PHYSICAL REVIEW D **73,** 025005 (2006)

Boltzmann collision term

Steffen Weinstock*

Fakultät für Physik, Universität Bielefeld, 33615 Bielefeld, Germany

(Received 8 November 2005; published 11 January 2006)

We derive the Boltzmann equation for scalar fields using the Schwinger-Keldysh formalism. The focus lies on the derivation of the collision term. We show that the relevant self-energy diagrams have a factorization property. The collision term assumes the Boltzmann-like form of scattering probability times statistical factors for those self-energy diagrams which correspond to tree level scattering processes. Our proof covers scattering processes with any number of external particles, which come from self-energy diagrams with any number of loops.

DOI: 10.1103/PhysRevD.73.025005 PACS numbers: 11.10.Wx

I. INTRODUCTION

Kinetic theory has proven to be a very successful tool for the description of systems out of equilibrium. Important applications can be found in many fields of physics, current examples are early cosmology, or the theory of heavy ion collisions, where one aims to understand how the matter produced in the collision evolves, and, in particular, whether it thermalizes or not. A very famous kinetic equation is the Boltzmann equation for the evolution of the particle distribution functions in a sufficiently dilute system. A first derivation of nonrelativistic kinetic theory from field theory was given by Kadanoff and Baym [1]. Using the Closed-Time-Path (CTP) formalism, also called Schwinger-Keldysh formalism, this derivation becomes simpler and can also be extended to relativistic theories (see for example [2–6]). In early treatments the collision term of the Boltzmann equation was simply modeled heuristically: consider each reaction the particle can undergo, compute the probability for this reaction by using the free particle cross section, and multiply with the appropriate statistical factors, that is the Bose-enhanced or Fermi-suppressed phase space distribution functions, respectively. The collision term for a scalar particle that can undergo 2-to-2 scattering, for example, is

$$\begin{aligned} d_t f_p = \frac{1}{4\omega_p} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3p'}{(2\pi)^3 2\omega_{p'}} \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \\ \times (2\pi)^4 \delta^4(p' + k' - p - k) \mathcal{P}_{pkp'k'} \\ \times [f_{p'} f_{k'} (1 + f_p)(1 + f_k) \\ - (1 + f_{p'})(1 + f_{k'}) f_p f_k]. \end{aligned} \tag{1}$$

We refer to a collision term of this form as a standard Boltzmann collision term. In Refs. [2–5] the collision term of the resulting Boltzmann equation was derived for a number of specific theories from the right-hand side of the Kadanoff-Baym equation considering self-energy diagrams with up to two loops. To our knowledge, the first paper with a computation that went beyond the 2-loop self-energy is [7]. The authors explicitly computed a selection of self-energy diagrams with up to three or four loops in real, scalar ϕ^3- and ϕ^4-theory, respectively, and managed to bring them into a form like (1). This computation was quite involved and required the assistance of a computer system to handle the big number of terms appearing in intermediate steps.

In the present paper we consider a scalar theory with an unspecified nonderivative self-interaction. We first show that the self-energies appearing in the collision term have a useful factorization property. In the strict on-shell limit a certain class of self-energy diagrams then indeed leads to a collision term of the form (1), namely, those that correspond to tree level scattering diagrams with any number of external particles. Not all contributions to the self-energy fit into this picture, which can already be expected from the problems that arise when extending the vacuum Cutkosky rules to finite temperature [8,9].

In Sec. II we provide the basics of nonequilibrium field theory as far as required for our purpose. We also show briefly how the flow term of the Boltzmann equation is obtained from the Schwinger-Dyson equation. This procedure is standard and has extensively been covered in literature already, so only the basic steps are given here. For more details, see for example [7] and references therein. Section III is the main part of this paper, where we study the collision term for a real scalar field. A remark on the extension to charged scalar fields can be found in Sec. IV, and in Sec. V we finally discuss our results.

II. BASICS

An appropriate framework to study the time evolution of physical quantities in nonequilibrium situations is given by the Schwinger-Keldysh or Closed-Time-Path formalism [10–12]. The basic technical point is that the time variable of all objects is defined on a path C on the real axis that leads from $-\infty$ to $+\infty$, and then back to $-\infty$. The definition of the Green function is still

$$\Delta(x, y) = -i\langle T_C \phi(x) \phi^\dagger(y) \rangle, \tag{2}$$

but time ordering is performed along the path C here. If we

*Electronic address: steffen@physik.uni-bielefeld.de

 © 2006 The American Physical Society

split the path into a "+"-branch from $-\infty$ to $+\infty$ and a "$-$"-branch going back to $-\infty$, we can distinguish four real time Green functions, differing by the branches on which the time arguments are placed:

$$\begin{aligned} i\Delta^{++}(x,y) &= i\Delta^{t}(x,y) = \langle T\phi(x)\phi^{\dagger}(y)\rangle, \\ i\Delta^{+-}(x,y) &= i\Delta^{<}(x,y) = \langle \phi^{\dagger}(y)\phi(x)\rangle, \\ i\Delta^{-+}(x,y) &= i\Delta^{>}(x,y) = \langle \phi(x)\phi^{\dagger}(y)\rangle, \\ i\Delta^{--}(x,y) &= i\Delta^{\bar{t}}(x,y) = \langle \bar{T}\phi(x)\phi^{\dagger}(y)\rangle. \end{aligned} \tag{3}$$

The $\bar{T}$ in the last line denotes antitime ordering. These functions are not independent of each other, but are connected via the relation

$$\Delta^{t} + \Delta^{\bar{t}} = \Delta^{<} + \Delta^{>}. \tag{4}$$

In addition, one defines the retarded and advanced Green functions:

$$\Delta^{R,A}(x,y) = \Delta^{t}(x,y) - \Delta^{<,>}(x,y). \tag{5}$$

In contrast to ordinary perturbation theory in vacuum, here each internal vertex can be either of type "+"or of type "$-$". The lines between vertices represent the Green functions defined in (3), depending on the types of the incident vertices. For each vertex of type "$-$" there is an additional factor -1.

The Schwinger-Dyson equation for the contour Green function (2) leads to the following equation of motion:

$$(\partial_x^2 + m^2)\Delta(x,y) = -\delta_C^4(x-y) + \int_C d^4z\,\Pi(x,z)\Delta(z,y). \tag{6}$$

The self-energy Π is defined as i times the sum of all one-particle irreducible two-point functions. We switch to index notation and obtain the equation of motion for $\Delta^{<,>}$:

$$\begin{aligned} (\partial_x^2 + m^2)\Delta^{<,>}(x,y) = \int d^4z (&\Pi^{<,>}(x,z)\Delta^{A}(z,y) \\ &+ \Pi^{R}(x,z)\Delta^{<,>}(z,y)). \end{aligned} \tag{7}$$

For simplicity we omit a potential tadpole contribution to the self-energy Π. We assume that its only effect is a shift in the mass of the particles which possibly introduces a force on the left-hand side of the resulting Boltzmann equation, but that does not change the argumentation concerning the collision term.

Our goal is an equation that describes the evolution of the phase space density of particles, which takes place on a macroscopic scale. In order to separate this from the quantum evolution on a microscopical scale, we perform a Wigner transformation. For any two-point function we introduce the average coordinate

$$X = \tfrac{1}{2}(x+y) \tag{8}$$

and carry out the Fourier transformation with respect to the relative coordinate:

$$\Delta(X,p) = \int d^4(x-y) e^{ip\cdot(x-y)}\Delta(x,y). \tag{9}$$

Note that the functions $i\Delta^{<,>}(X,p)$, called Wigner functions, are real, while $i\Delta^{t}(X,p)$ is the complex conjugate of $i\Delta^{\bar{t}}(X,p)$. The respective self-energies have the same properties.

We apply the Wigner transform to the equation of motion (7). The fact that we are dealing with two-point functions is reflected in the appearance of an infinite series of derivatives

$$\diamond\{(1)\}\{(2)\} = \tfrac{1}{2}(\partial^{(1)}\cdot\partial_p^{(2)} - \partial_p^{(1)}\cdot\partial^{(2)})\{(1)\}\{(2)\} \tag{10}$$

with respect to X and p:

$$\begin{aligned} (-p^2 + m^2 - ip\cdot\partial + \tfrac{1}{4}\partial^2)&\Delta^{<,>}(X,p) \\ &= e^{-i\diamond}(\{\Pi^{<,>}(X,p)\}\{\Delta^{A}(X,p)\} \\ &\quad + \{\Pi^{R}(X,p)\}\{\Delta^{<,>}(X,p)\}). \end{aligned} \tag{11}$$

Now some approximations are necessary. We assume that the functions of interest have a smooth macroscopic behavior; more precisely, we assume that the scale on which the functions change with respect to the average coordinate X is much bigger than the microscopical scale set by the de Broglie wavelength of the particles. Consequentially, the mixed derivative $\partial\cdot\partial_p$ is a small quantity and kept only up to linear order. We furthermore assume that the macroscopic scale is also large compared to the particles' Compton wavelength, so that the second order derivative on the left-hand side of (11) is negligible as well. Since the Wigner functions $i\Delta^{<,>}$ are real, we can extract the real and imaginary part of the equation of motion and find

$$\begin{aligned} (p^2 - m^2)i\Delta^{<,>} = &-\frac{i}{2}(\Pi^{<,>}\,\mathrm{Re}\,\Delta^{R} + \mathrm{Re}\,\Pi^{R}\Delta^{<,>}) \\ &-\frac{1}{2}\diamond(\Pi^{<}\Delta^{>} - \Pi^{>}\Delta^{<}), \end{aligned} \tag{12}$$

$$\begin{aligned} (-p\cdot\partial)i\Delta^{<,>} = &\frac{1}{2}(\Pi^{>}\Delta^{<} - \Pi^{<}\Delta^{>}) \\ &-\frac{i}{2}\diamond(\Pi^{<,>}\,\mathrm{Re}\,\Delta^{R} + \mathrm{Re}\,\Pi^{R}\Delta^{<,>}). \end{aligned} \tag{13}$$

The real part has the form of a constraint, the imaginary part has the form of a kinetic equation.

For a free field we can write down the solutions of these equations in the form

$$\begin{aligned} i\Delta_0^{<}(X,p) &= 2\pi\delta(p^2 - m^2)\,\mathrm{sgn}(p_0)n_0(X,p), \\ i\Delta_0^{>}(X,p) &= 2\pi\delta(p^2 - m^2)\,\mathrm{sgn}(p_0)(1 + n_0(X,p)). \end{aligned} \tag{14}$$

The solutions for the chronological and antichronological Green functions in the free case are

$$
\begin{aligned}
i\Delta_0^t(X,p) &= \frac{i}{p^2 - m^2 + i\,\mathrm{sgn}(p_0)\epsilon} \\
&\quad + 2\pi\delta(p^2 - m^2)\,\mathrm{sgn}(p_0) n_0(X,p), \\
i\Delta_0^{\bar{t}}(X,p) &= \frac{-i}{p^2 - m^2 + i\,\mathrm{sgn}(p_0)\epsilon} \\
&\quad + 2\pi\delta(p^2 - m^2)\,\mathrm{sgn}(p_0)(1 + n_0(X,p)).
\end{aligned}
\tag{15}
$$

In thermal equilibrium the Kubo-Martin-Schwinger relation determines n_0 to be the Bose-Einstein distribution, but in a nonequilibrium situation this function is not known *a priori*.

The next simplification is a small coupling expansion. Later this will be used for a detailed analysis of the collision term; here we need it to get rid of those terms on the right-hand side which are suppressed by both the mixed derivative $\partial \cdot \partial_p$ and the coupling constant. At this point we also demand that our system can be described in terms of (quasi)particles. To this end we assume that the right-hand side of the constraint equation (12) vanishes, turning this equation into a mass-shell condition. In the end, Eqs. (12) and (13) simplify to

$$
(p^2 - m^2) i\Delta^{<,>} = 0, \tag{16}
$$

$$
(-p \cdot \partial) i\Delta^{<,>} = \tfrac{1}{2}(\Pi^> \Delta^< - \Pi^< \Delta^>), \tag{17}
$$

and we can make an on-shell ansatz for the Wigner functions:

$$
\begin{aligned}
i\Delta^<(X,p) &= 2\pi\delta(p^2 - m^2)[\theta(p_0) f_+(X,\vec{p}) \\
&\quad + \theta(-p_0)(1 + f_-(X,-\vec{p}))], \\
i\Delta^>(X,p) &= 2\pi\delta(p^2 - m^2)[\theta(p_0)(1 + f_+(X,\vec{p})) \\
&\quad + \theta(-p_0) f_-(X,-\vec{p})].
\end{aligned}
\tag{18}
$$

Spectral sum rules that follow from the basic commutator relations for the scalar field operators and make a connection between $i\Delta^<$ and $i\Delta^>$ ensure that this ansatz is consistent. By comparison with the equilibrium functions and also by inserting the ansatz (18) into the expressions for the expectation values of current

$$
j^\mu(X) = 2 \int \frac{d^4p}{(2\pi)^4}\, p^\mu i\Delta^<(X,p) \tag{19}
$$

and energy-momentum

$$
T^{\mu\nu}(X) = \int \frac{d^4p}{(2\pi)^4}\, p^\mu p^\nu i\Delta^<(X,p), \tag{20}
$$

we finally can identify f_+ and f_- with the phase space densities of particles and antiparticles, respectively. In the case of real scalar fields there is an additional relation, $i\Delta^<(-p) = i\Delta^>(p)$, which leads to $f_+ = f_-$.

The constraint equation is satisfied identically with this ansatz, and all that is left over is the kinetic equation (17). We insert (18) and integrate over positive frequencies. The resulting equation reproduces the flow term of a relativistic Boltzmann equation for particles with phase space density f_+:

$$
\left(\partial_t + \frac{\vec{p}}{\omega} \cdot \vec{\partial}\right) f_+(X,\vec{p}) = \int_0^\infty \frac{dp_0}{\pi} C(X,p). \tag{21}
$$

Integration over negative momenta results in a similar equation for the corresponding antiparticles. The most important part for us is the right-hand side: the collision term so far is

$$
\begin{aligned}
C(X,p) &= \tfrac{1}{2}(i\Pi^>(X,p) i\Delta^<(X,p) \\
&\quad - i\Pi^<(X,p) i\Delta^>(X,p)).
\end{aligned}
\tag{22}
$$

In the next section we try to reexpress this in terms of particle densities and scattering amplitudes.

III. COLLISION TERM

Now we come to the main part of this paper, the collision term. We make a perturbative expansion of the self-energies in C and try to bring it to a form resembling the collision term of a Boltzmann equation as shown in the introduction. The collision term is local in our approximation, so we drop the argument X from now on in order to simplify the notation.

A. Self-energy

In a perturbative expansion the self-energy $i\Pi^<(p) = i\Pi^{+-}(p)$ is expressed as minus the sum of all amputated one-particle irreducible graphs with momentum p entering at a "+"-vertex and leaving at a "−"-vertex. We first classify the graphs in this expansion in the following way: take any graph and imagine all lines connecting a "+"-vertex with a "−"-vertex were cut. This would split the graph into a number of connected subgraphs, each containing only "+"- or "−"-vertices, respectively. We call these subgraphs *clusters*. Obviously there are at least two clusters, namely, one which is connected to the incoming line, called "+"-base, and one connected to the outgoing line, called "−"-base. Now we can distinguish two types of graphs:

(i) Graphs of type 1 have only direct connections between the "+"- and the "−"-base, i.e. there are no paths leading from the "+"-base to the "−"-base via some other clusters (see Fig. 1). The simplest example for this are graphs with only two clusters.
(ii) Graphs of type 2 have connections between the "+"-base and the "−"-base via other clusters (see Fig. 2).

There is another way to classify the graphs contributing to $i\Pi^{+-}(p)$: try to divide a given graph into two parts, one attached to the incoming line, the other part attached to the outgoing line, by cutting a suitable set of "+−"-lines. Only cut lines which are attached to a "+"-vertex in the

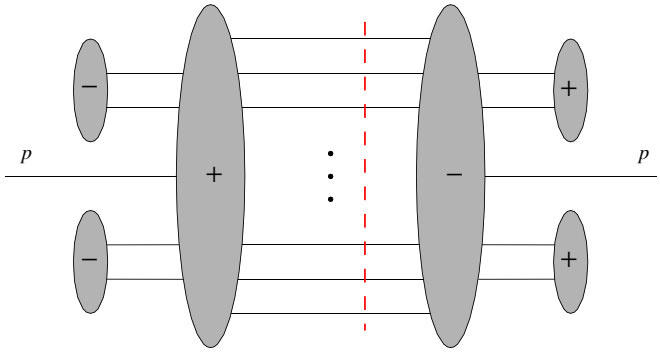


FIG. 1 (color online). A graph with only direct connections between "+" and "−"-base (type 1). There is a unique complete cut (dashed line).

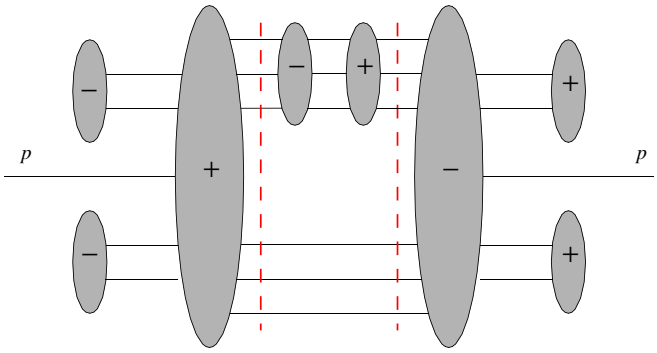


FIG. 2 (color online). A graph with indirect connections between "+" and "−"-base (type 2). Two different complete cuts are possible (dashed lines).

first part and to a "−"-vertex in the second part, and which are effective in separating the two parts. This works with every graph, since the incoming line ends in a "+"-vertex, while the outgoing line starts from a "−"-vertex. We call this a complete cut. It is not hard to see that a graph belongs to type 1 if and only if there is exactly one such complete cut. In this case the cut lines are exactly the ones connecting the "+"-base with the "−"-base. Otherwise the graph belongs to type 2.

1. Unique complete cut

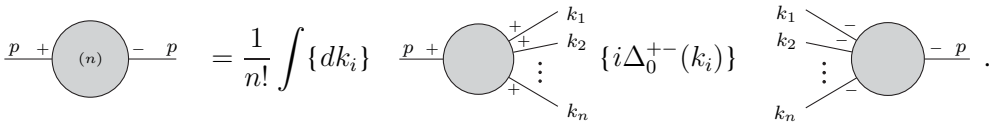
Let us first concentrate on the graphs of type 1, i.e. graphs which have a unique complete cut. We further classify these diagrams according to the number of cut lines:

$$i\Pi^{+-}_{(\text{type1})}(p) = \sum_{n=2}^{\infty} i\Pi^{+-}_{n}(p). \tag{23}$$

Now we claim that

$$-i\Pi^{+-}_{n}(p) = \frac{1}{n!}\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}(2\pi)^4\delta^4(p-k_1-\cdots-k_n)\mathcal{M}^{+}_{(1)}(p,-k_1,\ldots,-k_n)i\Delta^{+-}_{0}(k_1)\ldots i\Delta^{+-}_{0}(k_n) \times \mathcal{M}^{-}_{(1)}(-p,k_1,\ldots,k_n). \tag{24}$$

Symbolically we can write this equation as



The infinite sum of diagrams that contribute to the self-energy on the left-hand side factorizes into two other infinite sums which are identified as the diagrammatic expansions of certain n-point functions. The totally symmetric function $\mathcal{M}^{+}_{(1)}$ is the scattering amplitude belonging to the amputated, connected, out-of-equilibrium $(n+1)$-point function; positive momenta are entering, and all external momenta are attached to "+"-vertices. Basically it is the Fourier transform

$$\Delta^{+}_{(1)}(q_1,\ldots,q_{n+1}) = \prod_{j=1}^{n+1}\left[\int d^4x_j e^{iq_jx_j}\Box_{x_j}\right] \times \Delta^{+}_{(1)}(x_1,\ldots,x_{n+1}) \tag{25}$$

of the out-of-equilibrium $(n+1)$-point function (we use the same symbol for the function and its Fourier transform) with time ordered fields

$$\Delta^{+}_{(1)}(x_1,\ldots,x_{n+1}) = \langle T\phi(x_1)\ldots\phi(x_{n+1})\rangle_{\text{conn}}, \tag{26}$$

however, with the additional restrictions that the graphs contributing to this function must not have any corrections on the p-line, and for each k_i there must be a path connecting it to p which only contains "+"-vertices. The derivative operators $\Box = \partial^2 + m^2$ remove the external legs and we explicitly take out the momentum conservation δ-function,

$$\Delta^{+}_{(1)}(q_1,\ldots,q_{n+1}) = (2\pi)^4\delta^4(q_1+\ldots+q_{n+1}) \times i\mathcal{M}^{+}_{(1)}(q_1,\ldots,q_{n+1}), \tag{27}$$

so that $\mathcal{M}^{+}_{(1)}$ has the form of a scattering amplitude. The functions $\mathcal{M}^{-}_{(1)}$ and $\Delta^{-}_{(1)}$ are defined analogously with antitime ordering.

In order to prove (24), we have to show two things: first, the set of diagrams appearing in the perturbative expansion on the left-hand side is the same as the set of diagrams one finds on the right-hand side. Here we do not care about statistical factors of diagrams or how often they appear. This is done in a second step: it may be possible to obtain the same diagram by putting together several different contributions from Δ^+ and Δ^-. The number of such combinations together with the factor $1/n!$ and the symmetry factors of these contributions must match the symmetry factor of the resulting diagram on the left-hand side.

2. Diagrams

Any graph G contributing to Π_n^{+-} has a unique complete cut consisting of n lines. The free propagators $i\Delta_0^{+-}$ associated to the cut lines and the integrals over their momenta, named $k_1, \ldots, k_n$, are taken out of G and written down explicitly on the right-hand side of (24). The remainders of G to the "left" and to the "right" of the cut are called G^+ and G^-, respectively. Obviously G^+ is a Feynman diagram with $n+1$ external momenta attached to "+"-vertices. We took away the k_i-lines, so it is amputated. Since G is one-particle irreducible, there can be no corrections on the p-line, while there may be corrections on the k_i-lines. Because G is of type 1, the cut lines are exactly the ones that connect the "+"-base to the "−"-base, so all external momenta in G^+ are attached to the "+"-base and thus are connected with each other via paths that only include "+"-vertices. This shows that G^+ is a Feynman diagram that contributes to $\Delta^+_{(1)}$, and likewise G^- is a part of $\Delta^-_{(1)}$.

Conversely, take any contributions G^+ from $\Delta^+_{(1)}$ and G^- from $\Delta^-_{(1)}$. Together with the free propagators $i\Delta_0^{+-}$ and the integrals over their momenta they make up an amputated Feynman diagram G with a unique complete cut and with momentum p entering at a "+"-vertex and leaving at a "−"-vertex. Since there are at least two k_i-lines, the only way for G not to be one-particle irreducible would be to consist of a G^+ or a G^- which can be split into two parts, one connected to the p-line and one connected to the k_i-lines, by cutting a single line. But this is impossible, because G^+ and G^- do not have corrections on the p-line. So the diagram G contributes to the diagrammatic expansion of Π_n^{+-}, which eventually proves that the same diagrams are appearing on both sides of (24).

A comment about the momentum conservation δ-function is in order here. If one performs as many integrals over internal momenta as possible in the diagrams G^+ and G^-, in both cases a $(2\pi)^4\delta^4(p - k_1 - \cdots - k_n)$ is left over which is not part of the scattering amplitude. One appears explicitly in (24), the other one reduces to $(2\pi)^4\delta^4(0)$ and is dropped according to the usual definition of the self-energy.

3. Numbers

Let us first have a closer look at the symmetry factor of a diagram G contributing to Π_n^{+-}. The symmetry factor of a diagram is the order of the graph's symmetry group, which contains all permutations of lines and vertices that do not alter the diagram. G has a unique cut, and accordingly we can distinguish lines and vertices in the part G^+ left of the cut from lines and vertices in the part G^- to the right of the cut, and they all are topologically different from the cut lines. This means that there are no symmetries exchanging lines or vertices in G^+ with those in G^-, and neither are there symmetries that exchange cut lines with uncut ones. Therefore the symmetry group of G is the direct product of three groups: the symmetry group of G^+, the symmetry group of G^-, and the symmetry group S of the cut lines (in the context of G, i.e. permutations of cut lines that do not change G). We call the orders of these groups s_+, s_- and s, respectively, and so the symmetry factor of G is $1/(s_+ s s_-)$. In fact, things are a bit more complicated, since in graphs with tadpolelike structures, i.e. where both ends of some lines are attached to the same vertex, the above definition of the symmetry group is too narrow. However, such lines will never be cut, and therefore their symmetry properties are part of the symmetry groups of G^+ or G^- and do not change this discussion.

Given a graph G contributing to Π_n^{+-}, how many different combinations of a $G^+ \in \Delta^+$ and a $G^- \in \Delta^-$ produce this G? First of all, the topologies of G^+ and G^- are completely determined by G and its cut. But there are several possibilities to name their external lines, which determines how they are put together. Some namings result in topologies different from G, so they contribute to a different graph and are not relevant here. The rest of the namings corresponds exactly to the inequivalent renamings of the cut lines k_i of G, where inequivalent means that they are not symmetries of G. So the number of combinations of a G^+ and a G^- that produce G is equal to the number of inequivalent renamings or permutations of the cut lines in G.

The group P of all permutations σ_i of cut lines in G, i.e. all possibilities of renaming them, has the order $n!$ If we build the right cosets of P relative to S,

$$S\sigma_1, S\sigma_2, \ldots, S\sigma_{n!}, \tag{28}$$

then two sets $S\sigma_i$ and $S\sigma_j$ are either identical or disjoint [13]. So in fact there are only f different sets

$$S\sigma_{i_1}, \ldots, S\sigma_{i_f}, \tag{29}$$

each of which contains s elements. Since all elements within one $S\sigma_i$ belong to equivalent permutations of cut lines, the number of inequivalent permutations must be equal to f. But the step from (28) to (29) only removes redundant elements and leaves their total number unchanged: $n! = f \cdot s$. Together with the symmetry factors of the parts G^+ and G^- the complete combinatorial factor

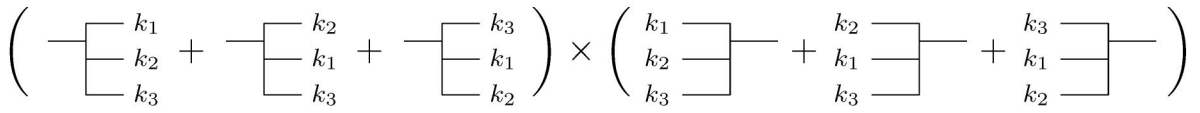
on the right-hand side then is

$$\frac{f}{n!}\frac{1}{s_+ s_-} = \frac{1}{s_+ s s_-} \tag{30}$$

and exactly matches the symmetry factor of G.

4. Example

As an example we consider two 2-loop contributions to the self-energy of a real scalar theory with a ϕ^3-interaction, displayed in Fig. 3. There is only one possible topology for the parts G^+ and G^-, namely, a 2-to-2 scattering diagram. There are three possibilities for naming the external lines of this diagram, so that the relevant contribution is



(31)

From the resulting nine graphs, three correspond to the inequivalent ways of naming the cut lines in diagram Fig. 3(a), while the remaining six graphs belong to diagram Fig. 3(b).

5. No unique cut

Next we have to deal with those contributions to the self-energy with more than one complete cut (type 2). In this case we decide to use the cut which makes the left portion G^+ of the diagram as small as possible. This means to cut all lines that leave the "+"-base, except when they are not efficient in separating the two parts. In the example of Fig. 2 this is the left cut. This prescription leads to an unambiguously defined cut and we can repeat all the arguments already used above. The only difference concerns the nature of the right-hand portion of the diagrams: the contributions to $\mathcal{M}^-_{(2)}$ must have the property that at least one of the external k_i-lines is connected to the outgoing p-line only via paths which contain at least one "+"-vertex. Note that $\mathcal{M}_{(1)}$ and $\mathcal{M}_{(2)}$ do not have any graphs in common.

B. Collision term

For real fields the n-point functions Δ^+ and Δ^- are related by complex conjugation:

$$\Delta^-(k_1, \ldots, k_n) = \Delta^{+*}(-k_1, \ldots, -k_n). \tag{32}$$

By referring to the corresponding diagrammatic representation we can check that this holds for the restricted functions $\Delta^{\pm}_{(1)}$ and $\Delta^{\pm}_{(2)}$, too. After sending $k_i \to -k_i$ the full collision term can thus be written

$$\begin{aligned} C(p) = \frac{1}{2}\sum_{a=1,2}\sum_{n=2}^{\infty}\frac{1}{n!}\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}(2\pi)^4\delta^4(p+k_1+\cdots+k_n) \\ \times[\mathcal{M}^+_{(1)}(p,k_1,\ldots,k_n)\mathcal{M}^{+\,*}_{(a)}(p,k_1,\ldots,k_n)i\Delta_0^>(k_1)\ldots i\Delta_0^>(k_n)i\Delta^>(p) \\ -\mathcal{M}^{+\,*}_{(1)}(-p,-k_1,\ldots,-k_n)\mathcal{M}^+_{(a)}(-p,-k_1,\ldots,-k_n)i\Delta_0^<(k_1)\ldots i\Delta_0^<(k_n)i\Delta^<(p)]. \end{aligned} \tag{33}$$

The sum over $a = 1, 2$ represents the contributions of self-energy diagrams of type 1 and type 2, respectively. We emphasize that this factorization is exact. The form (14) for the Wigner functions suggests that (33) is an expansion in the particle phase space density. But this is not the case, since the distribution function is also contained in the scattering matrix. A true expansion in the particle densities has been done in [14], which shares some technical similarities with our analysis.

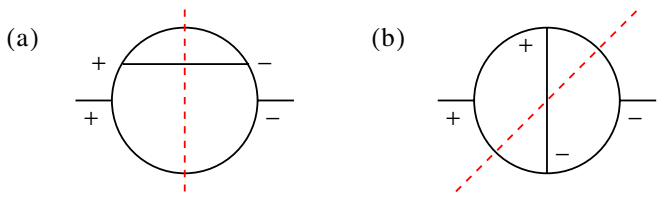


FIG. 3 (color online). Two 2-loop contributions to the self-energy of real scalar ϕ^3-theory with a unique complete cut comprising $n = 3$ lines. The symmetry factors of these diagrams are $1/2$ and 1, respectively.

In order to proceed towards the standard collision term of a Boltzmann equation, we still have to overcome three difficulties. First, a matrix element squared can obviously only be obtained for type 1 diagrams, where $a = 1$. Second, the propagators corresponding to the cut lines are free ones, while the propagator for the p-line is a full one. In the Boltzmann equation, however, all distribution functions are expected to be of the same type. Third, if we want to combine the scattering matrices of the last two lines, they have to be symmetric with respect to inverting

all momenta. The last point is satisfied if we assume that our system is symmetric under *CP*. Note that this refers to both the interaction and the initial conditions, since if the evolution starts with initial particle distributions f that are not *CP*-even, then the matrix elements will not be *CP*-even either, even if the interaction conserves *CP*. The first two points force us to restrict ourselves to the tree level, since then the difference between free and full propagators vanishes. In addition, the scattering diagrams include no internal "$-$"-vertices: diagrams with "$-$"-vertices have several "$+-$"-lines which cannot all be on-shell in a tree diagram, and according to (14) these diagrams vanish.

The fact that we have to exclude quantum corrections is not unexpected. After all, the Boltzmann equation is a classical equation, so our derivation definitely has to break down at some level when adding quantum effects. As we have seen, this breakdown takes place immediately beyond the classical level: we can recover a standard Boltzmann collision term of the form (1) only by the restriction to classical processes. Now we can simplify the collision term to

$$C(p) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n!}\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}(2\pi)^4\delta^4(p+k_1+\cdots+k_n)|\mathcal{M}^{+}_{(1)}(p,k_1,\ldots,k_n)|^2[i\Delta^{>}(k_1)\ldots i\Delta^{>}(k_n)i\Delta^{>}(p) - i\Delta^{<}(k_1)\ldots i\Delta^{<}(k_n)i\Delta^{<}(p)]. \tag{34}$$

The final step is to use the on-shell ansatz (18) for the propagators and to perform the integrals over the zero components of the momenta. The ansatz for $i\Delta^{<}(k_i)$ contains two δ-functions, one that corresponds to positive energies and one that corresponds to negative energies. In the case of a negative energy, we additionally invert the corresponding spatial momentum. This way we obtain contributions with all possible combinations of particles k_1 to k_n either going into or coming out of the scattering. Since the scattering amplitude is totally symmetric with respect to the order of its arguments, all particles going in are exchangeable, and so are all the particles that come out of the scattering. Thus we can group those terms that have the same number of ingoing and outgoing particles. For j outgoing particles (besides p), there are

$$N(j) = \frac{n!}{j!(n-j)!} = \binom{n}{j} \tag{35}$$

identical terms. The complete collision term is

$$C(p) = \frac{1}{2}\frac{\pi}{\omega_p}\delta(p_0-\omega_p)\sum_{n=2}^{\infty}\frac{1}{n!}\int\frac{d^3k_1}{(2\pi)^3 2\omega_1}\cdots\frac{d^3k_n}{(2\pi)^3 2\omega_n}\sum_{j=0}^{n}\binom{n}{j}(2\pi)^4\delta^4(p+k_1+\cdots+k_j-k_{j+1}-\cdots-k_n) \times|\mathcal{M}^{+}_{(1)}(p,k_1,\ldots,k_j,-k_{j+1},\ldots,-k_n)|^2[(1+f_1)\ldots(1+f_j)f_{j+1}\ldots f_n(1+f_p)-f_1\ldots f_j(1+f_{j+1})\ldots(1+f_n)f_p], \tag{36}$$

where $f_i \equiv f(\vec{k}_i)$, $\omega_i = (\vec{k}_i^2 + m^2)^{1/2}$, and all four-momenta are on-shell: $k_i = (\omega_i, \vec{k}_i)$. The collision term consists of two parts, according to the two terms in square brackets, referred to as gain and loss term, respectively. They describe the increase or decrease of the density of particles with momentum p in the plasma due to the scattering. Typically, several contributions vanish because of kinematical restrictions, for example, the one with all particles going in. How this works in detail depends on the type of the interaction and the particle masses; see also the example in the next section. Since in the kinetic equation (21) we only integrate over positive p_0, terms proportional to the negative energy $p_0 = -\omega_p$ were dropped here.

C. Example: ϕ^3-theory at the 2-loop level

As an example consider a real, massive scalar theory with a $\lambda\phi^3/3!$ self-interaction. The relevant 2-loop self-energy diagrams are shown in Fig. 3. The cuts in these diagrams comprise three lines, so the corresponding scattering processes in the Boltzmann collision term will have four external lines, and at tree level are shown in Fig. 4. The matrix element for these diagrams is

$$\mathcal{M}^{+}(p,k_1,k_2,k_3) = \lambda^2\left(\frac{1}{(p+k_2)^2}+\frac{1}{(p+k_3)^2}+\frac{1}{(p+k_1)^2}\right). \tag{37}$$

The collision term can then be read off from (36). In principle there are 4 contributions, corresponding to 0, 1,

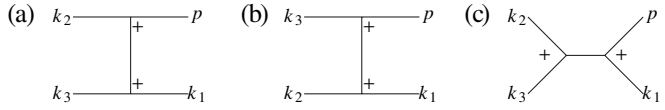


FIG. 4. The tree diagrams contributing to the scattering matrix for four external particles.

2 or 3 particles coming out together with p, but due to kinematic restrictions only 2-to-2 scattering can occur:

$$C(p) = \frac{\pi}{4\omega_p}\delta(p_0 - \omega_p)\int \frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}\frac{d^3k_3}{(2\pi)^3 2\omega_3}(2\pi)^4\delta^4(p + k_1 - k_2 - k_3)|\mathcal{M}^+(p, k_1, -k_2, -k_3)|^2 \times [(1 + f_p)(1 + f_1)f_2 f_3 - f_p f_1(1 + f_2)(1 + f_3)]. \tag{38}$$

One could have obtained this result directly from the self-energy computed with the CTP Feynman rules,

$$-i\Pi^{+-}(p) = \int \frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\Big[\frac{1}{2}i\Delta^{-+}(q)i\Delta^{++}(p + q)i\Delta^{+-}(p + q + k)i\Delta^{-+}(k)i\Delta^{--}(p + q) + i\Delta^{++}(p + q)i\Delta^{+-}(p + q + k)i\Delta^{--}(q + k)i\Delta^{-+}(q)i\Delta^{-+}(k)\Big], \tag{39}$$

but already for these comparatively simple diagrams this is not trivial, and it becomes rather involved for self-energies with more loops [7].

IV. COMPLEX SCALAR FIELD

In the case of a complex scalar field, when there are particles and antiparticles, we can adopt the general line of argumentation from the real case, but we have to make adjustments at a few points.

First of all, charge conservation constrains the cut: the net charge flow across the cut must be equal to the net charge flow through the whole self-energy diagram. Thus the cut must comprise $2n + 1$ lines, $n + 1$ leading from the left to the right, and n leading back from the right to the left. Then the analog to the basic claim (24) for complex fields is

$$-i\Pi_n^{+-}(p) = \frac{1}{(n + 1)!n!}\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{n+1}}{(2\pi)^4}\int \frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_n}{(2\pi)^4}(2\pi)^4\delta^4(p + q_1 + \cdots + q_n - k_1 - \cdots - k_{n+1}) \times \mathcal{M}^+_{(1)}(-k_1, \ldots, -k_{n+1}; p, q_1, \ldots, q_n)i\Delta_0^{+-}(k_1)\ldots i\Delta_0^{+-}(k_{n+1})i\Delta_0^{-+}(q_1)\ldots i\Delta_0^{-+}(q_n) \times \mathcal{M}^-_{(1)}(-p, -q_1, \ldots, -q_n; k_1, \ldots, k_{n+1}). \tag{40}$$

The proof runs almost exactly like in the real case, the only difference is that we have to distinguish the two sets of cut lines. Lines running from the left to the right cannot be interchanged with lines running from the right to the left, because they transport charge in different directions. This leads to the factor $1/(n + 1)!n!$ that represents one group of $n + 1$ lines carrying charge to the right and another group of n lines carrying charge to the left.

The final result is similar to (36), but both gain and loss term now consist of two parts. In one part the distribution functions represent a charge coming out of the reaction, where we have all combinations of these functions being either $(1 + f_+)$ for a particle coming out, or f_- for an antiparticle going in. The other part contains distribution functions that represent a charge going into the reaction, that is all combinations with either f_+ for a particle going in or $(1 + f_-)$ for an antiparticle coming out.

It is straightforward to generalize to the case of several scalar particle species in a similar way. Each species corresponds to a group of lines or distribution functions, respectively, where members of different groups cannot be interchanged with each other.

V. DISCUSSION

We start with the equation of motion for the out-of-equilibrium Green function for a real scalar field. Using gradient expansion, on-shell approximation and small coupling expansion the left-hand side leads to the flow term of a Boltzmann equation in a well-known way. We make a perturbative expansion of the self-energies on the right-hand side and give a diagrammatic proof that the relevant self-energies have a factorization property, and that for a certain type of contributions to the self-energy the right-hand side can be rewritten as the matrix element squared of scattering processes times the distribution functions of the particles involved in the reactions. This works for those self-energy contributions that correspond to tree level scattering diagrams. The scattering processes obtained include any number of external particles and thus correspond to self-energy diagrams with any number of loops.

If effects beyond the classical level are required, it is in general not correct to simply compute the scattering probabilities including quantum corrections and plug them into a standard Boltzmann collision term. In specific situations

it is possible to fit quantum corrections into the picture, however at some cost. In [15] for example, a model of scalar quarks and gluons was examined and a standard Boltzmann collision term was constructed from the self-energy up to two loops. In order to be able to do this, the scattering matrices were not computed simply by following the CTP rules, but for some diagrams Feynman propagators had to be used for some internal lines. Furthermore, kinematical arguments had to be used to get rid of some diagrams that did not match the picture. The approach presented there required an explicit study of each diagram and cannot be generalized to arbitrary diagrams or theories.

The simple structure (1) cannot be obtained beyond the classical level for two reasons. First, there are the self-energy diagrams that allow several complete cuts. This is precisely the type of diagrams that cause problems in the extension of the vacuum Cutkosky rules to finite temperature [8,9]. Second, there is the difference between the free propagators for the cut k_i-lines and the full propagator for the p-line. Since we know that the scattering diagrams we obtain must not contain corrections on the p-lines but do have corrections on the k_i-lines, it seems tempting to collect these corrections and attribute them to the free propagators, thus promoting them to full ones. In effect this means that we would have to do the perturbative analysis of the self-energy in terms of resummed and thus full propagators. As a consequence, all diagrams contributing to the self-energy must not have corrections to already full lines. Then the available self-energy diagrams are not sufficient to be rewritten as the square of a matrix element, however: certain diagrams are missing. For instance, the 2-loop diagram in Fig. 3(b) provides only the interference terms in (31), the squares of the individual amplitudes can only come from Fig. 3(a), which has a forbidden correction.

This shows that there is no consistent and systematic way to obtain a standard Boltzmann collision term beyond the tree level. At the tree level, a standard Boltzmann collision term is found that includes scattering processes with any number of external particles.

[1] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).
[2] S. P. Li and L. D. McLerran, Nucl. Phys. **B214**, 417 (1983).
[3] E. Calzetta and B. L. Hu, Phys. Rev. D **37**, 2878 (1988).
[4] S. Mrowczynski and U. W. Heinz, Ann. Phys. (N.Y.) **229**, 1 (1994).
[5] S. P. Klevansky, A. Ogura, and J. Hufner, Ann. Phys. (N.Y.) **261**, 37 (1997).
[6] T. Prokopec, M. G. Schmidt, and S. Weinstock, Ann. Phys. (N.Y.) **314**, 208 (2004); **314**, 267 (2004).
[7] M. E. Carrington and S. Mrowczynski, Phys. Rev. D **71**, 065007 (2005).
[8] R. L. Kobes and G. W. Semenoff, Nucl. Phys. **B260**, 714 (1985); **B272**, 329 (1986).
[9] F. Gelis, Nucl. Phys. **B508**, 483 (1997).
[10] J. S. Schwinger, J. Math. Phys. (N.Y.) **2**, 407 (1961).
[11] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].
[12] K. C. Chou, Z. B. Su, B. L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).
[13] This is a standard theorem of group theory. See, e.g., W. Ledermann and A. J. Weir, *Introduction to Group Theory* (Addison-Wesley, Reading, MA, 1997).
[14] S. Jeon and P. J. Ellis, Phys. Rev. D **58**, 045013 (1998).
[15] D. S. Isert and S. P. Klevansky, Eur. Phys. J. A **12**, 453 (2001).