Role of dipole charges in black hole thermodynamics

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Modern derivations of the first law of black holes appear to show that the only charges that arise are monopole charges that can be obtained by surface integrals at infinity. However, the recently discovered five dimensional black ring solutions empirically satisfy a first law in which dipole charges appear. We resolve this contradiction and derive a general form of the first law for black rings. Dipole charges do appear together with a corresponding potential. We also include theories with Chern-Simons terms and generalize the first law to other horizon topologies and more generic local charges.

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I. INTRODUCTION

One of the first indications that there was a connection between black holes and thermodynamics was the discovery of the laws of black hole mechanics in the early 1970's [1]. Not surprisingly, this work was in the context of four spacetime dimensions. Over the past decade there has been growing interest in higher dimensional black holes and black branes, so it is natural to ask how these laws extend. The second law extends trivially, since the argument that the horizon area always increases is independent of spacetime dimension or horizon topology. The zeroth law also has an extension, since one can compute the derivative of the surface gravity in any dimension [2] (although the condition for it to be constant requires a field equation or symmetries). We will focus here on the first law, which describes how stationary black holes respond to small perturbations.

There have been several previous derivations of the first law for higher dimensional black holes (see, e.g., [3–6]). However, most of these assume the horizon is topologically spherical, as in four dimensions. In addition, there have been several derivations which assume that the four dimensional uniqueness theorems extend to higher dimensions [7–9]. It has recently been shown that both of these properties can be violated. There are five dimensional vacuum solutions describing stationary black rings with horizon topology $S^2 \times S^1$ [10]. These black rings can have the same mass and angular momentum as spherical black holes. More importantly, in the presence of suitable matter, the nontrivial topology of the event horizon makes it possible for the black holes to carry a local dipole charge. Emparan has found a continuous family of nonvacuum black rings, all with the same asymptotic conserved charges and differing only by their dipole charge [11].

When thinking about the role of dipole charges in the first law, one is led to an apparent paradox. On the one hand, from the explicit form of the solutions, Emparan claims that the dipole charge does enter the first law, at least for perturbations from one stationary solution to another. On the other hand, a powerful and elegant derivation of the first law by Sudarsky and Wald [12] (which does not assume black hole uniqueness) seems to show that the only charges that can enter into the first law are (monopole) charges obtained by surface integrals at infinity.

We will review the Sudarsky-Wald argument in Sec. II and generalize it to five dimensions. We then review the solutions found by Emparan in Sec. III and finally resolve this paradox in Sec. IV. The net result is that dipole charges do appear in the general form of the first law in higher dimensions. In the next section, we extend this derivation to include a Chern-Simons term and derive a first law appropriate for, e.g., black rings in minimal 5D supergravity. The fact that dipole charges arise in the first law raises the question of whether other charges can arise in the first law, perhaps carried by some not-yet-discovered black hole solution in higher dimensions. We discuss this in Sec. VI, and argue that the answer is yes. Finally, Sec. VII contains some concluding remarks.

We will use Greek indices $\mu, \nu, \dots$ for spacetime tensors, and latin indices $a, b, \dots$ for purely spatial tensors.

II. SUDARSKY-WALD ARGUMENT FOR THE FIRST LAW

We first consider asymptotically flat solutions of the five dimensional theory

$$S = \beta \int d^5x\sqrt{-g}\Big[R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{12}e^{-\alpha\phi}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big], \tag{2.1}$$

where $H = dB$ is a three form field strength, ϕ is the dilaton, α is the dilaton coupling, and β is a normalization constant we choose to leave arbitrary for the present. This is the simplest theory which contains stationary black ring solutions with dipole charge. It is also of interest in string theory and M theory. If we parameterize α in terms of an integer N via $\alpha^2 = \frac{4}{N} - \frac{4}{3}$, then for $N = 1, 2, 3$ the solutions can be interpreted as arising from N intersecting

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branes in higher dimensions. In particular, $N = 1$ is the NS sector of low energy string theory (in the Einstein frame). For $N = 3$, the dilaton decouples and can be set to zero. In this case, the theory is equivalent to Einstein-Maxwell in five dimensions by a simple duality transformation.

Since we have a three form in five dimensions, the natural charge defined at infinity is the magnetic charge

$$Q_M = \frac{1}{4\pi} \int_{S^3} H. \tag{2.2}$$

However, if the horizon has topology $S^2 \times S^1$ one can also define an electric dipole charge

$$q_e = \frac{1}{4\pi} \int_{S^2} e^{-\alpha\phi} \star H, \tag{2.3}$$

where the integral is over any S^2 which can be continuously deformed to an S^2 on the horizon[1]. q_e is well defined due to the field equation $d(e^{-\alpha\phi} \star H) = 0$.

The Sudarsky-Wald derivation of the first law is based on the Hamiltonian formulation of general relativity. It was originally given in the context of four dimensional Einstein-Maxwell (or Einstein Yang-Mills) theory and goes as follows. The Hamiltonian for Einstein-Maxwell theory takes the "pure constraint" form

$$H = \int_\Sigma (\xi^\mu C_\mu + \xi^\mu A_\mu C) + \text{surface terms}, \tag{2.4}$$

where Σ is a spacelike surface, ξ^μ is the time evolution vector field, C_μ are the constraints from the Einstein equations, and C is the Maxwell constraint ($D_a E^a = 0$). Note we define the electric field

$$E^a = F^{\mu a} n_\mu \tag{2.5}$$

with n^μ denoting the unit normal to Σ. The surface terms are determined by the requirement that the variation of the Hamiltonian is well defined. In addition to the usual gravitational surface terms, one gets an additional surface term:

$$\frac{1}{4\pi} \int (\xi^\mu A_\mu) E_a dS^a. \tag{2.6}$$

Consider a stationary, axisymmetric, electrically charged black hole with bifurcate Killing horizon. Choose Σ to have boundaries at infinity and the bifurcation surface S. Let χ^μ denote the Killing field which vanishes on S and set $\xi^\mu = \chi^\mu$. Then the variation of the Hamiltonian must vanish, since this just yields the time derivative of the canonical variables in the direction χ, and χ is a symmetry.[2] However, as long as the perturbation satisfies the linearized constraints, the volume term in the Hamiltonian vanishes by itself. This means that the sum of the variation of the surface terms must vanish. This yields the first law

$$\delta M = \frac{\kappa}{8\pi} \delta A_H + \Omega \delta J + \Phi_E \delta Q_E, \tag{2.7}$$

where κ is the surface gravity, A_H and Ω are, respectively, the area and angular velocity of the horizon and Φ_E is the electrostatic potential (and we set $G = 1$). The origin of each term is the following. Since

$$\chi = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \tag{2.8}$$

the gravitational surface terms at infinity yield $\delta M - \Omega \delta J$. The fact that $\xi^\mu = 0$ on S implies that the only contribution from the Maxwell field comes from the surface integral at infinity (assuming all fields are regular) and yields the $\Phi_E \delta Q_E$ term where $\Phi_E = -A_t(\infty)$. The gravitational surface term on S does provide a nonzero contribution but this is only because the constraint involves the scalar curvature which has two derivatives of the metric. The surface term thus involves a derivative of ξ^μ and yields the $\frac{\kappa}{8\pi} \delta A_H$ term.

It is easy to generalize this to the five dimensional theory (2.1). The first step is to do a Hamiltonian decomposition of this theory. We denote the Lie derivative of a tensor in the ξ direction by a dot:

$$\dot{B} = \mathcal{L}_\xi B. \tag{2.9}$$

The momentum canonically conjugate to the spatial metric h_{ab} is, as usual

$$\pi_G^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}} = \beta \sqrt{h} (K^{ab} - h^{ab} K), \tag{2.10}$$

where K^{ab} is the extrinsic curvature and $K = K^{ab} h_{ab}$. The momentum conjugate to the dilaton ϕ is

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \beta \sqrt{h} n^\mu \nabla_\mu \phi, \tag{2.11}$$

while the momentum conjugate to the 2-form potential B is

$$\pi_B^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{B}_{ab}} = \frac{\beta \sqrt{h}}{2} e^{-\alpha\phi} H^{ab\mu} n_\mu. \tag{2.12}$$

In addition to the usual gravitational constraints, there is the additional constraint

$$D_a \left(\frac{\pi_B^{ab}}{\sqrt{h}} \right) = 0, \tag{2.13}$$

where D_a is the derivative compatible with spatial metric h_{ab}.

The general form of the Hamiltonian will be given in Sec. VI, but here we simply quote the surface terms coming from the matter fields. In addition to the usual gravitational surface terms, we obtain

[1] More generally, one need only require that the S^2 is cobordant to an S^2 on the horizon.

[2] We choose a gauge with $\mathcal{L}_\chi A = 0$.

$$-\beta \int dS_b \bigg[\bigg(ND^b\phi + N^b \frac{\pi_\phi}{\beta\sqrt{h}} \bigg) \delta\phi - 2\xi^\mu B_{\mu c} \delta\bigg(\frac{\pi_B^{bc}}{\beta\sqrt{h}} \bigg) + \bigg(\frac{Ne^{-\alpha\phi}}{2} H^{bcd} + \frac{3}{\beta\sqrt{h}} N^{[b}\pi_B^{cd]} \bigg) \delta B_{cd} \bigg] \qquad (2.14)$$

where N, N^a are the usual lapse and shift decomposition of the evolution vector ξ^μ.

Suppose there exists a stationary, axisymmetric solution with bifurcate Killing horizon. In five dimensions, one can have rotation in two orthogonal planes. If there are two rotational Killing fields, the null Killing field on the horizon takes the general form

$$\chi = \frac{\partial}{\partial t} + \Omega_\varphi \frac{\partial}{\partial \varphi} + \Omega_\psi \frac{\partial}{\partial \psi}. \qquad (2.15)$$

Choosing $\xi^\mu = \chi^\mu$ and assuming the metric is asymptotically flat in the sense that it approaches flat space at the same rate as the Myers-Perry black hole [7], the gravitational surface terms at infinity yield $\delta M - \Omega_\varphi \delta J^\varphi - \Omega_\psi \delta J^\psi$. On the horizon, they yield the usual $\frac{\kappa}{8\pi}\delta A_H$ term. The main object for us is determining the possible contributions from the matter fields.

Since N and N^a both vanish on S, one does not expect any contribution from the horizon. To evaluate the contribution at infinity, we must be more specific about the asymptotic behavior of the fields. We require the solutions to have finite energy and hence $T_{\mu\nu}n^\mu n^\nu = \mathcal{O}(r^{-4-2\epsilon})$. At leading order $T_{\mu\nu}n^\mu n^\nu$ is given by a sum of positive definite terms and hence we get the following restrictions:

$$H^{tr\theta_1} = \mathcal{O}(r^{-3-\epsilon}), \qquad (2.16)$$

$$H^{t\theta_1\theta_2} = \mathcal{O}(r^{-4-\epsilon}), \qquad (2.17)$$

$$H^{r\theta_1\theta_2} = \mathcal{O}(r^{-4-\epsilon}), \qquad (2.18)$$

$$H^{\theta_1\theta_2\theta_3} = \mathcal{O}(r^{-5-\epsilon}). \qquad (2.19)$$

The condition that the magnetic charge (2.2) be finite actually imposes the stronger condition

$$H^{\theta_1\theta_2\theta_3} = \mathcal{O}(r^{-6}). \qquad (2.20)$$

Any components of B of higher order than necessary to produce H are pure gauge and we choose a gauge where they do not appear. The above falloff are sufficient to show that all the $\delta\pi$ and δB terms vanish. Using again the finite energy requirement and the equation of motion for the dilaton we find:

$$\phi = C + \frac{a(\theta_i)}{r^{1+\epsilon}} + \frac{b(\theta_i, t)}{r^{3+\epsilon}}, \qquad (2.21)$$

where C is a constant. However, to obtain a finite asymptotic scalar charge one needs the stronger falloff $\phi = C + \mathcal{O}(r^{-2})$. If the perturbation is allowed to change the value of the constant at infinity, we get a scalar charge term, otherwise we do not; these conclusions match those found by Gibbons, Kallosh, and Kol [13]. We will assume that the dilaton vanishes at infinity and hence there is no contribution from the matter fields to the first law. In particular, dipole charges do not seem to appear.

III. EMPARAN'S DIPOLE RING SOLUTIONS

We now briefly review the stationary black ring solutions to (2.1) found by Emparan [11]. (We follow Emparan's convention and take $\beta = \frac{1}{16\pi G}$ in the next two sections.) The solutions depend on three parameters, but since only one component of the angular momentum is nonzero, there are only two conserved quantities, M, J. The third parameter is the dipole charge. It is easiest to start with four auxiliary parameters R, λ, μ, ν and later impose one constraint. These solutions are most conveniently expressed in terms of the following three functions

$$F(\xi) = 1 + \lambda\xi, \qquad G(\xi) = (1 - \xi^2)(1 + \nu\xi), \qquad H(\xi) = 1 - \mu\xi. \qquad (3.1)$$

The black rings are independent of time, t, and two orthogonal rotations parameterized by φ and ψ. Introducing two other spatial coordinates, $-1 \le x \le 1$ and $y \le -1$, the metric is

$$ds^2 = -\frac{F(y)}{F(x)}\bigg(\frac{H(x)}{H(y)}\bigg)^{N/3}\bigg(dt + C(\nu,\lambda)DR\frac{1+y}{F(y)}d\psi\bigg)^2 + \frac{R^2}{(x-y)^2}F(x)(H(x)H^2(y))^{N/3}\bigg[-\frac{D^2G(y)}{F(y)H^N(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{D^2G(x)}{F(x)H^N(x)}d\varphi^2\bigg], \qquad (3.2)$$

while the dilaton is given by

$$e^\phi = \bigg(\frac{H(x)}{H(y)}\bigg)^{N\alpha/2}, \qquad (3.3)$$

and the only nonzero component of the two-form potential is

$$B_{t\psi} = \frac{C(\nu, -\mu)\sqrt{N}DR(1+y)}{H(y)} + k, \qquad (3.4)$$

C and D are given by $C(\nu,\lambda) = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}$ and $D = \frac{\sqrt{1-\lambda}(1+\mu)^{N/2}}{1-\nu}$. The horizon is at $y = -1/\nu$ with topology $S^1 \times S^2$ where ψ parametrizes the S^1 and x and φ parametrize the S^2. The reader familiar with [11] should note we take ψ and φ to have period 2π. To avoid conical singularities along the φ-axis $x = \pm 1$, one requires a relation between λ, μ, ν.

The x, y coordinates break down near the axis and near infinity but making the following coordinate transformation one finds a manifestly asymptotically flat metric:

$$y = -1 - \frac{A\sin^2\theta}{r^2 + f(\theta)}, \qquad x = -1 + \frac{A\cos^2\theta}{r^2 + f(\theta)}, \quad (3.5)$$

where

$$A = \frac{2R^2(1-\lambda)(1+\mu)^N}{1-\nu} \quad (3.6)$$

and

$$f(\theta) = \frac{(1-3\nu)A\cos^2\theta}{2(1-\nu)} + c_0 \quad (3.7)$$

with c_0 an arbitrarily chosen constant. Then asymptotically the dilaton is

$$\phi = -\sqrt{\left(N - \frac{N^2}{3}\right)}\frac{\mu A}{(1+\mu)r^2} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad (3.8)$$

while the potential asymptotically is

$$B_{t\psi} = C(\nu, -\mu)D\sqrt{N}R\frac{1+y}{1-\mu y} + k$$
$$= -C(\nu, -\mu)D\sqrt{N}R\frac{A\sin^2\theta}{(1+\mu)r^2} + k + \mathcal{O}\left(\frac{1}{r^4}\right) \quad (3.9)$$

where $C(\nu, -\mu) = \sqrt{\mu(\mu+\nu)\frac{1-\mu}{1+\mu}}$.

It is easy to check that the dipole charge (2.3) is nonzero for this solution. The only angular momentum is in the ψ direction. Emparan computed the mass M, surface gravity κ, horizon area A_H, angular velocity Ω, and angular momentum J for these solutions and verified that they satisfy

$$\delta M = \frac{\kappa}{8\pi G}\delta A_H + \Omega\delta J + \phi_e\delta q_e, \quad (3.10)$$

where

$$\phi_e = \frac{\pi}{2G}(B_{t\psi}|_\infty - B_{t\psi}|_{\text{horizon}}) \quad (3.11)$$

In (3.10), the perturbations are restricted to go from one stationary solution to another. But this is certainly included in the Sudarsky and Wald argument which applies to an arbitrary perturbation that satisfies the constraints. Since the dipole charge clearly appears in Emparan's first law, we have an apparent contradiction. This is particularly puzzling since the dipole charge requires an integral over an S^2, and the Sudarsky-Wald derivation only produces integrals over the horizon and infinity which are three-surfaces.

Let us turn now to the Hamiltonian formalism and explicitly evaluate the surface terms at infinity. We take a surface of constant t, a vector n^μ normal to these surfaces (and hence having nonzero t and ψ components), and $\xi^\mu = \chi^\mu = (\frac{\partial}{\partial t})^\mu + \Omega(\frac{\partial}{\partial\psi})^\mu$. For Emparan's solutions, since the dilaton is independent of t and ψ, the momentum canonically conjugate to the dilaton vanishes. The dilaton also goes to zero at infinity and so we get no scalar charge terms. The nonvanishing components of the momentum conjugate to the two-form B are

$$\pi_B^{\psi r} = \frac{\sqrt{h}}{32\pi G}e^{-\alpha\phi}n_\mu H^{\mu\psi r}$$
$$= \frac{C(\nu, -\mu)D\sqrt{N}RA\sin\theta\cos\theta}{16\pi G(1+\mu)r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \quad (3.12)$$

and

$$\pi_B^{\psi\theta} = \frac{\sqrt{h}}{32\pi G}e^{-\alpha\phi}n_\mu H^{\mu\psi\theta}$$
$$= -\frac{C(\nu, -\mu)D\sqrt{N}RA\cos^2\theta}{16\pi G(1+\mu)r^3} + \mathcal{O}\left(\frac{1}{r^5}\right). \quad (3.13)$$

These fields falloff sufficiently quickly to eliminate any surface terms at infinity.

IV. RESOLUTION

The resolution to this apparent contradiction is an implicit assumption in the Sudarsky-Wald argument[3]: the potential $B_{\mu\nu}$ must be globally defined and nonsingular everywhere outside (and on) the horizon. Since we are dealing with an electric dipole charge which does not have any obvious topological obstruction, this seems reasonable. However we now show that this it is incompatible with our other assumptions that the dipole charge is nonzero and that B is invariant under the spacetime symmetries $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial\psi}$. We first consider the case where the only angular velocity is Ω_ψ (as in Emparan's solutions), and then comment on the generalization to $\Omega_\varphi \neq 0$. First note that $B_{\mu\psi}$ must vanish along the ψ-axis. This is simply because $B_{\mu\psi} = B_{\mu\nu}(\partial/\partial\psi)^\nu$ and, by definition, $\partial/\partial\psi = 0$ on the axis. If $B_{\mu\psi} \neq 0$, then $B_{\mu\nu}$ diverges, and this is not just a gauge effect. Set $A_\nu = \chi^\mu B_{\mu\nu}$. Then a nonzero $\oint A \cdot dl$ for arbitrarily small loops around the ψ axis indicates a δ-function flux of $H_{\mu\nu\rho}\chi^\rho$ along the axis (see Eq. (4.3) below). This means that the constant k in Emparan's solution for $B_{t\psi}$ is not arbitrary. In his solution, the ψ axis is $y = -1$, and k must be chosen so that $B_{t\psi}(y = -1) = 0$. However $B_{t\psi}$ must also vanish at the horizon [14]. This is because

$$B_{t\psi} = B_{\mu\nu}\chi^\mu\left(\frac{\partial}{\partial\psi}\right)^\nu \quad (4.1)$$

and $\chi^\mu = 0$ on S. It is clearly impossible to satisfy both of these conditions in Emparan's solution. As presented in the previous section (and in [11]) once k is chosen to avoid a δ-function flux of H along the ψ-axis, $B_{\mu\nu}$ necessarily diverges at the horizon. Unlike the axis, this IS purely a gauge effect: the physical field H remains finite at the horizon.

[3]To be fair, this condition was stated explicitly in [12], but its significance becomes clearer in the context of dipole charge.

The inability to have $B_{t\psi}$ vanish at both the axis and horizon is not just a feature of Emparan's solution, but will be present whenever the dipole charge is nonzero. Let us introduce a coordinate y (as in Emparan's solution) so that constant t, ψ, y label two-spheres which are continuously connected to the S^2 on the horizon. Aside from some factors involving the dilaton and metric, the dipole charge (2.3) involves an integral of $H_{t\psi y} = \partial_y B_{t\psi}$ over S^2. If $B_{t\psi}$ vanished at both the axis and horizon, then along every path connecting these surfaces, $H_{t\psi y}$ would have to change sign. But the charge is conserved and the factors of the dilaton and the metric in the integrand cannot change sign. Hence $B_{t\psi}$ cannot vanish on both the axis and horizon.

We will keep $B_{\mu\nu}$ regular on the axis, but allow it to diverge on the horizon[4]. Consider again the boundary terms (2.14) on S. We require the perturbations of the canonical variables to be finite on the horizon, so the vanishing of N and N^a on S still cause all terms to vanish except

$$2\int_S dS_b \chi^\mu B_{\mu c}\delta\left(\frac{\pi_B^{bc}}{\sqrt{h}}\right). \tag{4.2}$$

Since B diverges, its contraction with χ can now remain nonzero on S. To evaluate this term, we use the fact that

$$d(\xi \cdot B) = \mathcal{L}_\xi B - \xi \cdot H \tag{4.3}$$

for any vector ξ where a dot denotes contraction on the first index. If we take $\xi = \chi$ then the right hand side vanishes on S since B is invariant under χ and $\chi = 0$ on S. So on the horizon $\chi \cdot B$ is a closed one form. Hence it must be the sum of an exact form and a harmonic form. Since the S^2 in the horizon is simply connected, the only harmonic one form comes from the S^1. Hence

$$\chi \cdot B = df + c d\psi \tag{4.4}$$

where c is a constant. The first term gives no contribution since integrating the surface term by parts and using the constraint (2.13) and the symmetry of the extrinsic curvature (now using $\hat{n}^a$, the unit normal to the horizon and to n^μ) we see that it vanishes. Using the fact that B is independent of ψ we have $c = B_{t\psi}|_{\text{horizon}}$ and the surface term becomes

$$2c\int_S dS_b \delta\left(\frac{\pi_B^{b\psi}}{\sqrt{h}}\right) = \frac{c}{8G}\delta\int_{S^2} e^{-\alpha\phi} \star H = \frac{c\pi}{2G}\delta q_e. \tag{4.5}$$

Thus, the dipole charge does appear in the first law. Including the gravitational surface terms, we obtain

$$\delta M = \frac{\kappa}{8\pi G}\delta A_H + \Omega_\psi \delta J^\psi + \phi_e \delta q_e, \tag{4.6}$$

where $\phi_e = -\pi c/2G = -\frac{\pi}{2G} B_{t\psi}|_{\text{horizon}}$. This is identical to the first law found by Emparan, except for an apparent discrepancy in the definition of ϕ_e. However note that $B_{t\psi}$ must be constant over the sphere at infinity. If not, $H_{t\psi\theta}$ would be nonzero asymptotically contradicting the falloff (2.17). Since the ψ-axis goes off to infinity, the fact that $B_{t\psi} = 0$ on this axis implies that it must vanish everywhere at infinity. Thus our definition of ϕ_e indeed agrees with Emparan's (3.11). Note that the only symmetry of B that we needed to evaluate the surface term was that $\mathcal{L}_\chi B = 0$. To show that $B_{t\psi}$ could not vanish at both the horizon and the axis when the dipole charge is nonzero, we used that B was independent of both t and ψ separately. We never needed to assume that B was independent of φ.

To summarize: the contradiction is resolved in two steps. The first is that the B field cannot be finite at both the horizon and axis if the dipole charge is nonzero. Allowing B to diverge on the horizon produces a nonzero surface term. The second step is to use the cohomology of the horizon together with its symmetry to show that the surface term is indeed related to the dipole charge.

If $\Omega_\varphi \neq 0$, there are two changes to the first law (4.6). The obvious one is that one picks up a term $\Omega_\varphi \delta J^\varphi$ on the right hand side. The more subtle change is in the definition of the potential ϕ_e. Since χ^μ is now given by (2.15),

$$\phi_e = -\frac{\pi}{2G}(B_{t\psi} + \Omega_\varphi B_{\varphi\psi})|_{\text{horizon}}. \tag{4.7}$$

Solutions have not yet been found in which black rings with dipole charge have nonzero angular velocity in both the φ and ψ directions. However, there is no reason why they should not exist.

V. LOCAL CHARGES AND MINIMAL 5D SUPERGRAVITY

Recently, a family of black ring solutions were found in minimal 5D supergravity [15]. Unlike Emparan's original dipole rings discussed in Sec. III, these solutions were sufficiently complicated that a first law could not be found by inspection.[5] The general procedure of Sudarsky and Wald can be used to derive a first law. Our earlier analysis does not immediately apply to this case since the supergravity action contains a Chern-Simons term which we have not included. In this section we extend our analysis to include this term. Specifically, we consider the five dimensional action

[4]We emphasize again that this is purely a gauge effect. One could use more than one patch and keep the potential finite everywhere, but the argument appears to be more complicated in this case.

[5]Larsen [16] has recently given a first law based on a model for the microscopic degrees of freedom. However this only applies to the near extremal solutions and is formulated in terms of near horizon quantities (mass, charges, etc.) which do not agree with the usual asymptotically defined quantities.

$$S = \beta \int d^5x\sqrt{-g}\Big(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \gamma\epsilon^{\mu\nu\rho\sigma\eta}F_{\mu\nu}F_{\rho\sigma}A_\eta\Big). \qquad (5.1)$$

Minimal 5D supergravity corresponds to $\gamma = (12\sqrt{3})^{-1}$. (If we set $\gamma = 0$, this theory is equivalent to (2.1) with $\alpha = 0$ and the dilaton removed.) Before discussing the details it is worth pointing out an important difference between this case and the discussion in Sec. II. Since we are now working with a two-form F rather than a three form H, the dipole charge is a magnetic charge

$$q_m = \frac{1}{4\pi}\int_{S^2} F. \qquad (5.2)$$

A nonzero dipole charge clearly implies that we will not have a potential which is globally defined, so we must work in patches. However, in each patch, we can choose A so that it is finite on the horizon. Hence when we evaluate the surface terms from the variation of the Hamiltonian, we will have no Maxwell contributions from the horizon of the black hole. The dipole charge will appear through surface terms on the interface between the patches.

The Chern-Simons term clearly enters the field equation for F which now takes the form

$$\nabla_\mu F^{\mu\eta} - 3\gamma\epsilon^{\mu\nu\rho\sigma\eta}F_{\mu\nu}F_{\rho\sigma} = 0. \qquad (5.3)$$

Since the Gauss law constraint is just the time component of this equation, it too will be modified by the Chern-Simons term. However this term is independent of the spacetime metric, and so does not contribute to the stress energy tensor. Since the gravitational constraints involve the matter only through components of $T_{\mu\nu}$, one might expect that the gravitational constraints are unaffected by the Chern-Simons term. Unfortunately, this is not the case. The Hamiltonian must be expressed in terms of the canonical momenta and the momentum conjugate to A now has a Chern-Simons contribution:

$$\pi^a = \beta\sqrt{h}(F^{\mu a}n_\mu + 4\gamma\epsilon^{abcd}F_{bc}A_d), \qquad (5.4)$$

where we define the four dimensional $\epsilon^{abcd} = \epsilon^{abcd\mu}n_\mu$. Note, in particular, that since A is only defined in patches, so is π^a. After computing the canonical Hamiltonian, we find that it has a pure constraint form, as we expect on general grounds:

$$\mathcal{H}_V = \xi^\mu C_\mu + \xi^\mu A_\mu C, \qquad (5.5)$$

where the general relativity constraints C_μ and the gauge constraint C both contain Chern-Simons contributions. Explicitly

$$C = -\beta\sqrt{h}\Big(D_a\Big(\frac{\pi^a}{\beta\sqrt{h}}\Big) + \gamma\epsilon^{abcd}F_{ab}F_{cd}\Big), \qquad (5.6)$$

$$\begin{aligned} C_0 &= -2\sqrt{h}(G_{\mu\nu} - 8\pi T_{\mu\nu})n^\mu n^\nu \\ &= -\beta\sqrt{h}R^{(4)} + \frac{1}{\beta\sqrt{h}}\Big(\pi_G^{ab}\pi_{ab}^G - \frac{\pi_G^2}{3}\Big) + \frac{\pi^a\pi_a}{2\beta\sqrt{h}} \\ &\quad + \frac{\beta\sqrt{h}}{4}F_{ab}F^{ab} - 4\gamma\epsilon^{abcd}\pi_a F_{bc}A_d \\ &\quad + 8\gamma^2\beta\sqrt{h}\epsilon^{abcd}\epsilon_{ajkl}F_{bc}A_dF^{jk}A^l, \end{aligned} \qquad (5.7)$$

$$\begin{aligned} C_a &= -2\sqrt{h}(G_{a\mu} - 8\pi T_{a\mu})n^\mu \\ &= -2\sqrt{h}h_{ab}D_c\Big(\frac{\pi_G^{bc}}{\sqrt{h}}\Big) + F_{ab}(\pi^b \\ &\quad - 4\gamma\beta\sqrt{h}\epsilon^{bcde}F_{cd}A_e). \end{aligned} \qquad (5.8)$$

Much of the complication in these expressions arises from replacing the electric field $E^a = F^{\mu a}n_\mu$ which appears in $T_{\mu\nu}$ by the canonical momentum (5.4). This is necessary since to derive the appropriate surface terms we must vary the Hamiltonian with respect to the canonical variables. Requiring that the Hamiltonian have a well defined variation leads to the following surface terms (in addition to the usual gravitational terms)

$$\beta\int dS_b\Big[\xi^\mu A_\mu\delta\Big(\frac{\pi^b}{\beta\sqrt{h}}\Big) + (NF^{ab} - 2E^{[a}N^{b]})\delta A_a + 4\gamma\epsilon^{bcda}(\xi^\mu A_\mu F_{cd} - 2F_{c\mu}\xi^\mu A_d)\delta A_a\Big]. \qquad (5.9)$$

We now discuss the interpretation of these surface terms.

Suppose we have a black ring solution which is stationary and axisymmetric (in both orthogonal planes) and has a bifurcate Killing horizon. As before, let Σ be a spacelike surface which is asymptotically flat and has an inner boundary at the bifurcation surface S. Also, set the time evolution vector, ξ^μ, equal to the Killing field χ^μ which is tangent to the horizon and vanishes on S. There are no contributions from the inner boundary since $\xi^\mu = 0$ there. At infinity, our finite energy conditions ensure that only the first term contributes and we get the usual $\Phi_E\delta Q_E$ global electric charge term. To see this, recall that in this theory, the definition of a conserved electric charge depends on the Chern-Simons term in general. From the constraint, its clear that

$$Q_E = \frac{1}{4\pi}\int_{S^3} dS_a\Big[\frac{\pi^a}{\beta\sqrt{h}} + 2\gamma\epsilon^{abcd}A_bF_{cd}\Big] \qquad (5.10)$$

is independent of which S^3 it is evaluated on. However, with standard asymptotically flat boundary conditions, the Chern-Simons term does not contribute to the charge computed at infinity. Also, in five dimensions, A_φ and A_ψ must vanish asymptotically, so $\Phi_E = -4\pi\beta\chi^\mu A_\mu|_{r=\infty} = -4\pi\beta A_t|_{r=\infty}$.

The new complication arises from the fact that we have magnetic dipole charge associated with the S^2 of the horizon. This means that we must divide our surface Σ into two patches. The surface terms (5.9) will then arise on the interface between the two patches. Each patch produces surface terms of the same form (with their appropriate A), but with opposite sign, so we are interested in the difference between these two contributions. The new black ring solutions [15] were found in C-metric like coordinates similar to those used in Emparan's dipole rings. So we again use coordinates like those in the previous section (x, y, φ, ψ), where $-1 \leq x \leq 1$ and φ parameterize an S^2, ψ parameterizes the S^1 of the horizon, and y is like a radial coordinate. We choose $-1 < x_0 < 1$ and define our two patches to be $x < x_0$ and $x > x_0$. A_φ is discontinuous across $x = x_0$ with $\Delta A_\varphi = 2q_m$. The surface $x = x_0$ begins and ends on the horizon. It does not enter the asymptotic region.

We have previously assumed that $\xi^\mu A_\mu$ vanishes at the horizon, so we have no contributions there. To do this, however, in the presence of a magnetic dipole charge we need to be able to set $\xi^\mu A_\mu$ to zero in both patches. In particular, $\xi^\mu A_\mu$ must be continuous across the interface between patches. If $\Omega_\varphi \neq 0$ this means that A_t is discontinuous by $2\Omega_\varphi q_m$. (We will work in a gauge in which A_ψ is continuous.) For all presently known nonextremal black rings with dipole charges, $\Omega_\varphi = 0$, but this does not seem to be a fundamental restriction and we expect solutions with nonvanishing Ω_φ to be discovered in the future. Since the interface between our two patches does not enter the asymptotic region, A_t is continuous at large radius and there is no ambiguity in the $\Phi_E \delta Q_E$ term.

We can now evaluate the contribution from the surface terms on the interface between our two patches. At first sight, it appears that there will be terms proportional to the dipole charge q_m and not just its variation. Fortunately, those terms cancel. After a bit of algebra we find all the terms in (5.9) reduce down to a surprisingly simple $\phi_m \delta q_m$ where we define

$$\phi_m = -2\beta \int dS_b [N F^{bc} - 2E^{[b} N^{c]} - 12\gamma \xi^\mu A_\mu \epsilon^{bcde} F_{de}] D_c \varphi, \qquad (5.11)$$

where the integral is over the interface between the patches. To make this more covariant, one could replace the $D_c \varphi$ with $\frac{\Delta A_c}{2q_m}$. To be well defined, this potential must not change when we deform the surface of integration, since this just corresponds to choosing different gauge patches. In other words, the divergence of the integrand must vanish. This is far from obvious, but we have checked that the potential is indeed independent of surface whenever A_μ satisfies the field Eq. (5.3) and $\mathcal{L}_\xi A = 0$. This provides a highly nontrivial check of this potential.

The net result is a standard looking first law

$$\delta M = \frac{\kappa}{8\pi} \delta A_H + \Omega_i \delta J^i + \Phi_E \delta Q_E + \phi_m \delta q_m, \qquad (5.12)$$

where the magnetic dipole potential is given by (5.11).

VI. GENERALIZATION TO HIGHER DIMENSIONS

Having seen that dipole charges can appear in the first law, we now investigate whether other charges might arise. For simplicity, we will drop the Chern-Simons term and consider a higher dimensional generalization of (2.1) including a p-form potential and dilaton in d dimensions

$$S = \beta \int d^d x \sqrt{-g} \Big[R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2(p+1)!} e^{-\alpha\phi} H_{\mu_1 \cdots \mu_{p+1}} H^{\mu_1 \cdots \mu_{p+1}} \Big], \qquad (6.1)$$

where ϕ is the dilaton and $H = dB$ is a $(p+1)$-form field strength. Then let us perform the usual Hamiltonian decomposition with N the lapse function, N^a the shift vector, h_{ab} the induced metric on a surface Σ of constant time and n^μ the unit normal to Σ. The momentum canonically conjugate to the spatial metric h_{ab} and dilaton ϕ are again given by (2.10) and (2.11) respectively. The momentum conjugate to the p-form potential B is

$$\pi_B^{a_1 \ldots a_p} = \frac{\partial \mathcal{L}}{\partial \dot{B}_{a_1 \ldots a_p}} = \frac{\beta \sqrt{h}}{p!} e^{-\alpha\phi} n_\mu H^{\mu a_1 \ldots a_p}. \qquad (6.2)$$

We define the Hamiltonian volume density

$$\mathcal{H}_V = \pi_G^{ab} \dot{h}_{ab} + \pi_B^{a_1 \ldots a_p} \dot{B}_{a_1 \ldots a_p} + \pi_\phi \dot{\phi} - \mathcal{L}, \qquad (6.3)$$

where $\mathcal{L}$ is the Lagrangian density. Then performing integrations by parts to put the result in pure constraint form we obtain

$$H_V = \int_\Sigma (\xi^\mu C_\mu + \xi^{\mu_1} B_{\mu_1 \mu_2 \ldots \mu_p} C^{\mu_2 \cdots \mu_p}), \qquad (6.4)$$

where C_μ is the constraint from the Einstein equations and $C^{\mu_2 \cdots \mu_p}$ is the constraint from the p-form. Explicitly,

$$C_0 = -2\sqrt{h}(G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu n^\nu = -\beta\sqrt{h} R^{(d-1)} + \frac{1}{\beta\sqrt{h}} \Big(\pi_G^{ab} \pi^G_{ab} + \frac{\pi_G^2}{2-d} \Big) + \frac{\pi_\phi^2}{2\beta\sqrt{h}} + \frac{\beta\sqrt{h}}{2} (D\phi)^2 + \frac{p!}{2\beta\sqrt{h}} e^{\alpha\phi} \pi_B^2 + \frac{\beta\sqrt{h}}{2(p+1)!} e^{-\alpha\phi} H_{a_1 \ldots a_{p+1}} H^{a_1 \ldots a_{p+1}}, \qquad (6.5)$$

$$C_a = -2\sqrt{h}(G_{a\mu} - 8\pi T_{a\mu})n^\mu = -2\sqrt{h}D_c\left(\frac{\pi^c_{Ga}}{\sqrt{h}}\right) + \pi_\phi D_a\phi + \pi_B^{a_1\ldots a_p}H_{aa_1\ldots a_p}, \tag{6.6}$$

$$C^{a_2\ldots a_p} = -p\sqrt{h}D_a\left(\frac{\pi_B^{aa_2\ldots a_p}}{\sqrt{h}}\right). \tag{6.7}$$

Now the variation of H_V is well defined only if one adds appropriate surface terms. In addition to the usual gravitational terms, we must add

$$-\beta\int dS_{a_1}\left[\left(ND^{a_1}\phi + N^{a_1}\frac{\pi_\phi}{\beta\sqrt{h}}\right)\delta\phi - p\xi^\mu B_{\mu a_2\ldots a_p}\delta\left(\frac{\pi_B^{a_1\ldots a_p}}{\beta\sqrt{h}}\right) + \left(\frac{Ne^{-\alpha\phi}}{p!}H^{a_1\ldots a_{p+1}} + \frac{p+1}{\beta\sqrt{h}}N^{[a_1}\pi_B^{a2\ldots a_{p+1}]}\right)\delta B_{a_2\ldots a_{p+1}}\right]. \tag{6.8}$$

Let us again specify what we mean by asymptotically flat. We take the metric to be flat with order $\frac{1}{r^{d-3}}$ corrections, as in the Myers-Perry black holes. We again require the solutions to have finite energy and hence $T_{\mu\nu}n^\mu n^\nu = \mathcal{O}(r^{1-d-\epsilon})$. At leading order $T_{\mu\nu}n^\mu n^\nu$ is given by a sum of positive definite terms and hence we get the following restrictions:

$$H^{tr\theta_1\ldots\theta_{p-1}} = \mathcal{O}(r^{[(3-d)/2]-p-(\epsilon/2)}), \tag{6.9}$$

$$H^{t\theta_1\ldots\theta_p} = \mathcal{O}(r^{[(1-d)/2]-p-(\epsilon/2)}), \tag{6.10}$$

$$H^{r\theta_1\ldots\theta_p} = \mathcal{O}(r^{[(1-d)/2]-p-(\epsilon/2)}), \tag{6.11}$$

$$H^{\theta_1\ldots\theta_{p+1}} = \mathcal{O}(r^{[-(1+d)/2]-p-(\epsilon/2)}). \tag{6.12}$$

This is sufficient to ensure that all the $\delta\pi$ and δB terms vanish except the last one. Previously, we used the fact that the magnetic charge (2.2) must be finite to argue that this term must also vanish. But this charge is only defined when $p = d - 3$. For now we simply assume here that the last term also falls off sufficiently rapidly to give no contribution at infinity. Using again the finite energy requirement and the equation of motion for the dilaton we find:

$$\phi \to C + \frac{a(\theta_i)}{r^{[(d-3)/2]+(\epsilon/2)}} + \frac{a(\theta_i, t)}{r^{[(d+1)/2]+(\epsilon/2)}}, \tag{6.13}$$

where C is a constant. To have a well defined scalar charge, we require the faster falloff: $\phi = C + \mathcal{O}(r^{3-d})$. If the perturbation is allowed to change the constant value of ϕ, there is a scalar charge term in the first law, otherwise, there is not: these conclusions match those found by Gibbons, Kallosh, and Kol [13]. We will assume that ϕ vanishes asymptotically, so the scalar surface terms vanish.

Let us now consider generalizations of the first law in Sec. IV. The simplest generalization occurs when $p = 2$ (as in our previous example). Suppose there exists a black ring with horizon topology $\mathcal{M} \times S^1$ where $\mathcal{M}$ denotes any simply connected $d - 3$ dimensional manifold. Since $\star H$ is a $d - 3$ dimensional form, one can again define a dipole charge, $q_e \propto \int_{\mathcal{M}} e^{-\alpha\phi}\star H$, and proceed as before. An essentially identical argument yields the first law:

$$\delta M = \frac{\kappa}{8\pi}\delta A_H + \Omega_i\delta J^i + \phi_e\delta q_e, \tag{6.14}$$

where $\phi_e \propto B_{t\psi}$ evaluated on the horizon (and ψ is the coordinate along the S^1).

For $p > 2$, the general situation is the following. Suppose there is a bifurcate Killing horizon S and let χ be the Killing field which vanishes on S. The horizon can be an arbitrary $d - 2$ dimensional manifold, but we will assume there is a nontrivial (torsion-free) $d - p - 1$ cycle T. Then we can define a local charge (it is no longer a dipole charge) by

$$q_l \propto \int_T e^{-\alpha\phi}\star H. \tag{6.15}$$

It seems likely that it will again be impossible to find a potential B which is finite and globally defined outside the horizon, and invariant under χ. In this case, the surface term

$$p\int_S \xi^\mu B_{\mu a_2\ldots a_p}\delta\left(\frac{\pi_B^{a_1\ldots a_p}}{\sqrt{h}}\right)dS_{a_1} \tag{6.16}$$

can be nonzero. To evaluate this term, it is convenient to note that it is proportional to

$$\int_S (\chi\cdot B)\wedge\delta(e^{-\alpha\phi}\star H), \tag{6.17}$$

where the dot denotes contraction on the first index of B. Applying (4.3) we again see that $\chi\cdot B$ is a closed $p - 1$ form on the horizon, so it can be written as the sum of an exact and a harmonic form. An exact form does not contribute since $d(e^{-\alpha\phi}\star H) = 0$ by the field equation. (Actually, all we need is the constraint, which is the spatial projection of this equation.) So the only contribution comes from the harmonic part of $\chi\cdot B$. By the usual duality between homology and cohomology, there is a harmonic form ω which is dual to T in the sense that for any $d - p - 1$ form σ

$$\int_T\sigma = \int_S\sigma\wedge\omega. \tag{6.18}$$

It then follows that the surface term takes the form $\phi_l\delta q_l$ where the potential ϕ_l is just the constant relating the harmonic part of $\chi\cdot B$ to ω. The first law will then include this new local charge.

VII. DISCUSSION

We have resolved an apparent contradiction between a general derivation of the first law for higher dimensional black holes and an explicit family of solutions found by Emparan. The resolution is based on the realization that there does not exist a globally defined, nonsingular two-form potential B which respects the symmetry. This should not come as a surprise. Even in four dimensional Einstein-Maxwell theory one has a similar situation when considering more than one extremal black hole. Here again, one cannot find a globally defined potential A_μ which is static and finite on all of the horizons. For the five dimensional rotating black ring, one has a problem even for a single object since the rotation axis plays the role of one of the horizons in the sense that it imposes a constraint on the behavior of the potential.

We have also derived a first law for black ring solutions in minimal 5D supergravity. The Chern-Simons term produces an extra complication in the analysis, but in the end, a standard first law is obtained with a nontrivial potential for the magnetic dipole charge. It is likely that there also exist asymptotically flat black holes with horizon topology other than S^n or $S^1 \times S^2$. We have discussed some local charges that these holes might carry and their contribution to the first law.

Our derivation of the first law requires a bifurcate Killing horizon. Hence it does not immediately apply to extremal black holes such as the supersymmetric black rings [17–19]. However, the matter surface terms we derived are generic and could be used to derive a first law even in the absence of a bifurcation surface.

There are several possible generalizations of our work. We have so far considered just a single dipole charge. It should be straightforward to extend this to include several dipole charges and several global charges such as those which arise in dimensional reductions of ten and 11 dimensional supergravity. In four dimensions, the first law has been generalized to apply to isolated horizons in nonstationary spacetimes [20]. This has also been discussed in the context of higher dimensions [4], but without local charges. It would be interesting to extend our derivation of the first law with dipole charges to this case.

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