Particle decay in false vacuum

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We revisit the problem of decay of a metastable vacuum induced by the presence of a particle. For the bosons of the "master field" the problem is solved in any number of dimensions in terms of the spontaneous decay rate of the false vacuum, while for a fermion we find a closed expression for the decay rate in (1 + 1) dimensions. It is shown that in the (1 + 1) dimensional case an infrared problem of one-loop correction to the decay rate of a boson is resolved due to a cancellation between soft modes of the field. We also find the boson decay rate in the "sine-Gordon staircase" model in the limits of strong and weak coupling.

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I. INTRODUCTION

The decay of a metastable vacuum state is a quite universal problem in quantum field theory. The decay proceeds through nucleation and subsequent classical expansion of the bubbles of the true vacuum. The classical bubbles can exist only starting from a certain critical radius at which the energy loss due to the surface terms is compensated by the gain in the volume energy. The formation of the critical bubbles is thus a quantum tunneling process [1] which tunneling can be described by an Euclidean-space configuration of the field, called a "bounce" [2]. The space-time nucleation rate of the critical bubbles, w_0, i.e. the probability of such nucleation per unit time and per unit volume, is proportional to the exponent of the classical action on the bounce configuration: $w_0 \propto \exp(-S_{\rm cl})$, while the preexponential factor requires a calculation of the functional determinant at the bounce [3]. The exponential factor is readily found [1,2] in the so called thin wall limit, namely, when the radius of the critical bubble is much bigger than the effective thickness of its wall. This limit is always realized at a small difference ϵ between the vacuum energy density of the false and the true vacua, with the other parameter determining $S_{\rm cl}$ being the surface tension μ of the bubble wall, i.e. of the boundary between the metastable and the stable phases. The preexponential factor in the bubble nucleation rate is known in a closed form only in (1 + 1) dimensional models [4,5], where in the thin wall limit it is determined only by the parameter ϵ, with only partial results found in (2 + 1) dimensions [6–8], and virtually no result known in the (3 + 1) dimensional case.

Similarly to the behavior in the decay of a metastable phase in a thermal setting, the presence of matter in the false vacuum generally provides "centers of nucleation" for the bubbles of the true vacuum. Thus one can consider the false vacuum decay induced by the presence of a particle[9–11], by particle collisions [10,11], by matter with finite density [12], as well as by the matter being in a thermal equilibrium where the problem goes back to the more conventional thermodynamic setting [13]. In this paper we revisit the calculation of the bubble nucleation rate associated with the presence of a particle in the false vacuum. The particle-induced nucleation can also be viewed as the decay of the particle (albeit in the process the initial "vacuum" state also gets destroyed), whose rate Γ generically can be written in the form $\Gamma = K w_0$, where the constant K, which can be naturally called the "catalysis factor," is the main subject of our consideration. The catalysis is most efficient for the particles which have zero modes localized on the boundary between the false and the true vacua. The reason for this behavior is that in this case the energy corresponding to the mass of the particle m in the initial state is fully transferred to the bubble degrees of freedom, since in the final state the particle ends up as a zero mode localized on the bubble wall. This effectively corresponds to the upward shift by m of the energy at which the tunneling takes place [10], and results in K being proportional to the exponential factor $\exp(2m\tau)$, where τ is the (Euclidean) time on the tunneling trajectory. The exponential behavior due to the shift of the energy for the tunneling trajectory can be found explicitly both in (1 + 1) dimensions [10] and in higher-dimensional models [11]. However the preexponential factor has been calculated only for the bosons of the master field in a (3 + 1) dimensional case [9], for which bosons the existence of the zero mode is always true. Here we calculate the preexponential behavior of the catalysis factor for the same bosons in lower dimensions, and also find a closed formula for this factor in (1 + 1) dimensions for a fermion, whose field has a zero mode on the intervacua boundary. The existence of such fermionic mode is a generic phenomenon and is guaranteed in the case where the mass term for the fermions changes sign across the bubble wall [14].

Our consideration, similarly to Ref. [9], is generally limited to models with weak coupling, which implies that the masses m of the both types of considered particles are small in comparison with the scale of the surface tension μ. In this case the deformation of the tunneling trajectory due to the energy shift by m [10,11] can be neglected, so that, in particular, the tunneling time τ coincides with the radius $R \propto \mu/\epsilon$ of the critical bubble, $\tau = R$, as it does in the spontaneous vacuum decay [2]. Furthermore, we also assume the applicability of the thin wall limit, which implies the condition $mR \gg 1$, and which is always valid in the limit of small ϵ.

The catalysis factor K, as defined, has the dimension of the spatial volume. Thus it would be natural to compare the preexponential factor in K with the spatial volume of the critical bubble of the radius R. Under our assumptions we find that for a fermion in $(1 + 1)$ dimensions this factor is indeed of order R, while the catalysis factor for the bosons is enhanced in comparison with the volume of the bubble by inverse powers of the (small) coupling constant.

It should be noted that technically the bosonic catalysis factor in lower dimensions can be found by a straightforward application of the treatment of Ref. [9]. Such application is fully justified in a $(2 + 1)$ dimensional case. However in $(1 + 1)$ dimensions there is a potential complication in estimating the effect of the quantum fluctuations arising from an infrared behavior of the modes of the bosonic field over the bounce background. We demonstrate for this case that the large infrared terms in fact cancel due to the specific properties of the soft modes.

The material in the rest of the paper is organized as follows. In Sec. II we briefly review the calculation of the spontaneous decay rate of false vacuum and present a calculation of the decay rate induced by a boson of the scalar field, which defines the vacuum states. In Sec. III the problem of the infrared behavior of the one-loop correction to the calculated decay rate in $(1 + 1)$ dimensions is considered and it is shown that this problem is resolved due to a cancellation of the contributions to this correction between the negative mode and the sum over the positive soft modes of the field of the bounce. In Sec. IV the catalysis factor is calculated for a fermion in a $(1 + 1)$ model. We then discuss the decay of metastable states in the sine-Gordon model with added linear term, the so called "sine-Gordon staircase." Using the equivalence [15] of this bosonic model and the massive fermionic Thirring model in an external electric field, we find the induced decay rate for both the weak coupling limit (Sec. V) and for the strong coupling limit (Sec. VI), the latter corresponding to a weak coupling in the Thirring model. Finally, in Sec. VII we discuss possible implications of our calculation for other models.

II. SPONTANEOUS AND INDUCED DECAY OF FALSE VACUUM

In what follows we assume a situation where the energy density of a scalar field ϕ, the "master field," has a local minimum at $\phi = \phi_+$, which is higher than in a neighboring minimum at $\phi = \phi_-$. The vacuum state defined by the former minimum is referred to as the false vacuum, while the latter is the true vacuum. A typical example of such situation is provided by the well known model of a scalar field with the potential

$$V(\phi) = \frac{\lambda^2}{8}(\phi^2 - v^2)^2 + a\phi, \tag{1}$$

where λ, v, and a are constants. At $a = 0$ the potential has two degenerate minima at $\phi_\pm = \pm v$, while at small positive a the degeneracy is lifted in such a way that the minimum at ϕ_+ has energy density bigger than that of ϕ_- by the amount $\epsilon \approx 2av$. The vacuum state at ϕ_+, being stable at $a \leq 0$ becomes metastable at positive a and decays by nucleation and subsequent expansion of the bubbles filled with the phase ϕ_-. At small a the surface density of the bubble wall can be approximated [1,2] by the surface density of the soliton with the field profile

$$\phi(x) = v \tanh\frac{mx}{2} \tag{2}$$

interpolating between the two degenerate vacua in the limit $a \to 0$:

$$\mu = \int_{-v}^{+v} \sqrt{2V(\phi)}d\phi = \frac{2}{3}\lambda v^3. \tag{3}$$

The mass of the scalar particles of the field ϕ propagating in either of the vacua is given (also in the limit $a \to 0$) as $m = \lambda v$. In a model with the total space-time dimensions equal to d the ratio m^{d-1}/μ coincides with the dimensionless coupling constant for the perturbation theory in this model. We assume throughout this paper that this ratio is a small parameter, which thus corresponds to weak coupling.

In the Euclidean-space formulation of the problem of the false vacuum decay [2,3] the calculation of the spontaneous decay rate amounts to a semiclassical evaluation of the imaginary part of the energy of the false vacuum from the path integral

$$Z = \mathcal{N} \int e^{-S[\phi,\ldots]}\mathcal{D}\phi \ldots, \tag{4}$$

where the dots stand for other possible fields present in a specific model, $\mathcal{N}$ is the normalization factor, and the integration is performed with the condition that the field ϕ approaches its false vacuum value ϕ_+ at the boundaries of the space-time normalization box. The decay rate is then given by $w_0 = 2\,\mathrm{Im}(\ln Z)/VT$, where VT is the space-time volume of the normalization box.

The action functional S has a semiclassical saddle point at the configuration described by the bounce [2]. In the thin wall limit the bounce is an $O(d)$ symmetric bubble with the field ϕ_- inside and ϕ_+ outside, and the bubble wall, separating the two phases has the surface tension μ. The action for the bounce in this approximation is given by

$$S = \mu A_B - \epsilon V_B, \tag{5}$$

where V_B is the d dimensional volume of the bounce and A_B is its $(d - 1)$ dimensional surface area. The action (5) reaches its extremum on a spherical bounce with the radius

$R = (d-1)\mu/\epsilon$, which is also the radius of the critical bubbles capable of classical expansion in the Minkowski space-time.

The spectrum of small deformations of the bounce around the extremum contains exactly one negative mode, corresponding to an overall variation of the radius. This mode in fact gives rise to the imaginary part [2] of the path integral in Eq. (4). Furthermore this spectrum also contains d translational zero modes, the integration over which introduces the factor of the space-time volume VT in the contribution of the bounce to the energy of the vacuum state.

The decay rate of a particle of the field ϕ in the false vacuum can be calculated[9,10] by considering the imaginary part of the contribution of a bounce to the Euclidean-space propagator of the excitations $\sigma(x) = \phi(x) - \phi_+$ of the field ϕ:

$$D(x, y) = \frac{1}{Z}\int \sigma(x)\sigma(y)e^{-S[\phi,\ldots]}\mathcal{D}\phi \ldots \tag{6}$$

in the limit of large separation $L = |x - y|$. Indeed, the contribution of the bounce to the correlator (6), as shown in Fig. 1(a), has the generic form

$$\delta D(x, y) = \frac{i}{2}w_0\int d^d z F(x - z, y - z)D_0(x - z)D_0(y - z), \tag{7}$$

where $D_0(x)$ is the free-particle propagator in the vacuum ϕ_+, satisfying the equation

$$(-\partial^2 + m^2)D(x) = \delta^{(d)}(x) \tag{8}$$

and the factor $(i/2)w_0 d^d z$ is the proper measure of integration over the coordinate z of the center of the bounce, as follows from the consideration of the bounce contribution to (the imaginary part of) the vacuum energy.

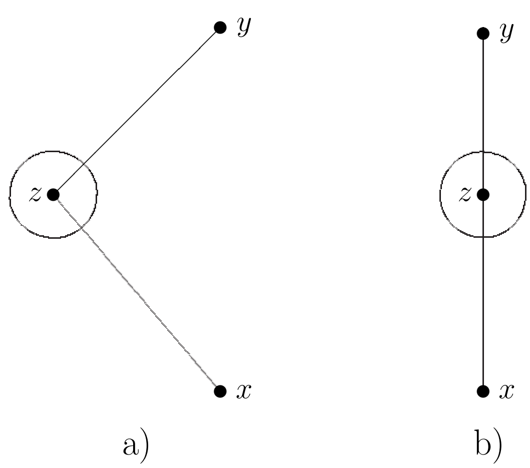

FIG. 1. The configurations for the bounce contribution to the propagator (Eq. (7)). A generic configuration (a) and the alignment of the bounce position (b), dominating the integral in Eq. (7) at large $|x - y|$.

Let us consider the contribution to the integral (7) arising from the configurations, where the bounce is far (in units of its radius) from either of the points x and y, i.e. where $|x - z| \gg R$ and $|y - z| \gg R$. The propagators $D_0(x - z)$ and $D_0(y - z)$ in the integral in Eq. (7) describe the exponential attenuation of the correlation ($D(x) \sim \exp(-m|x|)$ at large separations, while the form factor $F(x - z, y - z)$ does not have this exponential behavior. For this reason at $|x - y| = L \gg R$ the integrand in Eq. (7) is maximized for z lying on the straight line running between x and y: $z_\nu = s(x_\nu - y_\nu)/L$, and the integration can be split into the longitudinal, over the parameter s along this line, and the transversal, over $z_\perp$. The integration over $z_\perp$ can be done by the saddle point method, so that the form factor $F(x - z, y - z)$ can be replaced by its value at $z_\perp = 0$, and the essential configuration to be considered is the one shown in Fig. 1(b). As will be discussed few lines below, when the bounce is far from the endpoints of integration over s, i.e. $s \gg R$ and $L - s \gg R$, the value of the form factor in fact does not depend on s and is a constant F_0. Since the contribution of the excluded regions around the endpoints is only of relative order R/L, the integral in Eq. (7) can be replaced at large L by

$$\delta D(x, y) = \frac{i}{2}w_0 F_0\int d^d z D_0(x - z)D_0(y - z), \tag{9}$$

where F_0 should be calculated from the configuration shown in Fig. 1b.

The expression (9) for the modification δD of the propagator by the bounce can be compared with the first-order correction to the propagator due to a small shift of mass by δm^2, $m^2 \to m^2 + \delta m^2$, in Eq. (8). In the standard way one finds

$$\delta_m D(x, y) = -\delta m^2\int d^d z D_0(x - z)D_0(y - z). \tag{10}$$

Thus the contribution (9) of the bounce to the propagator of the boson in the false vacuum is equivalent to an imaginary shift of the boson mass: $\delta m^2 = -(i/2)w_0F_0$, which corresponds to the particle decay rate given by $\Gamma = F_0w_0/(2m)$, so that the catalysis factor K is found as

$$K = \frac{F_0}{2m}. \tag{11}$$

The factor F_0 can be readily found [9] for the discussed here case of the bosons of the classical field of the bounce. Indeed, consider the classical field just outside the bounce, i.e. at the distance $r > R$ from the center, such that $r - R \gg m^{-1}$, but still $r - R \ll R$. The former condition ensures that the field is described by its asymptotic approach to the vacuum value ϕ_+, while the latter implies that in this region the curvature of the bounce wall can be neglected in a calculation of this asymptotic behavior. Thus one can consider instead the asymptotic behavior of the field in the limit $\epsilon \to 0$, i.e. of the field of the stable soliton separating two degenerate vacua. This asymptotic behavior has the form $\phi(x) - \phi_+ = -2v \exp[-m(r-R)]$, where in the model described by the potential (1) v coincides with the corresponding parameter in the potential, while in a generic model $v \sim (\phi_+ - \phi_-)/2$. On the other hand in the $O(d)$-symmetric problem the asymptotic approach of the scalar field to its vacuum value is described by the solution of the linearized spherically-symmetric equation, equivalent to the homogeneous part of Eq. (8), and reads as

$$\phi(r) - \phi_+ = CD_0(r), \tag{12}$$

where the free boson propagator in d dimensions has the well known expression in terms of the modified Bessel function $K_\nu(mr)$:

$$D_0(r) = \frac{m^{d/2-1}}{(2\pi)^{d/2} r^{d/2-1}} K_{d/2-1}(mr). \tag{13}$$

The constant C in the asymptotic expression (12) is found by comparing the two expressions for $\phi(x) - \phi_+$ in the discussed region just outside the bounce and using the standard asymptotic formula for the function $K_\nu(mr)$. In this way one finds

$$C = -4(2\pi)^{d/2-1} m^{(3-d)/2} R^{(d-1)/2} v e^{mR}. \tag{14}$$

Using then the expression (12) for the field with thus determined constant C, one finds the product of the classical fields $\sigma(x)\sigma(y)$ in the integral in Eq. (6) in the configuration shown in Fig. 1(b), corresponding to the constant F_0 in Eq. (9) given by

$$F_0 = C^2 = 16(2\pi)^{d-2} m^{3-d} R^{(d-1)} v^2 e^{2mR}, \tag{15}$$

which indeed does not depend on the position of the bubble along the straight line connecting the points x and y as long as both these points are sufficiently outside the bounce. The catalysis factor thus can be found from the relations (11) and (15) in the form

$$K = 2^{d+1} \pi^{(d-3)/2} \Gamma\left(\frac{d+1}{2}\right) m^{2-d} v^2 V_{d-1} e^{2mR}, \tag{16}$$

where $V_{d-1} = \pi^{(d-1)/2} R^{d-1} / \Gamma[(d+1)/2]$ is the spatial ($d-1$ dimensional) volume of the critical bubble. As discussed in the introduction, it is natural to compare the catalysis factor with this volume. The result in Eq. (16) shows that besides the classical exponential factor the catalysis is additionally enhanced by the factor $m^{2-d} v^2$ in the preexponent, which is the inverse of the small dimensionless coupling in the theory.

III. BOSON-INDUCED DECAY IN (1 + 1) DIMENSIONS

The formula for the catalysis factor in Eq. (16) reduces in (3 + 1) dimensions to the result of Ref. [9], and in other dimensions it presents a rather straightforward generalization. There is however one point, of a special importance to a (1 + 1) dimensional case, related to the effect of the quantum fluctuations on the essentially classical result in Eq. (16). Generally, the effect of the quantum fluctuations (the loop correction) is expected to be suppressed by a power of the coupling constant as compared to the classical contribution. In the discussed problem this expectation is true at $d > 2$, however in a (1 + 1) dimensional problem this expectation is potentially jeopardized by an infrared behavior. Indeed the eigenvalues of the second variation of the action for the fluctuations of the shape of the bounce, described by the effective action (5), are proportional to R^{-2}. The modes with these eigenvalues are localized on the bounce boundary and describe the soft part of the spectrum of the modes of the field around the stationary bounce configuration, as opposed to the modes, whose eigenvalues start at $O(m)$, and those "hard" modes describe the excitations propagating in the bulk as well as possible deformations of the profile of the field across the bounce wall. Let us estimate the contribution of an individual soft mode with the eigenvalue c_n/R^2 to the correlator (6), with c_n being a number. All such modes originate from local shifts of the wall of the bounce, so that the field profile of an individual mode is proportional to the radial derivative of the field of the bounce, $\phi'(r)$. The field σ_n of a normalized to one mode in (1+1) dimensions is then parametrically estimated at the distance $r > R$, such that $r - R \gg m^{-1}$, but still $r - R \ll R$, as

$$\sigma_n(r) \sim \frac{mv}{\sqrt{\mu R}} e^{-m(r-R)} \sim \sqrt{\frac{m}{R}} e^{-m(r-R)}, \tag{17}$$

where it is taken into account that $\int(\phi')^2 dr \approx \mu \sim mv^2$, and any numerical factors are dropped for a parametrical estimate. The contribution of such mode to the correlator (6) is then proportional to

$$\frac{R^2}{c_n} \sigma_n(r_1)\sigma_n(r_2) \sim c_n^{-1}(mR) e^{2mR} e^{-m(r_1+r_2)}. \tag{18}$$

In this estimate the large factor (mR) stands in place of the factor v^2 in the similar product for the classical field. Thus the magnitude of an individual mode contribution to the correlator relative to the classical part is described by the parameter $\frac{(mR)}{v^2}$, which in spite of the expected suppression by the dimensionless coupling v^{-2} is infrared unstable at large R.

We will show however that the sum over the soft modes gives zero for this infrared contribution due to a cancellation between one negative and all positive modes. In order to demonstrate this we consider the parametrization of the shape of the bounce in the polar coordinates (r, θ) on a plane, so that the effective action (5) for the soft modes takes the form

$$S = \int_0^{2\pi}\left(\mu\sqrt{r^2+\dot r^2} - \frac{1}{2}\epsilon r^2\right)d\theta = \frac{\pi\mu^2}{\epsilon} + \int_0^{2\pi}\frac{\epsilon}{2}(\dot\rho^2-\rho^2)d\theta + O(\rho^4), \tag{19}$$

where the latter expression shows the two first terms of expansion in the small deviation $\rho = r - R$ of the radial variable r from its stationary value $R = \mu/\epsilon$, and the dot stands for the derivative over θ. The quadratic part in this expression has one negative eigenmode $\rho = 1/\sqrt{2\pi}$ and the spectrum of zero and positive double degenerate eigenmodes:

$$\rho_n^{(1)} = \frac{1}{\sqrt{\pi}}\cos n\theta, \quad \text{and} \quad \rho_n^{(2)} = \frac{1}{\sqrt{\pi}}\sin n\theta; (n = 1, 2, \ldots). \tag{20}$$

The spectrum of the eigenvalues is proportional[1] to $(n^2 - 1)$ with the negative mode corresponding to $n = 0$.

Let us consider now the configuration shown in Fig. 1(b) with the bounce located on the line connecting the points x and y. Let the angle θ be defined as measured counterclockwise from the downward vertical connecting the center of the bounce with the point x, so that $\theta = \pi$ corresponds to the upward vertical connecting the same center with the point y. Clearly, the contribution of the fluctuations of ρ to the propagator (6) is proportional to

$$\langle[\rho(0)+\rho(\pi)]^2\rangle \propto \sum_n \frac{[\rho_n(0)+\rho_n(\pi)]^2}{n^2-1}. \tag{21}$$

Note however that the sum $\rho(0) + \rho(\pi)$ is not vanishing only for the negative mode and for the positive modes of the first type, $\rho_n^{(1)}$, with even n, i.e. $n = 2k$. Thus the sum in Eq. (21) is proportional to the numeric sum

$$-\frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{4k^2-1} = 0, \tag{22}$$

where the first term is due to the negative mode and the sum runs over the positive modes. The arithmetic identity (22) explicitly demonstrates that the infrared contribution in a $(1 + 1)$ dimensional model cancels between the negative mode and the sum over the positive ones.

Let us also remark on the decay of a moving particle in the false vacuum. If particle moves with a constant velocity then the probability depends on the velocity through the standard Lorentz factor. However if it moves with the constant acceleration situation is more subtle since in the particle frame vacuum behaves as the thermal bath due to Unruh effect. The effective temperature is defined through the acceleration as

$$T_{\text{eff}} = \frac{a}{2\pi}, \tag{23}$$

hence probability of the particle decay is modulated by the thermal effects. The most essential effect corresponds to the possible deformation of the classical bounce. Since the temperature corresponds to the periodicity in the Euclidean time, the deformation of the bounce happens when the corresponding period is comparable to $2R$. Thus a deformation of the bubble is essential when the acceleration is larger than

$$a_{\text{crit}} = \frac{\pi\epsilon}{\mu}, \tag{24}$$

and our approximation fails.

IV. FERMION-INDUCED DECAY IN (1 + 1) DIMENSIONS

The very existence of fermions in a model is known to modify (without any fermions being present in the initial state) the preexponential factor in the rate of the false vacuum decay in the situation where the complex fermion field ψ has a zero mode on the boundary between the vacua (in the limit $\epsilon \to 0$). Such situation takes place when the mass term for the fermion changes sign between the two vacua [14]. In particular the rate w_0 for the spontaneous decay of the false vacuum in $(1 + 1)$ dimensions receives a factor of 2 in comparison with purely bosonic theory [16,17]. This doubling corresponds to the existence of two final states in the false vacuum decay in $(1 + 1)$ dimensions viewed as a spontaneous creation of a kink-antikink pair: one state where both the kink and the antikink are created with the fermion zero mode empty, and the other state is where is a zero-energy fermion on the kink and a zero-energy antifermion on the antikink.

In what follows we assume that the interaction of the fermion field with the scalar field of the bounce is such that there exists a zero fermion mode on the kink separating the two vacua. In order to find the effect of a fermion on the probability of nucleation of a critical bubble we consider the bounce contribution to the fermion propagator $G(x, y) = \langle\psi(x)\bar\psi(y)\rangle$ in the configuration shown in Fig. 1(b). Clearly, the exponentially enhanced factor $\exp(2m_f R)$, where m_f is the mass of the fermion, arises from the contribution of the zero mode, whose propagation from the lower point of the bounce to the upper one does not contain any exponential attenuation. Assuming for definiteness that the fermion mass is positive (and equals m_f) in the false vacuum, and choosing the γ matrices as $\gamma_1 = \sigma_1$ and $\gamma_2 = \sigma_2$, one finds the solution for the Dirac equation for the field of the zero mode of ψ in the background scalar field $\phi(r)$ of the bounce,

$$[\sigma_i\partial_i + m(\phi)]\psi_0 = 0, \tag{25}$$

in the form

[1] The proportionality coefficient is not important for this discussion. It can be noted however that in terms of the normalized modes for the field ϕ the eigenvalues are $(n^2 - 1)/R^2$.

$$\psi_0(r,\theta) = C_f\sqrt{\frac{R}{r}}\exp\left\{-\int_R^r m[\phi(r')]dr'\right\}\chi(\ell)\begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2}\end{pmatrix}, \tag{26}$$

where $\chi(\ell)$ is a one-dimensional fermion field living on the bounce boundary and (nominally) depending on the length parameter $\ell = R\theta$ along the boundary. Notice that the classical equation for χ reads $\dot\chi = 0$. Finally, the constant C_f in Eq. (26) is the normalization factor relating the normalization of ψ and χ and satisfying the condition

$$2C_f^2\int\exp\left[-2\int_R^r m(\phi)dr'\right]dr = 1. \tag{27}$$

In what follows we rather use a related factor $\tilde C_f$ defined as the coefficient in the expression

$$C_f\exp\left\{-\int_R^r m[\phi(r')]dr'\right\} \approx \tilde C_f\exp[m_f(R-r)], \tag{28}$$

which is valid sufficiently far outside the bounce where $m_{\min}(r-R) \gg 1$ with $m_{\min}$ being the minimal mass scale in the model. Generally the factor $\tilde C_f$ can be estimated as

$$\tilde C_f^2 = \frac{m_f}{2}f\left(\frac{m_f}{m}\right), \tag{29}$$

where f is a dimensionless function of the ratio of m_f to masses of other particles in the false vacuum. In the limit where m_f is much smaller than other masses one has $f(0) = 1$. In the model, where the scalar "master field" is described by the potential (1) and the mass of the fermion is proportional to ϕ, the function f can be found explicitly:

$$f(u) = \frac{2^{2u}}{\sqrt{\pi}}\frac{\Gamma(u+1/2)}{\Gamma(u+1)}. \tag{30}$$

The contribution of the fermion zero mode on the bounce configuration shown in Fig. 1(b) can be written, using the asymptotic behavior of the zero mode (26), in terms of the one-dimensional propagator of the field χ on the boundary, $g(\ell_1,\ell_2) = \langle\chi(\ell_1)\chi^\dagger(\ell_2)\rangle$, as

$$\delta G(x,y) = -\frac{i}{2}\frac{w_0}{2}d^2z\tilde C_f^2 e^{2m_fR}R\frac{e^{-|x-y|}}{\sqrt{|x-z||y-z|}} \times (1+\sigma_1)g(0,\pi R). \tag{31}$$

Notice that for a complex fermion there is only one path for propagation of χ along the boundary from the bottom of the bounce to its top (assumed here for definiteness to be counterclockwise in terms of Fig. 1(b)], corresponding to the final state, where the fermion is a bound state localized on the kink. The other path (clockwise) would be relevant for the vacuum decay induced by an antifermion, which in the final state is localized on the antikink. The expression in Eq. (31) contains an extra factor $1/2$, due to the fact that, as mentioned before, the spontaneous nucleation rate w_0 in the theory with fermions contains extra factor of 2, due to the existence of two final states in the decay, which is to be compensated in the proper measure of integration over the coordinate of the center of the bounce d^2z. The propagator g has a very simple explicit form in terms of the sign function: $g(\ell_1,\ell_2) = (1/2)\text{sign}(\ell_1-\ell_2)$, so that $g(0,\pi R) = -1/2$.

The expression in Eq. (31) can now be compared with the corresponding change of the free propagator G_0 under a shift δm_f of the fermion mass:

$$\delta_m G(x,y) = -\delta m_f d^2z G_0(x-z)G_0(z-y) \to -\delta m_f d^2z\frac{m}{4\pi}(1+\sigma_1)\frac{e^{-|x-y|}}{\sqrt{|x-z||y-z|}}, \tag{32}$$

where the asymptotic expression takes into account the explicit form of the free propagator:

$$G_0(x,y) = \frac{1}{2\pi}(-\sigma_i\partial_i + m)K_0(m_f|x-y|). \tag{33}$$

Using this comparison and Eq. (29) one finds the imaginary part of the fermion mass shift corresponding to the decay rate of the fermion

$$\Gamma_f = \frac{\pi}{2}f\left(\frac{m_f}{m}\right)Rw_0\exp(2m_fR) = \frac{\mu}{2}f\left(\frac{m_f}{m}\right)\exp\left(-\frac{\pi\mu^2}{\epsilon} + 2m_fR\right). \tag{34}$$

Here in the latter transition are used the explicit expressions: $w_0 = (\epsilon/\pi)\exp(-\pi\mu^2/\epsilon)$ and $R = \mu/\epsilon$. One can readily see that in the fermion case, as expected, the preexponent in the catalysis factor, $K_f = (\pi/2)f(m_f/m)R\exp(2m_fR)$, is indeed of the order of the spatial size of the critical bubble.

V. MESON DECAY IN WEAKLY COUPLED SINE-GORDON MODEL

In order illustrate the universality of the derived results let us discuss one more example where the particle decay in the false vacuum happens in two dimensions. We shall derive the probability of the decay of the electrically neutral meson bound state in the Thirring model with the preexponential accuracy.

Consider the sine-Gordon theory with the Lagrangian

$$L_{\rm SG} = \frac{1}{2}(\partial\phi)^2 + \frac{\alpha}{\beta^2}\cos(\beta\phi) \tag{35}$$

and add a term $(\epsilon\beta/2\pi)\phi$ which yields the situation with the metastable states. This theory upon the two-dimensional fermionization is equivalent to the massive Thirring model with the Lagrangian

$$L_{\rm Th} = i\bar{\psi}\partial_\nu\gamma^\nu\psi - \frac{1}{2}gj^\nu j_\nu + \mu\bar{\psi}\psi + A_0 j_0, \qquad (36)$$

where $\frac{\beta^2}{4\pi} = (1+\frac{g}{\pi})^{-1}$, and $j_\nu = \bar{\psi}\gamma_\nu\psi$. One can identify μ with the soliton mass in the sine-Gordon model, and $\partial_x A_0 = \epsilon$. In what follows we shall assume that $\beta^2 < 4\pi$ which is the condition for the bound state of fermions to exist in the Thirring model. The solitons in the sine-Gordon model get mapped into the fermions in the Thirring model while the field ϕ gets mapped into the fermion-antifermion meson bound state.

The linear perturbation term in the sine-Gordon model corresponds to the constant electric field in the Thirring model realization, so that the problem of false vacuum decay can be discussed in both formulations. In the Thirring model it corresponds to the Schwinger pair production. The probability of the spontaneous vacuum decay in the sine-Gordon model and equivalent Schwinger process in the Thirring model has been found in [15]. The one-loop result coincides with the general formula $w_{\rm Th} = (\epsilon/2\pi)\exp(-\pi\mu^2/\epsilon)$, while in the special case of $\beta^2 = 4\pi$, corresponding to $g = 0$, the exact result is found as

$$w_{\rm Th} = -\frac{\epsilon}{2\pi}\ln(1 - e^{-\pi\mu^2/\epsilon}). \qquad (37)$$

Note that in this case the bose-fermi equivalence allows to perform the summation over the multiple bounces in the sine-Gordon theory.

Now we can discuss the decay of the false vacuum in the presence of a particle corresponding to the field ϕ in the sine-Gordon model. In the weak coupling regime for the bosons, i.e. at small β, the soliton is much heavier than the boson particle, which corresponds to the situation where the external particle does not deform the classical bounce configuration. Hence the decay rate can be immediately read off Eq. (16). In this case the process corresponds in the Thirring model to the nonperturbative decay of the light electrically neutral meson in the electric field and the catalysis factor of this process is

$$K_{\rm Th} = \frac{32}{\beta^2}\frac{\mu}{\epsilon}e^{2m_b\mu/\epsilon}, \qquad (38)$$

where m_b is the meson mass. Note that this process is the two-dimensional counterpart of the induced Schwinger processes discussed in four dimensions in [18,19] in the exponential approximation.

VI. MESON DECAY IN STRONGLY COUPLED SINE-GORDON MODEL

The boson-fermion correspondence in this model actually allows to find the meson decay rate in the limit, opposite to what has been considered so far in this paper, namely, for strongly coupled bosons, when the boson mass is close to the kink-antikink threshold. This limit corresponds to a small positive g, and the boson mass (at $\epsilon \to 0$) is $m_b = 2\mu - \mu g^2$. The near-threshold dynamics of the soliton-antisoliton pair can be considered nonrelativistically as a motion of a pair with the reduced mass $\mu/2$ in the local potential $U(x) = -2g\delta(x)$, which correctly reproduces the energy of the bound state (the boson). For a nonzero ϵ the nonrelativistic Hamiltonian for this system takes the form:

$$H = \frac{p^2}{\mu} - \epsilon x - 2g\delta(x). \qquad (39)$$

The problem of the boson decay in the false vacuum is reduced in terms of this equivalent nonrelativistic system to that of ionization of the bound state in the external electric field ϵ.

In order to solve the ionization problem we start with considering the Euclidean time propagator ("the heat kernel") defined as $\mathcal{K}(x, y; \tau) = \langle x|\exp(-H\tau)|y\rangle$, and the corresponding energy dependent Green's function at the negative (i.e. below the threshold) energy $E = -\kappa^2/\mu$:

$$G\left(x, y; -\frac{\kappa^2}{\mu}\right) = \int_0^\infty \mathcal{K}(x, y; \tau)\exp\left(-\frac{\kappa^2}{\mu}\tau\right)d\tau \qquad (40)$$

at $x = 0$ and $y = 0$. We remind that if only the kinetic term is retained in the Hamiltonian (39), i.e. at $\epsilon = 0$ and $g = 0$, these functions are $\mathcal{K}_0(0, 0; \tau) = (4\pi\tau/\mu)^{-1/2}$ and $G_0(0, 0; -\kappa^2/\mu) = \mu/(2\kappa)$. At $\epsilon = 0$ and a nonzero g the Green's function for the Hamiltonian (39) is found as

$$G_{\epsilon=0}(0, 0; -\kappa^2/\mu) = \frac{G_0(0, 0; -\kappa^2/\mu)}{1 - 2gG_0(0, 0; -\kappa^2/\mu)} = \frac{\kappa}{2\mu}\frac{1}{1 - g\mu/\kappa}. \qquad (41)$$

The latter expression contains explicitly the pole at $\kappa = \mu g$ corresponding to the bound state.

When both the ϵ and g are nonzero the Eq. (41) is modified by replacing the Green's function $G_{\epsilon=0}$ by that for a nonvanishing ϵ: G_ϵ. The latter Green's function can be expressed in terms of the corresponding propagator $\mathcal{K}_\epsilon(0, 0; \tau)$ which can be found in the textbook [20]:

$$\mathcal{K}_\epsilon(x, y; \tau) = \sqrt{\frac{\mu}{4\pi\tau}}\exp\left[-\frac{\mu(x-y)^2}{4\tau} + \frac{\epsilon(x+y)}{2}\tau + \frac{\epsilon^2}{12\mu}\tau^3\right], \qquad (42)$$

$$G_\epsilon\left(0, 0; -\frac{\kappa^2}{\mu}\right) = \int_0^\infty \sqrt{\frac{\mu}{4\pi\tau}}\exp\left(\frac{\epsilon^2}{12\mu}\tau^3 - \frac{\kappa^2}{\mu}\tau\right)d\tau, \qquad (43)$$

and the pole position is thus determined from the equation

$$2gG_\epsilon\left(0, 0; -\frac{\kappa^2}{\mu}\right) = 1. \qquad (44)$$

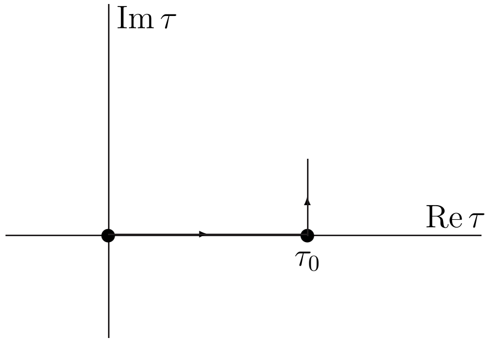

FIG. 2. The contour of integration for the integral in Eq. (43).

The peculiarity of the latter equation is that the integral in Eq. (43) is formally divergent, which is the usual situation in a calculation of the energy of an unstable state. In order to make physical sense, both that energy and the integral in Eq. (43) should be understood as a result of an analytical continuation in the parameters of the model from the region where the considered state is stable. In terms of Eq. (43) this corresponds to a continuation from the region of (formally) negative ϵ^2 where the integral is convergent. The result of such analytical continuation to physical positive ϵ^2 can be formulated as follows [2]: The integration runs along the real axis of τ from $\tau = 0$ to the value of τ where the integrand has minimum, i.e. to $\tau_0 = 2\kappa/\epsilon$. From that point the contour of integration should be turned parallel to the imaginary axis of τ, corresponding to the direction of the steepest descent (see Fig. 2). This contour rotation gives rise to an imaginary part of the integral, and hence to an imaginary part of the energy of the resonant state, corresponding to the decay width of the resonance. Following this procedure one can readily find the real and the imaginary parts of the integral in Eq. (43) and reduce the Eq. (44) for the position of the pole to the form

$$g\frac{\mu}{\kappa}\left[1 + \frac{i}{2}\exp\left(-\frac{4}{3}\frac{\kappa^3}{\mu\epsilon}\right)\right] = 1, \qquad (45)$$

which corresponds to the decay rate of the bound state

$$\Gamma = 2\mu g^2 \exp\left(-\frac{4}{3}g^3\frac{\mu^2}{\epsilon}\right). \qquad (46)$$

It can be noted that the exponential factor in this formula is the standard WKB tunneling exponent in a linear potential, while the preexponential factor is a new result. The described derivation of the formula (46) assumes that the integral in Eq. (43) can be evaluated in the saddle point approximation, which implies that the parameter in the exponent in Eq. (46) is large, i.e. that $g^3\mu^2 \gg \epsilon$.

VII. DISCUSSION

In this paper we have refined the calculations of the decay rate of the boson in the false vacuum and have found the decay rate of fermion in the false vacuum in (1 + 1) dimension with the preexponential factor. All calculations, except for the one in Sec. VI, have been performed in the approximation when the back reaction of the external particle on the Euclidean bounce solution can be neglected.

The account of the back reaction amount gives rise to several new effects which are different for $d = 2$ and $d > 2$. In the (1 + 1) dimensional case the back reaction deforms the classical solution which deformation has been described classically in Ref. [10] however the calculation of the preexponential factor is beyond our approximation. Such calculation could be potentially interesting from the stringy perspective. Indeed, the worldsheet theory on nonabelian string in several models can be identified with the CP^N model (see [21] for a recent review) which has one true vacuum and a set of metastable ones. At large N this theory can be treated perturbatively and the issue of the decay of metastable vacua or in other terns exited strings can be discussed, Similarly one can discuss the fate the different excitations on the exited vacua, or in other terms, of the excited strings can be discussed, thus considering the fate of excitations over a metastable string which is just the problem we have considered. In some situations the preexponential factor is of the prime importance, since in some range of parameters the N dependence disappear from the exponent [21]. However it is unclear if the regime with the negligible back reaction could exist in the worldsheet theory. It seems that the analysis similar to the one in Sec. VI could be applicable in this case.

In higher dimensions the situation is more complicated. The point is that the initial particle evolves along a trajectory in the complexified Minkowski space [11]. At the first stage of the process the initial particle "produces" the oscillating bubble in the Minkowski space which later develops the path in the Euclidean space. The overlap of the initial particle and the bubble happens out of the real axis. Let us also note that there is a possibility of the resonant decay of the particle in the false vacuum when the particle mass coincides with the energy levels of the quantized bubbles in the Minkowski space.

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