimate (4.1), we call the degree of bounding polynomial in the layer B_x.

Instead of the precise definition given above, one can just keep in mind the rough condition, more "strong": $f(\mathbf{x}, z) = o(|z|^{N+1})$ for all $\mathbf{x} \in D$ and large $|z|$, except small vicinities of singularities.

Condition (4.1) makes it natural to apply the Cauchy's integral formula for the function $f(z, \mathbf{x})/z^{N+1}$ on the closed contour formed by $C(i)$ (except small segments crossing the cuts), the corresponding parts of the contours C_k, $k = -m_i, \ldots, -1, 1, \ldots, n_i$, surrounding cuts, and a small circle around the origin[15] (the last one is not drawn in Fig. 3). In the limit $i \to \infty$ one obtains

$$f(z, \mathbf{x}) = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(0, \mathbf{x}) z^n + \frac{z^{N+1}}{2\pi i} \sum_{k=-\infty}^{+\infty} \oint_{C_k} \frac{f(\xi, \mathbf{x})}{\xi^{N+1}(\xi - z)} d\xi, \tag{4.2}$$

[15]We assume that f is regular at the origin. Therefore $f(z, \mathbf{x})/z^{N+1}$ may have a pole there, and to apply the Cauchy formula one should add a circle around the origin to the contour of integration. It is this part of the contour that gives the first sum on the right side of Eq. (4.2).

with $f^{(n)}(z, \mathbf{x}) \equiv \partial^n / \partial z^n f(z, \mathbf{x})$. It is essential to perform the last summation in order of increasing modulo of the singularities s_k which the contours C_k are drawn around. Equation (4.2) provides a mathematically correct form of the result. If the number of singular points is infinite, every contour integral on the right side should be considered as a single term of the series. The order of summation mentioned above provides a guarantee of the uniform (in both z and $\mathbf{x}$) convergence of the series. The formula (4.2) plays the key role in the renormalization program discussed below.

If the function f represents the tree-level amplitude, the summability principle formulated in Sec. III does not permit any brunch cut, because only the pole-type singularities appear in tree-level graphs. Hence, all the contours C_k are reduced to circles around poles and all the integrals in Eq. (4.2) can be expressed via the relevant residues. It is this way that the Cauchy forms introduced in [3] arise. For future reference we discuss this case in Appendix A.

V. MINIMAL PRESCRIPTIONS 1: TENTATIVE CONSIDERATION

The reason to construct resultant parameters shortly discussed in Sec. II is to single out the renormalization prescriptions (RP's) needed to calculate scattering amplitudes perturbatively. In turn, the results of Secs. III and IV lend a hand in forming the expressions for given loop order amplitude in terms of resultant parameters. The following three sections demonstrate how all this works together in explicit amplitude calculations. A very important result is formulated in Sec. VII. Namely, it is shown that, under certain assumptions suggested by phenomenology, in the effective scattering theory of strong interactions one only needs to know those minimal RP's which fix the resultant vertices with 1, 2, and 3 external legs. The other resultant couplings turn out to be fixed by certain self-consistency conditions. To show the origin of these conditions, we discuss below a simple example illustrating the main idea of our renormalization procedure.

Consider an elastic scattering process,

$$a(p_1) + b(k_1) \to a(p_2) + b(k_2), \tag{5.1}$$

where we took both a and b particles to be spinless: this considerably simplifies the purely technical details without changing the logical line of the analysis.

Along with the conventional kinematical variables $s = (k_1 + p_1)^2$, $t = (k_1 - k_2)^2$, and $u = (k_1 - p_2)^2$, we introduce three equivalent pairs of independent ones:

$$(x, \nu_x), \qquad x = s, t, u; \qquad \text{where } \nu_s = u - t, \quad \nu_t = s - u, \quad \nu_u = t - s. \tag{5.2}$$

The pair (x, ν_x) provides a natural coordinate system in three-dimensional (one complex and one real coordinate) hyperlayer $B_x\{\nu_x \in \mathbf{C}; x \in \mathbf{R}; x \sim 0\}$, while the pair

$(x, \mathrm{Re}\,\nu_x)$ does the same in the band parallel to the corresponding side $x = 0$ of the Mandelstam triangle: Fig. 4.

Let us suppose that in $B_t\{\nu_t \in \mathbf{C}; t \in \mathbf{R}, t \sim 0\}$ the full (nonperturbative) amplitude of the process (5.1) is described by the 0-bounded function $f(\nu_t, t)$ ($N_t \leq 0$). It is quite a typical experimental situation in hadron physics; in the end of this section we discuss more involved cases. According to the uniformity principle (Sec. III), we have to construct the perturbation series

$$f(\nu_t, t) = \sum_{l=0}^{\infty} f_l(\nu_t, t)$$

in such a way that each full sum $f_l(\ldots)$ of the lth loop order graphs also presents the 0-bounded function in B_t. Hence, the relation (4.2) in this layer reads

$$f_l(\nu_t, t) = f_l(0, t) + \frac{\nu_t}{2\pi i} \sum_{k=-\infty}^{+\infty} \oint_{C_k(t)} \frac{f_l(\xi, t)}{\xi(\xi - \nu_t)} d\xi. \quad (5.3)$$

Here the notation $C_k(t)$ is used to stress that the positions of singularities (and, hence, of the cuts) in the complex-ν_t plane depend on the other variable t, which now serves as a parameter.

When working at loop order l with 4-leg amplitude we, of course, imply that all the numerical parameters needed to fix the finite (renormalized) amplitudes of the previous loop orders, as well as those fixing 1-, 2-, and 3-leg l loops graphs, are known and one only needs to carry out the renormalization of the lth order 4-legs graphs. In the next section we will prove that the infinite sum of integrals in (5.3) depends solely on the parameters already fixed on the previous steps of the renormalization procedure. Thus, to obtain the complete renormalized expression for the lth order contribution in B_t, it only remains to specify the function $f_l(0, t)$. This can be done with the help of self-consistency requirement.

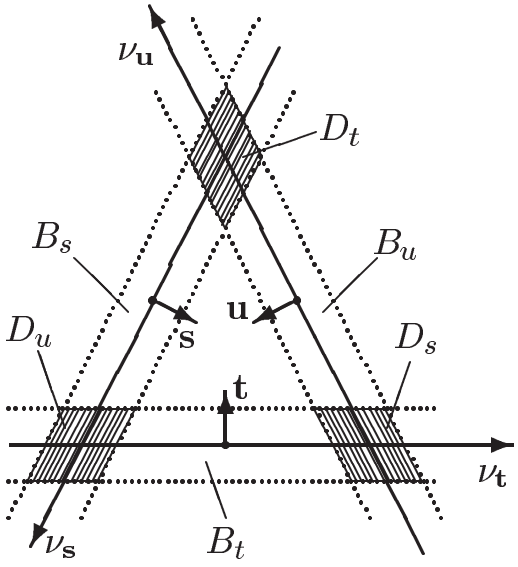


FIG. 4. Disposition of the bands B_x (bounded by dotted lines) and intersection domains D_x (hatched) ($x = s, t, u$).

To make use of this requirement, we consider the cross-conjugated process,

$$a(p_1) + \overline{a}(-p_2) \to \overline{b}(-k_1) + b(k_2), \quad (5.4)$$

and suppose that in $B_u\{\nu_u \in \mathbf{C}; u \in \mathbf{R}, u \sim 0\}$ it is described by the (-1)-bounded ($N_u \leq -1$; we discuss the other possibilities below) amplitude

$$\phi(\nu_u, u) = \sum_{l=0}^{\infty} \phi_l(\nu_u, u).$$

The uniformity principle tells us that every function $\phi_l(\nu_u, u)$, in turn, must be (-1)-bounded in B_u and, hence, (4.2) takes the form

$$\phi_l(\nu_u, u) = \frac{1}{2\pi i} \sum_{k=-\infty}^{+\infty} \oint_{C_k(u)} \frac{\phi_l(\xi, u)}{(\xi - \nu_u)} d\xi. \quad (5.5)$$

Again, it is implied (and proved in the next section) that the sum of integrals on the right side only depends on the parameters already fixed on the previous steps of the renormalization procedure.

Recalling that both expressions (5.3) and (5.5) follow from the same infinite sum of l-loop graphs (perturbative crossing symmetry) and attracting the summability principle, we conclude that in the intersection domain $D_s \equiv B_t \cap B_u$ they must coincide with one another:

$$f_l(0, t) + \frac{\nu_t}{2\pi i} \sum_{k=-\infty}^{+\infty} \oint_{C_k(t)} \frac{f_l(\xi, t)}{\xi(\xi - \nu_t)} d\xi = \frac{1}{2\pi i} \sum_{k=-\infty}^{+\infty} \oint_{C_k(u)} \frac{\phi_l(\xi, u)}{(\xi - \nu_u)} d\xi,$$

which means that in D_s

$$\begin{aligned} f_l(0, t) &= -\frac{\nu_t}{2\pi i} \sum_{k=-\infty}^{+\infty} \oint_{C_k(t)} \frac{f_l(\xi, t)}{\xi(\xi - \nu_t)} d\xi \\ &\quad + \frac{1}{2\pi i} \sum_{k=-\infty}^{+\infty} \oint_{C_k(u)} \frac{\phi_l(\xi, u)}{(\xi - \nu_u)} d\xi \\ &\equiv \Psi^{(0,-1)}(t, u). \end{aligned} \quad (5.6)$$

The relation (5.6) only makes sense in D_s. Given the asymptotics in B_s, it is not difficult to construct two more relations of this kind, one of them being valid in $D_t \equiv B_u \cap B_s$, and the other one in $D_u \equiv B_s \cap B_t$. These relations play a key role in our approach because they provide us with a source of an infinite system of bootstrap conditions. To explain bootstrap, let us consider (5.6) in more detail and make two statements.

First, despite the fact that (5.6) only makes sense in D_s, it allows one to express the function $f_l(0, t)$ in the layer B_t in terms of the parameters which, by suggestion, have been fixed on the previous steps of the renormalization proce-

dure. When translated to the language of Feynman rules, this means that in our model example there is no necessity in attracting special renormalization prescriptions fixing the finite part of the 4-leg counterterms. Instead, the relation (5.6) should be treated as that generating the relevant RP's iteratively—step by step. In what follows we call this (generating) part of self-consistency equations *bootstrap conditions of the first kind*.

Second, the relation (5.6) strongly restricts the allowed values of the parameters which are assumed to be fixed on the previous stages. To show this, it is sufficient to note that $f_l(0, t)$ only depends on the variable t while the function $\Psi^{(0,-1)}(t, u)$ formally depends on both variables t and u. Thus we are forced to require the dependence on u to be fictitious. It is this requirement that provides us with an additional infinite set of restrictions for the resultant parameters. We call these restrictions *the bootstrap conditions of the second kind*.

The proof of both statements is simple. Let us choose t and u as a pair of independent variables. As we just mentioned, $f_l(0, t)$ does not depend on the other variable u. Using the definitions (5.2) and the fact that $s + t + u = 2(m_a^2 + m_b^2)$ (m_a and m_b are the external particle masses), the variables ν_t and ν_u can be expressed via t and u. Then if we just take some value of u within the bounds given by D_s, say, $u = 0$, then the right side of Eq. (5.6),

$$f_l(0, t) = \Psi^{(0,-1)}(t, 0), \tag{5.7}$$

will give us $f_l(0, t)$ at $t \sim 0$, and, therefore, everywhere in B_t.

Further, since the domain $D_s\{t \sim 0, u \sim 0\}$ contains the point $(t = 0, u = 0)$, differentiating both sides of Eq. (5.6) one obtains an infinite system of bootstrap conditions of the second kind:

$$\frac{\partial^k}{\partial t^k}\frac{\partial^{m+1}}{\partial u^{m+1}}\Psi_l^{(0,-1)}\Big|_{t=0,u=0} = 0, \qquad k, m = 0, 1, \ldots. \tag{5.8}$$

They restrict the allowed values of the parameters fixed on the previous steps of the renormalization procedure.

We see that the system of bootstrap conditions of a given loop order[16] is naturally divided into two subsystems. Those of the first kind just allow one to express certain resultant parameters via the lower level parameters which, by condition, already have been expressed in terms of the fundamental observables[17] on the previous steps. In other words, they provide a possibility to express some parameters in terms of observable quantities. This subsystem does not restrict the admissible values of the latter quantities.

In contrast, the bootstrap conditions of the second kind do impose strong limitations on the allowed values of the physical (observable) couplings and masses[18] of effective scattering theory with certain asymptotic conditions. In fact, it provides us with the system of physical predictions which—at least, in principle—can be verified experimentally.

To make our analysis complete, we shall now explain the above-made choice of the bounding polynomials degrees. Besides, in the next section we discuss the parameter dependence of contour integrals appearing in Eqs. (5.3) and (5.5).

From experiment we know that the bounding polynomial degree N_{el} for the elastic scattering amplitude in B_t at $t \sim 0$ does not exceed the value $N = 1$. As noted in [3], if the system of contours appearing in the definition of polynomial boundedness is symmetric with respect to the origin of, say, the complex-ν_t plane and the amplitude in question is symmetric (antisymmetric) in $(s \leftrightarrow u)$, the bounding polynomials possess the same evenness property as the amplitude does. For simplicity, we have considered above this very situation which occurs, e.g., in the pion-nucleon elastic scattering. This explains why the term linear in ν_t is not present in (5.3).

In a more general situation, when the amplitude in B_t has a bounding polynomial degree $N_t > 0$ (while in B_u, as above, $N_u \leq -1$), the relation (5.6) is replaced by

$$\begin{aligned}\sum_{n=0}^{N_t}\frac{1}{n!}f_l^{(n)}(0, t)\nu_t^n = &-\frac{\nu_t^{N_t+1}}{2\pi i}\sum_{k=-\infty}^{+\infty}\oint_{C_k(t)}\frac{f_l(\xi, t)}{\xi^{N_t+1}(\xi - \nu_t)}d\xi \\ &+ \frac{1}{2\pi i}\sum_{k=-\infty}^{+\infty}\oint_{C_k(u)}\frac{\phi_l(\xi, u)}{(\xi - \nu_u)}d\xi \\ &\equiv \Psi(t, u),\end{aligned}$$

and the bootstrap conditions take a slightly different form as compared to (5.7) and (5.8). However, it is easy to see that our main conclusion remains unchanged: in this case there is no necessity in attracting additional RP's.

The situation when in both layers B_t and B_u the amplitude has non-negative bounding polynomial degrees is discussed in Appendix B. The analysis is similar, but one needs also to attract RP's for some 4-leg vertices.

Running a bit ahead, we shall explain why the just considered example deserves attention. The bootstrap equations analyzed in this section are valid in the intersection domain of two layers B_t and B_u, which contain points

[16]Equation (5.6) does not generate the full system: it only mirrors the self-consistency (crossing) requirements for the given order amplitude in certain domains of the complex space of kinematical variables. Another amplitudes/domains/orders can give additional constraints.

[17]These are the physical (measurable) parameters that appear in renormalization prescriptions.

[18]With respect to resultant couplings this system turns out to be homogeneous and the common scale factor remains undefined. With respect to mass parameters this system is highly nonlinear but, nevertheless, does not constrain the overall mass scale. This means that at least two scaling parameters must be fixed by the corresponding measurements (RP's).

with $t \sim 0$ and $u \sim 0$, respectively. Actually, the reason for this choice of layers is explained by the existence of experimental information on Regge intercepts. This choice is, however, also justified from the field-theoretic point of view [1,3,4]. Here are the arguments. A formal way to construct the $(L+1)$th order amplitude of the process $X \to Y$ is to close the external lines of the relevant Lth order graphs corresponding to the process $X + a(p_1) \longrightarrow Y + \overline{a}(p_2)$ with two additional particles carrying the momenta p_1 (let it be incoming) and p_2 (outgoing). This means that the latter graphs should be calculated at $p_1 = p_2 \equiv q$, dotted by the a-particle propagator and integrated over q (and then, of course, summed over all possible types of particles a). To ensure the correctness of this procedure (see [14]), one needs to require the polynomial boundedness (in p_1) of the Lth order amplitude of the process $X + a(p_1) \to Y + \overline{a}(p_2)$ at $t \equiv (p_1 - p_2)^2 = 0$ and, by continuity, in a small vicinity of this value. Clearly, this argumentation applies to arbitrary graphs with $N \geq 4$ external lines. That is why bootstrap conditions would arise and reduce the number of independent RP's even in the absence of phenomenological data on asymptotic behavior.

VI. MINIMAL PRESCRIPTIONS 2: CONTRIBUTION OF SINGULARITIES

As promised, here we show that all the contour integrals in (5.3) and (5.5) only depend on the parameters already fixed on the previous stages of the renormalization procedure. The proof is based on the structure of Eq. (4.2) and on the results of [1] briefly reviewed in Sec. II. Again, we consider first the elastic two body scattering (5.1), and then generalize the result to arbitrary scattering process.

The summability principle tells us that only parameters of graphs with singularities can appear in the contour integrals under consideration. Working with resultant parameters, the simplest way to trace which of them contribute to singular graphs is to picture the lth order amplitude via resultant Feynman rules—the recipe is given in the end of Sec. II. However, we find it instructive to demonstrate once more how the resultant vertices are built of the Hamiltonian ones: for this we shall look at the reduction procedure in action. This procedure does not change the structure of singularities of a given graph; it only reexpresses this graph via minimal parameters of various levels. When applied to a full sum of graphs forming a given order amplitude, it reexpresses this amplitude in terms of resultant vertices and minimal propagators. One of the great advantages of minimal (resultant) parametrization is that the singularities are explicitly seen when graphs are drawn—there are no more nonminimal structures that could cancel the propagator's denominator.

Let us look at one loop contribution to the process (5.1). Figure 5 schematically pictures the reduction procedure. The graphs on the left side of pictorial equation are drawn via initial Hamiltonian (effective) vertices,[19] that is why the self-closed lines appear. The right side is the result of the reduction procedure: the 1-loop contribution is presented in terms of resultant vertices of various levels (to save space we do not draw the graphs with resultant tadpoles). The numbers inside circles stand for the level indices of resultant vertices. One can easily verify that Eq. (2.2) and the drawing rules formulated thereafter are respected. Namely, we see that the resulting graphs are constructed from the minimal propagators and resultant 1-, 2-, 3-, and 4-leg vertices of levels 0 and 1. The graphs with explicit loops only contain the vertices of the lowest level $l = 0$, otherwise the loop counting would be violated. The 1-loop level resultant parameters appear only in the diagrams without explicit loops: these parameters come from vertices with self-closed lines and from 1-loop counterterms, thus no more loops are allowed.

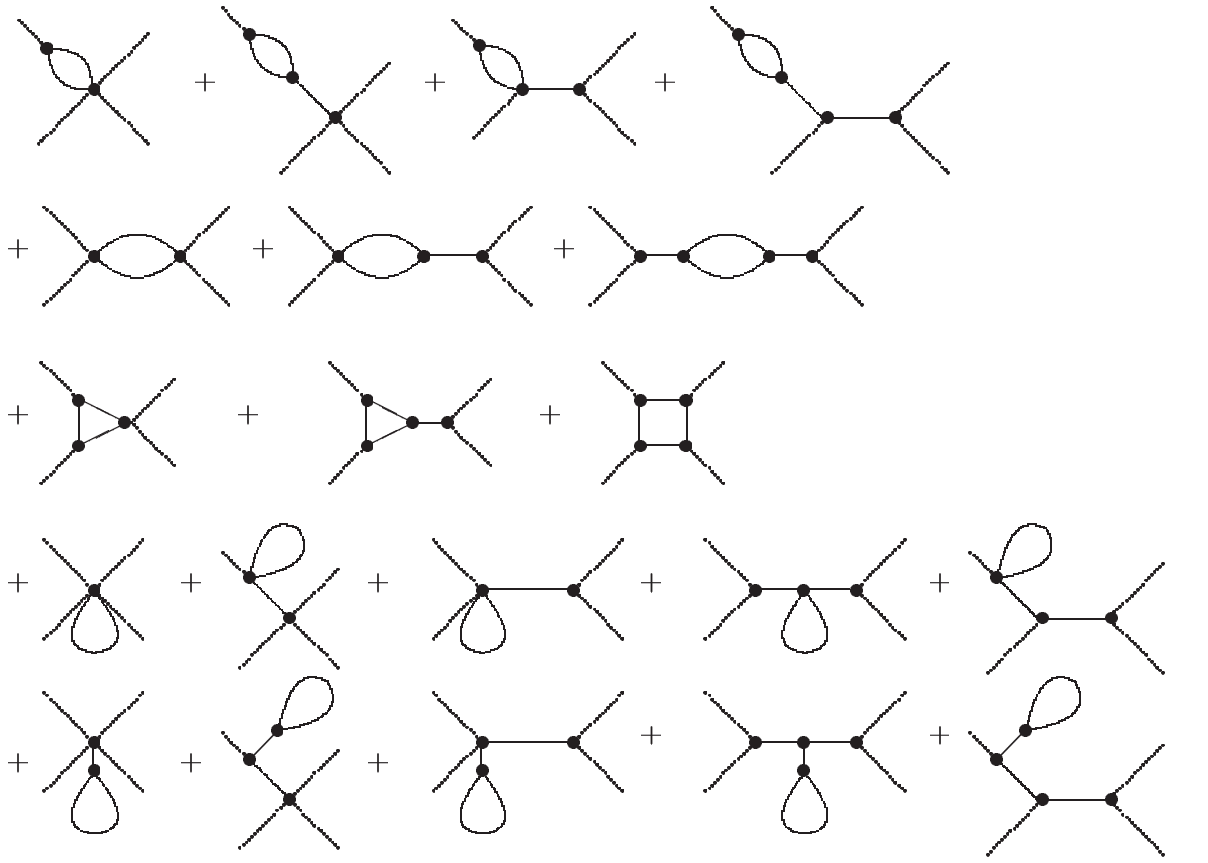
By the very logic of the renormalization procedure, at this step the 1-, 2-, and 3-leg one-loop counterterms were already adjusted to remove infinities from the corresponding subgraphs. Thus 1-, 2-, and 3-leg resultant vertices are fixed and there are no more subdivergencies.[20] The only parameters which remain free are those describing 4-leg 1-loop level resultant vertex. It is this vertex that absorbs the 4-leg 1-loop counterterms, and it is the only one which remains to be fixed by renormalization prescriptions. But this latter vertex appears in the graph with no singular structure (contact vertex) and thereby cannot contribute to contour integrals around cuts (or poles) in (5.3) and (5.5).

In the end of Sec. II we already said that in this paper we only consider the structure of minimal RP's needed to fix the finite parts of minimal counterterms. As shown in [1], fixing the latter counterterms completely determines the S-matrix at a given loop order. From the technical point of view, it is clear that those RP's are quite sufficient to perform the very last step of renormalization of S-matrix elements at a given loop order: S-matrix is calculated on shell and thus can be fixed by the minimal counterterms. However, the standard way one renormalizes a graph implies that divergencies from (off-shell) subgraphs are removed first. Of course, the latter divergencies are not necessarily minimal even for the subgraph built of minimal elements, and therefore may require nonminimal counterterms.[21] The source of this apparent confusion was pointed

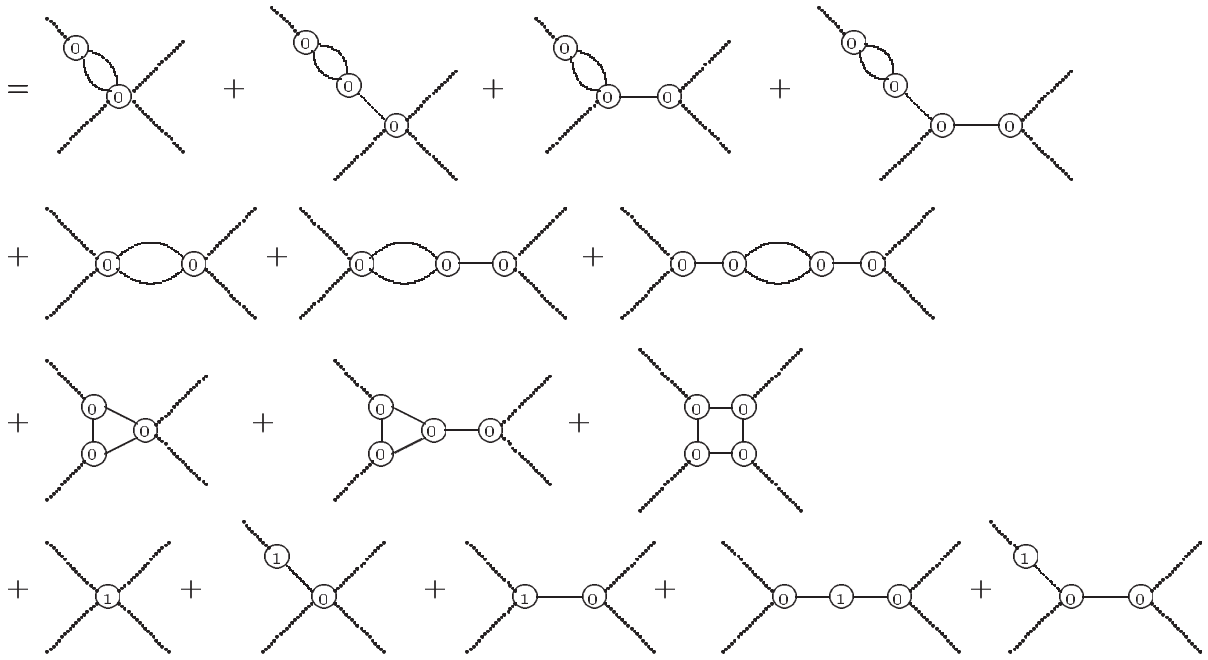
[19]The Hamiltonian effective vertex (not minimal) is just the sum of all bare Hamiltonian vertices (both minimal and nominimal) with a given set of legs [1]. Summation over all possible internal lines and vertices is implied in both sides of the equation.

[20]Nonminimal renormalization prescriptions may be needed to remove them (see the next paragraph).

[21]For example, consider the 2-leg off-shell graph in ϕ^4 theory (ϕ^4 is a minimal vertex, since it does not change its structure on shell), which has $\Lambda^2 + p^2 \log\Lambda$ behavior, so that the nonminimal—proportional to p^2—counterterm is needed.



+ [Cross terms] + [Graphs with 1-st order 1-, 2-, 3- and 4-leg counterterms]



+ [Tree graphs with resultant tadpole insertions] + [Cross terms].

FIG. 5. Reduction of 1-loop contribution to the $2 \to 2$ amplitude: numbers indicate loop levels of the resultant vertices.

out in Sec. II—the nonminimal counterterms were absorbed by minimal vertices during the reduction. Although all off-shell subdivergencies will cancel in the given loop order amplitude, we still have no explicit counterterm to kill each of them directly in graphs where they arise. In particular, this relates to the wave function renormalization. Keeping in mind what was said about possible solutions in Sec. II, we postpone the detailed discussion to the next publication (see, however, Appendix C). Here we will just tacitly imply that all subgraphs are made finite. The present analysis is quite sufficient to justify the comparison of tree-level computations with experimental data (for preliminary discussion see [3,7,8]).

In general, the renormalization of the l-loop amplitude of a process involving $n = 4, 5, \ldots$ particles consists of l stages. In turn, every l'th stage ($l' = 1, 2, \ldots, l-1$) consists of a certain (depending on n, l and l') number of steps, each one being the renormalization of l'-loop graphs with a given number $1, 2, \ldots, n'_{\max}(n, l, l')$ of external lines. The last (lth) stage consists of $(n-1)$ preliminary steps: renormalization of the l-loop resultant graphs with $1, 2, \ldots, (n-1)$ legs (tadpole, self-energy, etc.). At last, the final (nth) step—renormalization of n-leg l-loop graphs—requires attracting renormalization prescriptions that fix the values of lth order n-leg counterterm vertices. This is precisely the situation known from conventional renormalization theory. The only difference is that in effective scattering theory the number of counterterms needed for renormalization of all lth loop order S-matrix elements is infinite.

As stressed in the previous section, when writing down the lth order amplitudes (5.3) and (5.5), we imply that all the previous steps already have been passed and we only need to make the last step—to fix the lth level 4-leg counterterms or, equivalently, fix the coefficients in the formal series for 4-leg resultant vertex. To have a singularity and, thus, to contribute to contour integral in (5.3) and (5.5), a graph must have at least one internal line. Using the pictorial rules formulated in Sec. II, or just by direct analogy with Fig. 5, it is easy to understand that graphs with internal lines may only depend on resultant parameters of lower levels,[22] or on the lth level parameters from vertices with less number of legs: $n = 1, 2, 3$. Since all those parameters have already been fixed on the previous steps of renormalization procedure, the contour integrals in (5.3) and (5.5) should be, indeed, considered as known functions. This completes our proof.

Now we are in a position to review our analysis of the process (5.1) and put all the steps in logical order. That is, we started from the formal sum of l-loop 4-leg amplitude graphs and rewrote it in terms of resultant parameters of various levels. We suggested that all subgraphs are renormalized (finite) and, hence, all the resultant parameters of lower levels $l' = 1, 2, \ldots, l-1$ are fixed. Then we *required* that this formal sum results in a function with the following properties:

- In B_t it is 0-bounded in complex variable $\nu_t \equiv (s-u)$, while t is treated as a parameter.
- In B_u it is (-1)-bounded (decreasing) in complex variable $\nu_u \equiv (t-s)$, while u is treated as a parameter.
- In both layers the resulting function has only those singularities which are presented in the contributions of individual graphs of the formal sum under consideration (summability principle).

These requirements allowed us to rewrite the sum of graphs in a form of Cauchy-type integral (4.2) which, therefore, provides the mathematically correct expression for the S-matrix element as a function of the external parameters (or, the same, RP's) of the theory.

It is shown that the last (lth) stage of renormalization of S-matrix elements does not require attracting any minimal RP's in addition to those fixing the lth order resultant vertices with $n = 1, 2, 3$ legs; the RP's fixing minimal 4-leg counterterms of the lth order are automatically generated by the first kind bootstrap equations. Besides, the bootstrap equations of the second kind impose certain restrictions on the lth level parameters of 1-, 2-, and 3-leg vertices (and, possibly, on the parameters of lower levels). As noted in the end of Sec. V, this conclusion actually holds for any $2 \to 2$ amplitude which possesses a negative bounding polynomial degree in one of the intersecting layers, while the asymptotic in the other layer may be arbitrary, though also polynomial.

This result offers a hint on which requirements are sufficient for S-matrix renormalization in effective scattering theory.

VII. MINIMAL RENORMALIZATION PRESCRIPTIONS: GENERAL OUTLINE

The model example considered in two previous sections shows that, as long as there are two intersecting hyperlayers such that in both of them the amplitude of a given process $2 \to 2$ is polynomially bounded, and at least in one of them it is (-1)-bounded in the corresponding complex variable, there is no need in independent RP's for 4-leg amplitude graphs: the summability principle provides us with a tool for generating those (on shell) prescriptions order by order. This conclusion, however, implies, that all the previous renormalization steps are done, so that all subgraphs of previous loop orders $l' < l$ and 1-, 2-, and 3-leg graphs of the same loop order l are made finite. As to the 1-, 2-, and 3-leg minimal counterterms (which are just constants in terms of the resultant parameters), one does need to fix them by relevant RP's, but this cannot be done

[22] It could be wrong if there were 2-leg resultant vertices of the zeroth (tree) level, or, the same, if we chose masses in propagators to differ from the corresponding pole positions in tree-level amplitudes. We do not (see the next section).

arbitrarily—the bootstrap requirements must be taken into account.

It is well known that the amplitudes of inelastic processes involving $n > 4$ particles decrease with energy, at least in the physical area of other relevant variables: the phase space volume grows too fast to maintain unitarity of the S-matrix with nondecreasing inelastic amplitudes. Therefore it looks natural to suggest that in corresponding hyperlayers these amplitudes can be described with the help of at most (-1)-bounded functions of one complex energylike variable (and several parameters). Also, it is always possible to choose the variables in such a way that the domains of mutual intersection of every two hyperlayers are nonempty.[23] One then easily adjusts the analysis of Secs. V and VI to show that the amplitudes of processes with $n > 4$ particles are completely defined by contour integrals similar to (5.5), which depend only on the parameters already fixed on the previous steps of renormalization. In other words, RP's for 1-, 2- , 3-, and 4-leg graphs completely specify those for graphs with 5 legs, altogether these RP's give prescriptions for 6-leg graphs, and so on. Hence, the independent RP's for amplitude graphs with $n > 4$ legs are not required, so the system of (minimal) renormalization prescriptions needed to fix the physically interesting effective scattering theory only contains prescriptions for 1-, 2-, 3-, and, possibly, 4-leg resultant graphs.

The next step is to employ hadronic phenomenology. Consider first the SU_2 sector, where the only stable particles are the pion and the nucleon. As known, the high-energy behavior of elastic pion-pion, pion-nucleon, and nucleon-nucleon scattering amplitudes is governed by the Regge asymptotic law. Then it is not difficult to check that each of these amplitudes is described by scalar form factors with negative degrees of bounding polynomial, at least in one of three cross-conjugated channels. In fact, even for heavier flavors, we are not aware of the process with four stable (w.r.t. strong interactions) particles that violates this rule. That is why the analysis in Sec. V is relevant and 4-leg amplitude graphs with stable particles on external lines do not require formulating RP's.[24]

There are also 4-leg graphs with resonances on external legs. Here the situation looks more complicated owing to the absence of direct experimental information on processes with unstable hadrons. In other words, the choice of relevant bounding polynomial degrees is to a large measure nothing but a matter of postulate. The only way to check the correctness of the choice is to construct the corresponding bootstrap relations and compare them with existing data on resonance parameters. This work is in progress now. Here, however, we are tempted to consider the relatively simple situation when all the 4-leg "amplitudes" of the processes involving unstable particles decrease with energy, at least at sufficiently small values of the momentum transfer.[25] Then, again, according to Sec. V, the RP's for these 4-leg (on shell) graphs are not needed.

To summarize: the only minimal RP's needed to specify all S-matrix elements of the effective hadron scattering theory are those fixing finite parts of the resultant 1-, 2-, and 3-leg vertices. Moreover, this system cannot be taken arbitrarily—the relevant bootstrap constraints must be taken into account. Once these basic RP's are imposed, the minimal prescriptions for 4-, 5-, ...-leg graphs are automatically generated at any given loop order by summability principle. The possibility to do this for 4-leg graphs is provided by phenomenology, for 5-, ...-leg ones—by perturbative unitarity. Remember, however, that some nonminimal RP's (or another way to remove nonminimal divergencies) may also be needed.

To write the required minimal RP's explicitly, we need to introduce some notations. Consider the infinite sum C of all n-leg counterterm vertices (with a fixed set of legs) of the loop order l—we call it the lth order effective counterterm vertex. This vertex is pointlike but not minimal and should not be mixed with the lth level resultant vertex. We can write it as

$$C^{(l)}_{\dots}(i_1, \dots, i_n; p_1, \dots, p_n) = \delta(\Sigma p_k) \sum_a^{M+N} T^a_{\dots} C^{(l)i_1 \dots i_n}_a(\pi_1, \dots, \pi_n; \nu_1, \dots, \nu_{3n-10}), \tag{7.1}$$

where i_k marks the species of the kth particle (mass M_k, spin, etc.) and p_k is its four momentum. The index a numbers tensor/matrix structures $T^a_{\dots}$ and ellipses stand for corresponding tensor/matrix indices (if needed). The structures with $a = 1, \dots, M$ are minimal, while those with $a = M + 1, \dots, N$ are nonminimal—they vanish on the mass shell when dotted by the wave function of relevant particle. Since we work in effective theory, all such structures allowed by Lorentz invariance (and eventual linear symmetry) are present and do not depend on the loop order in question. The scalar functions $C^{(l)i_1 \dots i_n}_a$ stand for the formal power series

$$C^{(l)i_1 \dots i_n}_a(\dots) = \sum_{s_1, \dots, s_n, r_1, \dots, r_d = 0}^{\infty} C^{(a,l)i_1 \dots i_n}_{r_1 \dots r_d; s_1 \dots s_n} \nu_1^{r_1} \dots \nu_d^{r_d} \pi_1^{s_1} \dots \pi_n^{s_n}, \quad d \equiv 3n - 10, \tag{7.2}$$

in "mass variables" $\pi_k \equiv p_k^2 - M_k^2$ $(k = 1, 2, \dots, n)$ and

[23] This follows from the fact that the number of pair energies is much larger than that of independent kinematic variables.

[24] If a process with four stable hadrons with the amplitude asymptotics violating the Regge law will be found, additional RP's discussed in Appendix B may be needed.

[25] Surely, this is just a model suggestion which, however, seems to us quite reasonable. One of the arguments in its favor is that unstable particles cannot appear in true asymptotic states (Sec. VIII).

"on-shell variables" $\nu_1, \ldots, \nu_{3n-10}$, the latter ones are the scalar functions of 4-momenta chosen in a way that they provide a coordinate system on the mass shell; the concrete choice is not essential for the current discussion.

Next, using the same notations as in (7.1) and (7.2), we can present the full sum $\mathcal{G}^{(l)}$ of (amputated) resultant l-loop[26] graphs with n external (off-shell) particles of the types $i_1, \ldots, i_n$ as follows:

$$\mathcal{G}^{(l)}_{\ldots}(i_1, \ldots, i_n; p_1, \ldots, p_n) = \delta(\Sigma p_k) \sum_{a=1}^{M+N} T^a_{\ldots} \mathcal{G}^{(l)i_1 \ldots i_n}_a(\pi_1, \ldots, \pi_n; \nu_1, \ldots, \nu_{3n-10}). \tag{7.3}$$

Here $\mathcal{G}^{(l)i_1 \ldots i_n}_a$ stand for true lth loop order form factors which, in contrast to $C^{(l)i_1 \ldots i_n}_a$, are not just formal series but complex functions of π's and ν's.

Suppose that we are going to perform the last step of renormalization—to fix the lth loop order S-matrix element for the given n-particle process, while the computation of higher order terms is not assumed. Hence, we only need RP's for the on-shell ($\pi_k = 0$) value of that part of the sum (7.3) which survives when the external legs are multiplied by the relevant wave functions. Therefore only the minimal tensor structures contribute:

$$\mathcal{G}^{(l)}_{\ldots}(i_1, \ldots, i_n; p_1, \ldots, p_n)\big|_{\substack{\text{relevant} \\ \text{for } S\text{-matrix}}} = \delta(\Sigma p_k) \prod_{k=1}^{n} (Z_{i_k})^{-(1/2)} \sum_{a=1}^{M} T^a_{\ldots} G^{(l)i_1 \ldots i_n}_a(\nu_1, \ldots, \nu_{3n-10}), \tag{7.4}$$

where for $a = 1, \ldots, M$ we have introduced

$$G^{(l)i_1 \ldots i_n}_a \equiv \prod_{k=1}^{n} (Z_{i_k})^{1/2} \mathcal{G}^{(l)i_1 \ldots i_n}_a\big|_{\pi_i = 0}, \tag{7.5}$$

and Z_{i_k} stands for corresponding field-strength renormalization constant.

It is the prescriptions for $G^{(l)i_1 \ldots i_n}_a$ that we call minimal. As it was stressed several times in the text, fixing the latter quantities is sufficient for specification of all the S-matrix elements, though some nonminimal (off-shell) RP's may be needed to remove infinities from subgraphs. In particular, the RP's for derivatives of two-point functions—field-strength renormalization—are exactly of nonminimal type. However, the analysis of possible structure of nonminimal prescriptions is beyond the scope of this article and postponed until following publications. Here we just assume that all subdivergencies are removed from $\mathcal{G}^{(l)}$ and Z's are known.

[26]Recall that the true number of loops should be calculated in accordance with (2.2).

The discussion in Sec. VI shows that the sum (7.3) consists of:

(a) Graphs (not pointlike) built of the lower level resultant vertices and of resultant vertices of the lth level with lower number of legs. Couplings at these vertices are considered fixed on the previous steps of renormalization. As we just mentioned, all subdivergencies are removed, thereby these graphs introduce only superficially divergent terms to be eliminated and properly normalized by corresponding RP's.

(b) The lth level n-leg resultant vertex $V^{(l)}$ with the same set of external particles. By definition (Sec. II), the latter has the same tensor structure as the sum of graphs (7.4). This vertex consists of the lth level pointlike effective vertex formed after the reduction, and the minimal (surviving on the mass shell) part of effective counterterm vertex (7.1). Hence, this resultant vertex can be written as

$$V^{(l)}_{\ldots}(i_1, \ldots, i_n; p_1, \ldots, p_n) = \delta(\Sigma p_k) \sum_{a=1}^{M} T^a_{\ldots} V^{(l)i_1 \ldots i_n}_a(\nu_1, \ldots, \nu_{3n-10}), \tag{7.6}$$

where $V^{(l)}_a$ are just formal power series:

$$V^{(l)i_1 \ldots i_n}_a(\cdots) = \sum_{r_1, \ldots, r_d = 0}^{\infty} g^{(a,l)i_1 \ldots i_n}_{r_1 \ldots r_d} \nu_1^{r_1} \ldots \nu_d^{r_d}, \qquad d = 3n - 10, \tag{7.7}$$

and the numerical coefficients $g^{(a,l)i_1 \ldots i_n}_{r_1 \ldots r_{3n-10}}$ (the lth level n-leg resultant parameters) are combined from the lth level n-leg coupling constants (given by products of initial Hamiltonian couplings and bubble factors) and the corresponding (surviving on shell) counterterm couplings:

$$g^{(a,l)i_1 \ldots i_n}_{r_1 \ldots r_{3n-10}} = \begin{pmatrix} \text{coupling of (secondary)} \\ \text{vertex with } l \text{ bubbles} \end{pmatrix} + C^{(a,l)i_1 \ldots i_n}_{r_1 \ldots r_{3n-10}; 0 \ldots 0}.$$

To specify the S-matrix contribution (7.4) one needs only to fix the latter sum—as we just mentioned, all other parameters are considered known (in fact, we just rely on the mathematical induction method). Note however, that the off-shell sum of graphs (the Green function) will not be completely renormalized in this way: in general, to compensate the off-shell superficially divergent terms, one will need to attract all the off-shell ($\pi_k \neq 0$) counterterms in (7.1), which may be absorbed later by the higher level resultant parameters during the reduction of $(l + 1)$ order graphs. It is exactly the subtlety with nonminimal RP's that we are not going to discuss further in this paper

So, apart from the field-strength renormalization and other nonminimal RP's, the S-matrix is renormalized by adjusting the (finite part of) resultant couplings g in (7.7), which is equivalent to imposing RP's on $G_a^{(l)i_1\ldots i_n}$ in (7.4).

In the framework of renormalized perturbation theory, one has to identify the tree-level (physical) parameters: mass parameters and coupling constants. We define the *mass parameter* (or, simply, mass) M of a particle as the number that fixes the pole position of (free) Feynman propagator. For a stable particle (like pion or nucleon), this number must coincide with the mass of the corresponding asymptotic state or, the same, with the eigenvalue of the momentum squared operator. Resonance masses do not have such an interpretation, and in Sec. VIII we explain how they arise. Their values should be deduced from fit with experimental data and may depend on various conventions (see the discussion in Sec. IX).

In terms of resultant parameters, the *physical coupling constants* are naturally identified with tree-level resultant couplings

$$g^{(a,0)i_1\ldots i_n}_{r_1\ldots r_{3n-10}} \tag{7.8}$$

from (7.7). For $n = 1, 2, 3$ there are no lower indices, because, as mentioned in Sec. II, the corresponding resultant form factors are just constants. Corresponding true form factors $\mathcal{G}_a^{(l)i}$, $\mathcal{G}_a^{(l)i_1i_2}$, $\mathcal{G}_a^{(l)i_1i_2i_3}$ are, of course, not constants off-shell, but with our choice of variables can only depend on $\pi_k \equiv p_k^2 - M_k^2$.

In the first part of this section, we have shown that the only minimal RP's needed (at least in SU_2 sector) are those fixing the values of the resultant vertices with 1, 2, and 3 external particles. Conventionally, a renormalization prescription is imposed on the sum of graphs with a given set of external legs computed at a certain kinematical point. Typically, the sum of all 1-particle irreducible (1PI) graphs[27] up to a given loop order is taken [9,10]. Equivalently, one may impose RP's on the sum of 1PI graphs of a fixed loop order, as we do. Namely, below we imply that the functions $\mathcal{G}_a^{(l)i_1\ldots i_n}$ in Eq. (7.5) only acquire contributions from 1PI l-loop graphs, and, therefore, the Z-factors are dropped. Keeping in mind that the momentum conservation delta function $\delta(\Sigma p_k)$ stands as an overall factor in Eq. (7.3), we can now write down the required system of minimal RP's ($l = 0, 1, \ldots$):

(i) For $n = 1$ (absence of tadpole contributions):

$$\mathcal{G}_a^{(l)i_1}(p_1)|^{(1\mathrm{PI})}_{p_1^2=M_1^2} \equiv G_a^{(l)i_1}|^{(1\mathrm{PI})} = 0. \tag{7.9}$$

In particular, at $l = 0$ it reads as $g^{(a,0)i_1} = 0$ so that there are no tadpoles at tree level.

(ii) For $n = 2$ (absence of mixing and real mass shift),

$$\widetilde{\mathrm{Re}}\,\mathcal{G}_a^{(l)i_1i_2}(p_1, p_2)|^{(1\mathrm{PI})}_{p_k^2=M_k^2} \equiv \widetilde{\mathrm{Re}}\,G_a^{(l)i_1i_2}|^{(1\mathrm{PI})} = 0. \tag{7.10}$$

At $l = 0$ it reads as $g^{(a,0)i_1i_2} = 0$ so that at tree level there is neither mixing nor correction to the particle mass.

(iii) For $n = 3$ at $l = 1, 2, \ldots$:

$$\widetilde{\mathrm{Re}}\,\mathcal{G}_a^{(l)i_1i_2i_3}(p_1, p_2, p_3)|^{(1\mathrm{PI})}_{p_k^2=M_k^2} \equiv \widetilde{\mathrm{Re}}\,G_a^{(l)i_1i_2i_3}|^{(1\mathrm{PI})} = 0, \tag{7.11}$$

while at tree level ($l = 0$) this form factor is, of course, equal to the triple physical coupling:

$$\mathcal{G}_a^{(0)i_1i_2i_3}(p_1, p_2, p_3)|^{(1\mathrm{PI})}_{p_k^2=M_k^2} \equiv G_a^{(0)i_1i_2i_3}|^{(1\mathrm{PI})} = g^{(a,0)i_1i_2i_3},$$

the latter we define to be real, thus attributing eventual complex phases to the tensor structures.

Equations (7.10) and (7.11) adjust mass shifts and three leg amplitudes. In fact, one is only allowed to constrain the real parts of corresponding loop integrals, which is indicated by the $\widetilde{\mathrm{Re}}$ symbol. The triple couplings $g^{(a,0)i_1i_2i_3}$ and masses M appearing in (7.9), (7.10), and (7.11) are the physical observables. In general, their values must be fitted by comparison with experimentally measured amplitudes.

These prescriptions are sufficient to perform the last step of the S-matrix renormalization under the above-specified conditions, of which the most important one is the Regge-like asymptotic behavior. In that situation when some phenomenological 4-leg amplitude has no hyperlayers with decreasing asymptotics (with negative value of bounding polynomial degree), one should also add prescriptions for 4-point amplitudes written down in Appendix B. However, as far as we know, for any process involving four stable (w.r.t. strong interactions) particles such hyperlayers are always present so that additional prescriptions are not needed.

The above RP's have to be discussed. As to the tadpoles (7.9), in the resultant parametrization they can have only a scalar particle on the external leg: the covariant structures of type $\partial_\mu \ldots \phi^{\mu\ldots}$ built of tensor field $\phi^{\mu\ldots}$ do not survive on shell and thus do not contribute to resultant parameters. So, the "structure" index a in (7.9) can be omitted. The remaining scalar resultant tadpoles give just a constant factor to the graph they appear in and can be absorbed by the relevant coupling constants. In fact, this way we followed in [1]. Here, instead, to avoid the formal problems

[27] A reducible (in conventional sense) graph constructed from the initial Hamiltonian vertices may become 1PI after the reduction and switching to minimal parametrization (Sec. II): some propagator denominators may be canceled so that corresponding lines disappear. When working with resultant (minimal) parameters we, of course, assume the reduction done, thereby no confusion may arise. It is also possible to formulate the RP's for one-particle reducible graphs (so-called oversubtractions, see [10]); here we do not consider this possibility.

with one-particle irreducibility, we just accept Eq. (7.9). For calculations of amplitudes both ways are equivalent and in the future tadpoles are dropped.

Next, Eq. (7.10) provides a definition of what we call the mass parameter—mass, appearing in Feynman propagator. If, as suggested, the tadpoles are absent, Eq. (7.10) forbids any 2-leg resultant vertex at zeroth (tree) level. In the case of stable particles this looks natural because their mass terms are attributed to the free Hamiltonian. As to the resonances, the situation is not so transparent; we discuss it in the next two sections. Note also that the only tensor structure surviving in the 2-leg resultant vertex is the (symmetrized product of) metric tensor $g^{\mu\nu}$ (or unit matrix for fermions) and the metric tensor for the eventual linear symmetry group, therefore the structure index a in (7.10) takes the only value and can be dropped. If there are many particles with the same quantum numbers except masses, one needs also to check that it is possible to apply RP's avoiding mixing. This is nontrivial; we shall discuss it in the next publication.

At last, the prescription (7.11) guarantees the absence of loop corrections to the physical (real) triple coupling constants.

Looking now at the system (7.9), (7.10), and (7.11) and recalling the step-stage description of the renormalization procedure given in Sec. VI, it is easy to understand that at the very first step—calculation of tree-level amplitudes of the processes $2 \to 2$—one just has to substitute the triple physical coupling constants in residues and the physical mass parameters in propagators. Therefore the bootstrap conditions of the second kind obtained at tree level restrict the allowed values of physical, in principle measurable quantities. In this sense, the bootstrap conditions of the second kind are invariant with respect to renormalization, which justifies the legitimacy of the data analysis presented in [3,7,8]. Note also, that the bootstrap equations obtained at higher loop orders may differ from the tree-level ones, thus providing additional constraints, again, for physical quantities. In fact, the way we obtained bootstrap in Sec. V ensures that bootstrap conditions of any loop order present the constraints imposed on the input parameters by the type of perturbation scheme.

As we demonstrated in Sec. V, using the first kind bootstrap equations,[28] one can obtain well-defined expressions for, say, tree-level amplitude in three intersecting hyperlayers. These expressions obey all the restrictions imposed by fundamental postulates of quantum field theory. In principle, they can be analytically continued to every point of the space of kinematical variables without introducing any new parameters. However, there is no guarantee that this analytic continuation will not generate new singularities in addition to those already contained in relevant Feynman graphs. Actually, to provide such a guarantee is to satisfy the relevant (second kind) bootstrap equations. In principle, it may turn out that all the bootstrap equations for processes with $n \geq 4$ particles at all loop orders should be taken into account. In other words, the system of RP's (7.9), (7.10), and (7.11) is overdetermined and the question whether the (numerical) solution exists remains open. We do not discuss this, extremely complicated, problem. Instead, we guess that the solution of all these bootstrap equations does exist, so that one can consider every separate equation as a relation between physical observables. In other words, our results are only concerned with a part of necessary bootstrap conditions.

Perhaps, one more detail is noteworthy in connection with Eqs. (7.4) and (7.5): to extract the values of S-matrix elements from the full sum (7.3) of n-leg l-loop graphs, one needs to adjust the wave function normalization constants Z. As we already mentioned, these RP's are of nonminimal (off-shell) type which we do not discuss here. Nevertheless, we have to be sure that one does not need to consider graphs with nonminimal vertices to compute these constants, or, the same, that our resultant parametrization is consistent with these nonminimal RP's. In the next section it is argued that we need wave function renormalization for stable particles only, and in Appendix C we show that Z's for stable particles are not affected by nonminimal graphs.

VIII. LOCALIZABILITY AND THE FIELDS OF RESONANCES

The main problem we would like to solve (or, at least, to understand better) is that of constructing the field-theoretic perturbation scheme suitable for the case of strong coupling, which is closely connected to the renormalization of canonically nonrenormalizable theories. The results of our previous papers allow us to expect that the effective field theory concept will be fruitful for finding a solution.

In this section we discuss the philosophy underlying our approach—the concept of *localizable* effective scattering theory. Below we explain this term and qualitatively describe the relevant extended perturbation scheme which, in fact, was considered in previous sections. We do not claim to be rigorous here and just try to give an idea how our technique introduced in [1–4] can be matched with the general ideas of effective theory formalism.

It is pertinent to mention that the now most popular approach based on the classical Lagrangian and the canonical quantization procedure looks impracticable for Lagrangians containing arbitrary high powers (and orders) of time derivatives of fields: the weight induced in the functional integral leads to non-Hermitian terms in the effective Lagrangian and can be calculated only in relatively simple cases—see e.g. [19], or [6], Chap. 9.3, where relevant calculations are done for the nonlinear σ-model.

[28]They are not needed if the bounding polynomial degrees for the considered amplitude are negative, like, e.g., in the processes involving unstable particles.

That is why we rely upon the alternative, intrinsically quantum, approach proposed in [20] (see also [6]). In that approach the structure of Fock space of asymptotic states is postulated, and the free field operators are constructed in accordance with symmetry properties of these states. This fixes the free Hamiltonian structure. The interaction Hamiltonian is also postulated as an *ab initio* interaction picture operator built from those free fields and (if necessary) their derivatives.

Strictly speaking, there is no room for unstable particles in conventional understanding of this scheme. However, as we argue below, in the framework of effective theory it is convenient to introduce ''fictitious'' resonance fields in order to avoid problems with divergencies of perturbation series. In a sense, resonances resemble the ghosts widely known in modern gauge theories. Their fields do not create asymptotic states and only manifest themselves inside the S-matrix graphs for scattering of stable particles.

For simplicity, in this section we consider an effective theory which contains the only field π—that of massive pseudoscalar particle which we shall refer to as ''pion''.[29] So, the space of asymptotic states is created by the pion creation operator and the free Hamiltonian is just the free pion energy.

In accordance with the ideology of effective theories, the interaction Hamiltonian $\mathcal{H}_I^{\text{initial}}$ (in the interaction picture) is constructed from the free pion field and its derivatives; it contains all the terms consistent with Lorentz symmetry requirements. No limitation on the degree and order of time derivatives is implied. The perturbation scheme for calculating S-matrix elements is based on the famous Dyson formula:

$$S = T_D \exp\left(-i \int d^4x \mathcal{H}_I^{\text{initial}}(x)\right),$$

where T_D stands for the so-called Dyson's T-product. By construction, such a theory is renormalizable just because all kinds of counterterms needed to absorb the divergencies of individual loop graphs are present in the Hamiltonian.

To avoid confusion, we should probably recall one important circumstance. To ensure Lorentz covariance of the above expression for the interaction containing field derivatives, one has to include certain noncovariant ''compensating'' terms in the interaction Hamiltonian. They cancel the influence of noncovariant propagator terms appearing due to noncovariance of Dyson's T-product. Fortunately, as one infers e.g. from the discussion in [6] (Chaps. 6.2 and 7.5), the noncovariant terms can be always thought dropped from both propagator and from interaction Hamiltonian. That is, we can construct the most general Lorentz-invariant amplitudes using the most general Lorentz-invariant interaction Hamiltonian in the Dyson formula written via the manifestly covariant Wick's T-product (see, e.g., [21]):

$$S = T_W \exp\left(-i \int d^4x \mathcal{H}_I^{\text{initial}}(x)\right), \tag{8.1}$$

so that there is no need to take account of any noncovariant terms.

As mentioned in Sec. III, the main problem reveals itself when one uses (8.1) for computing the S-matrix elements: the expressions turn out to be purely formal. Namely, the interaction Hamiltonian contains an infinite number of items with an unlimited number of field derivatives (of arbitrary high degree and order), so that already at tree level the amplitudes are represented by infinite functional series. As long as we have no guiding principle to fix the order of summation, we can say nothing about the sums of such series. Likewise, the higher loop orders are ill-defined. Therefore it looks natural to single out the special class of localizable effective theories consistent with certain summability conditions.

Our requirement of localizability can be understood as a system of restrictions for the Hamiltonian coupling constants. However, as will become clear from the discussion below, there is no need in explicit form of those restrictions. Instead, the results of [1] allow one to put them in a form of bootstrap equations for resultant parameters.

To have an idea what localizability is, let us consider the tree-level amplitude of the elastic ''pion-pion'' scattering. At tree level the relevant part $\mathcal{H}_{\pi\pi}^{\text{tree}}$ of the effective interaction Hamiltonian $\mathcal{H}_I^{\text{initial}}$ appears as follows:

$$\begin{aligned} \mathcal{H}_{\pi\pi}^{\text{tree}} &= \sum_{i,j=0}^{\infty} G_{ij} \pi (D^{\mu_i}\pi)(D^{\nu_j}\pi)(D_{\mu_i} D_{\nu_j}\pi), \\ D_{\mu_i} &\equiv \partial_{\mu_1} \ldots \partial_{\mu_i}, \end{aligned} \tag{8.2}$$

where G_{ij} stand for the Hamiltonian coupling constants. It follows from (8.2) that the tree-level amplitude can be formally presented as a double series in each pair of independent Mandelstam variables:

$$\begin{aligned} A(s,t,u) &= \sum_{i,j} \alpha_{ij} (s - s_u)^i (t - t_u)^j \\ &= \sum_{i,j} \beta_{ij} (t - t_s)^i (u - u_s)^j \\ &= \sum_{i,j} \gamma_{ij} (u - u_t)^i (s - s_t)^j, \end{aligned} \tag{8.3}$$

where α_{ij}, β_{ij}, γ_{ij}, s_u, t_u, t_s, u_s, u_t, and s_t are some constants which depend on the pion mass and couplings G_{ij}.

We make an assumption (or, better, require) that there is a domain $\mathcal{D}$ in the space of kinematical variables s, t, u, where at least one of the formal expressions (8.3) is well defined (convergent).

[29] Since we do not imply presence of any other symmetry but Lorentz invariance, this pion has no isospin.

For example, suppose that at fixed t the amplitude in the domain $\mathcal{D}$ is represented by the converging power series in s (take $s_u = 0$ for brevity):

$$A(s, t) = \sum_{i=0}^{\infty} a_i(t) s^i \tag{8.4}$$

with the coefficients a_i depending[30] on t. However, outside $\mathcal{D}$ the tree-level approximation loses its meaning, leave alone the higher loop order terms. Similar assumptions are made about tree-level amplitudes of all other processes (elastic and inelastic).

We see that in the domain $\mathcal{D}$ the tree-level amplitude is now defined by uniformly converging power series. As known, the convergency of a power series is limited by the singularities located on the border of the circle of convergency.[31] The initial power series cannot provide a satisfactory description of the function in the vicinity of singular points. However, if we have some additional information about these singularities, it may be possible to single out the singular terms and present our series in a form that permits the direct analytic continuation to a wider domain. On the other hand, this new form must also be consistent with conventional field-theoretic interpretation, otherwise the connection with basic principles may be lost.

The principles like crossing symmetry, unitarity, etc. are saved, if one finds a kind of ''auxiliary interaction Hamiltonian'' $\mathcal{Q}(x)$ which, when inserted in (8.1), produces the tree-level series with the following properties. First, for every process in initial theory, in every area $\mathcal{D}$, where tree-level series given by (true) Hamiltonian $\mathcal{H}_I^{\text{initial}}$ converges, the relevant series produced by $\mathcal{Q}$ shall also converge and result in the same function (amplitude)—this is just to be consistent with the initial theory. Second, these new series shall converge in wider domains thus providing the analytic continuation of the initial amplitudes. Once this ''auxiliary Hamiltonian'' results in the tree-level series that converge almost everywhere in the complex space of relevant variables, the construction of the higher loop order amplitudes is reduced to the singular integration (in fact, one will need uniform convergence on compacts, excluding arbitrarily small vicinities of singularities).

Now, if a series of analytic functions (which converges to an analytic sum) has an ''isolated'' point of divergence (a singular point, surrounded by convergence area), then at least some items of the series must also be singular in that point—that is what complex analysis tells us. Roughly speaking, ''inside'' a domain of convergence functional series may diverge only in singular points of its items. So, the ''border'' singularities of the initial tree-level amplitudes should be reproduced by singular terms in the tree-level series generated by $\mathcal{Q}$. In fact, this is the reason for the summability requirement formulated in Sec. III. Next, the general features of the field-theoretic formalism reduce severely the types of singularities in tree-level series: only simple poles are allowed. Thereby we also have to assume (this is one of implicit restrictions that single out the class of localizable theories) the initial Hamiltonian to be constructed in a way that all the singularities of tree-level amplitudes in relevant variable are just simple poles which can be interpreted as a result of particle exchange in appropriate channel. In other words, every singularity (pole) shall come from a propagator of a (auxiliary) particle with spin J and mass M appearing in the relevant exchange graph of the tree-level series generated by the ''auxiliary Hamiltonian'' $\mathcal{Q}$. Accepting the latter requirement together with summability principle for loop amplitudes, we can develop the extended perturbation scheme based on $\mathcal{Q}$, rather than $\mathcal{H}_I^{\text{initial}}$.

The set of assumptions (the localizability hypothesis) just declared is nothing but an attempt to develop a perturbation scheme in a spirit of famous quasiparticle method [22]. In this approach the divergencies of the initial Born series are cured by introducing a new state (quasiparticle) in such a way that the corresponding singularity (pole) appears as a single item in the modified (reconstructed) perturbation series. This method is widely used in the nonrelativistic many body problem for potentials that are too strong to permit the use of perturbation theory over a certain energy scale. In this approach the fictitious elementary particles are introduced into the Hamiltonian in correspondence with the spectrum of bound states of the potential. The potential is also modified, so that in the domain where the initial Born series converges, the extended one gives the same transition probabilities. The modified potential is weaker, since it does not produce those bound states which appear now as elementary particles. As a result, the modified potential may be sufficiently weak to allow perturbation methods in a broader energy domain.

Shortly, the general idea may be formulated as follows: the tree-level amplitude produced by the extended perturbation scheme (with auxiliary Hamiltonian $\mathcal{Q}$) must realize an analytic continuation of the tree-level amplitude obtained in the initial perturbation scheme to a wider domain. To ensure this requirement, all the singularities of the initial tree-level series must be reproduced as separate items of the new series following from the extended perturbation scheme. Thus the extended perturbation scheme presents just an auxiliary construction only needed to properly define the tree-level series of the initial effective theory in a wider domain. Once the tree-level amplitude is defined in the whole space of variables, the

[30]The continuity in t is implied at least in a small interval. Note also that $\mathcal{D}$ does not necessarily belong to the physical area of the reaction.

[31]The radius of this circle should be finite: otherwise the series would result in a polynomially unbounded function not allowed by the general axiomatic requirements (see, e.g., [14]).

calculation of higher orders of Dyson series becomes just a matter of machinery.

Of course, the feasibility of such an extension strongly depends on the structure of the initial Hamiltonian. Simply speaking, localizable Hamiltonians are those for which the described above procedure is possible. We cannot formulate any sufficient condition for existence of such a Hamiltonian, but we can investigate the consequences.

It is clear that in the field theory such a modified perturbation scheme should take account of the interaction of unstable[32] particles (resonances). It is in this connection that the problem of the field-theoretic description of unstable particles in a finite order of perturbation theory becomes crucial in our approach. Let us look closer at how they arise.

The only known way to ensure the consistency of the S-matrix defined by (8.1) with Lorentz covariance requirements and cluster decomposition principle is to construct the interaction Hamiltonian density out of free causal quantum fields (see e.g. [6]). The free causal field transforms according to a certain representation of inhomogeneous Lorentz group and coefficients of its Fourier transform (creation and annihilation operators) satisfy standard commutation relations which, in turn, uniquely define the corresponding propagator (or, rather, its pole part). Suppose there is such a field $\psi_R(x)$, which does not correspond to any asymptotic state of the initial effective theory: the action of its creation operator on the Fock vacuum does not form an eigenstate of the free Hamiltonian[33] $\mathcal{H}_0$. Let this field transform according to an irreducible representation of the inhomogeneous Lorentz group with mass M_R, spin J, and internal quantum numbers of certain "$\pi\pi$" channel. Let us formally add to the interaction Hamiltonian $\mathcal{H}_I^{\text{initial}}$ all possible terms constructed from ψ_R (and its derivatives) and from the pion fields describing the true asymptotic states of the initial theory, while all the coupling constants at these terms are treated as free parameters. The obtained operator[34] appears as follows:

$$\mathcal{Q}(x) = \mathcal{H}_I^{\text{initial}}(x) + \mathcal{H}_I^{\text{extra}}(x).$$

Note also that the free Hamiltonian structure remains unchanged. Consider now the extended perturbation scheme, in which the operator $\mathcal{Q}(x)$ replaces the Hamiltonian in the matrix elements of (8.1) between the asymptotic states of the initial effective theory (containing only "pions"). Hence, in addition to diagrams of the initial theory there will appear diagrams with internal exchanges by (fictitious) particles ψ_R. It is crucial for us that the new tree-level amplitude of $\pi\pi$ scattering will have an explicit pole at $s = M_R^2$ (as well as, of course, the poles in t and u). Now, to require the localizability is to require that the new tree-level amplitude with explicit pole part is equal to the initial tree-level amplitude (8.4) in the domain $\mathcal{D}$:

$$A(s,t) = \sum_{i=0}^{\infty} b_i(t)s^i + \frac{r(t)}{s - M_{R_s}^2} = \sum_{i=0}^{\infty} a_i(t)s^i \quad \text{for } (s,t) \in \mathcal{D}, \tag{8.5}$$

but the new series is valid in a wider domain.[35] This equation can be interpreted as a system of matching conditions for the residue $r(t)$ and the coefficients $b_i(t)$ of the new series, which, in turn, can be expressed through the coupling constants of the new "Hamiltonian" $\mathcal{Q}$ and the quantum numbers of field ψ_R.

In principle, one can imagine a certain step-by-step extension of perturbation scheme with the resonance fields introduced in accordance with singularities that limit the convergency of the initial tree-level series for amplitudes. These steps should be repeated until the tree-level amplitude is defined everywhere in the space of kinematical variables. However, there is another way which looks much more promising. One can consider the most general extended perturbation scheme containing an arbitrary number of resonances of arbitrary high spin and mass and then analyze the general structure of such a scheme.[36] This is precisely the way which we follow in our studies.

Employing a few additional principles (summability and uniformity, see Sec. III), we have shown that the structure of such a perturbation scheme is far from being arbitrary. The minimal parametrization described in [1] proved to be especially convenient for the classification of coupling constants appearing in calculations. Besides, as shown in Secs. V, VI, and VII (for preliminary discussion see [4]), the necessary conditions of self-consistency of the tree-level approximation impose strong limitations on the possible set of the parameters of extended perturbation scheme. Altogether, this gives a hope that the concept of localizable effective theory may prove to be fruitful in studies of the resonance physics.

Let us make a summary, just repeating what has been said about the localizability. The effective theory is called localizable if: (i) its formal series for all tree-level amplitudes converge at certain small domains in relevant spaces

[32]Recall that, by construction, all the stable particle states are already taken into account. This means that the corresponding poles are present in tree-level amplitudes.

[33]Moreover, the resulting vector does not belong to the initial Fock space. This will not bother us because we never need to deal with such vectors: we only need the propagators of auxiliary fields.

[34]In our previous papers we called it "extended Hamiltonian," which is not quite correct. The point is that this operator cannot be obtained as a result of transition to the interaction picture of the Heisenberg picture Hamiltonian describing pions. Nevertheless, in what follows we sometimes use the term "extended Hamiltonian" just for the sake of brevity.

[35]For definiteness we assumed that it is the $s = M_R^2$ pole lying at the border of the convergency circle of the initial series (8.4).

[36]A similar ideology is used, for example, in [12] in perturbation theory for nonlinear oscillations.

of kinematical variables, and (ii) in those domains they can be reproduced by (convergent in wider domains) tree-level series of the extended perturbation scheme. This scheme contains auxiliary resonance fields with spins J_i and "masses" [37] M_i which do not belong to the spectrum of one-particle asymptotic states. The auxiliary fields may be interpreted as those describing particles which are unstable with respect to decays into the true asymptotic states of initial theory.

One of the immediate consequences of this is that in the extended perturbation scheme the only S-matrix elements one needs to compute are those between the true asymptotic states.[38] The normalization condition remains the same as that in the initial effective theory and the unit operator $\hat{\mathbf{1}}$ in the Fock space takes a form of infinite sum over the states only containing stable particles:

$$\hat{\mathbf{1}} = \sum_{n=0}^{\infty} |n(\text{stable!})\rangle\langle n(\text{stable!})|.$$

Therefore the only RP's needed to provide the correct normalization of the wave functions are those for the self-energy graphs of stable particles (pions in the case in question). As mentioned in [1], the computation of relevant counterterms does not require operating with nonminimal vertices. The proof of the latter statement is given in Appendix C.

The last, but very important remark to be made here is that explaining our usage of term "strong coupling" or "strong interaction." Let us call the interaction described by certain localizable effective theory as strong, if the number of steps needed for complete localization is actually infinite. To put it another way, in the case of strong coupling one needs an infinite number of resonances to construct the extended perturbation scheme: it is not possible to point out the upper limit for M_i. Surely, we imply that the number of resonances falling into arbitrary finite energy interval is finite, so that the tree-level amplitudes of strong processes belong to the class of meromorphic functions. In contrast, the theories which require just a finite number of localization steps produce the tree-level amplitudes expressed by the rational functions having a finite number of poles. This latter interaction type we may call weak. The immediate example of such a theory is the celebrated electroweak model considered at energies below the mass of Higgs or Z-boson (depending on which one is taken to be lighter).

The important physical difference between strong and weak interactions is that in the strong case the degree of the bounding polynomial for an amplitude depends on the hyperlayer under consideration. It is this kind of amplitudes that our technique is developed to handle. On the other hand, it is easy to understand that in a theory with a finite number of resonances (weak case) the amplitude asymptotics is given by the highest degree of kinematical variable present, and, hence, does not depend on the layer. So, perhaps, the most interesting information which can be extracted from hadron experiments is that allowing to fix the high-energy asymptotics of the amplitudes of exclusive processes at various fixed values of the other kinematical variables.

IX. ON THE PARAMETRIZATION OF RESONANCE

The way resultant parameters were introduced in [1] and the way we impose renormalization prescriptions in Sec. VII assume the use of renormalized perturbation theory with on-mass-shell renormalization conditions. So, when we speak about the stable particles (like e.g. pion or nucleon), the mass is the quantum number of the corresponding asymptotic state, and it is the same quantity that appears in the Feynman propagator. In contrast, the terms mass and width often used to describe a resonance are not so well defined. Here we shall trace their relation to the parameters used in our approach.

The most important fact proven in [1] is that every N-loop S-matrix element can be represented as a sum of graphs constructed from resultant vertices. The proof implies that each amplitude is calculated precisely at a given loop order and no kind of partial resummation (like, e.g., the Dyson resummation of the propagator) is allowed. Hence, at any finite loop order, every pole of the amplitude is located at real values of the momentum squared. This means that the customary Breit-Wigner description of resonances (mass and width as real and imaginary parts of the pole position, respectively) loses its meaning in terms of the finite level resultant parameters.[39] This creates a problem when one needs to compare the (finite loop order) theoretical expression with experimental data.

One of the ways to circumvent this difficulty is to exclude the small vicinities of resonances from the data analysis. Technically, this is just a problem of appropriate organization of the fitting procedure; it can be easily solved. In contrast, from the purely theoretical viewpoint there is a problem of suitable definition for the parameters describing a resonance. We shall stress that the latter may only make sense in the framework of a particular perturbation scheme. If we could construct the complete nonperturbative expressions for S-matrix elements we would never need to use this term.

As explained in Sec. VII, throughout the article the term "physical mass" refers to the parameter M_i in the denomi-

[37] Resonance masses are discussed in Sec. IX below.

[38] In this respect the philosophy of the approach based on the extended perturbation scheme is quite consistent with that developed in [23].

[39] Actually, this is just a consequence of the fact that the notion of resonance is ill-defined in the framework of perturbative quantum field theory.

nator of the (stable or unstable) particle propagator. The conventional term width is closely related to our definition (7.8) of ''physical coupling constants.'' That is, when fitting data with the finite loop order amplitude, one obtains the values of relevant physical couplings which, in turn, can be used to compute (formally) the decay amplitudes and, hence, the resonance widths. Therefore, in principle, it is possible to avoid using the term width by operating only with the values of physical coupling constants and mass parameters.

However, when comparing our bootstrap equations with experiment we are forced to use the numbers quoted, say, in [24]. Those numbers are obtained by data fitting with amplitudes constructed in terms of complex poles and smooth background. For example, in [24] the resonance characteristics are given in terms of mass and width, which are the parameters of the T-matrix poles at an unphysical sheet of the complex energy plane. In many other sources the data on resonances are quoted in terms of Breit-Wigner or K-matrix parameters. One should realize that these phenomenological constants never appear in the finite loop order expressions for field-theoretic perturbative amplitudes produced by the formula (8.1). Moreover, there is no guarantee at all that the partial sums of Dyson series (or any other partial summation) can provide a reasonable sequence of approximants for the amplitude at complex values of kinematical variables: this is a purely mathematical problem still awaiting a solution. All this means that the most reliable way to verify predictions of field-theoretic models is to fit data with the parameters that appear in field-theoretic formulas. The use of any other kind of parametrization (say, Breit-Wigner plus background) may lead to uncontrollable errors. In particular, there is no direct correspondence between the Breit-Wigner resonance parameters (mass and width as real and imaginary parts of the pole position) and field-theoretic ones discussed above (mass as the Feynman propagator pole position and width as the value of corresponding ''decay amplitude'' calculated in terms of physical coupling constants at a given loop order).

Nevertheless, the use of numerical data from [24] for approximate checking of selected bootstrap equations (see [7,8]) looks quite justified, at least for rough estimates. Indeed, for well-separated narrow resonance the different methods of parametrization (including those based on field theory models operating directly with coupling constants) result in approximately the same values of masses and couplings. However, this is not true for broad resonances like famous light scalars.[40] In this latter case the values of mass and coupling constant extracted from the same data set with the help of different theoretical amplitudes may differ considerably. To avoid inconsistencies, in [7,8] we only rely upon the data [24] as long as considered bootstrap equations are well saturated by a set of relatively narrow resonances, while the remaining equations are used to make certain estimates and predictions.

X. EFFECTIVE THEORY VERSUS ANALYTIC *S*-MATRIX

Wide use of complex analysis and some terminological analogies may turn the reader to think that the physical ideas underlying our approach are similar to those widely known as analytic S-matrix philosophy (see, e.g. [18]). It is not so, and here we discuss the main differences.

First, the analytic S-matrix approach (ASM) is based on the idea of nuclear democracy: all the particles (together with all their composites like nuclei) are considered on the same footing. For this reason the methods based on the Hamiltonian (Lagrangian) were rejected by ASM ideologists. In fact, neither Fock space nor Dyson's formula for the S-matrix play a role there. In contrast, our scheme uses the field-theoretic (Dyson's) construction of S-matrix as the operator acting on the Fock space of true asymptotic states. These states are created by field operators that describe just ''aristocratic''—stable species of particles (pions and nucleons in SU_2 sector). It is implied that the quantum numbers (mass, spin, etc.) of these particles can be explained by some underlying fundamental theory (say, QCD). For us those numbers are just external parameters which (perhaps, along with few another ones) should be taken from experiment. These and only these particles (or, better, the corresponding operators) appear in the free Hamiltonian.[41] So, from the very beginning we do not consider resonances on the same ground as stable particles. In our approach the resonance fields only appear in the structure of the extended interaction Hamiltonian $\mathcal{Q}$: they are nothing but a convenient tool for presenting the analytic continuation of tree-level amplitudes in the form consistent with Dyson's formula (8.1). In other words, in our approach the free Hamiltonian is assumed to have the same spectrum as that of the full one. It is this feature which gives us a hope to develop the efficient field-theoretic perturbation scheme.

The main reason why we prefer to deal with S-matrix elements and not with Green functions is that in the former case we can point out the full set of numerical parameters that appear in the process of perturbative calculations (resultant parameters). Besides, as stressed in [26], the essential parameters (which can be built of resultant ones) play a special role in canonically nonrenormalizable theories. It is interesting to note that our system of RP's (7.9), (7.10), and (7.11) looks precisely like that conven-

[40] An excellent discussion of this point is presented in the series of papers [25].

[41] This statement is not quite correct. Along with pion and nucleon terms one should add to free Hamiltonian also the items that correspond to their stable composites (nuclei). For now we just close our eyes to the existence of such objects.

tionally used in ordinary renormalizable theories (like, say, ϕ^3), though, of course, in our case the number of field species is infinite. This result checks well with Weinberg's conjecture made in Ref. [26].

In fact, all the other distinctive features of the approach based on the effective theory concept are just consequences of Dyson's formal construction of the S-matrix. The localizability, as well as uniformity and summability requirements (Secs. III and VIII), are only needed to handle individual terms of Dyson series in mathematically correct way. These requirements allow one, first, to single out the set of essential parameters and, second, to put this set in order of increasing loop level. At last, the natural self-consistency conditions (that mirroring the requirement of crossing symmetry at a given loop order) make it possible to understand the origin of regularity known as bootstrap restrictions.[42]

One of the most important technical differences between the philosophy of ASM and the effective field theory approach is that in the former case the coupling constants at a given vertex are considered independent of the loop order under consideration. As we have shown, this is not so in the framework of effective theory. Instead, it is the dependence of coupling constants on the loop order that—as shown in Secs. VI and VII—makes it possible to write down the explicit form of the lowest (tree) level bootstrap conditions for RP's. The fact that these conditions restrict physical parameters is a direct consequence of the resultant parametrization, which implies the use of renormalized perturbation theory. So, while we never know how many loops in the Dyson series should be considered to get reasonable coincidence with experimental amplitude, the bootstrap equations, when written in the form (5.8), are exact equalities at any loop order, though, of course, in numerical tests one is limited by the set of established resonances and experimental precision.

XI. CONCLUSION

At first glance, the concept of effective theory might seem too general to be of practical use in computing the amplitudes of strong processes. However, this is not quite true. The famous chiral perturbation theory (see [5,27]) provides an example that disproves this opinion. Unfortunately, the problem of infiniteness of the required set of renormalization prescriptions, the inseparable feature of nonrenormalizable theories, remains a stumbling block for this approach. The above-discussed concept of localizable effective scattering theory provides a way to a solution. The localization procedure along with subsequent reduction of all the lines of relevant graphs allow one to single out the well-ordered (though still infinite) subset of minimal RP's needed for complete renormalization of S-matrix elements. In physically interesting cases this subset only contains the RP's for 1-, 2-, 3-, and, possibly, 4-leg minimal vertices. Moreover, as shown above, the prescriptions collected in this set cannot be chosen arbitrarily because they must fulfil the infinite number of bootstrap conditions. So, the question on the full number of independent prescriptions takes on great importance. Unfortunately, it still remains unanswered. At the moment we only can say that the number of independent RP's needed for complete renormalization of a given effective scattering theory equals that of solutions of the full system of the second kind of bootstrap conditions. In the next paper we will show that the set of independent non-minimal prescriptions is governed by similar bootstrap equations.

In this connection it is pertinent to stress that Gross's note [28] on the essential infiniteness of independent solutions of the bootstrap constraints only relates to the case of finite-component theories. According to the definition in Sec. VIII, the theories of such a kind correspond to weak interaction. They cannot describe the situation when the asymptotic behavior (more precisely, the bounding polynomial degree) of an amplitude depends on the momentum transfer. One of the most important properties of strong interaction is that the degrees of bounding polynomials (which characterize the high-energy behavior) are strongly correlated with the values of remaining kinematical variables. Perhaps, it is this correlation which could provide a key idea on the general structure of the full system of bootstrap conditions and, hence, to elucidate the question on its solvability and the full number of solutions. This is a problem for the future investigations. As to the results obtained thus far, it seems to us most important to carry out the systematic approximate comparison of known experimental data with the second kind of bootstrap conditions for the parameters of resonances appearing in concrete elastic scattering processes. This will be done in two separate publications on pion-nucleon and kaon-nucleon reactions.

ACKNOWLEDGMENTS

We are grateful to V. Cheianov, H. Nielsen, P. Osland, S. Paston, J. Schechter, A. Vasiliev, and especially to M. Vyazovski for stimulating discussions. The work was supported in part by INTAS (project 587, 2000) and by Ministry of Education of Russia (Programme ''Universities of Russia''). The work by A. Vereshagin was supported by L. Meltzers Høyskolefond (Student-prosjektstipend, 2004).

APPENDIX A: THE CAUCHY FORMS

The existence of the Cauchy form for a meromorphic (no cuts, only poles) polynomially bounded function is a

[42] As far as we know, the explicit form of those restrictions or, at least, of their part has never been obtained in the framework of the ASM approach.

special case of the famous Mittag-Leffler theorem valid for any meromorphic function of one complex variable [29]. We used this representation in a way adjusted for two variables when working with tree-level amplitudes in [3]. To have a feeling of how this technique works with concrete mathematical examples, one is referred to [3], Sec. 4, where it is shown that the commonly accepted interpretation of so-called "duality hypothesis"[43] is not correct even in the case of the Veneziano model [31]. The bootstrap equations for that model obtained with the help of Cauchy forms are further explored in [4]. The Cauchy form (or, the same, Cauchy series) can be easily derived from Eq. (4.2) of Sec. IV; that is why we found it natural to discuss them here.

Consider the situation when all the singularities $s_k(\mathbf{x})$ of the N-bounded function $f(z, \mathbf{x})$ investigated in that section are just simple poles. Hence, the cuts are not needed anymore and all the contours C_k in Fig. 3 are transformed into small circles around the poles. Let us denote the residue at the pole s_k as $r_k(\mathbf{x})$. Using the residue theorem, the integrals on the right side of Eq. (4.2) are easily performed and the expression takes the following form:

$$f(z,\mathbf{x}) = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(0,\mathbf{x}) z^n + \sum_{k=-\infty}^{+\infty} \left\{ \frac{r_k(\mathbf{x})}{z - s_k(\mathbf{x})} - h_k^N(\mathbf{x}, z) \right\}, \tag{A1}$$

where the correcting polynomials

$$h_k^N(z,\mathbf{x}) \equiv -\frac{r_k(\mathbf{x})}{s_k(\mathbf{x})} \sum_{n=0}^{N} \left[\frac{z}{s_k(\mathbf{x})} \right]^n \tag{A2}$$

ensure the convergence of resulting series. We have stressed in Sec. IV that one should sum the items in this series in order of increasing $|s_k|$. As long as this rule is kept, the series converge uniformly in the layer B_x. Once this rule is violated or correcting polynomials are dropped, the convergence may be lost[44] and the series loses its meaning. This is, of course, governed by the estimate (4.1).

The relation (A1) is precisely what we call the Cauchy form (series) for the function $f(z, \mathbf{x})$. In [3] we called the first term on the right-hand side of (A1) the external part, while the first terms in curly brackets are conventionally called the principal parts of the function.

When N becomes negative, both the correcting polynomial and external part disappear, and the Cauchy form is reduced to the simple sum of principal parts—the pole terms. More generally, it is easy to show that for the N-bounded in the layer B_x function presented by (A1) certain "collapsing" conditions are valid: the correcting polynomial degrees higher than N converge separately to the values of corresponding derivatives:

$$\sum_{k=-\infty}^{+\infty} \frac{r_k(\mathbf{x})}{[s_k(\mathbf{x})]^{N'+1}} = \frac{1}{N'!} f^{(N')}(0,\mathbf{x}), \qquad N' = N+1, N+2, \ldots. \tag{A3}$$

To put it another way, suppose that one uses some higher degree $N' > N$ in Eq. (A1). Since estimate (4.1) holds for N', the Cauchy form certainly converges but can be reduced to one with true degree N—the superfluous degrees of the external part and correcting polynomials just cancel. So, roughly speaking, the appearance of Cauchy's form is strongly correlated to asymptotics.

In the case of physical interest the function f represents the tree-level amplitude of a process, while z and x play a role of kinematical variables. At first glance at Eq. (A1) it is tempting to separate the correcting polynomials from contribution of the poles, sum up two resulting series independently, and combine the sum of correcting polynomials with the external part. This could give a polynomial in z plus a sum of poles. However, this is not correct until the amplitude f possesses the decreasing asymptotics $N < 0$. Instead, as long as Eq. (4.1) does not hold for negative N, the sum of poles and the sum of correcting polynomials taken separately diverge,[45] and the resummation procedure is illegal. Therefore, in the case $N \geq 0$, there is a crucial difference between approximating some amplitude by a polynomial plus a finite sum of pole terms (as it is often done when the amplitude is saturated by resonances), and approximating it by the finite sum of terms of the corresponding Cauchy form. The first way is nothing but an attempt to approximate by the first few terms of the divergent series and balance the situation by adjusting the polynomial. Therefore, as it should be with the divergent series, every next pole term taken into account forces one to change the polynomial significantly. This leads to instability of approximation. In contrast, this does not happen if one uses first terms of the relevant Cauchy series with correct asymptotics, because these series converge by construction.

APPENDIX B: THE CASE OF NONDECREASING ASYMPTOTICS

For completeness, here we consider the situation when the process $2 \to 2$ is described by the amplitude (or, more precisely, scalar form factor) characterized by non-

[43] Namely, it is widely believed that to calculate the direct channel amplitude one has to drop out the contributions of the pointlike graphs ("background") and of the graphs with the cross-channel poles, see e.g. [30].

[44] There are exceptions (see the next footnote).

[45] Except the trivial case when poles do not take part in forming the nondecreasing asymptotics which, thus, turns out to be caused by the properties of the external part. Then the infinite series of poles converges independently, as well as the series of correcting polynomials. Such a situation does not present any interest for constructing the effective theory of strong interactions because, as shown in [3], it does not occur even in the Veneziano model.

negative degrees of the bounding polynomials in each of the three layers: B_s, B_t, and B_u (see Fig. 4), though, as we have noted in Sec. VII, such a process (with the hadrons stable w.r.t. strong decays) is forbidden by known phenomenology. As mentioned in Sec. V, in addition to the renormalization prescriptions for 3-leg vertices, we will also need here RP's for the 4-leg one. What is significant is that in this case to ascertain the number (and the form) of required RP's we will need to employ the bootstrap conditions of the second kind.

To be specific, let us take $N_t = N_u = 0$ (we use the same notations as those in Sec. V). Hence, in the layer B_t the amplitude M of the process can be presented as

$$M|_{B_t} = f(t) + F(t, \nu_t), \tag{B1}$$

where $F(t, \nu_t)$ is an infinite sum of contour integrals appearing in (5.3) and $f(t)$ is an unknown function only depending on t. Similarly, for the same amplitude in B_u, we have

$$M|_{B_u} = \phi(u) + \Phi(u, \nu_u), \tag{B2}$$

with another unknown function $\phi(u)$ (only depending on u) and an analogous sum of contour integrals $\Phi(u, \nu_u)$. Recall that each of the above expressions is well defined (convergent) only in the corresponding hyperlayer. Both forms (B1) and (B2) are written for the same amplitude, therefore in $D_s \equiv B_t \cap B_u$ they must identically coincide with one another:

$$f(t) + W(t, u) = \phi(u), \qquad (t, u) \in D_s, \tag{B3}$$

where

$$W(t, u) \equiv F(t, \nu_t) - \Phi(u, \nu_u).$$

The self-consistency requirement (B3) provides us with a source of an infinite system of bootstrap conditions.

Let us first derive the conditions of the second kind. From (B3) we obtain

$$\partial_t \partial_u W(t, u) \equiv 0, \qquad (t, u) \in D_s. \tag{B4}$$

According to the results of Sec. VI, both series $F(t, \nu_t)$ and $\Phi(u, \nu_u)$ should be considered as known functions completely determined by the resultant parameters fixed on the previous steps of renormalization procedure. Then in D_s their difference $W(t, u)$ is also a well-defined known function[46] of t and u. Hence, the subsystem (B4) restricts the allowed values of those fixed resultant parameters or, the same, restricts the allowed values of the relevant RP's.

Now, differentiating Eq. (B3) separately w.r.t t or u, we get

$$\partial_t f(t) + \partial_t W(t, u) = 0, \qquad \partial_u \phi(u) - \partial_u W(t, u) = 0,$$

which, when solved, gives

$$\begin{aligned} f(t) &= -W(t, u) + W(t_0, u) + f(t_0), \\ \phi(u) &= W(t, u) - W(t, u_0) + \phi(u_0), \end{aligned} \tag{B5}$$

for any $(t_0, u_0) \in D_s$. Substituting these relations into (B3) one obtains

$$f(t_0) = [W(t, u) - W(t, u_0) - W(t_0, u)] + \phi(u_0). \tag{B6}$$

On the other hand, again from (B3), we know that

$$f(t_0) = -W(t_0, u_0) + \phi(u_0). \tag{B7}$$

Combining (B6) with (B7), we obtain the universal form of the subsystem of second kind bootstrap conditions:

$$\begin{aligned} &W(t, u) = W(t, u_0) + W(t_0, u) - W(t_0, u_0), \\ &\text{for } (t, u) \in D_s, (t_0, u_0) \in D_s, \end{aligned} \tag{B8}$$

which could also be derived directly from (B4). The forms of this type always appear during analysis of the bootstrap conditions in the intersection of two hyperlayers.

We can now answer the question on how many RP's are needed to fix the amplitude under consideration. From the relations (B5) it follows that both functions $f(t)$ and $\phi(u)$ are known if we specify the values $f(t_0)$ and $\phi(u_0)$. Further, (B7) tells us that it is enough to know one of the latter constants. Hence, we need to add only one additional RP to the system (7.9), (7.10), and (7.11). It is natural to fix the amplitude value at one point arbitrary chosen either in B_t or in B_u. For example, one may do it in B_u as follows:

$$M|_{u=0;\nu_u=0} = g_{00},$$

where g_{00} may be considered as a parameter given, say, by fit with experimental data. The latter equality, in principle, can be understood as an indirect fixing of corresponding 4-leg resultant vertex in that point of variable space. However, it is not easy to trace the contribution of that vertex to the series (B2), that is why an explicit fixing may be tricky and inconvenient for calculations.

We conclude that, in the case of ''constant asymptotics'' in both intersecting layers ($N_t = N_u = 0$), one needs to add only one new renormalization prescription for 4-leg amplitude, in addition to those for 3-leg couplings considered in Sec. VI. Note also that it may happen convenient to fix an amplitude value in some chosen point, as we just did, and not directly the value of relevant resultant 4-leg vertex.

By the same method, one can examine an amplitude with arbitrary high asymptotic in both layers. In particular, for the situations ($N_t = 1, N_u = 0$) and ($N_t = 1, N_u = 1$), one will need to fix two and three numbers, respectively. The corresponding RP's can be written, say, as follows:

($N_t = 1, N_u = 0$):

$$M|_{u=0;\nu_u=0} = g_{00}, \qquad \partial_u M|_{u=0;\nu_u=0} = g_{10};$$

[46]Using the equality $s + t + u = \sum_{i=1}^{4} m_i^2$ (m_i's are the masses of external particles), express $\nu_t \equiv s - u$ and $\nu_u \equiv t - s$ via t and u.

$(N_t = 1, N_u = 1)$:

$$M|_{u=0;\nu_u=0} = g_{00}, \qquad \partial_u M|_{u=0;\nu_u=0} = g_{10},$$
$$\partial_u \partial_{\nu_u} M|_{u=0;\nu_u=0} = g_{11}.$$

APPENDIX C: SELF-ENERGY GRAPHS

In [1] very little was said about the wave function (field-strength) renormalization. In this appendix we explain why the computation of the wave function renormalization constants does not require operating on nonminimal graphs.[47] To put it another way, when calculating the S-matrix one can simply neglect the residual graphs with nonminimal external lines that certainly remain after the reduction described in [1].

The problem is that this reduction produces graphs of two different kinds. All vertices of the first kind graphs are minimal w.r.t. internal and (if connected to) external lines. Therefore these graphs depend only on minimal parameters: Fig. 6 (I). In contrast, at least one vertex connected with external line of a second kind graph is nonminimal: Fig. 6 (II). All the internal vertices are minimal in both cases, and, according to the footnote in Sec. II, all the propagators are always taken minimal. Surely, every graph of the second kind vanishes when viewed as a part of an S-matrix element because the latter is always computed on the mass shell. However, it is not a good reason to drop these graphs out of consideration: their parameters may still contribute to the wave function renormalization constants. It may seem then that to take account of this effect one needs to calculate the second kind graphs and, hence, to deal with nonminimal parameters. In other words, the renormalization program discussed in Secs. V and VI fails. Fortunately, this is not the case. The reason was pointed out in [1]; here we only would like to discuss details.

As argued in Sec. VIII, in the extended perturbation scheme one never needs to compute S-matrix elements between the states with unstable particles as the Fock space is created by free field operators of the stable particles only. Technically it means that there is no need in the "wave function" renormalization for resonances and only field-strength renormalization constants for true asymptotic states may be of interest. Therefore we can safely refer to the conventional derivation of the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula.[48]

The field-strength renormalization is caused by 1PI self-energy (2-leg) insertions in external lines of graphs that contribute to a given scattering process. Suppose that the reduction is done and, hence, all the vertices are minimal

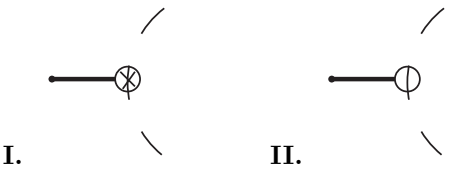


FIG. 6. Graph after reduction: some vertices, nonminimal w.r.t. corresponding external lines, may have remained (case "II"). The rest of the graph (dashed circle segment) contains only minimal vertices (we label them by crosses).

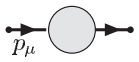
w.r.t. internal lines. The external line without insertions can be amputated and put on shell immediately, thus the relevant vertex can always be taken minimal. So, let us look at the external leg with self-energy insertions. According to [1], we can then distinguish two possibilities: Fig. 7. The first one suits us, since relevant 2-leg graph contains only minimal vertices (**m**-type graphs). The second is uncomfortable since the very left vertex (that without cross) is nonminimal (**n**-type graphs).

To see that the latter insertion does not affect the S-matrix, we recall that the only part of two-point function contributing to the LSZ formula is the residue in the one-particle exchange pole, proportional to the wave function renormalization constant Z (see e.g. [11], Chaps. 7.1–7.2). Hadronic phenomenology tells us that the highest spin we may need for the external (stable) particle is $\frac{1}{2}$. So let us for definiteness work with the nucleon propagator. The analysis below (we do not discuss mixing) can be easily adjusted for any spin; the scalar case is much simpler. The isospin structure is irrelevant for current discussion and, therefore, will be neglected.

For the spin-$\frac{1}{2}$ particle the (parity conserving) amputated 1PI two-point function $\Sigma(p)$ (that contains all the contributions of both **m** and **n** types) can always be expressed via matrices $\hat{\mathbf{1}}$ (the unit matrix) and $\not{p}$ as

$$\Sigma(p) = \sigma_1(\pi)\hat{\mathbf{1}} + \sigma_2(\pi)(\not{p} - m), \qquad \pi \equiv p^2 - m^2,$$

so that the full propagator takes the form



$$\equiv \frac{\not{p}+m}{\pi} + \frac{\not{p}+m}{\pi}\,\Sigma\,\frac{\not{p}+m}{\pi} + \ldots$$
$$= \frac{(\not{p}+m)(1-\sigma_2)+\sigma_1}{\pi(1-\sigma_2)^2 - 2m\sigma_1(1-\sigma_2) - \sigma_1^2}\,. \quad \text{(C1)}$$

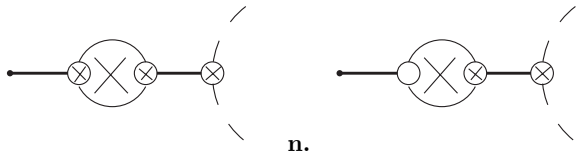


FIG. 7. Self-energy insertion in external leg of a reduced graph. Minimal vertices are indicated by crosses: only the very left vertex can be nonminimal (case "**n**").

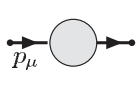
[47] As usual the absence of massless particles with spin $J \geq 1$ is implied.

[48] See, e.g. [6,11]; it should not be mixed with the reduction to minimal form (reduction procedure or reduction theorem [1]) that we discuss here.

To keep the pole position in the physical mass m we, therefore, need the prescription

$$\sigma_1(0) = 0, \tag{C2}$$

while σ_2 does not affect the mass shift. Further, once the condition (C2) holds, the full propagator near the pole is



$$= \frac{1}{1 - \sigma_2(0) - 2m\sigma_1'(0)} \frac{\not{p} + m}{\pi} + \dots , \tag{C3}$$

and the residue $Z_f = 1$ is provided by the prescription

$$\sigma_2(0) + 2m\sigma_1'(0) = 0. \tag{C4}$$

First of all, let us show that the considered nonminimal insertions do not affect the pole position of the propagator (C1). Indeed, for the spin-$\frac{1}{2}$ case the only nonminimal tensor (matrix) structure is $\not{p} - m$. Hence, by its very definition, a nonminimal (w.r.t. external nucleon line) vertex brings the extra factor π or $\not{p} - m$, so the contribution to Σ of the **n**-type graphs has the form

$$\Sigma^{\mathbf{n}} = \Sigma_{\mathrm{s}} \cdot \pi + \Sigma_{\mathrm{t}} \cdot (\not{p} - m).$$

Besides, it is generally accepted that Σ is regular near $p^2 = m^2$ (nucleon is massive and stable); at any finite loop order it is justified by the very procedure of loop calculation. Then, decomposing the (matrices) Σ_{s} and Σ_{t} through $\hat{\mathbf{1}}$ and $\not{p} \pm m$, one easily shows that the last expression can always be rewritten via some scalar (and regular in $\pi = 0$) functions σ_s and σ_t:

$$\Sigma^{\mathbf{n}} = \pi\sigma_{\mathrm{s}}(\pi)\hat{\mathbf{1}} + \sigma_{\mathrm{t}}(\pi)(\not{p} - m),$$

so that the only contribution it may give to σ_1 is proportional to π and, thereby, does not affect the condition (C2). In fact, only the minimal piece of the on-shell contribution of **m**-type graphs [cf. Eq. (7.3) and (7.4)] can affect $\sigma_1(0)$, and it is set to zero by the RP (7.10) in Sec. VII (there is no imaginary part as the particle is stable). That is, to keep the correct pole position one just imposes the RP (7.10) on the graphs with only minimal (resultant) vertices and there is no need in the corresponding RP's for graphs containing nonminimal vertices.

However, as seen from Eq. (C3), the nonminimal vertices do affect the residue as they may contribute to σ_1' and σ_2. Here we shall follow another logic. Separating the contributions of **m**- and **n**-type graphs we can write the prescription (C4) as

$$\sigma_2^{\mathbf{m}} + 2m(\sigma_1^{\mathbf{m}})' = -[\sigma_2^{\mathbf{n}} + 2m(\sigma_1^{\mathbf{n}})'] \text{ at } p^2 = m^2. \tag{C5}$$

As usual, to fulfill it we must introduce counterterms. Working in effective theory at some fixed loop order, we can separately adjust counterterms for graphs of two different types, to be collectively denoted as $C_Z^{\mathbf{m}}$ and $C_Z^{\mathbf{n}}$, respectively. Now, according to the reduction theorem, the self-energy subgraphs with nonminimal vertices are not present inside the graph after the reduction, the only possibility is pictured in Fig. 7**n**. Hence, the adjusted $C_Z^{\mathbf{n}}$ can appear only on external legs replacing corresponding self-energy insertions and are never put inside the graph. In turn, any 2-leg insertion inside the graph is regulated by $C_Z^{\mathbf{m}}$ only and not affected by $C_Z^{\mathbf{n}}$. Simply speaking, the (nonminimal) RP's for two-point graphs with nonminimal vertices decouple from the other set of RP's. Only on external legs both $C_Z^{\mathbf{m}}$ and $C_Z^{\mathbf{n}}$ contribute. But, as it is seen from Eq. (C5), whatever the **m**-type counterterms may be, their contribution on the left side can always be compensated by **n**-type counterterms on the right side so that the equality, and, thus, the condition (C4) holds for external leg propagator. One can treat it as a manifestation of the well-known fact that the wave function renormalization constant is a redundant parameter of a theory.

Practically it allows one to set $Z = 1$ on the external legs automatically, having still a freedom in relevant (off-shell) counterterms for resultant (sub)graphs. Then the LSZ formula says that we need to calculate only the resultant (amputated) graphs without self-energy insertions on external legs (S-matrix resultant graphs). In particular, graphs with nonminimal vertices can be discarded in the S-matrix calculations—that is exactly what we wanted to show.

Note also that any additional mixing in external propagators due to nonminimal vertices can be excluded by the same arguments; though, as mentioned in Sec. VII, the problem of mixing which unavoidably arises whenever a two-point function is discussed is beyond the scope of this article.

Unfortunately, the above reasoning does not give a hand to avoid nonminimal RP's fixing the off-shell counterterms for graphs built of minimal vertices. In particular, corresponding 2-leg subgraphs will appear and may introduce off-shell subdivergencies [the on-shell behavior is, of course, regulated by RP (7.10)]. Moreover, while in conventional renormalizable theories we are basically limited by the first derivative of Σ, in effective theory other derivatives can also diverge. In a forthcoming publication we will show that, accepting certain off-shell asymptotic conditions for the full sums of given type graphs (e.g. self-energy), one can avoid calculation of an infinite number of nonminimal counterterms and work directly with finite parts of those sums. In fact, the corresponding technique is already described in Secs. IV, V, VI, and VII.

[1] A. Vereshagin and V. Vereshagin, Phys. Rev. D **69**, 025002 (2004).
[2] V. Vereshagin, Phys. Rev. D **55**, 5349 (1997).
[3] A. Vereshagin and V. Vereshagin, Phys. Rev. D **59**, 016002 (1999).
[4] A. Vereshagin, V. Vereshagin, and K. Semenov-Tian-Shansky, J. Math. Sci. **125**, 144 (2005). This article originally appeared in Zap. Nauchn. Semin. POMI **291**, 78 (2002) (in Russian).
[5] S. Weinberg, Physica A (Amsterdam) **96**, 327 (1979).
[6] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 2000), Vols. 1–3.
[7] A. Vereshagin and V. Vereshagin, πN Newsletter **15**, 288 (1999); A. Vereshagin, πN Newsletter **16**, 426 (2002).
[8] K. Semenov-Tian-Shansky, A. Vereshagin, and V. Vereshagin, *Tenth International Conference on Hadron Spectroscopy, Aschaffenburg, Germany, 2003*, edited by E. Klempt, H. Koch, and H. Orthin, AIP Conf. Proc. No. 717 (AIP, New York, 2004). See also Proceedings of the International Workshop ''Hadron Structure and QCD: from LOW to HIGH Energies'', St. Petersburg, Repino, Russia, 2004 (to be published); Proceedings of the XVIIIth International Workshop ''High Energy Physics and Quantum Field Theory'', St. Petersburg, Peterhof, Russia, 2004.
[9] A. N. Vasil'ev, *The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics* (Chapman and Hall/CRC, New York, 2004).
[10] J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, England, 1984).
[11] M. E. Peskin and D. V. Schroeder, *Quantum Field Theory* (Addison-Wesley, New York, 1997).
[12] N. N. Bogoliubov and Y. A. Mitropolski, *Asympthotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Breach, New York, 1961).
[13] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985).
[14] N. N. Bogoliubov, A. A. Logunov, and I. T. Todorov, *Foundations of the Axiomatic Method in Quantum Field Theory* (Nauka, Moskow, 1969).
[15] S. Weinberg, Phys. Rev. **177**, 2604 (1969).
[16] C. Cronström and M. Noga, Nucl. Phys. **B15**, 61 (1970); A. Sudbery, Nucl. Phys. **B20**, 1 (1970); V. I. Ogievetskii, Sov. J. Nucl. Phys. **13**, 105 (1971); S. R. Beane, Phys. Rev. D **64**, 116010 (2001).
[17] W. Balser, *From Divergent Power Series to Analytic Functions*, Lecture Notes in Mathematics (Springer-Verlag, Berlin, 1994).
[18] J. F. Chew, *The Analytic S-Matrix* (Benjamin, New York, 1966).
[19] A. Salam and J. Strathdee, Phys. Rev. D **2**, 2869 (1970).
[20] S. Weinberg, Phys. Rev. **133**, B1318 (1964); **134**, B882 (1964); **135**, B1049 (1964); **138**, B988 (1965); **181**, 1893 (1969).
[21] A. N. Vasiliev, *Functional Methods in Quantum Field Theory and Statistical Physics* (Gordon and Breach, Amsterdam, 1998).
[22] S. Weinberg, Phys. Rev. **130**, 776 (1963); **131**, 440 (1963); **133**, B232 (1964); M. Scadron and S. Weinberg, Phys. Rev. **133**, B1589 (1964); M. Scadron, S. Weinberg, and J. Wright, Phys. Rev. **135**, B202 (1964).
[23] M. Veltman, Physica (Amsterdam) **29**, 186 (1963).
[24] S. Eidelman *et al.*, Phys. Lett. B **592**, 1 (2004).
[25] M. Harada, F. Sannino, and J. Schechter, Phys. Rev. D **54**, 1991 (1996); D. Black, A. Fariborz, F. Sannino, and J. Schechter, Phys. Rev. D **58**, 054012 (1998); **59**, 074026 (1999); D. Black, A. Fariborz, S. Moussa, S. Nasri, and J. Schechter, Phys. Rev. D **64**, 014031 (2001); D. Black, M. Harada, and J. Schechter, hep-ph/0303223.
[26] S. Weinberg, in *General Relativity—An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[27] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); Nucl. Phys. **B250**, 465 (1985).
[28] D. J. Gross, Nucl. Phys. B, Proc. Suppl. **74**, 426 (1999).
[29] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists* (Academic, New York, 1995); E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1927).
[30] P. H. Frampton, *Dual Resonance Models and Superstrings* (World Scientific, Singapore, 1986); M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).
[31] G. Veneziano, Nuovo Cimento A **57**, 190 (1968).