

Metric of a tidally perturbed spinning black holeNicolás Yunes^{1,*} and José A. González^{2,†}¹*Institute for Gravitational Physics and Geometry, Center for Gravitational Wave Physics, Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802-6300, USA*²*Theoretical Physics Institute, University of Jena, Max-Wien-Platz 1, 07743, Jena, Germany*

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We explicitly construct the metric of a Kerr black hole that is tidally perturbed by the external universe in the slow-motion approximation. This approximation assumes that the external universe changes slowly relative to the rotation rate of the hole, thus allowing the parameterization of the Newman-Penrose scalar ψ_0 by time-dependent electric and magnetic tidal tensors. This approximation, however, does not constrain how big the spin of the background hole can be and, in principle, the perturbed metric can model rapidly spinning holes. We first generate a potential by acting with a differential operator on ψ_0 . From this potential we arrive at the metric perturbation by use of the Chrzanowski procedure in the ingoing radiation gauge. We provide explicit analytic formulas for this metric perturbation in Kerr coordinates, where the perturbation is finite at the horizon. This perturbation is parametrized by the mass and Kerr spin parameter of the background hole together with the electric and magnetic tidal tensors that describe the time evolution of the perturbation produced by the external universe. In order to make the metric accurate far away from the hole, these tidal tensors should be determined by asymptotically matching this metric to another one valid far from the hole. The tidally perturbed metric constructed here could be useful in initial data constructions to describe the metric near the horizons of a binary system of spinning holes. This perturbed metric could also be used to construct waveforms and study the absorption of mass and angular momentum by a Kerr black hole when external processes generate gravitational radiation.

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I. INTRODUCTION

Gravitational wave observatories, such as LIGO and VIRGO, have the potential to study black holes in the strong field regime [1]. These black holes are expected to be immersed in a sea of gravitational perturbations that will alter the gravitational field of the background hole. Even though tidal perturbations are expected to be small relative to the background, they will be important in some astrophysical scenarios when attempting to provide an accurate description of the nonlinear dynamical orbital evolution of bodies around this background. The need for high accuracy in the description of the orbital evolution derives from the fact that gravitational wave observatories are extremely sensitive to the phase of the gravitational waves emitted by the system. Therefore, since this phase is directly related to the orbital evolution, in some astrophysical scenarios it is necessary to take these tidal effects into consideration.

The study of gravitational perturbations around Kerr black holes is important for several reasons. First, it is of astrophysical interest to study the flux of mass and angular momentum across a perturbed Kerr horizon [2–5], which can be calculated through manipulations of the tidally perturbed metric computed in this paper. Although this flux might be small for equal-mass binaries, in extreme-mass ratio inspirals (EMRI) up to 5% of the total energy might be absorbed by the background hole. This absorption

might slow down the orbital evolution increasing the duration of the gravitational wave signal [6,7]. Space-based detectors, such as LISA, will be able to observe and measure the gravitational waveforms of EMRIs, since they have particularly low noise in the low frequency band where such inspirals are common. Therefore, precise knowledge of the gravitational waveform including the tidal perturbations effects might be important in data analysis [8]. Finally, the explicit formulas of this paper might be useful to compute initial data near the horizons of a binary system of spinning holes. For example, Refs. [9,10] make use of such explicit formulas for the nonspinning case to construct initial data via asymptotic matching. This data might be useful to the numerical relativity community because it derives from an approximate solution to the Einstein equation and, thus, we expect it to accurately describe the gravitational field of the system up to uncontrolled remainders.

In this paper, we analytically construct explicit formulas for the metric of a tidally perturbed Kerr hole, where the perturbations of the external universe are assumed to vary slowly in a well-defined sense. Metric perturbations for nonspinning holes have been studied in Refs. [9,11–14] using the Regge-Wheeler formalism [15]. However, this method is difficult to implement for spinning holes because the metric will now depend on both radius and angle θ in a nontrivial way, rendering the Einstein equations very difficult to solve. For this reason, we use the Chrzanowski procedure [16,17] to construct the metric perturbation from the Newman-Penrose (NP) scalar ψ_0 . This procedure

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allows us to calculate the metric in the so-called ingoing radiation gauge (IRG), which is suitable to study gravitational perturbations near the outer horizon (r_+), since there the metric is transverse and traceless.

We will work in the slow-motion approximation, described in detail in Refs. [2,11–14,18–20], where we assume that the rate of change of the curvature of the external universe is small relative to the rotation rate of the background black hole, which in principle could be extremal. The external universe is completely arbitrary in that sense, as long as it respects the slow-motion approximation. For example, in the case where the external universe is given by a second black hole in a quasicircular orbit around the background hole, this approximation is valid as long as their orbital separation is sufficiently large. In particular, this separation must be at least greater than the inner most stable circular orbit (ISCO)[21,22], so that the binary is still in a quasicircular orbit. In that case, the curvature generated by the second hole would correspond to the external universe, which will change slowly as long as the orbital velocity is sufficiently small. In this sense, the slow-motion approximation will hold for astrophysically realistic binaries as long as they are sufficiently separated.

Assuming this approximation to be valid, Poisson [2] has computed ψ_0 in the neighborhood of a spinning hole. First, the Weyl tensor of the spacetime is re-expressed in terms of the electric and magnetic tidal tensors of the external universe. Using the slow-motion approximation, Poisson argues that these tensors will be spatially coordinate independent if evaluated at sufficiently large distances from the worldline of the hole. With this tensor, the asymptotic form of ψ_0 , denoted by $\tilde{\psi}_0$, is computed far from the hole by projecting the Weyl tensor onto the Kinnersly tetrad. This scalar will be a combination of slowly-varying functions of time, which will be parametrized via the electric and magnetic tidal tensors, and scalar functions of the spatial coordinates, which will be given by the tetrad. The asymptotic form of ψ_0 now allows for the construction of an ansatz for ψ_0 , which consists of its asymptotic form $\tilde{\psi}_0$ multiplied by a set of undetermined function of radius $R_m(r)$. These functions must satisfy the asymptotic condition $R_m \rightarrow 1$ as $r \gg r_+$, as well as the Teukolsky equation. This last condition is a differential constraint on $R_m(r)$ which can be solved for explicitly, thus allowing for the full determination of ψ_0 . In this manner, the final expression for ψ_0 is obtained and is now valid close to the horizon as well, in particular, approaching the perturbations generated by the external universe sufficiently far from the hole's worldline.

Once ψ_0 has been calculated, we can apply the Chrzanowski procedure to compute the metric perturbation, still in the slow-motion approximation. This calculation contains two parts: the computation of a potential (Ψ) and the determination of the metric perturbation (h_{ab}) from Ψ . In principle, one might think that it would be easier to

try to compute h_{ab} directly from ψ_0 . Chrzanowski [16] attempted this by applying a differential operator onto ψ_0 . However, Wald [23] discovered that doing this leads to a physically different gravitational perturbation from that represented by ψ_0 . Cohen and Kegeles [24] showed that by constructing a Hertz-like potential Ψ from ψ_0 first and then applying Chrzanowski's differential operator to Ψ instead leads to the real metric perturbation. This is the procedure we will follow to construct the metric perturbation in this paper.

The construction of Ψ requires the action of a fourth order differential operator on ψ_0 , where here we follow Ori [17]. This potential is simplified by the use of the slow-motion approximation that allows us to neglect any time derivatives of the electric and magnetic tidal tensors. Once Ψ is calculated, we can apply Chrzanowski's differential operator to this potential [24,25]. In this manner, we compute the metric of a perturbed spinning hole from a tidal perturbation described by ψ_0 in terms of tidal tensors. These tensors are unknown functions of time that represent the external universe and which should be determined by asymptotically matching this metric to another approximation valid far from the holes [9,10].

The metric computed here, however, has a limited applicability given by the validity of the slow-motion approximation and the Chrzanowski procedure. The slow-motion approximation implies that we can neglect all derivatives of the tidal tensors. Furthermore, since we are working only to first nonvanishing order in this approximation, it suffices to consider only the quadrupolar perturbation of the metric, since the monopolar and dipolar perturbations are identically zero. In perturbation theory, any l mode in the decomposition of the perturbation is one order larger than the $l + 1$ mode. Therefore, any higher modes or couplings of the quadrupole to other modes will be of higher order. The Chrzanowski procedure also possesses a limited region of validity, given by a region sufficiently close to the event horizon so that the spatial distance from the horizon to the radius of curvature of the external universe is small. This restriction is because the Chrzanowski procedure builds the metric as a linear perturbation of the background and neglects any nonlinear interactions with the external universe. In particular, this restriction implies that this metric cannot provide valid information on the dynamics of the entire spacetime. However, if this metric is asymptotically matched to another approximation that is valid far from the background hole, then the combined global metric will describe the 3-manifold accurately.

We verify the validity of our calculations in several ways. First, we check that ψ_0 indeed satisfies the Teukolsky equation. After computing Ψ , we also check that it satisfies the Teukolsky equation and the differential constraint that relates Ψ to ψ_0 . Finally, we check that the metric perturbation constructed with this potential satisfies

all of the Einstein equations to the given order. We further check that this perturbation is indeed transverse and traceless in the tetrad frame so that it is suitable for the study of gravitational perturbations near the horizon.

This paper is divided as follows. Section II describes the slow-motion approximation in detail, summarizes some relevant results from Ref. [2] and establishes some notation. Section III computes Ψ from ψ_0 , while Sec. IV calculates h_{ab} from Ψ . Finally, Sec. V presents some conclusions and points toward future work. In the appendix, we provide an explicit transformation to Kerr-Schild coordinates that might be more amenable to numerical implementation.

In the remaining of the paper, we use geometrized units ($G = 1$, $c = 1$) and the symbol $O(a)$ stands for terms of order a , where a is dimensionless. Latin indices range from 0 to 3, where 0 is the time coordinate. The Einstein summation convention is assumed all throughout the paper, where repeated indices are to be summed over unless otherwise specified. Tetrad notation will be used, where indices with parenthesis refer to the tetrad and those without parenthesis to the components of the tensor. The relational symbol \sim stands for ‘‘asymptotic to’’ as defined in [26], while the symbols \ll and \gg are also to be understood in the asymptotic sense. In particular, note that if $f(r)$ is valid for $r \ll b$, then this function is not valid as $r \rightarrow b$. In this paper we have relied heavily on the use of symbolic manipulation software, such as MAPLE and MATHEMATICA.

II. THE SLOW-MOTION APPROXIMATION AND THE NP SCALAR

In this section we will describe the slow-motion approximation in more detail and discuss the construction of ψ_0 due to perturbations of the external universe. Both the slow-motion approximation and ψ_0 have already been explained in detail and computed by Poisson in Ref. [2]. Therefore, here we follow this reference and summarize the most relevant results for this paper while establishing some notation.

Let us begin by discussing the slow-motion approximation. Consider a nonspinning black hole of mass m_1 immersed in an external universe, with radius of curvature \mathcal{R} . This external universe could be given by any object that lives in the exterior of the hole’s horizon, such as a scalar field or another black hole. The slow-motion approximation requires that the external universe’s length scales be much larger than the hole’s scales. In other words, for the nonspinning case we must have $m_1/\mathcal{R} \ll 1$, since these are the only scales available.

For concreteness, let us assume that the external universe can be described by another object of mass M_{ext} and that our hole is in a quasicircular orbit around it. Then, we have

$$\frac{m_1}{\mathcal{R}} \sim \frac{m_1}{m_1 + M_{\text{ext}}} V^2, \quad V = \sqrt{\frac{m_1 + M_{\text{ext}}}{b}}, \quad (1)$$

where V is the orbital velocity and b is the orbital separation. There are two ways of enforcing $m_1/\mathcal{R} \ll 1$: the small-body approximation, where we let $m_1/M_{\text{ext}} \ll 1$; and the slow-motion approximation, where $V \ll 1$. However, in a future paper we might want to asymptotically match the metric perturbation computed in this paper to a post-Newtonian (PN) expansion [27], which requires small velocities. For this reason, we will restrict our attention to the slow-motion approximation, which implies that we can only investigate systems that are sufficiently separated. The ISCO is not a well-defined concept for black hole binaries, but it has been estimated for nonspinning binary numerically [21,22] to be given approximately by $\omega_{\text{ISCO}}M/10$, where M and ω_{ISCO} are the total mass of the system and its angular velocity at the ISCO, respectively. For spinning binaries the holes can get closer without plunging, where the value of the ISCO becomes a function of the spin parameter of the holes. Regardless of the type of binary system, the slow-motion approximation will hold as long as we consider systems that are separated by at least more than their ISCO so that they are still in a quasicircular orbit.

The above considerations suffice for nonspinning holes because there is only one length scale associated with it, m_1 . However, for spinning ones we must also take into account the time scale associated with the intrinsic spin of the hole. The slow-motion approximation does not constraint how large the spin of the background hole could be. However, in the standard theory, it is usually assumed that isolated holes will obey $a < m_1$, where a is the rotation parameter of the hole and m_1 is its mass. If the previous inequality did not hold, then the hole would be tidally disrupted by centrifugal forces. When the background hole is surrounded by an accretion disk, however, some configuration might lead to a violation of the previous condition [28], but we will not consider those in this paper. Let us then define a dimensionless rotation parameter $\chi = a/m_1$, which is now restricted to $0 < \chi < 1$. The mass of the hole and this rotation parameter now define a new timescale, related to the rotation rate of the horizon and given by

$$\tau_H = \frac{1}{\Omega_H} = \frac{2m_1}{\chi} (1 + \sqrt{1 - \chi^2}), \quad (2)$$

where Ω_H is the angular velocity of the hole’s horizon [29]. The slow-motion approximation then requires this time scale to be much smaller than the time scale associated with the change of the radius of curvature of the external universe, τ_{ext} , i.e. $\tau_H/\tau_{\text{ext}} \ll 1$. This condition then becomes

$$m_1/\mathcal{R} \ll \chi, \quad (3)$$

since $\tau_{\text{ext}} \approx \mathcal{R}$. We take the above relation as the precise definition of the slow-motion approximation for spinning black holes. Following the reasoning that lead to Eq. (1), we must then have $V^2 \ll \chi$, which, for a binary system, means that the orbital velocity of the system cannot exceed the rotation rate of the background hole. Thus, the slow-motion approximation implies that, to this order and for spinning holes, we can also only consider systems that are sufficiently separated and where the black holes have relatively rapid spins. This reasoning does not imply that the Schwarzschild limit is incompatible with the slow-motion approximation. Actually, Poisson [2] showed that when computing certain quantities, such as ψ_0 , this limit can be recovered if we work to higher order in the slow-motion approximation. The precise radius of convergence of the slow-motion approximation to first order remains unknown, but an approximate measure of how small a separation the approximation can tolerate will be studied in a later section.

Now that the slow-motion approximation has been explained in detail, let us proceed with the construction of ψ_0 , which is defined by

$$\psi_0 = C_{abcd} l^a m^b l^c m^d, \quad (4)$$

where C_{abcd} is the Weyl tensor of the spacetime and l^a and m^a are the first and third tetrad vectors. This null vector m^a is not to be confused with the m that we will introduce later in this section to denote the angular mode of the perturbation. Poisson works with the Kinnersly tetrad in advanced Eddington-Finkelstein (EF) coordinates, also known as Kerr coordinates, which are well-behaved at the outer horizon, given by $r_+ = m_1 + (m_1^2 - a^2)^{1/2}$.

The calculation of ψ_0 is based on making an ansatz guided by its asymptotic form far from the worldline of the hole but less than the radius of curvature of the external universe, i.e. $r_+ \ll r \ll \mathcal{R}$. In this region, the Weyl tensor can be decomposed into electric \mathcal{E}_{ab} and magnetic \mathcal{B}_{ab} tidal fields, which are slowly-varying functions of advanced time v only. Thus, in this region, the only spatial coordinate dependence in ψ_0 is given by the tetrad, namely

$$\tilde{\psi}_0 \sim -\sum_m z_m(v) {}_2Y_2^m(\theta, \phi), \quad (5)$$

where the tilde is to remind us that this quantity is the asymptotic form of ψ_0 and where ${}_2Y_2^m(\theta, \phi)$ are spin-weighted spherical harmonics, given by

$$\begin{aligned} {}_2Y_2^0(\theta, \phi) &= -\frac{3}{2}\sin^2\theta, \\ {}_2Y_2^{\pm 1}(\theta, \phi) &= -\sin\theta(\cos\theta \mp 1)e^{\pm i\phi}, \\ {}_2Y_2^{\pm 2}(\theta, \phi) &= \frac{1}{4}(1 \mp 2\cos\theta + \cos^2\theta)e^{\pm 2i\phi}. \end{aligned} \quad (6)$$

In Eq. (5), the quantities $z_m(v)$ are complex combinations of the tidal fields given by $z_m(v) = \alpha_m(v) + i\beta_m(v)$, where

$$\begin{aligned} \alpha_0(v) &= \mathcal{E}_{11}(v) + \mathcal{E}_{22}(v), \\ \alpha_{\pm 1}(v) &= \mathcal{E}_{13}(v) \mp i\mathcal{E}_{23}(v), \\ \alpha_{\pm 2}(v) &= \mathcal{E}_{11}(v) - \mathcal{E}_{22}(v) \mp 2i\mathcal{E}_{12}(v), \\ \beta_0(v) &= \mathcal{B}_{11}(v) + \mathcal{B}_{22}(v), \\ \beta_{\pm 1}(v) &= \mathcal{B}_{13}(v) \mp i\mathcal{B}_{23}(v), \\ \beta_{\pm 2}(v) &= \mathcal{B}_{11}(v) - \mathcal{B}_{22}(v) \mp 2i\mathcal{B}_{12}(v). \end{aligned} \quad (7)$$

Note that in this region, ψ_0 is independent of radial coordinate.

With Eq. (5) at hand, Poisson makes an ansatz for the functional form of ψ_0 , namely

$$\psi_0 = -\sum_m z_m(v) R_m(r) {}_2Y_2^m(\theta, \phi), \quad (8)$$

where $R_m(r)$ is an undetermined function of radius that must satisfy $R_m(r) \rightarrow 1$ as $r \gg r_+$. If Eq. (8) is inserted into the Teukolsky equation, one obtains a differential equation for $R_m(r)$. The angular part of the Teukolsky equation is automatically satisfied by the angular decomposition of ψ_0 in spin-weighted spherical harmonics with eigenvalue $E = 4$ (Eq. 2.10 in Ref. [30]). The spatial part of the Teukolsky equation yields a differential constraint for $R_m(r)$ namely,

$$\left\{ x(1+x) \frac{d^2}{dx^2} + [3(2x+1) + 2im\gamma] \frac{d}{dx} + 4im\gamma \frac{2x+1}{x(1+x)} \right\} R_m(x) = 0, \quad (9)$$

where x is a rescaled version of the radial coordinate given by

$$x = \frac{r - r_+}{r_+ - r_-}, \quad (10)$$

and where the inner and outer horizons are given, respectively, by $r_{\pm} = m_1^2 \pm (m_1^2 - a^2)^{1/2}$. We should note that r_+ is an event horizon, while r_- is actually an apparent horizon. Solving this equation [2] one obtains

$$R_m(r) = A_m x^{-2} (1+x)^{-2} F(-4, 1, -1 + 2im\gamma; -x), \quad (11)$$

where A_m is a normalization constant given by

$$A_m = -\frac{i}{6} m\gamma (1 + im\gamma) (1 + 4m^2\gamma^2). \quad (12)$$

In Eq. (11), the function $F(a, b, c; x)$ is the hypergeometric function

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (13)$$

where $(a)_n$ is the Pochhammer symbol defined as

$$(a)_n = a(a+1)(a+2)\dots(a+n-1) = \frac{(a+n-1)!}{(a-1)!},$$

$$(a)_0 = 1. \quad (14)$$

For the present case, the series gets truncated at the fourth power and we obtain

$$F(-4, 1, -1 + 2im\gamma; -x) = \left(1 + \frac{4}{2im\gamma - 1}x + \frac{6}{(2im\gamma - 1)im\gamma}x^2 + \frac{12}{(2im\gamma - 1)im\gamma(2im\gamma + 1)}x^3 \right. \\ \left. + \frac{12}{(2im\gamma - 1)im\gamma(2im\gamma + 1)(2im\gamma + 2)}x^4 \right), \quad (15)$$

where γ is a constant given by

$$\gamma = \frac{a}{r_+ - r_-}. \quad (16)$$

In this manner, Poisson calculates ψ_0 , which encodes the gravitational perturbations of the external universe on a spinning hole in the slow-motion approximation. The full expression for ψ_0 is then given by

$$\psi_0 = - \sum_{m \neq 0} B_m x^{-2} (1+x)^{-2} \left(1 + \frac{4}{2im\gamma - 1}x + \frac{6}{(2im\gamma - 1)im\gamma}x^2 + \frac{12}{(2im\gamma - 1)im\gamma(2im\gamma + 1)}x^3 \right. \\ \left. + \frac{12}{(2im\gamma - 1)im\gamma(2im\gamma + 1)(2im\gamma + 2)}x^4 \right) {}_2Y_2^m(\theta, \phi), \quad (17)$$

where we have used the final abbreviation

$$B_m = A_m z_m(v), \quad (18)$$

without summing over repeated indices here, to group all terms that are spatially coordinate independent. We have checked that Eq. (17) indeed satisfies the Teukolsky equation [30] for the $s = 2$ mode that corresponds to this scalar. Note that, in this final expression, summation over the $m = 0$ mode is removed because Poisson [2] has shown that it corresponds to the Schwarzschild limit and, thus, it contributes at a relative $O(m_1/\mathcal{R})$ higher than all other modes. Also note that the final expression for ψ_0 only contains the quadrupolar $l = 2$ mode, once more because higher multipoles will be smaller by a relative factor of m_1/\mathcal{R} in the slow-motion approximation. For this reason, there are no mode couplings in the ψ_0 presented above.

The time-evolution of the tidal perturbation will be exclusively governed by the electric and magnetic tidal fields. These tensors should be determined by asymptotically matching the metric perturbation generated by this ψ_0 to another approximation valid far from the hole. However, in Refs. [9,10,14,20] it has been shown that when the external universe is given by another nonspinning black hole in a quasicircular orbit, these tensors scale approximately as

$$z_m \approx \frac{m_2}{b^3}, \quad (19)$$

where m_2 is the mass of the other hole (the external universe) and b is the orbital separation (approximately equal to the radius of curvature of the external universe).

Note that the factor of b^3 in the denominator is necessary to make the tidal tensors dimensionally correct. Also note that in Eq. (19) we have neglected the time dependence of the tidal fields, which generally is given by a trigonometric function, since we are interested in a spatial hypersurface of constant time. For the case where the other hole is spinning, Eq. (19) will contain corrections proportional to χ , but these terms will not change the overall scale of the tidal tensors.

Throughout the rest of the paper we will generate plots of physical quantities, such as ψ_0 , Ψ and h_{ab} . For the purpose of plotting, we will have to make two choices: one regarding the physical scenario that produces the perturbation of the external universe; and another regarding the parameters of the background black hole. As for the physical scenario, we will choose the external universe to be given by another orbiting black hole in a quasicircular orbit. This choice allows us to represent the tidal fields with the scaling given in Eq. (19). This scaling is not the exact functional form of the tidal fields and, therefore, the plots generated will not be exact. However, this scaling will allow us to provide plots accurate enough to study the general features of the global structure of the quantities plotted, as well as some local features near the horizons. Regarding the parameter choice, we will assume $m_1 = m_2$, $a = 0.99m_1$, and $b = 10m_1$, where $M = m_1 + m_2 = 1$ is the total mass. These choices are made in accordance with the slow-motion approximation, while making sure the system is not inside its ISCO. However, the formulas presented in this paper should apply to other choices of physical scenarios and background parameters as well, as

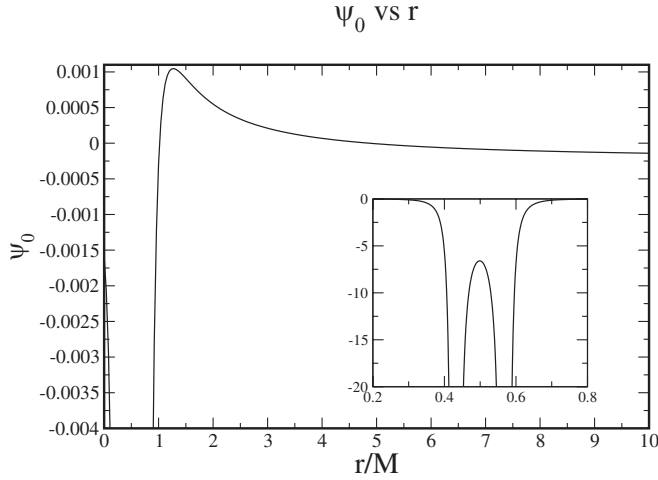


FIG. 1. Plot of the real part of the NP scalar ψ_0 along the x axis with the plotting parameters described in Sec. II.

long as these do not conflict with the slow-motion approximation.

Given the chosen physical scenario, the formulas in this paper should apply to other mass ratios and separations, as long as the orbital velocity does not become too large. For the background parameters chosen, the orbital velocity is approximately $V \approx 0.3$, which indicates that, for fixed masses, we cannot reduce the orbital separation by much more without breaking the slow-motion approximation. However, since we provide explicit analytic formulas for all relevant physical quantities, we can estimate their error by considering the uncontrolled remainders, i.e. the neglected terms in the approximation. In particular, in a later section, we will see that the uncontrolled remainders are still much smaller than the perturbation itself even at $b = 10M$ as long as we restrict ourselves to field points sufficiently near the outer horizon of the background hole.

In Fig. 1 we plot the real part of ψ_0 with the plotting choices described earlier. Observe that as r becomes large the scalar asymptotes to a constant given by $\tilde{\psi}_0$. Also observe that the functional behavior of the scalar is drastically different as r becomes small. In this figure, as well as in future figures, we have chosen to include an inset where we zoom to a region close to the horizons, so that we can observe its local and global behavior. For the orbital parameter chosen, the inner and outer horizons are given by $r_- \approx 0.43M$ and $r_+ \approx 0.57M$. Observe from the inset in Fig. 1 that the scalar diverges at the horizons and as $r \rightarrow 0$, which is due to the choice of tetrad. Finally, observe that, except for where it diverges, the real part of ψ_0 is of $O(V^2)$ in the entire 3-manifold.

III. THE POTENTIAL

In this section we will use ψ_0 to construct the potential Ψ by acting some differential operators on the NP scalar. This potential must satisfy the vacuum Teukolsky equation for

the $s = -2$ mode together with the differential equation

$$\psi_0 = D^4[\bar{\Psi}], \quad (20)$$

where the overbar stands for complex conjugation and where the differential operator is given by $D = l^a \partial_a$ [17,23,31].

Ori [17] has shown that the above differential equations can be inverted with use of the Teukolsky-Starobinsky relation to obtain

$$\Psi = \frac{1}{p} \Delta^2 (D^\dagger)^4 [\Delta^2 \bar{\psi}_0], \quad (21)$$

where here Δ is given by

$$\Delta = r^2 - 2m_1 r + a^2, \quad (22)$$

while p is a constant that for the time-independent case reduces to

$$p = [l(l+1) - s^2 + |s|^2][l(l+1) - s^2 + |s| + 2]^2. \quad (23)$$

In our case, since s and l refer to ψ_0 , $s = 2$ and $l = 2$ so that this constant becomes $p = 576$.

The differential operator D^\dagger is given in spherical Brill-Lindquist (BL) coordinates $(t_{\text{BL}}, r, \theta, \phi_{\text{BL}})$ by

$$D_{\text{BL}}^\dagger = \partial_r - \frac{a}{\Delta} \partial_{\phi_{\text{BL}}}, \quad (24)$$

neglecting any time dependence, since time derivatives will only contribute at a higher order. In order to compute Ψ in Kerr coordinates, we must transform the above differential operator. The transformation between Kerr and BL coordinates is given by

$$\begin{aligned} dv &= dt_{\text{BL}} + dr_{\text{BL}} \left(\frac{2m_1 r}{\Delta} + 1 \right), & dr &= dr_{\text{BL}}, \\ d\theta &= d\theta_{\text{BL}}, & d\phi &= d\phi_{\text{BL}} + dr_{\text{BL}} \frac{a}{\Delta}. \end{aligned} \quad (25)$$

After transforming the differential operator D_{BL}^\dagger we obtain

$$D^\dagger = \partial_r, \quad (26)$$

because the $dr_{\text{BL}}/d\phi$ term in the transformation cancels the ϕ dependence. Note that r and θ do not change in this transformation and, thus, Δ remains unchanged. Also note that Ψ is a scalar constructed from differential operators on ψ_0 and, since the latter is a scalar, Ψ will also be gauge invariant.

Before plugging in Eq. (17) into Eq. (21) to compute Ψ , let us try to simplify these expressions. In the previous section, we defined the inner and outer horizons r_+ and r_- , as well as the new variable x . We can invert the definition of x so that it becomes a definition for r as a function of x and then insert this into Eq. (22). We then obtain

$$\Delta = 4\eta x(1+x), \quad (27)$$

where we have defined $\eta = m_1^2 - a^2$. It is clear now that

when we combine the square of this expression with Eq. (17) some cancellations will occur that will simplify all future calculations.

We are now ready to compute Ψ , but first let us rewrite the function we want to differentiate, namely

$$\Delta^2 \bar{\psi}_0 = \sum_{m \neq 0} \bar{C}_m \bar{F}(x)_2 \bar{Y}_2^m(\theta, \phi_{BL}), \quad (28)$$

where $F(x)$ is shorthand for the aforementioned hypergeometric function and where C_m is a new function of advanced time only given by

$$C_m = -16B_m \eta^2 = -16A_m \eta^2 z_m(v). \quad (29)$$

Note that the angular dependence occurs in the spherical harmonics, while the only x dependence is in the hypergeometric function. We can transform the D^\dagger operator to x space to obtain

$$D^\dagger = \partial_r = \frac{1}{2\eta^{1/2}} \partial_x. \quad (30)$$

Applying all these simplifications, Ψ becomes

$$\Psi = \frac{\Delta^2}{576} \sum_{m \neq 0} \bar{C}_{m2} \bar{Y}_2^m(\theta, \phi) \frac{1}{16\eta^2} \partial_x^4 [\bar{F}(x)]. \quad (31)$$

We can now apply all derivatives to obtain

$$\Psi = \frac{\Delta^2}{576} \sum_{m \neq 0} \bar{C}_{m2} \bar{Y}_2^m(\theta, \phi) \frac{\bar{F}^{(4)}}{16\eta^2}, \quad (32)$$

where we have used the shorthand $\bar{F}^{(n)}$ which stands for the n th derivative of the complex conjugate of the hypergeometric function. This derivative is given by

$$\bar{F}^{(4)} = \frac{288}{(2im\gamma + 1)im\gamma(-2im\gamma + 1)(-2im\gamma + 2)}, \quad (33)$$

Note that we can re-express the constant γ in terms of η as $\gamma = a/(2\eta^{1/2})$. Interestingly this constant combines with \bar{C}_m to return an overall constant that is purely real so that Ψ is given by

$$\Psi = -\frac{1}{24} \Delta^2 \sum_{m \neq 0} Y^m(\theta) e^{-im\phi} \bar{z}_m(v), \quad (34)$$

where here Y^m stands for the $l=2$ spherical harmonics with zero ϕ dependence. We have checked that the potential of Eq. (34) indeed satisfies the definition of Eq. (20) as well as the Teukolsky equation for the $s=-2$ mode with angular eigenvalue $E=10$. Note that Ψ has units of mass squared because the electric and magnetic tidal fields scale as the inverse of the mass squared. Furthermore, note that even though ψ_0 is singular at the horizon, Ψ is finite and actually vanishes there.

Next, we proceed to decompose Ψ into real and imaginary parts. The potential contains 2 complex terms,

Re Ψ vs r

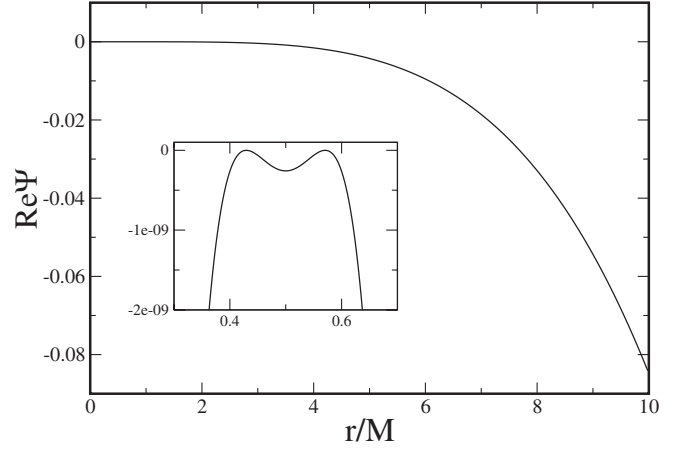


FIG. 2. Plot of the potential Ψ_R along the x axis with the plotting parameters described in Sec. II.

namely, the ϕ part of the spherical harmonics and the electric and magnetic tidal tensors. Decomposing Ψ we obtain

$$\Psi_R = -\frac{\Delta^2}{24} \sum_{m \neq 0} Y^m(\alpha_m(v) \cos m\phi - \beta_m(v) \sin m\phi),$$

$$\Psi_I = \frac{\Delta^2}{24} \sum_{m \neq 0} Y^m(\alpha_m(v) \sin m\phi + \beta_m(v) \cos m\phi). \quad (35)$$

Note that the entire radial dependence is encoded in Δ^2 , whereas the angular dependence is hidden in the spherical harmonics. The time dependence occurs only in the tidal fields that should be determined via asymptotic matching, as mentioned previously. This is the potential in Kerr coordinates associated with the ψ_0 calculated in the previous section in the slow-motion approximation.

In Fig. 2 we plot Ψ_R with the plotting choices described in Sec. II. Observe from the inset that the potential has nodes at both horizons. Also observe that the potential does not asymptote to a constant, but instead it grows quartically. This growth is due to the factor of Δ^2 that dominates at large radius. Finally, note that the potential is still of $O(V^2)$ for radii sufficiently close to the hole (roughly for $r < 8m$).

IV. THE METRIC PERTURBATION

In this section we compute the metric perturbation by applying Chrzanowski's differential operator to the potential calculated in the previous section. The full metric of the spacetime is given by

$$g_{ab} = g_{ab}^B + h_{ab}, \quad (36)$$

where g_{ab}^B is the background metric and h_{ab} is the perturbation. Since ψ_0 and Ψ were computed in Kerr coordinates, the background should also be in this coordinate

system. This background is given then by

$$\begin{aligned} g_{00}^B &= -\left(1 - \frac{2m_1 r}{\Sigma}\right), & g_{01}^B &= 1, \\ g_{03}^B &= -\frac{m_1 r}{\Sigma}(2a \sin^2 \theta), & g_{13}^B &= -a \sin^2 \theta, \\ g_{22}^B &= \Sigma, & g_{33}^B &= (r^2 + a^2) \sin^2 \theta + \frac{2m_1 r}{\Sigma}(a^2 \sin^4 \theta), \end{aligned} \quad (37)$$

where m_1 is the mass of the background black hole, a is its spin parameter, related to the angular momentum vector by $\vec{S} = m\vec{a}$, and where $\Sigma = r^2 + a^2 \cos^2 \theta$.

Let us now construct the metric perturbation. We will work with the form of the differential operator presented in Ref. [25], namely

$$\begin{aligned} h_{ab} &= 2\Re(\{-l_a l_b (\delta + \bar{\alpha} + 3\beta - \tau)(\delta + 4\beta + 3\tau) \\ &\quad - m_a m_b (D - \rho)(D + 3\rho) \\ &\quad + l_{(a} m_{b)} [(D - 2i\rho_l)(\delta + 4\beta + 3\tau) \\ &\quad + (\delta + 3\beta - \bar{\alpha} - \bar{\pi} - \tau)(D + 3\rho)]\}\Psi), \end{aligned} \quad (38)$$

where we have replaced $\bar{\rho} - \rho = -2i\rho_l$. The metric perturbation constructed in this fashion will be in the ingoing radiation gauge (IRG), which is defined by the conditions

$$h_{ll} = h_{ln} = h_{lm} = h_{l\bar{m}} = h_{m\bar{m}} = 0. \quad (39)$$

In Eq. (38) there are terms that depend on the differential operators $D = l^a \partial_a$ and $\delta = m^a \partial_a$, which in turn depends on the tetrad. To be consistent, we will continue to work with the Kinnersly tetrad in Kerr coordinates given by

$$\begin{aligned} e_{(1)}^a &= l^a = \left(2\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{2a}{\Delta}\right), \\ e_{(2)}^a &= n^a = \left(0, -\frac{\Delta}{2\Sigma}, 0, 0\right), \\ e_{(3)}^a &= m^a = \frac{1}{\sqrt{2}(r + ia \cos \theta)} \left[ia \sin \theta, 0, 1, \frac{i}{\sin \theta}\right], \\ e_{(4)}^a &= \bar{m}^a, \end{aligned} \quad (40)$$

where the overbar stands for complex conjugation. Note that the m^a vector is the same as the one in BL coordinates, but l^a and n^a are different. The differential operators associated with this tetrad in Kerr coordinates are

$$\begin{aligned} D &= \partial_r + \frac{2a}{\Delta} \partial_\phi, \\ \delta &= \frac{1}{\sqrt{2}(r + ia \cos \theta)} \left(\partial_\theta + \frac{i}{\sin \theta} \partial_\phi\right), \end{aligned} \quad (41)$$

where once more we neglect the time derivatives by use of the slow-motion approximation. The covariant form of the tetrad in these coordinates is given by

$$\begin{aligned} l_a &= l_a^R = \left(1, -\frac{2\Sigma}{\Delta}, 0, -a \sin^2 \theta\right), \\ n_a &= n_a^R = \frac{1}{2\Sigma} (\Delta, 0, 0, -a \Delta \sin^2 \theta), \\ m_a^R &= \frac{a \sin \theta}{\sqrt{2}\Sigma} \left[a \cos \theta, 0, -\frac{r\Sigma}{a \sin \theta}, -\cos \theta (r^2 + a^2)\right], \\ m_a^I &= \frac{1}{\sqrt{2}\Sigma} [ar \sin \theta, 0, a\Sigma \cos \theta, -r \sin \theta (r^2 + a^2)], \end{aligned} \quad (42)$$

where the superscript I and R stand for the imaginary and real parts, respectively. One can show that if this tetrad is used we can recover the background metric g_{ab}^B with the formula $g_{ab}^B = 2l_{(a} n_{b)} + 2m_{(a} \bar{m}_{b)}$.

Equation (38) contains terms that depend on the spin coefficients of the background [32]. These coefficients, also called Ricci rotation coefficients for the case where the tetrad is non-null, are simply contraction of the tetrad with its derivatives. In the tetrad formalism, these quantities can also be related to the Riemann tensor. Let us decompose the spin coefficients into real and imaginary parts, i.e.

$$\begin{aligned} \rho_R &= -\frac{r}{\Sigma}, & \rho_I &= -\frac{a}{\Sigma} \cos \theta, & \beta_R &= \frac{\sqrt{2}}{4} \frac{r}{\Sigma} \cot \theta, \\ \beta_I &= -\frac{\sqrt{2}}{4} \frac{a}{\Sigma} \cot \theta \cos \theta, & \pi_R &= -\sqrt{2} \frac{a^2}{\Sigma^2} r \sin \theta \cos \theta, \\ \pi_I &= \frac{\sqrt{2}}{2} \frac{a}{\Sigma^2} (r^2 - a^2 \cos^2 \theta) \sin \theta, & \tau_I &= -\frac{\sqrt{2}}{2} \frac{a}{\Sigma} \sin \theta, \\ \alpha_R &= -\sqrt{2} \frac{a^2}{\Sigma^2} r \sin \theta \cos \theta - \frac{\sqrt{2}}{4} \frac{r}{\Sigma} \cot \theta, \\ \alpha_I &= \frac{\sqrt{2}}{2} \frac{a}{\Sigma^2} \sin \theta (r^2 - a^2 \cos^2 \theta) - \frac{\sqrt{2}}{4} \frac{a}{\Sigma} \cot \theta \cos \theta. \end{aligned} \quad (43)$$

These spin coefficients are the same as those obtained with the Kinnersly tetrad in BL coordinates. This invariance is due to the fact that the spin coefficients are only tetrad dependent but still gauge invariant.

We will split Eq. (38) into 4 terms in order to make calculations more tractable. The split is as follows: the term proportional to $l_a l_b$ will be denoted term A ; the term proportional to $m_a m_b$ will be referred to as term B ; the first half of the term proportional to $l_{(a} m_{b)}$ will be denoted term C ; and the remaining of this term will be referred to as term D . In this manner we have

$$\begin{aligned} h_{ab}^A &= [-l_a l_b (\delta + \bar{\alpha} + 3\beta - \tau)(\delta + 4\beta + 3\tau)]\Psi, \\ h_{ab}^B &= [-m_a m_b (D - \rho)(D + 3\rho)]\Psi, \\ h_{ab}^C &= [l_{(a} m_{b)} (D - 2i\rho_l)(\delta + 4\beta + 3\tau)]\Psi, \\ h_{ab}^D &= [l_{(a} m_{b)} (\delta + 3\beta - \bar{\alpha} - \bar{\pi} - \tau)(D + 3\rho)]\Psi, \end{aligned} \quad (44)$$

and the full metric perturbation is given by

$$h_{ab} = 2\Re(h_{ab}^A + h_{ab}^B + h_{ab}^C + h_{ab}^D). \quad (45)$$

With this split we can now proceed to simplify each expression more easily. One such simplification is to operate with the differential operators. Expanding all terms we obtain

$$\begin{aligned} h_{ab}^A &= -l_a l_b [\delta^2 \Psi + \Psi (4\delta\beta + 3\delta\tau) + (7\beta + 2\tau + \bar{\alpha})\delta\Psi + (\bar{\alpha} + 3\beta - \tau)(4\beta + 3\tau)\Psi], \\ h_{ab}^B &= -m_a m_b [D^2 \Psi + 2\rho D\Psi + 3\Psi D\rho - 3\rho^2 \Psi], \\ h_{ab}^C &= l_{(a} m_{b)} [D\delta\Psi + 4\Psi D\beta + 3\Psi D\tau + (4\beta + 3\tau)D\Psi - 2i\rho_l (4\beta + 3\tau)\Psi - 2i\rho_l \delta\Psi], \\ h_{ab}^D &= l_{(a} m_{b)} [\delta D\Psi + 3\Psi \delta\rho + 3\rho \delta\Psi + (3\beta - \bar{\pi} - \tau - \bar{\alpha})D\Psi + (3\beta - \bar{\pi} - \tau - \bar{\alpha})3\rho\Psi], \end{aligned} \quad (46)$$

These expressions give the metric perturbation in terms of the action of the differential operators on the spin coefficients and the potential.

A. Action of the differential operators

In order to provide explicit formulas for the metric perturbation we must investigate how the differential operators act on the spin coefficients and on the potential. Let us first concentrate on the action of the differential operators on the spin coefficients. After taking the necessary derivatives and decomposing the result into imaginary and real parts, we obtain

$$\begin{aligned} D\rho &= D_{\rho,R} + iD_{\rho,I} = \frac{r^2 - a^2 \cos^2 \theta}{\Sigma^2} + i \left[2 \frac{ra}{\Sigma^2} \cos \theta \right], \\ D\tau &= iD_{\tau,I} = i\sqrt{2} \frac{ar}{\Sigma^2} \sin \theta, \\ D\beta &= D_{\beta,R} + iD_{\beta,I} = -\frac{\sqrt{2} \cot \theta (r^2 - a^2 \cos^2 \theta)}{4\Sigma^2} + i \left[\frac{\sqrt{2}}{2} \frac{ra}{\Sigma^2} \cot \theta \cos \theta \right], \\ \delta\rho &= \delta_{\rho,R} + i\delta_{\rho,I} = -\frac{\sqrt{2}}{2} \frac{a^2}{\Sigma^2} \sin \theta \cos \theta + i \left[\frac{\sqrt{2}}{2} \frac{ar}{\Sigma^2} \sin \theta \right], \\ \delta\beta &= \delta_{\beta,R} + i\delta_{\beta,I} \\ &= \frac{a^4 \cos^6 \theta - r^4 + 3r^2 a^2 \cos^2 \theta - 3\cos^4 \theta r^2 a^2}{4\Sigma^3 \sin^2 \theta} + i \left[ra \cos \theta \frac{3a^2 \cos^4 \theta - r^2 \cos^2 \theta + 3r^2}{4\sin^2 \theta \Sigma^3} - ra \cos \theta \frac{a^2 \cos^2 \theta}{4\sin^2 \theta \Sigma^3} \right], \\ \delta\tau &= \delta_{\tau,R} + i\delta_{\tau,I} = \frac{1}{2} a^2 \cos^2 \theta \frac{-r^2 + a^2 \cos^2 \theta - 2a^2}{\Sigma^3} + i \left[\frac{1}{2} ra \cos \theta \frac{-r^2 + a^2 \cos^2 \theta - 2a^2}{\Sigma^3} \right]. \end{aligned} \quad (47)$$

We now need to act with the differential operators on the potential itself. We can separate the ϕ dependence from these operators to obtain

$$D\Psi = D_m \Psi, \quad \delta\Psi = \delta_m \Psi, \quad (D\delta)\Psi = (D\delta)_m \Psi, \quad (\delta D)\Psi = (\delta D)_m \Psi, \quad (48)$$

where

$$D_m = \partial_r - 2i \frac{ma}{\Delta} = \frac{1}{2\eta^{1/2}} \partial_x - \frac{ima}{2\eta} \frac{1}{x(1+x)}, \quad \delta_m = \delta_0 \left(\partial_\theta + \frac{m}{\sin \theta} \right), \quad \delta_0 = \frac{-\bar{\rho}}{\sqrt{2}}, \quad \delta_{0,r} = -\sqrt{2} \delta_0^2. \quad (49)$$

We can also compute the square of these operators acting on $\Delta^2 Y^m$, where Y^m stands for the spherical harmonics with no ϕ

dependence. Doing so we obtain

$$\begin{aligned}
 D_m^2[\Delta^2] &= 8(r - m_1)^2 + 4\Delta - 4m^2a^2 - 12ima(r - m_1), \\
 \delta_m^2[Y^m] &= \delta_0^2 Y_{,\theta\theta}^m + \left[\delta_0 \left(\frac{ia \sin\theta}{\sqrt{2}\Sigma} + \frac{\delta_0}{\Sigma} a^2 \sin 2\theta \right) + 2\delta_0^2 \frac{m}{\sin\theta} \right] Y_{,\theta}^m + \left[\frac{\delta_0 m}{\sin\theta} \left(\frac{ia \sin\theta}{\sqrt{2}\Sigma} + \frac{\delta_0}{\Sigma} a^2 \sin 2\theta \right) \right. \\
 &\quad \left. + \frac{\delta_0^2 m}{\sin^2\theta} (m - \cos\theta) \right] Y^m, \\
 (\delta D)_m[\Delta^2 Y^m] &= \delta_0 \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) [4\Delta(r - m_1) - 2ima\Delta], \\
 (D\delta)_m[\Delta^2 Y^m] &= -\sqrt{2}\delta_0^2 \Delta^2 \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) + (\delta D)_m[\Delta^2 Y^m],
 \end{aligned} \tag{50}$$

where the commas stand for partial differentiation.

We now have all the ingredients to compute the action of the differential operators on the potential. Doing so we obtain

$$\begin{aligned}
 D_m \Psi &= -\frac{\Delta}{12} \sum_{m \neq 0} Y^m \bar{z}_m e^{-im\phi} [2(r - m_1) - ima], \\
 \delta_m \Psi &= -\frac{\Delta^2}{24} \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) e^{-im\phi} \bar{z}_m \delta_0, \\
 (D_m)^2 \Psi &= -\frac{1}{6} \sum_{m \neq 0} Y^m e^{-im\phi} \bar{z}_m [2(r - m_1)^2 + \Delta - m^2 a^2 - 3ima(r - m_1)], \\
 (\delta_m)^2 \Psi &= -\frac{\Delta^2}{24} \sum_{m \neq 0} e^{-im\phi} \bar{z}_m \left\{ \delta_0^2 Y_{,\theta\theta}^m + \left[\delta_0 \frac{ia \sin\theta}{\sqrt{2}\Sigma} + \delta_0^2 \left(\frac{a^2 \sin 2\theta}{\Sigma} + 2 \frac{m}{\sin\theta} \right) \right] Y_{,\theta}^m \right. \\
 &\quad \left. + \left[\frac{\delta_0 m}{\sin\theta} \left(\frac{ia \sin\theta}{\sqrt{2}\Sigma} + \frac{\delta_0}{\Sigma} a^2 \sin 2\theta \right) + \frac{\delta_0^2 m}{\sin^2\theta} (m - \cos\theta) \right] Y^m \right\}, \\
 (\delta D)_m \Psi &= -\frac{\Delta}{12} \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) e^{-im\phi} \bar{z}_m [2\delta_0(r - m_1) - \delta_0 ima], \\
 (D\delta)_m \Psi &= (\delta D)_m \Psi + \frac{\sqrt{2}}{24} \Delta^2 \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) e^{-im\phi} \bar{z}_m \delta_0^2,
 \end{aligned} \tag{51}$$

In order to complete the calculation, we need to provide explicit formulas for the first and second derivatives of the spherical harmonics. These derivatives are given by

$$\begin{aligned}
 Y_{,\theta}^{\pm 1} &= -2\cos^2\theta + 1 \pm \cos\theta, \\
 Y_{,\theta\theta}^{\pm 1} &= \sin\theta(4\cos\theta \mp 1), \\
 Y_{,\theta}^{\pm 2} &= -\frac{1}{2}\sin\theta(\cos\theta \mp 1), \\
 Y_{,\theta\theta}^{\pm 2} &= -\cos^2\theta \pm \frac{1}{2}\cos\theta + \frac{1}{2}.
 \end{aligned} \tag{52}$$

Note that the spherical harmonics and all of its derivatives are purely real.

B. Decomposition into real and imaginary parts

We will conclude this section by explicitly taking the real part of the metric perturbation, so as to have explicit formulas for the metric in terms of only the real and imaginary parts of the spin coefficients, the potential and the action of the differential operators on these quantities.

Before decomposing Eq. (46), however, we must decompose the action of the differential operators on the potential, i.e., Eq. (51). Let us first note that the action of any differential operator on the potential always contains the product of 3 complex terms, the first 2 of which are always $e^{-im\phi}$ and \bar{z}_m . The third term varies depending on the differential operator. Let us define the third term with superscripts as

$$\begin{aligned}
c^{(D)} &= -\frac{\Delta}{12} \sum_{m \neq 0} Y^m [2(r - m_1) - ima], \\
c^{(\delta)} &= -\frac{\Delta^2}{24} \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) \delta_0, \\
c^{(D^2)} &= -\frac{1}{6} \sum_{m \neq 0} Y^m [2(r - m_1)^2 + \Delta - m^2 a^2 - 3ima(r - m_1)], \\
c^{(\delta^2)} &= -\frac{\Delta^2}{24} \sum_{m \neq 0} \left\{ \delta_0^2 Y_{,\theta\theta}^m + \left[\delta_0 \frac{ia \sin\theta}{\sqrt{2}\Sigma} + \delta_0^2 \left(\frac{a^2 \sin 2\theta}{\Sigma} + 2 \frac{m}{\sin\theta} \right) \right] Y_{,\theta}^m \right. \\
&\quad \left. + \left[\frac{\delta_0 m}{\sin\theta} \left(\frac{ia \sin\theta}{\sqrt{2}\Sigma} + \frac{\delta_0}{\Sigma} a^2 \sin 2\theta \right) + \frac{\delta_0^2 m}{\sin^2\theta} (m - \cos\theta) \right] Y^m \right\}, \\
c^{(\delta D)} &= -\frac{\Delta}{12} \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) [2\delta_0(r - m_1) - \delta_0 ima], \\
c^{(D\delta)} &= c^{(\delta D)} + \frac{\sqrt{2}}{24} \Delta^2 \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) \delta_0^2,
\end{aligned} \tag{53}$$

In general, if we want to decompose the product of 3 complex quantities a , b and c we will obtain

$$(abc)_R = c_R(a_R b_R - a_I b_I) - c_I(a_R b_I + a_I b_R), \quad (abc)_I = c_R(a_R b_I + a_I b_R) + c_I(a_R b_R - a_I b_I). \tag{54}$$

Since we have identified $a = e^{-im\phi}$ and $b = \bar{z}_m$, their real and imaginary parts are $a_R = \cos m\phi$, $a_I = -\sin m\phi$, $b_R = \alpha_m$ and $b_I = -\beta_m$. Finally, if we further decompose c we obtain

$$\begin{aligned}
c_R^{(D)} &= -\frac{\Delta}{6} \sum_{m \neq 0} Y^m (r - m_1), & c_I^{(D)} &= \frac{\Delta}{12} \sum_{m \neq 0} Y^m ma, & c_R^{(\delta)} &= -\frac{\Delta^2}{24} \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) \delta_{0,R}, \\
c_I^{(\delta)} &= -\frac{\Delta^2}{24} \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) \delta_{0,I}, & c_R^{(D^2)} &= -\frac{1}{6} \sum_{m \neq 0} Y^m [2(r - m_1) + \Delta - m^2 a^2], \\
c_I^{(D^2)} &= \frac{1}{2} \sum_{m \neq 0} Y^m ma(r - m_1), & c_R^{(\delta D)} &= -\frac{\Delta}{12} \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) [2\delta_{0,R}(r - m_1) + \delta_{0,I} ma], \\
c_I^{(\delta D)} &= -\frac{\Delta}{12} \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) [2\delta_{0,I}(r - m_1) - \delta_{0,R} ma], \\
c_R^{(D\delta)} &= c_R^{(\delta D)} + \frac{\sqrt{2}}{24} \Delta^2 \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) (\delta_{0,R}^2 - \delta_{0,I}^2), & c_I^{(D\delta)} &= c_I^{(\delta D)} + \frac{\sqrt{2}}{12} \Delta^2 \sum_{m \neq 0} \left(Y_{,\theta}^m + \frac{m}{\sin\theta} Y^m \right) \delta_{0,R} \delta_{0,I}, \\
c_R^{(\delta^2)} &= -\frac{\Delta^2}{24} \sum_{m \neq 0} \left\{ Y_{,\theta\theta}^m (\delta_{0,R}^2 - \delta_{0,I}^2) + \left[-\delta_{0,I} \frac{a \sin\theta}{\sqrt{2}\Sigma} + (\delta_{0,R}^2 - \delta_{0,I}^2) \left(\frac{a^2 \sin 2\theta}{\Sigma} + \frac{2m}{\sin\theta} \right) \right] Y_{,\theta}^m \right. \\
&\quad \left. + \left[(\delta_{0,R}^2 - \delta_{0,I}^2) \left(\frac{ma^2 \sin 2\theta}{\sin\theta \Sigma} + \frac{m}{\sin\theta^2} (m - \cos\theta) \right) - \delta_{0,I} \frac{ma}{\sqrt{2}\Sigma} \right] Y^m \right\}, \\
c_I^{(\delta^2)} &= -\frac{\Delta^2}{24} \sum_{m \neq 0} \left\{ Y_{,\theta\theta}^m 2\delta_{0,R} \delta_{0,I} + \left[\delta_{0,R} \frac{a \sin\theta}{\sqrt{2}\Sigma} + 2\delta_{0,R} \delta_{0,I} \left(\frac{a^2 \sin 2\theta}{\Sigma} + \frac{2m}{\sin\theta} \right) \right] Y_{,\theta}^m \right. \\
&\quad \left. + \left[2\delta_{0,R} \delta_{0,I} \left(\frac{ma^2 \sin 2\theta}{\sin\theta \Sigma} + \frac{m}{\sin\theta^2} (m - \cos\theta) \right) + \delta_{0,R} \frac{ma}{\sqrt{2}\Sigma} \right] Y^m \right\},
\end{aligned} \tag{55}$$

and where

$$\delta_{0,R} = \frac{r}{\sqrt{2}\Sigma} \quad \delta_{0,I} = \frac{-a}{\sqrt{2}\Sigma} \cos\theta. \tag{56}$$

From these equations it is simple to reconstruct the real and imaginary parts of the action of the differential operators on the potential by combining Eq. (54) and (55). For example, the real part of $D_m \Psi$ is then given by

$$\begin{aligned} (D_m \Psi)_R &= c_R^{(D)}(a_R b_R - a_I b_I) - c_I^{(D)}(a_R b_I + a_I b_R) \\ &= \frac{\Delta}{12} \sum_{m \neq 0} Y^m [-2(r - m_1)(\alpha_m(v) \cos m\phi - \beta_m(v) \sin m\phi) + ma(\alpha_m(v) \sin m\phi + \beta_m(v) \cos m\phi)]. \end{aligned} \quad (57)$$

We are now finally ready to get a final expression for the metric perturbation by taking the real part of Eq. (46). Doing so we obtain

$$\begin{aligned} \Re[h_{ab}^{(A)}] &= -l_a l_b [\Psi_R (4\delta_{\beta R} + 4\alpha_R \beta_R + 3\alpha_I \tau_I + 3\tau_I^2 + 12\beta_R^2 - 5\beta_I \tau_I + 3\delta_{\tau R} + 4\alpha_I \beta_I - 12\beta_I^2) \\ &\quad + \Psi_I (-24\beta_R \beta_I - 4\delta_{\beta I} - 3\delta_{\tau I} - 5\tau_I \beta_R + 4\alpha_I \beta_R - 3\alpha_R \tau_I - 4\alpha_R \beta_I) + (\delta_m \Psi)_R (7\beta_R + \alpha_R) \\ &\quad + (\delta_m \Psi)_I (-7\beta_I - 2\tau_I + \alpha_I) + [(\delta_m)^2 \Psi]_R], \\ \Re[h_{ab}^{(B)}] &= 2[(D_m)^2 \Psi]_I m_a^R m_b^I + [(D_m)^2 \Psi]_R (m_a^I m_b^I - m_a^R m_b^R) + (D_m \Psi)_R [2(m_a^I m_b^I - m_a^R m_b^R) \rho_R + 4m_a^R m_b^I \rho_I] \\ &\quad + (D_m \Psi)_I [4m_a^R m_b^I \rho_R + 2(m_a^R m_b^R - m_a^I m_b^I) \rho_I] + 3\Psi_R [(m_a^I m_b^I - m_a^R m_b^R)(\rho_I^2 - \rho_R^2) - 4m_a^R m_b^I \rho_R \rho_I \\ &\quad + D_{\rho R} (m_a^I m_b^I - m_a^R m_b^R) + 2m_a^R m_b^I D_{\rho I}] + 6\Psi_I [(m_a^I m_b^I - m_a^R m_b^R) \rho_R \rho_I + m_a^R m_b^I (\rho_I^2 - \rho_R^2) \\ &\quad + \frac{1}{2}(m_a^R m_b^R - m_a^I m_b^I) D_{\rho I} + m_a^R m_b^I D_{\rho R}], \\ \Re[h_{ab}^{(C)}] &= l_a (m_b^R) \{[(D\delta)_m \Psi]_R + 2(\delta_m \Psi)_I \rho_I\} - m_b^I \{[(D\delta)_m \Psi]_I - 2(\delta_m \Psi)_R \rho_I\} + (D_m \Psi)_R [4\beta_R m_b^R - (4\beta_I + 3\tau_I) m_b^I] \\ &\quad - (D_m \Psi)_I [4\beta_R m_b^I + (4\beta_I + 3\tau_I) m_b^R] + \Psi_R \{[3D_{\tau R} + (8\beta_I + 6\tau_I) \rho_I + 4D_{\beta R}] m_b^R \\ &\quad + (-3D_{\tau I} - 4D_{\beta I} + 8\rho_I \beta_R) m_b^I\} + \Psi_I \{(-3D_{\tau I} - 4D_{\beta I} + 8\rho_I \beta_R) m_b^R \\ &\quad - [3D_{\tau R} + (8\beta_I + 6\tau_I) \rho_I + 4D_{\beta R}] m_b^I\}, \\ \Re[h_{ab}^{(D)}] &= l_a \{3(\delta_m \Psi)_R (-m_b^I \rho_I + m_b^R \rho_R) - 3(\delta_m \Psi)_I (m_b^I \rho_R + m_b^R \rho_I) + (D_m \Psi)_R [m_b^I (-\alpha_I - \pi_I + \tau_I - 3\beta_I) \\ &\quad + m_b^R (-\alpha_R + 3\beta_R - \pi_R)] + (D_m \Psi)_I [m_b^I (-3\beta_R + \pi_R + \alpha_R) + m_b^R (-3\beta_I - \alpha_I + \tau_I - \pi_I)] \\ &\quad + 3\Psi_R [m_b^R (-\pi_R \rho_R + \delta_{\rho R} - \alpha_R \rho_R - \alpha_I \rho_I + 3\rho_R \beta_R - \pi_I \rho_I + \rho_I \tau_I - 3\rho_I \beta_I) \\ &\quad + m_b^I (\alpha_R \rho_I - \delta_{\rho I} - 3\beta_I \rho_R + \pi_R \rho_I - \pi_I \rho_R - 3\rho_I \beta_R + \tau_I \rho_R - \alpha_I \rho_R)] \\ &\quad + 3\Psi_I [m_b^R (-\alpha_I \rho_R + \pi_R \rho_I - \delta_{\rho I} - \pi_I \rho_R + \tau_I \rho_R - 3\beta_I \rho_R + \alpha_R \rho_I - 3\rho_I \beta_R) \\ &\quad + m_b^I (-\delta_{\rho R} + \alpha_I \rho_I + \alpha_R \rho_R + \pi_R \rho_R - 3\beta_R \rho_R + 3\rho_I \beta_I - \rho_I \tau_I + \pi_I \rho_I)] \\ &\quad - m_b^I [(\delta D)_m \Psi]_I + m_b^R [(\delta D)_m \Psi]_R\}. \end{aligned} \quad (58)$$

The full metric perturbation is then given by

$$h_{ab} = 2[\Re[h_{ab}^{(A)}] + \Re[h_{ab}^{(B)}] + \Re[h_{ab}^{(C)}] + \Re[h_{ab}^{(D)}]]. \quad (59)$$

This is the metric of a tidally perturbed Kerr black hole in Kerr coordinates. We can transform this metric to Kerr-Schild coordinates, but this is left to the Appendix. We have checked that this metric indeed satisfies the Einstein equations by linearizing the Ricci tensor and verifying that all components vanish to first order. We have further checked that the metric perturbation is transverse and traceless ($h_a^a = 0$ and $h_{ab} l^a = 0$) in the tetrad frame making it suitable to study gravitational perturbations near the horizon. Furthermore, we have checked that the conditions that define the IRG [Eq. (39)] are also satisfied. Another

feature of this metric is that its determinant is zero, which renders it noninvertible. However, the full metric $g_{ab} = g_{ab}^B + h_{ab}$ is invertible and, thus, the calculation of the Einstein tensor is straightforward.

The metric perturbation has now been expressed entirely in terms of quantities explicitly defined in this paper. These quantities are the real and imaginary parts of the spin coefficients, the potential and the action of the differential operators on the spin coefficients and the potential. The spin coefficients were decomposed in Eq. (43); the potential was decomposed in Eq. (35); the action of the differential operators on the spin coefficients is given in Eq. (47); and the action of these operators on the potential is decomposed in Eq. (54) and (55).

The metric perturbation possesses the general global features that it diverges as $r \rightarrow \infty$ and it either diverges or converges to a finite value as $r \rightarrow 0$. The behavior as $r \rightarrow \infty$ is to be expected because the Chrzanowski procedure ceases to be valid far from the hole. On the other hand, the behavior as $r \rightarrow 0$ is a bit more surprising. In this region there are two different types of behavior: either the perturbation remains finite or it diverges. These different types of behavior depend on the component and axis we are investigating. On the one hand, there are some components that either are finite and of $O(V^2)$ or vanish as $r \rightarrow 0$ for all angles, such as h_{01} , h_{11} , h_{12} , h_{13} and h_{22} . On the other hand, there are other components that diverge along certain axis as $r \rightarrow 0$. For example, h_{00} , h_{03} and h_{33} diverge along the x axis, the y axis and the x - y diagonal, while h_{02} and h_{23} diverge along the y axis and x - y diagonal. This divergence is due to the choice of tetrad, since the fourth Kinnersly tetrad vector clearly diverges as $r \rightarrow 0$. Note, however, that since the divergences occur well inside the inner horizon they will be causally disconnected with all physical processes occurring outside the outer horizon and, thus, these divergences are irrelevant to most physical applications. This divergent behavior could nonetheless be avoided if a different tetrad, such as the Hawking-Hartle one, is used to compute the perturbation, but this will not be discussed here further.

In order to illustrate this global behavior, we have plotted h_{00} in Fig. 3 with the plotting choices described in Sec. II along the x axis and the y - z diagonal ($\theta = \pi/4$ and $\phi = \pi/2$). Observe that the perturbation diverges as $r \rightarrow \infty$ and as $r \rightarrow 0$ along the x axis, but it remains finite as $r \rightarrow 0$ along the y - z diagonal. Everywhere else, and, in particular, near the outer horizon, the perturbation is of $O(V^2)$, where $V = (M/b)^{1/2}$ is the orbital velocity. Finally,

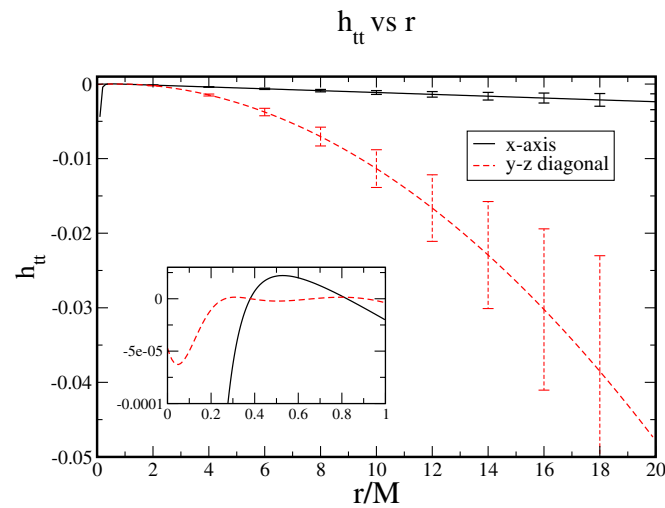


FIG. 3 (color online). Plot of the 00 component of the metric perturbation along the x axis (solid line) and along the y - z diagonal (dashed line) with the plotting parameters described in Sec. II.

observe that the perturbation vanishes close but not really at either horizon, remaining finite through them.

The divergence of the perturbation can be used to approximately determine the region of validity of the approximation. As we have discussed previously and seen in Fig. 3, the perturbation will be valid inside a shell centered at the background hole. The inner radius of this shell can be approximately determined by graphically studying the radius where $h_{ab} > O(V^2)$, approximately given by $r_{\text{inner}} \approx 0.01M$. This value for the inner radius of the shell is only an order of magnitude estimate, but it suffices to see that $r_{\text{inner}} \ll r_-$. The causal structure of the region $r < r_-$ is extremely complex and many of its features are known to be unstable under small perturbations that destroy the symmetries of the spacetime [29]. However, note that the perturbation is well-behaved for $r - r_- \ll M$ and, in particular, it is of the order predicted by the approximation [$O(V^2)$]. In any case, the region $r < r_-$ is hidden inside the event horizon and most astrophysical applications will be concerned with regions of small spacelike separations from the outer horizon, and not the inner one.

The outer radius of the shell can be estimated by studying the fractional error in the perturbation, which is determined by comparing the perturbation to the uncontrolled remainders in the approximation. The approximate error bars in Fig. 3 are given by an estimate of these uncontrolled remainders, δh_{ab} , which are due to truncating the formal series solution at a finite order. In the present case, this truncation is done at $O(V^2)$, where the tidal fields provide this scaling. The uncontrolled remainders then will be of $O(V^3)$ and should come from time derivatives of the tidal tensors, i.e. $\delta h_{ab} \propto \dot{h}_{ab}$. The argument of the tidal tensors is ωv , where $\omega = V/b$ is the angular velocity and where v is the advanced time coordinate. Any time derivative will pull out a factor of ω , which will in turn increase the order of that term. However, in order for the uncontrolled remainders to be dimensionally consistent (δh_{ab} must have the same dimensions as h_{ab}), we need to multiply the time derivatives of the tidal tensors by r , so that

$$\delta h_{ab} \approx r \dot{h}_{ab} \approx r(V/b)h_{ab}. \quad (60)$$

This line of reasoning, however, only leads to an order of magnitude estimate of the uncontrolled remainder. In principle, any dimensionless scalar function could be multiplying this estimate as long as it does not change the scaling. For the case of nonspinning holes, the metric perturbation has been computed to $O(V^3)$ [14], which allows us to compare these terms to Eq. (60). This comparison suggests that the multiplicative scalar function is roughly unity and, thus, unnecessary so that Eq. (60) is indeed a good approximation to the scaling of the uncontrolled remainders. Clearly, this estimate is not the exact error in the approximation, which can formally only be determined if we know the exact functional form of the

next order term. This estimate, however, is a physically well-motivated approximation for the uncontrolled remainders.

The outer radius of the region of validity of this approximation can be approximately determined by studying the behavior of these error bars. From Fig. 3 we see that the error bars become considerable large (approximately 50% as big as the perturbation itself) roughly at $r = 13M$. However, for $r < 4M$ the estimated error bars are less than 10% relative to the perturbation. This seems to indicate that, even at $b = 10M$ where the slow-motion approximation begins to become inaccurate, the approximate solution is still valid sufficiently close to the outer horizon. This estimation of the fractional errors is by no means a formal proof of the existence or size of the region of validity of the approximation. However, this estimation does provide a strong argument that the approximation is indeed valid sufficiently close to the outer horizon. If 10% fractional error is tolerable, which would correspond to neglecting terms of $O(V^3)$, then the outer radius of the shell is approximately given by $r_{\text{outer}} \approx 4M$. Clearly, as b is increased, the slow-motion approximation will become more accurate and, thus, the perturbation will be valid inside a bigger shell with larger outer radii.

The results of this paper are clearly valid close to the outer horizon of the background hole, thus allowing the study of physical processes of interest to the relativity community. For instance, we can use the perturbation presented here to construct initial data for a binary system near either hole. In this case, we are not interested in the behavior of the perturbation inside the inner horizon because that region can be excised and does not belong to the computational domain. Furthermore, the calculations of angular momentum and mass flux across the horizon can still be performed because they only depend on the behavior of the perturbation near or at the outer horizon.

V. CONCLUSIONS

We have computed a tidally perturbed metric for a spinning black hole in the slow-motion approximation. This approximation allows us to parameterize the NP scalar ψ_0 [Eq. (17)] in terms of the electric and magnetic tidal tensors of the external universe. With this scalar we can then construct a potential Ψ [Eq. (34)], by applying certain differential operators to it. From this potential, we can then apply the Chrzanowski procedure to construct a metric perturbation [Eqs. (35), (43), (47), (54), (55), (58), and (59)].

The metric is naturally computed in the ingoing radiation gauge and in Kerr coordinates, which are suitable to study perturbation near the horizon due to its horizon penetrating properties. This metric is given explicitly in terms of scalar functions of the coordinates and is parametrized by the mass of the background hole, its Kerr spin parameter and electric and magnetic tidal tensors. The

mass and the spin parameters of the background hole do not have to be small relative to each other, so in principle the metric presented here is capable of representing tidally perturbed extremal Kerr holes. The tidal tensors describe the time-evolution of the perturbation produced by the external universe and, thus, are functions of time that should be determined by matching the metric to another approximation valid far from the holes.

The slow-motion approximation constrains what kind of perturbations are allowed, thus limiting the external universe that is producing them. In this approximation, the radius of curvature of the external universe must be changing sufficiently slowly relative to the scales of the background hole. One consequence of this restriction is that the tidal fields must be slowly-varying functions of time, thus allowing us to neglect their time derivatives. In this sense, this approximation leads to a quasistatic limit, where in this paper we have calculated the first nonvanishing deviations from staticity. Another consequence of this approximation is that we can parametrize the perturbation in terms of multipole moments, where here we have only considered the first nonvanishing one (the quadrupolar perturbation). In perturbation theory, the $l + 1$ mode will be one order smaller than the l mode, which allows us to neglect the octopole and any other higher modes, as well as any mode beating between the quadrupole and higher modes.

Because of these restrictions, if we allow the external universe to be given by another hole in a quasicircular orbit around the background (a binary system), we are limited to those whose separation is sufficiently large. In this case, we must have sufficiently large orbital separations so that the Riemann curvature produced by the companion is large relative to the scales of the background hole. In other words, this approximation will break if we consider systems that are close to their innermost stable orbit and ready to plunge. We have seen, however, that for a separation of $b = 10M$, the metric presented here is valid inside a shell given approximately by $0.01M < r < 4M$.

The reason why the region of validity of the approximation is a shell can be traced back to the choice of tetrad and to the limitations of the Chrzanowski procedure. The perturbation cannot be valid too close to the background hole because in that region the Kinnersly tetrad, from which the perturbation was constructed, is also divergent. This perturbation is also divergent as $r \rightarrow \infty$ because the Chrzanowski procedure builds the perturbation as a linear expansion of the metric. In other words, we cannot analyze the dynamics of the entire spacetime with this metric, since its validity is limited to field points sufficiently close to the outer horizon of the background hole. However, if this metric were to be asymptotically matched to another metric valid far from the background hole, then the resultant metric would be accurate on the entire 3-manifold up to uncontrolled remainders thus, capable of reproducing the dynamics of the entire spacetime.

This metric might be useful to different areas in general relativity. On the one hand it might be useful in the construction of astrophysically realistic initial data for binary systems of spinning black holes. Its importance relies in that it could accurately represent the metric field in the neighborhood of a black hole tidally disrupted by a companion. These tides are analytic and arise as a true approximate solution to the Einstein equations in the slow-motion approximation. Initial data, however, requires a metric that is valid in the entire 3-manifold. Therefore, in order to use these results as initial data they will first have to be asymptotically matched [9,10] to a PN expansion valid far from the background holes [27]. In this manner, initial data could be constructed that satisfies the full set of the Einstein equation, including the constraints, to a high order of accuracy.

Another use for the metric computed in this paper relates to the flux of mass and angular momentum through a perturbed Kerr horizon. This flux will be important for EMRIs, where the effect of tidal perturbations could be large enough to lead to large fluxes, which in turn could affect the gravitational wave signal emitted by the system [6,7]. Recent investigations [2] have used a curvature formalism to compute this flux directly from ψ_0 . However, there exists a metric formalism to obtain this flux directly from the metric itself. An interesting research direction would be to compute this flux and compare to the results obtained with the curvature formalism.

Finally, the perturbed metric computed here can also be of use to the data analysis community to construct gravitational waveforms. EMRIs are particularly good candidates to be observed by LISA, but such observations require extremely accurate formulas for the phasing of the gravitational waves due to the use of matched filtering. Recently, Ref. [8] studied how to use and implement a quasi-Kerr metric (a perturbed Kerr metric in the limit of slow rotation of the background hole) to detect EMRIs with LISA. A similar study could be performed with the perturbed metric computed in this paper, which can also describe rapidly rotating black holes.

Future work will concentrate on performing the necessary asymptotic matching to shape this metric into useful initial data for numerical relativity applications. The matching procedure will provide expressions for the tidal tensors in terms of PN quantities, as well as a coordinate transformation between Kerr coordinates and the coordinate system used in the PN approximation. In this manner, a piecewise global solution can be computed, which will contain small discontinuities inside the matching region that could be eliminated by the introduction of transition function. Since these discontinuities will be small due to the matching, the transition functions will not alter the content of the data to the order of the approximation used. After the matching is completed, we will have obtained an approximate analytic global metric that will

contain the tidal fields of one hole on the other near the outer horizon of the former, where these fields come directly from solutions to the Einstein equations.

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APPENDIX

In this appendix we provide explicit formula for the transformation from Kerr coordinates to Kerr-Schild coordinate. This transformation can be found in Refs. [33–35] and is given by

$$\begin{aligned} x &= \sin\theta(r \cos\phi - a \sin\phi), \\ y &= \sin\theta(r \sin\phi + a \cos\phi), \quad z = r \cos\theta. \end{aligned} \quad (\text{A1})$$

The inverse transformation is given by

$$\begin{aligned} r &= \sqrt{\frac{R^2 - a^2 + w}{2}}, \quad w = \sqrt{(R^2 - a^2)^2 + 4a^2 z^2}, \\ \theta &= \arccos\frac{z}{r}, \quad \phi = \arctan\frac{ry - ax}{rx + ay}. \end{aligned} \quad (\text{A2})$$

Other useful relations are

$$\begin{aligned} \sin\theta &= \left(\frac{x^2 + y^2}{r^2 + a^2}\right)^{1/2}, \\ \sin\phi &= \frac{ry - ax}{[(r^2 + a^2)(x^2 + y^2)]^{1/2}}, \\ \cos\phi &= \frac{rx + ay}{[(r^2 + a^2)(x^2 + y^2)]^{1/2}}. \end{aligned} \quad (\text{A3})$$

Note that these transformation reduce to the usual transformation from spherical polar coordinates to Cartesian coordinates in the limit $a \rightarrow 0$.

The Jacobian of the transformation, $\Lambda_b^a = \partial x^a / \partial x^b$, is given explicitly by

$$\begin{aligned}
\Lambda^r_x &= \frac{x}{2r} \left(1 + \frac{R^2 - a^2}{w} \right), & \Lambda^r_y &= \frac{y}{2r} \left(1 + \frac{R^2 - a^2}{w} \right), & \Lambda^r_z &= \frac{z}{2r} \left(1 + \frac{R^2 + a^2}{w} \right), \\
\Lambda^\theta_x &= \frac{zx}{2r^2} \left(1 + \frac{R^2 - a^2}{w} \right) (r^2 - z^2)^{-1/2}, & \Lambda^\theta_y &= \frac{zy}{2r^2} \left(1 + \frac{R^2 - a^2}{w} \right) (r^2 - z^2)^{-1/2}, \\
\Lambda^\theta_z &= -(r^2 - z^2)^{-1/2} \left[1 - \frac{z^2}{2r^2} \left(1 + \frac{R^2 + a^2}{w} \right) \right], \\
\Lambda^\phi_x &= \frac{rx + ay}{(r^2 + a^2)(x^2 + y^2)} \left[\left(y - x \frac{ry - ax}{rx + ay} \right) \left(1 + \frac{R^2 - a^2}{w} \right) \frac{x}{2r} - a - r \frac{ry - ax}{rx + ay} \right], \\
\Lambda^\phi_y &= \frac{rx + ay}{(r^2 + a^2)(x^2 + y^2)} \left[\left(y - x \frac{ry - ax}{rx + ay} \right) \left(1 + \frac{R^2 - a^2}{w} \right) \frac{y}{2r} + r - a \frac{ry - ax}{rx + ay} \right], \\
\Lambda^\phi_z &= \frac{rx + ay}{(r^2 + a^2)(x^2 + y^2)} \left[\left(y - x \frac{ry - ax}{rx + ay} \right) \left(1 + \frac{R^2 + a^2}{w} \right) \frac{z}{2r} \right].
\end{aligned} \tag{A4}$$

Note that this Jacobian reduces to the standard Jacobian of the transformation between spherical polar and Cartesian coordinates in the limit $a \rightarrow 0$. There is a more elegant way to write this Jacobian in tensor notation as

$$\begin{aligned}
\Lambda^r_a &= \frac{1}{2r} \left\{ \delta_{ai} x^i + \frac{1}{w} [\delta_{ia} x^i (R^2 - a^2) + 2a^2 z \delta_{az}] \right\}, & \Lambda^\theta_a &= \frac{z}{r(r^2 - a^2)^{1/2}} \Lambda^r_a, \\
\Lambda^\phi_a &= \frac{rx + ay}{(r^2 + a^2)(x^2 + y^2)} \left[\Lambda^r_a \left(y - \frac{ry - ax}{rx + ay} x \right) + \left(r - \frac{ry - ax}{rx + ay} a \right) \delta_{ya} - \left(a + \frac{ry - ax}{rx + ay} r \right) \delta_{xa} \right].
\end{aligned} \tag{A5}$$

With this Jacobian, the metric in Kerr-Schild coordinates is given by

$$g_{ab} = g_{a'b'} \Lambda^{a'}_a \Lambda^{b'}_b, \tag{A6}$$

where here the primed indices refer to spherical coordinates and the unprimed indices to Cartesian coordinates.

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