

Radiation reaction on charged particles in three-dimensional motion in classical and quantum electrodynamics

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(Received 7 October 2005; published 27 January 2006)

We extend our previous work [A. Higuchi and G. D. R. Martin, *Found. Phys.* **35**, 1149 (2005)], which compared the predictions of quantum electrodynamics concerning radiation reaction with those of the Abraham-Lorentz-Dirac theory for a charged particle in linear motion. Specifically, we calculate the predictions for the change in position of a charged-scalar particle, moving in three-dimensional space, due to the effect of radiation reaction in the one-photon-emission process in quantum electrodynamics. The scalar particle is assumed to be accelerated for a finite period of time by a three-dimensional electromagnetic potential dependent only on one of the spacetime coordinates. We perform this calculation in the $\hbar \rightarrow 0$ limit and show that the change in position agrees with that obtained in classical electrodynamics with the Lorentz-Dirac force treated as a perturbation. We also show for a time-dependent but space-independent electromagnetic potential that the forward-scattering amplitude at order e^2 does not contribute to the position change in the $\hbar \rightarrow 0$ limit after the mass renormalization is taken into account.

DOI: [10.1103/PhysRevD.73.025019](https://doi.org/10.1103/PhysRevD.73.025019)

PACS numbers: 03.65.-w, 12.20.-m

I. INTRODUCTION

In classical electrodynamics, when a point charge e with mass m is accelerated by a 4-force F_{ext}^μ , it produces radiation and the equation of motion needs to be altered to incorporate the radiation-reaction force. In the Abraham-Lorentz-Dirac theory [1–3] we have

$$m \frac{d^2 x^\mu}{d\tau^2} = F_{\text{ext}}^\mu + F_{\text{LD}}^\mu, \quad (1)$$

where x^μ are the spacetime coordinates of the charge at proper time τ and where the Lorentz-Dirac radiation-reaction force is given in the units $c = 1$ by

$$F_{\text{LD}}^\mu \equiv \frac{2\alpha_c}{3} \left[\frac{d^3 x^\mu}{d\tau^3} + \frac{dx^\mu}{d\tau} \left(\frac{d^2 x^\nu}{d\tau^2} \frac{d^2 x_\nu}{d\tau^2} \right) \right]. \quad (2)$$

Here we have defined $\alpha_c \equiv e^2/4\pi$, and our metric convention is $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. We assume in this paper that the external force F_{ext}^μ is a Lorentz force resulting from a background electromagnetic field. (See Ref. [4] for an interesting derivation of this equation and Ref. [5] for a recent review.) The Abraham-Lorentz-Dirac theory has a number of problems if taken without modification, e.g. the existence of unphysical runaway solutions in which the charge accelerates under its own radiation. The suppression of this problem requires the existence of causality-violating preacceleration. (See, e.g., Refs. [6,7] for a discussion of these issues.) However, the treatment of the Lorentz-Dirac force as a perturbation, as used and justified in Refs. [8,9] for example, is free of these difficulties. Since classical electrodynamics is now meaningful only as the $\hbar \rightarrow 0$ limit of quantum electrodynamics (QED), which is the more fundamental theory, it is natural

to ask how the perturbation theory of QED compares in the classical limit to the Abraham-Lorentz-Dirac theory. Indeed we found recently that for a linearly accelerated charged particle the Lorentz-Dirac force can be recovered from the one-photon-emission amplitude in QED [10–12]. In this work we studied the position of a charged-scalar wave packet in linear motion and computed its change due to the presence of radiation reaction in the $\hbar \rightarrow 0$ limit. The result was found to be in complete agreement with that obtained by using the Lorentz-Dirac force as a perturbation in classical electrodynamics.

In this paper we generalize the work in Refs. [10–12] by extending the calculations to a charged particle in three-dimensional motion under an external electromagnetic potential dependent on one of the spacetime coordinates. We also study the position change due to the forward-scattering amplitude at order e^2 . (Note that Moniz and Sharp derived the Lorentz-Dirac force from QED by considering a charge of finite size and taking the zero-size limit [13]. Johnson and Hu obtained the Lorentz-Dirac-like force for a classical point charge interacting with a massless scalar field by integrating out the scalar field [14]. See also Ref. [15] on the relation between QED and the Lorentz-Dirac force.) The structure of this paper is as follows. Section II outlines the model in the context of classical electrodynamics. We then turn to the corresponding model for QED in Sec. III and give the expressions for the position shift of a radiating particle relative to non-radiating particles in terms of the one-photon-emission amplitude in the $\hbar \rightarrow 0$ limit. In Sec. IV we calculate the emission amplitudes for potentials dependent on one spacetime coordinate and then proceed to calculate the quantum position shift in Sec. V. We give in Sec. VI an expression for the classical position shift due to the addition of a radiation-reaction force treated as a perturbation. We conclude this section by giving our main result: the

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classical position shift is identical with the $\hbar \rightarrow 0$ limit of the quantum position shift. We summarize and discuss our results in Sec. VII. In the Appendix we demonstrate that the forward-scattering amplitude at order e^2 does not affect the position of the particle in the $\hbar \rightarrow 0$ limit after the mass renormalization is performed.

II. EXPLANATION OF THE MODEL

In this section we shall describe the model used for the investigation in this article in the context of classical electrodynamics. The model is the three-dimensional extension of that used in Refs. [11,12] with the potential dependent only on one of the spacetime coordinates, x^a say. The particle is accelerated by an external force arising from an electromagnetic potential V^μ . That is, the external force in Eq. (1) is given by

$$F_{\text{ext}}^\mu = (\partial^\mu V^\nu - \partial^\nu V^\mu) \frac{dx_\nu}{d\tau}. \quad (3)$$

We wish to calculate the position shift, i.e. the change in the position of the accelerated charge due to radiation reaction, after a finite period of acceleration. Thus, we shall assume that the potential V^μ behaves as follows: $V^\mu(x^a) = V_{(0)}^\mu$ (where $V_{(0)}^\mu$ is a constant vector) for $x^a < -X_1$, and $V^\mu(x^a) = 0$ for $x^a > -X_2$, with X_1 and X_2 being positive constants ($X_1 > X_2$). Hence, the acceleration occurs only in the region $-X_1 < x^a < -X_2$. We also require that the coordinate x^a of the particle increases as a function of time. (This condition is automatically satisfied if x^a is the time coordinate.) For the sake of simplicity we set up the model so that the charged particle would pass the space origin at time $t = 0$ in the absence of radiation reaction. (The spacetime origin is singled out as a convenient point of reference because the quantum fields are expanded in terms of the functions $e^{\pm ip \cdot x/\hbar}$, which take the value 1 at the spacetime origin. See Ref. [12] for an analysis of the model in one dimension with a reference point other than the origin.) The conditions on the external electromagnetic potential imply that at $t = 0$ the particle has already gone through the region of acceleration. The position shift is defined to be the change in the space coordinates x^i ($i = 1, 2, 3$) of this particle away from $x^i = 0$ at $t = 0$ due to radiation reaction.

It will be convenient for later use to write down the space components of the Lorentz-Dirac force explicitly in terms of the velocity $v^i \equiv \dot{x}^i$ and acceleration $a^i \equiv \dot{v}^i$, where the dot represents differentiation with respect to the coordinate time t . We write the equation of motion for the Abraham-Lorentz-Dirac theory (1) in a slightly different form as

$$\frac{d}{dt} \left(m \frac{dx^i}{dt} \right) = F_{\text{ext}}^i(x^a) \frac{d\tau}{dt} + \mathcal{F}_{\text{LD}}^i. \quad (4)$$

Then, defining \mathbf{x} , \mathbf{v} , and \mathbf{a} to be the vectors with i th components x^i , v^i , and a^i , respectively, and introducing

the definition $\gamma \equiv dt/d\tau = (1 - \mathbf{v}^2)^{-1/2}$ as usual, we obtain after a straightforward calculation

$$\mathcal{F}_{\text{LD}}^i = \frac{2\alpha_c}{3} \left\{ \frac{d}{dt} [\gamma^4 (\mathbf{a} \cdot \mathbf{v}) v^i + \gamma^2 a^i] - \gamma^6 (\mathbf{a} \cdot \mathbf{v})^2 v^i - \gamma^4 \mathbf{a}^2 v^i \right\}. \quad (5)$$

III. POSITION-SHIFT FORMULA IN QED

In this section we present the expression for the $\hbar \rightarrow 0$ limit of the quantum position shift in terms of the one-photon-emission amplitude. The Lagrangian density for the quantum-field-theoretic model of a charged-scalar field φ with mass m and charge e coupled to the electromagnetic field A_μ in the Feynman gauge is given by

$$\mathcal{L} = [(D_\mu + ieA_\mu/\hbar)\varphi]^\dagger (D^\mu + ieA^\mu/\hbar)\varphi - (m/\hbar)^2 \varphi^\dagger \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2, \quad (6)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \equiv \partial_\mu + ieV_\mu/\hbar$. The function V_μ is the external potential which accelerates the charged-scalar particle as in the previous section. For $e = 0$ the charged-scalar field is expanded as

$$\varphi(x) = \hbar \int \frac{d^3 \mathbf{p}}{2p_0(2\pi\hbar)^3} [A(\mathbf{p})\Phi_{\mathbf{p}}(x) + B^\dagger(\mathbf{p})\Phi_{\mathbf{p}}^*(x)] \quad (7)$$

with $p_0 = \sqrt{\mathbf{p}^2 + m^2}$, where the nonzero commutators are $[A(\mathbf{p}), A^\dagger(\mathbf{p}')] = [B(\mathbf{p}), B^\dagger(\mathbf{p}')] = 2p_0(2\pi\hbar)^3 \delta^3(\mathbf{p} - \mathbf{p}')$. (8)

The mode functions $\Phi_{\mathbf{p}}(x)$ satisfy the Klein-Gordon equation in the presence of the external field V_μ :

$$(D^\mu D_\mu + m^2)\Phi_{\mathbf{p}}(x) = 0. \quad (9)$$

If the potential is purely t dependent, then we require that $\Phi_{\mathbf{p}}(x)$ take precisely the form $e^{-ip \cdot x/\hbar}$, where $p \cdot x \equiv p^0 t - \mathbf{p} \cdot \mathbf{x}$, near the hypersurface $t = 0$. (This is possible because the potential V_μ vanishes there by assumption.) If the potential is space-dependent, then we require that $\Phi_{\mathbf{p}}(x)$ take precisely the form $e^{-ip \cdot x/\hbar}$ in a region including the spacetime origin [16]. Similarly, the electromagnetic field is expanded as

$$A_\mu(x) = \int \frac{d^3 \mathbf{k}}{2k(2\pi)^3} [a_\mu(\mathbf{k})e^{-ik \cdot x} + a_\mu^\dagger(\mathbf{k})e^{ik \cdot x}] \quad (10)$$

with $k = \|\mathbf{k}\|$ and $k \cdot x \equiv kt - \mathbf{k} \cdot \mathbf{x}$, where the annihilation and creation operators, $a_\mu(\mathbf{k})$ and $a_\mu^\dagger(\mathbf{k})$, for the photons with momenta $\hbar\mathbf{k}$ satisfy

$$[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')] = -g_{\mu\nu}(2\pi)^3 2\hbar k \delta^3(\mathbf{k} - \mathbf{k}'). \quad (11)$$

Notice that the scalar field φ is expanded in terms of the momentum \mathbf{p} whereas the electromagnetic field A_μ is

expanded in terms of the wave number \mathbf{k} . We adopt this convention because the vectors \mathbf{p} and \mathbf{k} are regarded as classical rather than \mathbf{p}/\hbar , the wave number of the scalar particle, and $\hbar\mathbf{k}$, the momentum of the electromagnetic field.

Let the initial wave-packet state be given by

$$|i\rangle = \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} f(\mathbf{p}) A^\dagger(\mathbf{p})|0\rangle, \quad (12)$$

where the function $f(\mathbf{p})$ is sharply peaked about a given momentum $\bar{\mathbf{p}}$ with width of order \hbar . If the potential depends on a space coordinate x^a , then we require that the a th component of this momentum be positive so that the wave packet is near the spacetime origin *after* it has undergone acceleration. The normalization of the operators $A^\dagger(\mathbf{p})$ is such that the condition $\langle i|i\rangle = 1$ leads to

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 = 1. \quad (13)$$

This shows that the function $f(\mathbf{p})$ can heuristically be regarded as the one-particle wave function in the momentum representation. The position expectation value is given by

$$\langle x^i \rangle = \int d^3\mathbf{x} x^i \langle \rho(\mathbf{x}, t) \rangle, \quad (14)$$

where the density operator for the scalar field is

$$\rho(x) \equiv \frac{i}{\hbar} : \varphi^\dagger \partial_t \varphi - \partial_t \varphi^\dagger \cdot \varphi :. \quad (15)$$

In Ref. [12] it was shown that for the nonradiating particle the position expectation value at $t = 0$ is given to lowest order by the expression

$$\langle x^i \rangle_0 = \frac{i\hbar}{2} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) \overleftrightarrow{\partial}_{p^i} f(\mathbf{p}), \quad (16)$$

where $\overleftrightarrow{\partial}_{p^i} = \overrightarrow{\partial}_{p^i} - \overleftarrow{\partial}_{p^i}$. This expression can heuristically be regarded as the expectation value of the position operator $i\hbar\partial_{p^i}$ in the momentum representation. It also agrees with the expectation value of the Newton-Wigner operator [17]. We let the function $f(\mathbf{p})$ satisfy $\langle x^i \rangle_0 = 0$ so that we measure the position shift for the particle relative to the origin as we required in the previous section.

Now, let us consider the radiating particle. We need to retain only the zero- and one-photon sectors in the final state to calculate physical quantities to order e^2 in the interaction picture. Thus, to this order the evolution of the state takes the following form:

$$\begin{aligned} A^\dagger(\mathbf{p})|0\rangle &\rightarrow [1 + i\mathcal{F}(\mathbf{p})]A^\dagger(\mathbf{p})|0\rangle + \frac{i}{\hbar} \\ &\times \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k} \mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) a_\mu^\dagger(\mathbf{k}) A^\dagger(\mathbf{P})|0\rangle, \end{aligned} \quad (17)$$

where $\mathcal{F}(\mathbf{p})$ is the forward-scattering amplitude and where $\mathcal{A}^\mu(\mathbf{p}, \mathbf{k})$ is the amplitude for the one-photon emission. If the potential is purely t dependent, i.e. if it is \mathbf{x} independent, then the momentum is conserved, i.e. the initial and final momenta are equal: $\mathbf{p} = \mathbf{P} + \hbar\mathbf{k}$. If the potential is only x^3 dependent, say, then the momentum \mathbf{P} is determined by energy conservation $\sqrt{\mathbf{p}^2 + m^2} = \sqrt{\mathbf{P}^2 + m^2} + \hbar k$ and transverse-momentum conservation $p^i = P^i + \hbar k^i$ ($i = 1, 2$). Thus, the final state for the initial wave-packet state $|i\rangle$ given by Eq. (12) is

$$\begin{aligned} |f\rangle &= \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} [1 + i\mathcal{F}(\mathbf{p})] f(\mathbf{p}) A^\dagger(\mathbf{p})|0\rangle \\ &+ \frac{i}{\hbar} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \\ &\times \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} \mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) f(\mathbf{p}) a_\mu^\dagger(\mathbf{k}) A^\dagger(\mathbf{P})|0\rangle. \end{aligned} \quad (18)$$

The position expectation value for this state in the $\hbar \rightarrow 0$ limit can be calculated by using Eq. (14). It consists of three terms identified as the nonradiating position expectation value, $\langle x^i \rangle_0$, which we have set to zero for simplicity, the forward-scattering contribution (which is not due to the radiation) and a third term identified as the position shift due to radiation reaction, which we denote by δx_Q^i . (The calculation is almost identical with that for linear acceleration given in Ref. [11] and in Sec. 4 of Ref. [12]. See Eq. (16) in Ref. [11], in particular.) Thus, the position expectation value at $t = 0$ is

$$\langle x^i \rangle_{t=0} = -\hbar \frac{\partial}{\partial p^i} \text{Re} \mathcal{F}(\bar{\mathbf{p}}) + \delta x_Q^i \quad (19)$$

with

$$\delta x_Q^i = -\frac{i}{2} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^{\mu*}(\bar{\mathbf{p}}, \mathbf{k}) \overleftrightarrow{\partial}_{\bar{p}^i} \mathcal{A}_\mu(\bar{\mathbf{p}}, \mathbf{k}) \quad (20)$$

in the $\hbar \rightarrow 0$ limit, where the wave packet becomes narrowly peaked about the momentum $\bar{\mathbf{p}}$. (From now on, the peak momentum $\bar{\mathbf{p}}$ will be denoted by \mathbf{p} .) Thus, the position shift δx_Q^i can be found from the emission amplitude $\mathcal{A}_\mu(\mathbf{p}, \mathbf{k})$ to be calculated for a given external electromagnetic potential that accelerates the particle. Although this expression was considered only for linear acceleration in Ref. [12], it is equally valid for motion in three dimensions: the details of the potential and the path of the particle are unnecessary for its derivation.

It will be shown in the Appendix that the forward-scattering contribution in Eq. (19) vanishes in the $\hbar \rightarrow 0$ limit after the mass renormalization is taken into account if the potential depends only on time, and we expect that this is also the case for potentials dependent on one space coordinate. For this reason we concentrate on the evalu-

ation of the radiation-reaction contribution δx_Q^i in the rest of this paper.

IV. EMISSION AMPLITUDE

The photon-emission part of the evolution of the state in Eq. (17) is given in terms of the emission amplitude by

$$A^\dagger(\mathbf{p})|0\rangle \rightarrow \cdots + \frac{i}{\hbar} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) a_\mu^\dagger(\mathbf{k}) A^\dagger(\mathbf{P})|0\rangle. \quad (21)$$

The evolution of this state to first order in time-dependent perturbation theory is

$$A^\dagger(\mathbf{p})|0\rangle \rightarrow \cdots - \frac{i}{\hbar} \int d^4x \mathcal{H}_I(x) A^\dagger(\mathbf{p})|0\rangle, \quad (22)$$

where $\mathcal{H}_I(x)$ is the interaction Hamiltonian density. Comparing these two evolution expressions, we can write the emission amplitude in terms of the interaction Hamiltonian density as

$$\begin{aligned} \mathcal{A}_\mu(\mathbf{p}, \mathbf{k}) &= \frac{1}{\hbar} \int \frac{d^3\mathbf{p}'}{2p'_0(2\pi\hbar)^3} \\ &\times \int d^4x \langle 0| a_\mu(\mathbf{k}) A(\mathbf{p}') \mathcal{H}_I(x) A^\dagger(\mathbf{p}) |0\rangle. \end{aligned} \quad (23)$$

The Hamiltonian density can be obtained from the Lagrangian density (6) by the standard procedure. Thus, we find the following interaction Hamiltonian density:

$$\begin{aligned} \mathcal{H}_I(x) &= \frac{ie}{\hbar} A_\mu : [\varphi^\dagger D^\mu \varphi - (D^\mu \varphi)^\dagger \varphi] : + \frac{e^2}{\hbar^2} \\ &\times \sum_{i=1}^3 A_i A_i : \varphi^\dagger \varphi : - \frac{\delta m^2}{\hbar^2} : \varphi^\dagger \varphi :, \end{aligned} \quad (24)$$

where $D_\mu \equiv \partial_\mu + iV_\mu/\hbar$ as before. We have normal-ordered the scalar-field operators to drop the vacuum polarization diagram automatically. [Note that the second term is different from what might be naively expected, $-(e^2/\hbar^2)A_\mu A^\mu : \varphi^\dagger \varphi :$. This difference is due to the presence of interaction terms involving $\dot{\varphi}$ or $\dot{\varphi}^\dagger$ in the Lagrangian density.] The last term in Eq. (24) is the mass counterterm. The counterterm for the wave-function renormalization will not be necessary in our calculations. A photon is emitted due to the first term in the Hamiltonian density (24). Thus, we have

$$\begin{aligned} \mathcal{A}_\mu(\mathbf{p}, \mathbf{k}) &= \frac{ie}{\hbar^2} \int \frac{d^3\mathbf{p}'}{2p'_0(2\pi\hbar)^3} \int d^4x \langle 0| a_\mu(\mathbf{k}) A(\mathbf{p}') \\ &\times \{ A_\nu : [\varphi^\dagger D^\nu \varphi - (D^\nu \varphi)^\dagger \varphi] : \} A^\dagger(\mathbf{p}) |0\rangle. \end{aligned} \quad (25)$$

By using the expansion of the fields A_μ and φ , and the commutation relations for the annihilation and creation

operators, one readily finds

$$\begin{aligned} \mathcal{A}_\mu(\mathbf{p}, \mathbf{k}) &= -ie\hbar \int \frac{d^3\mathbf{p}'}{2p'_0(2\pi\hbar)^3} \\ &\times \int d^4x e^{ik \cdot x} \{ \Phi_{\mathbf{p}'}^*(x) D_\mu \Phi_{\mathbf{p}}(x) \\ &- [D_\mu \Phi_{\mathbf{p}'}(x)]^* \Phi_{\mathbf{p}}(x) \}. \end{aligned} \quad (26)$$

Next we shall find a simple expression for the $\hbar \rightarrow 0$ limit of the emission amplitude given by Eq. (26) for an electromagnetic potential which depends only on one spacetime coordinate. We present the details of the calculation only for a t -dependent potential. The final result for an x -, y -, or z -dependent potential is identical, and the techniques involved are mostly the same. Thus, we consider a time-dependent potential $V^i(t)$ ($i = 1, 2, 3$) with $V^0(t) = 0$. [If $V^0(t) \neq 0$, one can gauge away this component.] The system is translationally invariant in the spatial directions and, hence, we can let

$$\Phi_{\mathbf{p}}(\mathbf{x}, t) = \phi_{\mathbf{p}}(t) \exp(i\mathbf{p} \cdot \mathbf{x}/\hbar). \quad (27)$$

The amplitude in a spatial direction for the t -dependent potential is then

$$\begin{aligned} \mathcal{A}^i(\mathbf{p}, \mathbf{k}) &= -e \int \frac{d^3\mathbf{p}'}{2p'_0(2\pi\hbar)^3} \int d^4x \phi_{\mathbf{p}'}^*(t) \phi_{\mathbf{p}}(t) \\ &\times [p^i + p'^i - 2V^i(t)] e^{i[(\mathbf{p}-\mathbf{p}') \cdot \mathbf{x}/\hbar]} e^{i(k t - \mathbf{k} \cdot \mathbf{x})} \\ &= -e \int dt e^{ikt} \phi_{\mathbf{p}}^*(t) \phi_{\mathbf{p}}(t) \frac{p^i - V^i(t)}{p_0} \end{aligned} \quad (28)$$

with $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$, where we have let $P^i + p^i - 2V^i(t) = 2[p^i - V(t)]$ as the difference $p^i - P^i$ is of order \hbar [18]. It would be wrong to equate $\phi_{\mathbf{p}}(t)$ with $\phi_{\mathbf{p}'}(t)$ because these functions oscillate with periods of order \hbar^{-1} . For the time component we have $D_t = \partial_t$ and simply obtain

$$\mathcal{A}^0(\mathbf{p}, \mathbf{k}) = -\frac{ie\hbar}{2p_0} \int dt [\phi_{\mathbf{p}}^* \partial_t \phi_{\mathbf{p}} - (\partial_t \phi_{\mathbf{p}}^*) \phi_{\mathbf{p}}] e^{ikt}. \quad (29)$$

To proceed further we need to approximate the function $\phi_{\mathbf{p}}(t)$ in a way suitable for taking the $\hbar \rightarrow 0$ limit. To this end we use the semiclassical WKB approximation. By substituting Eq. (27) in the Klein-Gordon equation (9) we find

$$\{\hbar^2 \partial_t^2 + [\mathbf{p} - \mathbf{V}(t)]^2 + m^2\} \phi_{\mathbf{p}}(t) = 0, \quad (30)$$

where $\mathbf{V}(t)$ is the vector with its i th component given by $V^i(t)$. The standard WKB approximation gives the following positive-frequency solution:

$$\phi_{\mathbf{p}}(t) = \sqrt{\frac{p_0}{\sigma_{\mathbf{p}}(t)}} \exp\left[-\frac{i}{\hbar} \int_0^t \sigma_{\mathbf{p}}(\zeta) d\zeta\right], \quad (31)$$

where

$$\sigma_{\mathbf{p}}(t) \equiv \sqrt{[\mathbf{p} - \mathbf{V}(t)]^2 + m^2}. \quad (32)$$

We note that the local momentum and energy of the point particle corresponding to the wave packet considered here are

$$m \frac{d\mathbf{x}}{d\tau} = \mathbf{p} - \mathbf{V}(t), \quad (33)$$

$$m \frac{dt}{d\tau} = \sigma_{\mathbf{p}}(t). \quad (34)$$

Now, the product of two wave functions in the emission amplitude (28) can be written

$$\phi_{\mathbf{p}}^*(t)\phi_{\mathbf{p}}(t) = \frac{p_0}{\sigma_{\mathbf{p}}(t)} \exp\left[-\frac{i}{\hbar} \int_0^t [\sigma_{\mathbf{p}}(\zeta) - \sigma_{\mathbf{p}}(\zeta)] d\zeta\right], \quad (35)$$

where we have replaced P_0 and $\sigma_{\mathbf{p}}(t)$ in the prefactor by p_0 and $\sigma_{\mathbf{p}}(t)$, respectively, because we are interested only in the $\hbar \rightarrow 0$ limit. The integrand in the exponent can be evaluated to lowest order in \hbar by using Eqs. (33) and (34) as

$$\sigma_{\mathbf{p}} - \sigma_{\mathbf{p}} = \frac{\partial \sigma_{\mathbf{p}}}{\partial p^i} (P^i - p^i) = \frac{dx^i}{dt} \hbar k^i, \quad (36)$$

where the repeated indices i are summed over. By substituting this approximation in Eq. (35) we find

$$\begin{aligned} \phi_{\mathbf{p}}^*(t)\phi_{\mathbf{p}}(t) &= \frac{p_0}{\sigma_{\mathbf{p}}} \exp\left(-i \int_0^t dt \frac{dx^i}{dt} k^i\right) \\ &= \frac{p_0}{\sigma_{\mathbf{p}}} \exp(-i\mathbf{k} \cdot \mathbf{x}), \end{aligned} \quad (37)$$

where we have used the fact that the particle passes through the spacetime origin. By substituting this formula in Eq. (28) and noting Eqs. (33) and (34) we obtain

$$\mathcal{A}^i(\mathbf{p}, \mathbf{k}) = -e \int dt e^{i\mathbf{k} \cdot \mathbf{x}} \frac{dx^i}{dt} = -e \int d\xi \frac{dx^i}{d\xi} e^{i\mathbf{k} \cdot \xi}, \quad (38)$$

where we have defined $\xi \equiv t - \mathbf{n} \cdot \mathbf{x}$ with $\mathbf{n} \equiv \mathbf{k}/k$. We emphasize that \mathbf{x} and ξ here are functions of t evaluated on the world line of the corresponding classical particle passing through the spacetime origin.

Let us now consider the time component $\mathcal{A}^0(\mathbf{p}, \mathbf{k})$ of the emission amplitude given by Eq. (29). Note that from the WKB expression (31) for $\phi_{\mathbf{p}}(t)$ we have, to lowest order in \hbar ,

$$\partial_t \phi_{\mathbf{p}}(t) = -\frac{i}{\hbar} \sigma_{\mathbf{p}}(t) \phi_{\mathbf{p}}(t). \quad (39)$$

By substituting this formula in Eq. (29) we obtain

$$\begin{aligned} \mathcal{A}^0(\mathbf{p}, \mathbf{k}) &= -\frac{e}{2p_0} \int dt [\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{p}}(t)] \phi_{\mathbf{p}}^*(t) \phi_{\mathbf{p}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} \\ &= -e \int dt e^{-i\mathbf{k} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{x}} \\ &= -e \int d\xi \frac{dt}{d\xi} e^{i\mathbf{k} \cdot \xi}, \end{aligned} \quad (40)$$

where we have let $\sigma_{\mathbf{p}}(t) = \sigma_{\mathbf{p}}(t)$ and used Eq. (37). By combining this formula and Eq. (38) we obtain the following concise expression for the $\hbar \rightarrow 0$ limit of the emission amplitude:

$$\mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) = -e \int d\xi \frac{dx^\mu}{d\xi} e^{i\mathbf{k} \cdot \xi}. \quad (41)$$

One can proceed in a similar manner for the case where the potential depends on any one of the spatial coordinates. One obtains Eq. (41) in this case as well. The emission amplitude (41) is identical to that for a classical particle corresponding to the wave packet considered here [11].

V. QUANTUM POSITION SHIFT

The expression (41) for the emission amplitude is ill-defined, as noted in Ref. [12] for the one-dimensional case, because the integrand does not tend to zero as $\xi \rightarrow \pm\infty$. For this reason we introduce a smooth cutoff function $\chi(\xi)$ which takes the value 1 while the acceleration is nonzero and has the property $\lim_{\xi \rightarrow \pm\infty} \chi(\xi) = 0$, viz.

$$\mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) = -e \int_{-\infty}^{+\infty} d\xi \frac{dx^\mu}{d\xi} \chi(\xi) e^{i\mathbf{k} \cdot \xi}. \quad (42)$$

One can show exactly as in Sec. 7 of Ref. [12] that the substitution of this expression into the quantum position shift given by Eq. (20) leads to an expression independent of the cutoff function as follows:

$$\delta x_Q^i = -\frac{\alpha_c}{4\pi} \int d\Omega \int dt \frac{d^2 x_\mu}{d\xi^2} \frac{d}{dt} \left(\frac{\partial x^\mu}{\partial p^i} \right)_\xi, \quad (43)$$

where $d\Omega$ is the solid angle in the wave-number space of the photon emitted. The variable appearing as the subscript for a partial derivative—the variable ξ in this equation—is held fixed. [Note that $dx^\mu/dt = (\partial x^\mu/\partial t)_{p^i}$.]

To evaluate δx_Q^i in Eq. (43) we first need to put the integrand in the form amenable to the solid-angle integration by rewriting it in terms of t instead of ξ . One can readily write $d^2 x^\mu/d\xi^2$ in terms of t derivatives by using $d/d\xi = (1 - n^i \dot{x}^i)^{-1} d/dt$ as follows:

$$\frac{d^2 x^\mu}{d\xi^2} = \dot{\xi}^{-3} [(1 - n^i \dot{x}^i) \ddot{x}^\mu + n^i \dot{x}^i \dot{x}^\mu], \quad (44)$$

where $\dot{\xi} = 1 - n^i \dot{x}^i$. Here and in the rest of this section, Latin indices take the spatial values 1 to 3, and are summed over when repeated. The time and space components of Eq. (44) can separately be given as

$$\frac{d^2 t}{d\xi^2} = \dot{\xi}^{-3} n^i \ddot{x}^i, \quad (45)$$

$$\frac{d^2 x^j}{d\xi^2} = \dot{\xi}^{-3} [(1 - n^i \dot{x}^i) \ddot{x}^j + n^i \ddot{x}^i \dot{x}^j]. \quad (46)$$

Next we express $(\partial x^\mu / \partial p^i)_\xi$ in Eq. (43) in the form involving t rather than ξ as follows. Note first

$$dx^\mu = \frac{dx^\mu}{dt} dt + \left(\frac{\partial x^\mu}{\partial p^i} \right)_t dp^i. \quad (47)$$

[The zeroth component of this equation is trivial because $(\partial t / \partial p^i)_t = 0$.] By substituting $dt = d\xi + n^k dx^k$ in this equation with $\mu = j$ and solving for dx^i , we obtain

$$dx^i = \frac{v^i}{1 - n^j v^j} d\xi + \frac{[\delta^{ik}(1 - n^l v^l) + n^k v^i]}{1 - n^l v^l} \left(\frac{\partial x^k}{\partial p^j} \right)_t dp^j, \quad (48)$$

where $v^i \equiv \dot{x}^i$. Hence

$$\left(\frac{\partial x^i}{\partial p^j} \right)_\xi = \frac{[\delta^{ik}(1 - n^l v^l) + n^k v^i]}{1 - n^l v^l} \left(\frac{\partial x^k}{\partial p^j} \right)_t. \quad (49)$$

With ξ fixed, we have $dt - n^i dx^i = d\xi = 0$. Thus,

$$\left(\frac{\partial t}{\partial p^j} \right)_\xi = n^i \left(\frac{\partial x^i}{\partial p^j} \right)_\xi = \frac{n^i}{1 - n^l v^l} \left(\frac{\partial x^i}{\partial p^j} \right)_t. \quad (50)$$

By substituting Eqs. (45), (46), (49), and (50) in Eq. (43) we find the following expression after a tedious but straightforward calculation:

$$\begin{aligned} \delta x_Q^i &= -\frac{\alpha_c}{4\pi} \int dt \left\{ [I_2^{kj} \gamma^{-2} a^j - I_0 a^k - I_1^j a^j v^k \right. \\ &\quad - I_1^k (\mathbf{a} \cdot \mathbf{v})] \frac{d}{dt} \left(\frac{\partial x^k}{\partial p^i} \right)_t + [I_3^{kjl} \gamma^{-2} a^j a^l \\ &\quad \left. - 2I_2^{kj} a^j (\mathbf{a} \cdot \mathbf{v}) - I_1^k \mathbf{a}^2] \left(\frac{\partial x^k}{\partial p^i} \right)_t \right\}, \end{aligned} \quad (51)$$

where

$$I_0 \equiv \int d\Omega \frac{1}{\xi^2} = 4\pi \gamma^2, \quad (52)$$

$$I_1^i \equiv \int d\Omega \frac{n^i}{\xi^3} = 4\pi \gamma^4 v^i, \quad (53)$$

$$I_2^{ij} \equiv \int d\Omega \frac{n^i n^j}{\xi^4} = \frac{16}{3} \pi \gamma^6 v^i v^j + \frac{4}{3} \pi \gamma^4 \delta^{ij}, \quad (54)$$

$$\begin{aligned} I_3^{ijk} &\equiv \int d\Omega \frac{n^i n^j n^k}{\xi^5} \\ &= 8\pi \gamma^8 v^i v^j v^k + \frac{4}{3} \pi \gamma^6 (v^i \delta^{jk} + v^j \delta^{ik} + v^k \delta^{ij}). \end{aligned} \quad (55)$$

Evaluation of these solid-angle integrals is facilitated by noting that the last three integrals are proportional to partial derivatives of I_0 with respect to v^i . Substitution of Eqs. (52)–(55) in Eq. (51) yields

$$\begin{aligned} \delta x_Q^i &= \frac{2\alpha_c}{3} \int dt \left\{ [\gamma^4 (\mathbf{a} \cdot \mathbf{v}) v^k + \gamma^2 a^k] \frac{d}{dt} \left(\frac{\partial x^k}{\partial p^i} \right)_t \right. \\ &\quad \left. + [\gamma^6 (\mathbf{a} \cdot \mathbf{v})^2 v^k + \gamma^4 \mathbf{a}^2 v^k] \left(\frac{\partial x^k}{\partial p^i} \right)_t \right\}. \end{aligned} \quad (56)$$

By comparing this equation with the expression (5) for the Lorentz-Dirac force, we note that the quantum position shift can be written as

$$\delta x_Q^i = - \int_{-\infty}^0 dt \mathcal{F}_{\text{LD}}^j \left(\frac{\partial x^j}{\partial p^i} \right)_t. \quad (57)$$

We have used the fact that $\mathbf{a}(t) \neq 0$ only for $t < 0$ and for a finite interval of time to integrate the first term in Eq. (56) by parts. In the next section we shall demonstrate that this expression for the position shift agrees with that in the Abraham-Lorentz-Dirac theory. We recall here that p^i is the momentum of the particle at $t = 0$, which is after it has gone through the acceleration.

VI. CLASSICAL POSITION SHIFT

In this section we discuss the classical perturbative solution to the Lorentz-Dirac equation (4) in general terms in order to show that the classical and quantum position shifts coincide. We do not need the assumption that the external electromagnetic potential depends only on one coordinate in this section; this assumption was necessary only for the calculation of the quantum position shift δx_Q^i .

Recall that the classical motion of a charge e with mass m in an electromagnetic potential $V^\mu = (V^0, \mathbf{V})$, which depends on t and \mathbf{x} , is described by the following Hamiltonian:

$$H = \sqrt{(\mathbf{P} - \mathbf{V})^2 + m^2} + eV^0, \quad (58)$$

where \mathbf{P} is the momentum conjugate to \mathbf{x} . One can readily show that the Lorentz-Dirac equation (4) is equivalent to the following set of equations:

$$\dot{x}^i = \frac{\partial H}{\partial P^i}, \quad (59)$$

$$\dot{P}^i = -\frac{\partial H}{\partial x^i} + \mathcal{F}_{\text{LD}}^i. \quad (60)$$

Now, let $(\mathbf{x}, \mathbf{P}) = (\mathbf{x}_0(t), \mathbf{P}_0(t))$ be the solution to these coupled equations with $\mathcal{F}_{\text{LD}}^i = 0$ satisfying $\mathbf{x}_0(0) = 0$ and $\mathbf{P}_0(0) = \mathbf{p}$. This solution gives the classical trajectory of the particle passing through the spacetime origin with final momentum \mathbf{p} in the absence of radiation reaction. If $(\mathbf{x}, \mathbf{P}) = (\mathbf{x}_0(t) + \delta \mathbf{x}(t), \mathbf{P}_0(t) + \delta \mathbf{P}(t))$ is the retarded solution to these equations with $\mathcal{F}_{\text{LD}}^i \neq 0$ to first order in

$\mathcal{F}_{\text{LD}}^i$, then $\delta\mathbf{x}$ and $\delta\mathbf{P}$ will have the property that $(\delta\mathbf{x}, \delta\mathbf{P}) \rightarrow (0, 0)$ as $t \rightarrow -\infty$ and satisfy

$$\frac{d}{dt} \delta x^i = \frac{\partial^2 H}{\partial x^j \partial P^i} \delta x^j + \frac{\partial^2 H}{\partial P^j \partial P^i} \delta P^j, \quad (61)$$

$$\frac{d}{dt} \delta P^i = -\frac{\partial^2 H}{\partial x^j \partial x^i} \delta x^j - \frac{\partial^2 H}{\partial x^i \partial P^j} \delta P^j + \mathcal{F}_{\text{LD}}^i, \quad (62)$$

where the partial derivatives of H are evaluated at $(\mathbf{x}, \mathbf{P}) = (\mathbf{x}_0(t), \mathbf{P}_0(t))$ and, hence, are functions of t alone. The quantity $\delta x_C^i \equiv \delta x^i(0)$ is identified as the position shift in the classical Abraham-Lorentz-Dirac theory. Thus, our task is to show that $\delta x_C^i = \delta x_Q^i$. To this end we define a set of solutions to the homogeneous equations obtained by letting $\mathcal{F}_{\text{LD}}^i = 0$ in Eqs. (61) and (62). Thus, we define the solutions $(\delta x^i(t), \delta P^i(t)) = (\Delta x_{(j)}^i(t; s), \Delta P_{(j)}^i(t; s))$, with labels $j = 1, 2, 3$ and $s \in (-\infty, +\infty)$, to these homogeneous equations by the following initial conditions:

$$\Delta x_{(j)}^i(s; s) = 0, \quad (63)$$

$$\Delta P_{(j)}^i(s; s) = \delta^{ij}. \quad (64)$$

Then, the solution $(\Delta x_{(j)}^i(t; s), \Delta P_{(j)}^i(t; s))$ represents the particle trajectory which coincides with $\mathbf{x}_0(t)$ at $t = s$ and has an excess momentum solely in the j direction at this value of t . Then it can readily be seen that the retarded solution to the inhomogeneous equations (61) and (62) is given by

$$\delta x^i = \int_{-\infty}^t ds \mathcal{F}_{\text{LD}}^j(s) \Delta x_{(j)}^i(t; s), \quad (65)$$

$$\delta P^i = \int_{-\infty}^t ds \mathcal{F}_{\text{LD}}^j(s) \Delta P_{(j)}^i(t; s), \quad (66)$$

where the index j is summed over. Hence, the classical position shift is

$$\delta x_C^i = \int_{-\infty}^0 dt \mathcal{F}_{\text{LD}}^j(t) \Delta x_{(j)}^i(0; t). \quad (67)$$

To obtain a similar expression for the quantum position shift δx_Q^i we note first that

$$\left(\frac{\partial x^j}{\partial P^i} \right)_t = \Delta x_{(i)}^j(t; 0). \quad (68)$$

This is because $\epsilon \Delta x_{(i)}^j(t; 0)$ with $\epsilon \ll 1$ is the change in the position of the charged particle in the j direction at time t caused by a change in the i th component of the momentum at $t = 0$ by ϵ . Thus, the quantum position shift δx_Q^i in Eq. (43) can be written

$$\delta x_Q^i = - \int_{-\infty}^0 dt \mathcal{F}_{\text{LD}}^j(t) \Delta x_{(i)}^j(t; 0), \quad (69)$$

where the index j is summed over.

The classical and quantum position shifts given by Eqs. (67) and (69), respectively, can be shown to be equal by using conservation of the symplectic product for the homogeneous equations (61) and (62) with $\mathcal{F}_{\text{LD}}^i = 0$. The symplectic product between two solutions, $(\Delta \mathbf{x}^A, \Delta \mathbf{P}^A)$ and $(\Delta \mathbf{x}^B, \Delta \mathbf{P}^B)$, to these homogeneous equations is defined by $\Delta \mathbf{x}^A \cdot \Delta \mathbf{P}^B - \Delta \mathbf{x}^B \cdot \Delta \mathbf{P}^A$. It is well known and can easily be verified that the time derivative of this product vanishes, i.e. that it is conserved. By equating the symplectic products of the two solutions $(\Delta x_{(i)}^k(t; s), \Delta P_{(i)}^k(t; s))$ and $(\Delta x_{(j)}^k(t; u), \Delta P_{(j)}^k(t; u))$ at $t = s$ and u we have

$$\begin{aligned} & \Delta x_{(i)}^k(s; s) \Delta P_{(j)}^k(s; u) - \Delta x_{(j)}^k(s; u) \Delta P_{(i)}^k(s; s) \\ &= \Delta x_{(i)}^k(u; s) \Delta P_{(j)}^k(u; u) - \Delta x_{(j)}^k(u; u) \Delta P_{(i)}^k(u; s). \end{aligned} \quad (70)$$

This equation and the initial conditions (63) and (64) imply

$$-\Delta x_{(j)}^i(s; u) = \Delta x_{(i)}^j(u; s). \quad (71)$$

In particular, we have $-\Delta x_{(j)}^i(0; t) = \Delta x_{(i)}^j(t; 0)$. This equality and Eqs. (67) and (69) imply that $\delta x_C^i = \delta x_Q^i$.

VII. CONCLUSION

In this paper we compared the change in position of a charged-scalar particle due to radiation reaction in both classical and quantum electrodynamics in three spatial dimensions. We found that, for a charged particle accelerated for a finite period of time by an electromagnetic potential dependent on one of the spacetime coordinates, the change in position in classical electrodynamics using the Lorentz-Dirac force as a perturbation is the same as that given by the $\hbar \rightarrow 0$ limit of the one-photon-emission calculation in quantum electrodynamics. This extended the results of Ref. [12] from one to three dimensions. Since the forward-scattering amplitude at order e^2 does not contribute to the change in the position in the $\hbar \rightarrow 0$ limit if the potential depends only on time as shown in the Appendix, one can conclude that the $\hbar \rightarrow 0$ limit of quantum field theory coincides with the Abraham-Lorentz-Dirac theory at least in this case. We expect that this will be true also with a potential dependent on one space coordinate.

In our calculations it was crucial that a plane wave corresponded to a set of parallel classical trajectories with the same motion. This property was guaranteed by the assumption that the external electromagnetic potential depended only on one spacetime coordinate. If the potential depends on more than one coordinate, classical particles with the same initial or final momentum can have many different motions depending on their initial or final positions. (Consider, for example, scattering by a central potential.) It would be interesting if one could overcome this difficulty and extend our results to cases with more general potentials. It would also be interesting to extend

our results to radiation reaction in curved spacetime or to the backreaction to gravitational radiation [19,20].

APPENDIX: FORWARD-SCATTERING AMPLITUDE

In this appendix we show that the contribution of the forward-scattering amplitude at order e^2 to the position change of the particle vanishes in the $\hbar \rightarrow 0$ limit if the external electromagnetic potential depends only on t . Since

$$2ip_0(2\pi\hbar)^3 \mathcal{F}(\mathbf{p})\delta^3(\mathbf{p} - \mathbf{p}') = -\frac{i}{\hbar} \int d^4x \langle 0|A(\mathbf{p}')\mathcal{H}_I(x)A^\dagger(\mathbf{p})|0\rangle - \frac{1}{2\hbar^2} \int d^4x' d^4x \langle 0|A(\mathbf{p}')T[\mathcal{H}_I(x')\mathcal{H}_I(x)]A^\dagger(\mathbf{p})|0\rangle, \quad (\text{A1})$$

where T denotes time ordering, to second order in the standard time-dependent perturbation theory in the interaction picture.

We first consider the contribution of the mass counterterm $-(\delta m^2/\hbar^2):\varphi^\dagger\varphi$: in the interaction Hamiltonian density to the forward-scattering amplitude. This contribution, which we denote by $\mathcal{F}^{\text{mass}}(\mathbf{p})$, can readily be found at first order in perturbation theory as

$$\mathcal{F}^{\text{mass}}(\mathbf{p}) = \frac{1}{2\hbar p_0} \int dt |\phi_{\mathbf{p}}(t)|^2 \delta m^2, \quad (\text{A2})$$

where $\phi_{\mathbf{p}}(t)$ is the time-dependent part of the scalar mode function defined by Eq. (27). [The quantity $\mathcal{F}^{\text{mass}}(\mathbf{p})$ is obviously divergent in the t integration. In the end all terms with this divergence property will cancel out.] The mass parameter in the counterterm in the standard covariant perturbation theory reads

$$\delta m^2 = \frac{e^2}{\hbar} \int \frac{d^4q}{(2\pi)^4 i} \left\{ \frac{(p+q)^2}{[q^2 - m^2 + i\epsilon][(p-q)^2 + i\epsilon]} - \frac{4}{[(p-q)^2 + i\epsilon]} \right\}, \quad (\text{A3})$$

$$2ip_0(2\pi\hbar)^3 \mathcal{F}_2(\mathbf{p})\delta^3(\mathbf{p} - \mathbf{p}') = -\frac{ie^2}{\hbar^3} \sum_{i=1}^3 \int d^4x \langle 0|A(\mathbf{p})A_i(x)A_i(x):\varphi^\dagger(x)\varphi(x):A^\dagger(\mathbf{p}')|0\rangle. \quad (\text{A5})$$

One can readily evaluate this expression with the following result:

$$\mathcal{F}_2(\mathbf{p}) = -\frac{3e^2}{2\hbar^2 p_0} \int_{-\infty}^{+\infty} dt |\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\mathbf{q}}{2K(2\pi)^3}, \quad (\text{A6})$$

where we have used $d^3\mathbf{k}/k = \hbar^{-2}d^3\mathbf{q}/K$.

The first term in Eq. (24) contributes to the forward-scattering amplitude in second order in time-dependent perturbation theory. Writing this contribution as $\mathcal{F}_1(\mathbf{p})$, we find

$$\begin{aligned} \mathcal{F}_1(\mathbf{p}) = & \frac{ie^2}{2p_0} \int \frac{d^3\mathbf{q}}{2q_0(2\pi\hbar)^3} \frac{1}{2K} \int dt_1 dt_2 \{ \theta(t_1 - t_2) [\phi_{\mathbf{p}}^*(t_1)\phi_{\mathbf{p}}(t_2)\overleftrightarrow{\mathcal{D}}_1(t_1, t_2, \mathbf{p}, \mathbf{q})\phi_{\mathbf{q}}(t_1)\phi_{\mathbf{q}}^*(t_2)] e^{-iK(t_1-t_2)/\hbar} \\ & + \theta(t_2 - t_1) [\phi_{\mathbf{p}}^*(t_1)\phi_{\mathbf{p}}(t_2)\overleftrightarrow{\mathcal{D}}_1(t_1, t_2, \mathbf{p}, \mathbf{q})\phi_{\mathbf{q}}^*(t_1)\phi_{\mathbf{q}}(t_2)] e^{iK(t_1-t_2)/\hbar} \}, \end{aligned} \quad (\text{A7})$$

this contribution has an explicit factor of \hbar in Eq. (19), we can neglect any term of order higher than \hbar^{-1} in the forward-scattering amplitude. What we demonstrate here is that the one-loop contribution of order \hbar^{-1} or lower is canceled exactly by that from the mass counterterm, which is formally of order \hbar^{-2} .

We use a noncovariant Hamiltonian formulation with the Hamiltonian density (24). The forward-scattering amplitude $\mathcal{F}(\mathbf{p})$ for the state with final momentum \mathbf{p} is given by

It is convenient to integrate over the time component q_0 of q^μ for later purposes. The result can be given as follows:

$$\begin{aligned} \delta m^2 = & \frac{e^2}{\hbar} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left\{ -\frac{(\mathbf{p} + \mathbf{q})^2}{4Kq_0} \left[\frac{1}{q_0 + K - p_0} \right. \right. \\ & \left. \left. + \frac{1}{K + p_0 + q_0} \right] + \frac{3}{2K} + \frac{1}{4Kq_0} \left[\frac{(p_0 - q_0)^2}{q_0 + K + p_0} \right. \right. \\ & \left. \left. + \frac{(p_0 + q_0)^2}{q_0 + K - p_0} \right] \right\} \end{aligned} \quad (\text{A4})$$

with $\mathbf{K} \equiv \mathbf{p} - \mathbf{q}$ and $K \equiv \|\mathbf{K}\|$, where q_0 is now defined to be $\sqrt{\mathbf{q}^2 + m^2}$.

The second term in Eq. (24) contributes to the forward-scattering amplitude at first order in time-dependent perturbation theory. By denoting this contribution by $\mathcal{F}_2(\mathbf{p})$ we have

where

$$\vec{\mathcal{D}}_1(t_1, t_2, \mathbf{p}, \mathbf{q}) \equiv -\hbar^2 \vec{\partial}_{t_1} \vec{\partial}_{t_2} + [\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t_1)] \cdot [\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t_2)]. \quad (\text{A8})$$

It is convenient to define the variables t and η by $t_1 = t - \hbar\eta/2$ and $t_2 = t + \hbar\eta/2$. Then

$$\mathcal{F}_{1+}(\mathbf{p}) = \frac{ie^2}{2\hbar^2 p_0} \int \frac{d^3\mathbf{q}}{2q_0(2\pi)^3} \frac{1}{2K} \int dt [G_-(\mathbf{p}, \mathbf{q}, t) + G_+(\mathbf{p}, \mathbf{q}, t)], \quad (\text{A9})$$

with

$$G_-(\mathbf{p}, \mathbf{q}, t) = \int_{-\infty}^0 d\eta [\phi_{\mathbf{p}}^*(t_1) \phi_{\mathbf{p}}(t_2) \vec{\mathcal{D}}_2(t, \eta, \mathbf{p}, \mathbf{q}) \times \phi_{\mathbf{q}}(t_1) \phi_{\mathbf{q}}^*(t_2)] e^{iK\eta}, \quad (\text{A10})$$

$$G_+(\mathbf{p}, \mathbf{q}, t) = \int_0^{\infty} d\eta [\phi_{\mathbf{p}}^*(t_1) \phi_{\mathbf{p}}(t_2) \vec{\mathcal{D}}_2(t, \eta, \mathbf{p}, \mathbf{q}) \times \phi_{\mathbf{q}}^*(t_1) \phi_{\mathbf{q}}(t_2)] e^{-iK\eta}, \quad (\text{A11})$$

where

$$\vec{\mathcal{D}}_2(t, \eta, \mathbf{p}, \mathbf{q}) \equiv -\frac{\hbar^2}{4} \vec{\partial}_t^2 - \vec{\partial}_\eta^2 + [\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t_1)] \cdot [\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t_2)]. \quad (\text{A12})$$

We write the time-dependent part of the mode function, $\phi_{\mathbf{q}}(t)$, as

$$\phi_{\mathbf{q}}(t) = \sqrt{\frac{q_0}{\sigma_{\mathbf{q}}(t)}} \psi_{\mathbf{q}}(t) \exp\left[-\frac{i}{\hbar} \int_0^t \sigma_{\mathbf{q}}(t') dt'\right], \quad (\text{A13})$$

where $\psi_{\mathbf{q}}(t) \rightarrow 1$ as $\hbar \rightarrow 0$. By substituting this expression in Eq. (30) we can expand $\psi_{\mathbf{q}}(t)$ in terms of \hbar as

$$\psi_{\mathbf{q}}(t) = 1 + i\hbar\psi_{\mathbf{q}}^{(1)}(t) + \mathcal{O}(\hbar^2). \quad (\text{A14})$$

The explicit form of $\psi_{\mathbf{q}}^{(1)}(t)$ is unnecessary though it can easily be found. Note also that

$$\phi_{\mathbf{q}}^*(t_1) \phi_{\mathbf{q}}(t_2) = \frac{q_0 \psi_{\mathbf{q}}^*(t_1) \psi_{\mathbf{q}}(t_2)}{\sqrt{\sigma_{\mathbf{q}}(t_1) \sigma_{\mathbf{q}}(t_2)}} \times \exp\left[-i \int_{-\eta/2}^{\eta/2} \sigma_{\mathbf{q}}(t + \hbar\zeta) d\zeta\right]. \quad (\text{A15})$$

It can readily be shown that the functions $G_{\pm}(\mathbf{p}, \mathbf{q}, t)$ are of the form

$$G_{\pm}(\mathbf{p}, \mathbf{q}, t) = \pm \int_0^{\pm\infty} d\eta [f_{\pm}(\mathbf{p}, \mathbf{q}, t) + \mathcal{O}(\hbar^2)] \times \exp\left\{\mp i \int_{-\eta/2}^{+\eta/2} d\zeta [\pm\sigma_{\mathbf{p}}(t + \hbar\zeta) + \sigma_{\mathbf{q}}(t + \hbar\zeta) + K]\right\}, \quad (\text{A16})$$

where the function $f_{\pm}(\mathbf{p}, \mathbf{q}, t)$ can be found as

$$f_{\pm}(\mathbf{p}, \mathbf{q}, t) = \{-[\sigma_{\mathbf{p}}(t) \mp \sigma_{\mathbf{q}}(t)]^2 + [\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t)]^2\} |\phi_{\mathbf{p}}(t)|^2 |\phi_{\mathbf{q}}(t)|^2. \quad (\text{A17})$$

An important fact to note here is that there are no terms of order \hbar in the prefactor inside the first square brackets in Eq. (A16). This fact is a simple consequence of the equations $\psi_{\mathbf{q}}^*(t_1) \psi_{\mathbf{q}}(t_2) = 1 + \mathcal{O}(\hbar^2)$ and

$$[\mathbf{p} + \mathbf{q} - \mathbf{V}(t_1)] \cdot [\mathbf{p} + \mathbf{q} - \mathbf{V}(t_2)] = [\mathbf{p} + \mathbf{q} - \mathbf{V}(t)]^2 + \mathcal{O}(\hbar^2). \quad (\text{A18})$$

Let us first consider the integral $G_+(\mathbf{p}, \mathbf{q}, t)$. We change the integration variable from η to β defined by the following relation:

$$[\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K]\beta \equiv \int_{-\eta/2}^{\eta/2} [\sigma_{\mathbf{p}}(t + \hbar\zeta) + \sigma_{\mathbf{q}}(t + \hbar\zeta) + K] d\zeta. \quad (\text{A19})$$

Expanding the integrand and integrating, we solve for η as a function of β for small \hbar and find

$$\eta = \left[1 - \frac{1}{24} \frac{\ddot{\sigma}_{\mathbf{p}}(t) + \ddot{\sigma}_{\mathbf{q}}(t)}{\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K} \hbar^2 \beta^2 + \mathcal{O}(\hbar^4 \beta^4)\right] \beta \quad (\text{A20})$$

and

$$d\eta = \left[1 - \frac{1}{8} \frac{\ddot{\sigma}_{\mathbf{p}}(t) + \ddot{\sigma}_{\mathbf{q}}(t)}{\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K} \hbar^2 \beta^2 + \mathcal{O}(\hbar^4 \beta^4)\right] d\beta. \quad (\text{A21})$$

Then we have

$$G_+(\mathbf{p}, \mathbf{q}, t) = \int_0^{\infty} d\beta [f_+(\mathbf{p}, \mathbf{q}, t) + \mathcal{O}(\hbar^2)] \times \exp\{-i[\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K]\beta\}. \quad (\text{A22})$$

Introducing a convergence factor by replacing K by $K - i\epsilon$, we obtain

$$G_+(\mathbf{p}, \mathbf{q}, t) = -\frac{if_+(\mathbf{p}, \mathbf{q}, t)}{\sigma_{\mathbf{q}}(t) + \sigma_{\mathbf{p}}(t) + K} + \mathcal{O}(\hbar^2). \quad (\text{A23})$$

The corresponding term in the forward-scattering amplitude is

$$\mathcal{F}_{1+}(\mathbf{p}) = \frac{e^2}{2\hbar^2 p_0} \int dt \int \frac{d^3\mathbf{q}}{2q_0(2\pi)^3} \frac{1}{2K} \times \frac{f_+(\mathbf{p}, \mathbf{q}, t)}{\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K} + \mathcal{O}(\hbar^0). \quad (\text{A24})$$

The amplitude $\mathcal{F}_{1+}(\mathbf{p})$ can readily be seen to be ultraviolet divergent.

Next we analyze the contribution from $G_-(\mathbf{p}, \mathbf{q}, t)$. One cannot proceed as above because of the infrared divergence in the \mathbf{q} integration as we shall see. We define the variable

$\tilde{\beta}$ in analogy with the variable β in Eq. (A19) by

$$[-\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K]\tilde{\beta} \equiv \int_{-\eta/2}^{\eta/2} [-\sigma_{\mathbf{p}}(t + \hbar\zeta) + \sigma_{\mathbf{q}}(t + \hbar\zeta) + K]d\zeta. \quad (\text{A25})$$

Now, for small $K = \|\mathbf{p} - \mathbf{q}\|$, we have

$$-\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K \approx K - \mathbf{v}(t) \cdot \mathbf{K}, \quad (\text{A26})$$

where $\mathbf{v}(t) = [\mathbf{p} - \mathbf{V}(t)]/\sigma_{\mathbf{p}}(t)$ is the velocity of the classical particle with final momentum \mathbf{p} [see Eq. (36)]. Hence, in the limit $K \rightarrow 0$ one finds

$$\tilde{\beta} = \frac{1}{1 - \mathbf{v}(t) \cdot \mathbf{n}} \int_{-\eta/2}^{\eta/2} [1 - \mathbf{v}(t + \hbar\zeta) \cdot \mathbf{n}]d\zeta, \quad (\text{A27})$$

where $\mathbf{n} \equiv \mathbf{K}/K$. Thus, if we write $d\eta = J(\mathbf{p}, \mathbf{q}, t, \hbar\tilde{\beta})d\tilde{\beta}$, then the function $J(\mathbf{p}, \mathbf{q}, t, \hbar\tilde{\beta})$ is finite as $K \rightarrow 0$. Hence, the expression corresponding to Eq. (A22) can be given in the following form:

$$\begin{aligned} G_{-}(\mathbf{p}, \mathbf{q}, t) &= \int_{-\infty}^0 d\tilde{\beta} \left[f_{-}(\mathbf{p}, \mathbf{q}, t) + \sum_{n,d} \hbar^n \tilde{\beta}^d f_{nd-}(\mathbf{p}, \mathbf{q}, t) \right] \\ &\quad \times \exp\{i[-\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K]\tilde{\beta}\} \\ &= \frac{-if_{-}(\mathbf{p}, \mathbf{q}, t)}{-\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K} \\ &\quad + \sum_{n,d} \frac{(-i)^d d! \hbar^n f_{nd-}(\mathbf{p}, \mathbf{q}, t)}{[-\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K]^{d+1}} \end{aligned} \quad (\text{A28})$$

with $n \geq 2$ and $n \geq d \geq 0$, where $f_{nd-}(\mathbf{p}, \mathbf{q}, t)$ are finite as $K \rightarrow 0$.

Now, if we substitute Eq. (A28) in Eq. (A9), then the terms with $d \geq 1$ are infrared divergent in the \mathbf{q} integration because $\lim_{K \rightarrow 0} [-\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t) + K] \rightarrow 0$ as can be seen from Eq. (A26). (Note that $d^3\mathbf{q} = d^3\mathbf{K}$.) To deal with this difficulty we cut off the integral over \mathbf{q} by requiring $K \geq K_0 = \hbar^\alpha \lambda$ with $\frac{3}{4} < \alpha < 1$, where λ is a positive constant, and postpone the analysis of the contribution from $K \leq K_0$ till later. (The reasoning for this particular choice of limits for α will be seen later). Then, we find that the small- K contribution of each term in Eq. (A28) to the \mathbf{q} integral behaves like $\hbar^n K_0^{1-d} = \hbar^{n+(1-d)\alpha} \lambda^{1-d}$ if $d \geq 2$ and $\hbar^n \log(\hbar^\alpha \lambda)$ if $d = 1$. Now, since $1 - \alpha > 0$, $n \geq 2$, and $n \geq d$, we have $n + (1 - d)\alpha \geq 2 - \alpha$. Therefore, in the $\hbar \rightarrow 0$ limit, the f_{nd-} terms will not contribute above the cutoff, leaving only the first term. Let us thus combine the leading order terms from \mathcal{F}_1 (using G_+ and the first term of G_- , above and below the cutoff) and the result for \mathcal{F}_2 to define

$$\begin{aligned} \mathcal{F}^{\text{leading}}(\mathbf{p}) &= \frac{\hbar^{-2} e^2}{2p_0} \int dt |\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\mathbf{q}}{2\sigma_{\mathbf{q}}(t)(2\pi)^3} \frac{1}{2K} \\ &\quad \times \left\{ -6\sigma_{\mathbf{q}}(t) \right. \\ &\quad + \frac{-[\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t)]^2 + [\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t)]^2}{\sigma_{\mathbf{q}}(t) + K - \sigma_{\mathbf{p}}(t)} \\ &\quad \left. + \frac{-[\sigma_{\mathbf{p}}(t) - \sigma_{\mathbf{q}}(t)]^2 + [\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t)]^2}{\sigma_{\mathbf{q}}(t) + K + \sigma_{\mathbf{p}}(t)} \right\}. \end{aligned} \quad (\text{A29})$$

(For f_{\pm} we have used the relation $|\phi_{\mathbf{q}}|^2 = q_0/\sigma_{\mathbf{q}}(t)[1 + \mathcal{O}(\hbar^2)]$). What remains is the contribution of G_- , minus the first term (which is in $\mathcal{F}^{\text{leading}}$), from below the cutoff. Thus we define

$$\begin{aligned} \mathcal{F}^{<}(\mathbf{p}) &= \frac{ie^2}{2p_0} \int_{K \leq \hbar^\alpha \lambda} \frac{d^3\mathbf{q}}{2q_0(2\pi\hbar)^3} \frac{1}{2K} \\ &\quad \times \int dt_1 dt_2 \theta(t_1 - t_2) e^{-iK(t_1 - t_2)/\hbar} \phi_{\mathbf{p}}^*(t_1) \\ &\quad \times \phi_{\mathbf{p}}(t_2) \overleftrightarrow{\mathcal{D}}_1(t_1, t_2, \mathbf{p}, \mathbf{q}) \phi_{\mathbf{q}}(t_1) \phi_{\mathbf{q}}^*(t_2), \end{aligned} \quad (\text{A30})$$

$$\begin{aligned} \mathcal{F}^{<,0}(\mathbf{p}) &= \frac{\hbar^{-2} e^2}{2p_0} \int dt |\phi_{\mathbf{p}}(t)|^2 \int_{K \leq \hbar^\alpha \lambda} \frac{d^3\mathbf{q}}{2\sigma_{\mathbf{q}}(t)(2\pi)^3} \frac{1}{2K} \\ &\quad \times \frac{-[\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{q}}(t)]^2 + [\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t)]^2}{\sigma_{\mathbf{q}}(t) + K - \sigma_{\mathbf{p}}(t)}, \end{aligned} \quad (\text{A31})$$

with the operator $\overleftrightarrow{\mathcal{D}}_1(t_1, t_2, \mathbf{p}, \mathbf{q})$ defined by Eq. (A8). Hence the forward-scattering amplitude can now be written as follows:

$$\begin{aligned} \mathcal{F}(\mathbf{p}) &= \mathcal{F}^{\text{mass}}(\mathbf{p}) + \mathcal{F}^{\text{leading}}(\mathbf{p}) + \mathcal{F}^{<}(\mathbf{p}) \\ &\quad - \mathcal{F}^{<,0}(\mathbf{p}) + \mathcal{O}(\hbar^{-\alpha}), \end{aligned} \quad (\text{A32})$$

with the contribution from the mass counterterm, $\mathcal{F}^{\text{mass}}(\mathbf{p})$, given by Eq. (A2). Below we show the equality $\mathcal{F}^{\text{leading}}(\mathbf{p}) = -\mathcal{F}^{\text{mass}}(\mathbf{p})$ and demonstrate that the quantity $\mathcal{F}^{<}(\mathbf{p}) - \mathcal{F}^{<,0}(\mathbf{p})$ is of order \hbar^{-1} and is purely imaginary at this order. Then one can conclude that the real part of the forward-scattering amplitude is of order higher than \hbar^{-1} , and hence does not contribute to the position change, as claimed in this paper.

The equality $\mathcal{F}^{\text{leading}}(\mathbf{p}) = -\mathcal{F}^{\text{mass}}(\mathbf{p})$ can readily be obtained by letting $\tilde{\mathbf{q}} \equiv \mathbf{q} - \mathbf{V}(t)$ and $\tilde{\mathbf{p}} \equiv \mathbf{p} - \mathbf{V}(t)$ in Eq. (A29). Then writing $\sigma_{\mathbf{p}}(t) = \sqrt{\tilde{\mathbf{p}}^2 + m^2} = \tilde{p}_0$ and $\sigma_{\mathbf{q}}(t) = \sqrt{\tilde{\mathbf{q}}^2 + m^2} = \tilde{q}_0$, we have

$$\begin{aligned} \mathcal{F}^{\text{leading}}(\mathbf{p}) &= \frac{\hbar^{-2}e^2}{2p_0} \int dt |\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\tilde{\mathbf{q}}}{2\tilde{q}_0(2\pi)^3} \frac{1}{2K} \\ &\times \left\{ -6\tilde{q}_0 + \frac{-(\tilde{p}_0 + \tilde{q}_0)^2 + (\tilde{\mathbf{p}} + \tilde{\mathbf{q}})^2}{\tilde{q}_0 + K - \tilde{p}_0} \right. \\ &\left. + \frac{-(\tilde{p}_0 - \tilde{q}_0)^2 + (\tilde{\mathbf{p}} - \tilde{\mathbf{q}})^2}{\tilde{q}_0 + K + \tilde{p}_0} \right\}. \end{aligned} \quad (\text{A33})$$

This can be seen to be the negative of $\mathcal{F}^{\text{mass}}(\mathbf{p})$ in Eq. (A2) with δm^2 given by Eq. (A4) with \mathbf{q} and \mathbf{p} replaced by $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}$, respectively. Clearly, one can freely rename the integration variable $\tilde{\mathbf{q}}$ as \mathbf{q} . Also, it is well known that δm^2 does not explicitly depend on \mathbf{p} , but only on $p_0^2 - \mathbf{p}^2 = m^2$. Since we have $\tilde{p}_0^2 - \tilde{\mathbf{p}}^2 = m^2$, we may replace $\tilde{\mathbf{p}}$ by \mathbf{p} . Hence we can conclude that $\mathcal{F}^{\text{leading}}(\mathbf{p}) = -\mathcal{F}^{\text{mass}}(\mathbf{p})$.

Next we analyze the contribution $\mathcal{F}^<(\mathbf{p})$. By changing the integration variable from \mathbf{q} to $\mathbf{K} = \mathbf{p} - \mathbf{q}$ and then to $\mathbf{k} = \mathbf{K}/\hbar$, we find

$$\begin{aligned} \mathcal{F}^<(\mathbf{p}) &= \frac{ie^2}{2\hbar p_0} \int dt_1 dt_2 \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{d^3\mathbf{k}}{2q_0(2\pi)^3} \frac{1}{2k} \\ &\times \theta(t_1 - t_2) [\phi_{\mathbf{p}}^*(t_1) \phi_{\mathbf{p}}(t_2) \overleftrightarrow{\mathcal{D}}_1(t_1, t_2, \mathbf{p}, \mathbf{q}) \\ &\times \phi_{\mathbf{q}}(t_1) \phi_{\mathbf{q}}^*(t_2)] e^{-ik(t_1 - t_2)}. \end{aligned} \quad (\text{A34})$$

Now we have $\mathbf{q} = \mathbf{p} - \hbar\mathbf{k}$. (Note also that the upper limit $\hbar^{\alpha-1}\lambda$ of integration for k becomes infinite as \hbar tends to zero.) Hence, we have $\mathbf{q} \rightarrow \mathbf{p}$ for all \mathbf{k} as $\hbar \rightarrow 0$ because $\hbar \cdot \hbar^{\alpha-1}\lambda \rightarrow 0$. The exponential factor takes the form

$$\begin{aligned} &\exp\left\{i \int_0^t d\zeta [K + \sigma_{\mathbf{q}}(\zeta) - \sigma_{\mathbf{p}}(\zeta)]/\hbar\right\} \\ &= \exp\left\{i \int_0^t \left[k - \frac{\partial \sigma_{\mathbf{p}}(\zeta)}{\partial \mathbf{p}} \cdot \mathbf{k} + \frac{1}{2} \frac{\partial^2 \sigma_{\mathbf{p}}(\zeta)}{\partial p^i \partial p^j} \hbar k^i k^j \right. \right. \\ &\left. \left. + \dots \right] d\zeta\right\}. \end{aligned} \quad (\text{A35})$$

Thus, to truncate the series in the exponent at the second term for all \mathbf{k} in the integration range, we need $\hbar(\hbar^{\alpha-1})^2 \rightarrow 0$ as $\hbar \rightarrow 0$. This is satisfied due to the requirement $\alpha > \frac{3}{4}$. Thus we have

$$\begin{aligned} \mathcal{F}^<(\mathbf{p}) &= \frac{ie^2}{\hbar} \int dt_1 dt_2 \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{d^3\mathbf{k}}{2k(2\pi)^3} \theta(t_1 - t_2) \\ &\times \left\{ -1 + \frac{[\mathbf{p} - \mathbf{V}(t_1)] \cdot [\mathbf{p} - \mathbf{V}(t_2)]}{\sigma_{\mathbf{p}}(t_1)\sigma_{\mathbf{p}}(t_2)} \right\} \\ &\times \exp\left[ik(t_2 - t_1) - \int_{t_1}^{t_2} \frac{\mathbf{p} - \mathbf{V}(\zeta)}{\sigma_{\mathbf{p}}(\zeta)} \cdot \mathbf{k} d\zeta \right] \\ &+ \mathcal{O}(\hbar^{4\alpha-4}). \end{aligned} \quad (\text{A36})$$

Since $4\alpha - 4 > -1$ because of the requirement $\alpha > \frac{3}{4}$, the nonleading terms do not contribute to the position shift in the limit $\hbar \rightarrow 0$. We drop this contribution from now on for this reason. Recalling that $[\mathbf{p} - \mathbf{V}(t)]/\sigma_{\mathbf{p}}(t)$ is the velocity of the corresponding classical particle, $d\mathbf{x}/dt$, we obtain at

leading order, in analogy with Eq. (38),

$$\begin{aligned} \mathcal{F}^<(\mathbf{p}) &= \frac{ie^2}{\hbar} \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{d^3\mathbf{k}}{(2\pi)^3 2k} \int_{-\infty}^{+\infty} d\xi \\ &\times \int_{-\infty}^{+\infty} d\xi' \theta(\xi' - \xi) \frac{dx^\mu}{d\xi} \frac{dx_\mu}{d\xi'} e^{ik(\xi' - \xi)}, \end{aligned} \quad (\text{A37})$$

where we have defined $\xi \equiv t_1 - \mathbf{n} \cdot \mathbf{x}(t_1)$ and $\xi' \equiv t_2 - \mathbf{n} \cdot \mathbf{x}(t_2)$ with $\mathbf{n} \equiv \mathbf{k}/k$. If we write the Heaviside function as $\theta(\xi - \xi') = 1/2 + \epsilon(\xi - \xi')/2$, where $\epsilon(\xi' - \xi) = 1$ if $\xi' > \xi$ and $\epsilon(\xi' - \xi) = -1$ if $\xi' < \xi$, then the first and second terms give the imaginary and real parts of $\mathcal{F}^<(\mathbf{p})$, respectively. Thus, twice the imaginary part is

$$\begin{aligned} 2\text{Im}\mathcal{F}^<(\mathbf{p}) &= \frac{ie^2}{\hbar} \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{d^3\mathbf{k}}{(2\pi)^3 2k} \\ &\times \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\xi' \frac{dx^\mu}{d\xi} \frac{dx_\mu}{d\xi'} e^{ik(\xi' - \xi)}. \end{aligned} \quad (\text{A38})$$

This coincides with the emission probability found in Ref. [12] after we let $\hbar^{\alpha-1}\lambda \rightarrow \infty$ as required by unitarity. The real part is given by

$$\begin{aligned} \text{Re}\mathcal{F}^<(\mathbf{p}) &= \frac{ie^2}{2\hbar} \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{kdkd\Omega}{2(2\pi)^3} \int_{-\infty}^{+\infty} d\xi \\ &\times \int_{-\infty}^{+\infty} d\xi' \epsilon(\xi' - \xi) \frac{dx^\mu}{d\xi} \chi(\xi) \frac{dx_\mu}{d\xi'} \\ &\times \chi(\xi') e^{ik(\xi' - \xi)}. \end{aligned} \quad (\text{A39})$$

We have introduced the cutoff function $\chi(\xi)$ as in Eq. (42). Now, we integrate by parts with respect to the variable ξ and add the result of integrating by parts with respect to ξ' , then divide by two. Thus we find

$$\begin{aligned} \text{Re}\mathcal{F}^<(\mathbf{p}) &= -\frac{e^2}{4\hbar} \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{kdkd\Omega}{2(2\pi)^3} \int_{-\infty}^{+\infty} d\xi \\ &\times \int_{-\infty}^{+\infty} d\xi' \left\{ 4\delta(\xi' - \xi) \frac{dx^\mu}{d\xi'} \frac{dx_\mu}{d\xi} \right. \\ &+ \epsilon(\xi' - \xi) \left[\left(\frac{d}{d\xi'} - \frac{d}{d\xi} \right) \frac{dx^\mu}{d\xi'} \chi(\xi') \right. \\ &\left. \left. \times \frac{dx_\mu}{d\xi} \chi(\xi) \right] \right\} e^{ik(\xi' - \xi)}. \end{aligned} \quad (\text{A40})$$

The factor multiplying $e^{ik(\xi' - \xi)}$ in the second term is symmetric in ξ and ξ' . Hence the k integral of $e^{ik(\xi' - \xi)}$ can be made into $\delta(\xi' - \xi)$ by extending the integration range of k to $(-\infty, +\infty)$ and dividing by two in the limit $\hbar \rightarrow 0$. The factor inside the square brackets becomes zero if one lets $\xi' = \xi$. Hence the contribution to $\text{Re}\mathcal{F}^<(\mathbf{p})$ from the second term is of order higher than \hbar^{-1} . Evaluating the first term with the use of $(dx^\mu/dt) \times (dx_\mu/dt) = 1 - \mathbf{v}^2$ and $\dot{\xi} = 1 - \mathbf{n} \cdot \mathbf{v}$, we find

$$\text{Re } \mathcal{F}^{<}(\mathbf{p}) = -\frac{e^2 \lambda}{16\pi^3 \hbar^{2-\alpha}} \int_{-\infty}^{+\infty} dt \int d\Omega \frac{1 - \mathbf{v}^2}{1 - \mathbf{n} \cdot \mathbf{v}} \quad (\text{A41})$$

to order \hbar^{-1} . Now, let us consider $\mathcal{F}^{<,0}(\mathbf{p})$. By substituting the small- K approximation (A26) in Eq. (A31) and noting the equations $|\phi_{\mathbf{p}}(t)|^2 = p_0/\sigma_{\mathbf{p}}(t)$, $\sigma_{\mathbf{p}}(t) = m/\sqrt{1 - \mathbf{v}^2}$, $[\mathbf{p} - \mathbf{V}(t)]/\sigma_{\mathbf{p}}(t) = \mathbf{v}$, and $[\sigma_{\mathbf{p}}(t)]^2 - [\mathbf{p} - \mathbf{V}(t)]^2 = m^2$,

we find that $\mathcal{F}^{<,0}(\mathbf{p})$ is indeed equal to the leading term of $\text{Re}\mathcal{F}^{<}(\mathbf{p})$ given by Eq. (A41).

Thus, we have shown the equality $\mathcal{F}^{\text{mass}}(\mathbf{p}) + \mathcal{F}^{\text{leading}}(\mathbf{p}) = 0$ and demonstrated that $\mathcal{F}^{<}(\mathbf{p}) - \mathcal{F}^{<,0}(\mathbf{p})$ is of order \hbar^{-1} but is purely imaginary at this order. Therefore, from Eq. (A32) we conclude that the real part of the forward-scattering amplitude, $\text{Re}\mathcal{F}(\mathbf{p})$, is of order higher than \hbar^{-1} , and hence does not contribute to the change of position of the scalar particle given by Eq. (19).

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