

# Massive vector field perturbations in the Schwarzschild background: Stability and quasinormal spectrum

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We consider the perturbations of the massive vector field around Schwarzschild, Schwarzschild-de Sitter, and Schwarzschild-anti-de Sitter black holes. Equations for a spherically symmetric massive vector perturbation can be reduced to a single wavelike equation. We have proved the stability against these perturbations and investigated the quasinormal spectrum. The quasinormal behavior for Schwarzschild black hole is quite unexpected: the fundamental mode and higher overtones show totally different dependence on the mass of the field  $m$ : as  $m$  is increasing, the damping rate of the fundamental mode is decreasing, what results in appearing of the infinitely long living modes, while, on the contrary, damping rate of all higher overtones are increasing, and their real oscillation frequencies gradually go to tiny values. Thereby, for all higher overtones, almost nonoscillatory, damping modes can exist. In the limit of asymptotically high damping,  $\text{Re}\omega$  goes to  $\ln 3/(8\pi M)$ , while imaginary part shows equidistant behavior with spacing  $\text{Im}\omega_{n+1} - \text{Im}\omega_n = 1/4M$ . In addition, we have found quasinormal spectrum of massive vector field for Schwarzschild-anti-de Sitter black hole.

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## I. INTRODUCTION

The existence of the charged black hole described by the Reissner-Nordstrom solution reflects the fact that a black hole can possess a massless monopole vector (electromagnetic) field. It gets rid of all higher multipoles through radiative processes dominated by quasinormal ringing at intermediately late times and by power-low or exponential tails at asymptotically late times.

Since Bekenstein's paper [1], it is well known that a black hole cannot possess even a monopole massive vector field. Therefore the black hole has to radiate away the massive vector field with some quasinormal frequencies governing this radiation. Nevertheless, the massive vector quasinormal modes of a Schwarzschild black hole were not studied so far, and, as we shall show in this paper, the problem is qualitatively different from that for a massive scalar field, leading to quite unusual quasinormal behavior. First of all, let us briefly review what we know about massive scalar and massless vector field perturbations.

The massless vector perturbations of the Schwarzschild background was considered for the first time in [2]. There the Maxwell field perturbations were reduced to a single wavelike equation for some gauge invariant function  $\Psi = \Psi(r, t)$ ,

$$\Psi_{,r_*r_*} - \Psi_{,tt} - \left(1 - \frac{2M}{r}\right) \frac{\ell(\ell+1)}{r^2} \Psi = 0. \quad (1)$$

Here  $M$  is the black hole mass and  $\ell$  is the multipole number. Note that this equation is valid only for  $\ell > 0$ , while for  $\ell = 0$  (monopole, or spherically symmetrical perturbations) the Maxwell equations in Schwarzschild background do not exhibit dynamical degrees of freedom.

This signifies about existence of nonradiative electromagnetic monopole hair, i.e. about existence of a black hole charge. The quasinormal modes and late-time behavior stipulated by this effective potential were found in a lot of papers (see recent papers [3–6] and references therein), and are well studied. In particular, we know that massless vector quasinormal modes [3] are qualitatively similar to those of scalar or gravitational fields [7], except for limit of asymptotically high overtones:  $\text{Re}\omega$  approaches zero for vector field and is  $\ln 3/8\pi M$  for scalar and gravitational fields [8].

On the other hand, the massive term corrects the effective potential, and for simplest case of scalar field it leads to the wave equation

$$\Psi_{,r_*r_*} - \Psi_{,tt} - \left(1 - \frac{2M}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + m^2\right) \Psi = 0. \quad (2)$$

Here one can take  $m = 0$  and recover the massless case. The corresponding quasinormal frequencies were found in [3] for massless and in [9] for massive case. Massive scalar quasinormal modes proved to show quite peculiar properties. Thus when one increases the mass of the field  $m$ , the damping rates of the quasinormal (QN) modes decrease strongly, so that existence of infinitely long living modes called “quasiresonances” [10] becomes possible. When increasing  $m$ , lower overtones, one by one, transform into quasiresonances, while all the other higher modes remain “ordinary,” i.e. damped [11]. (For this to happen one needs to deal with relatively large values of  $m$ , so that in a more realistic picture considering backreaction of the scalar particle onto a black hole, existence of such quasiresonances is questionable.) On the other hand, in 1992 Coleman, Preskill, and Wilczek stated that the classical

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vector monopole field is determined by the mass of the field itself, and not by the mass of the black hole [12]. This stimulated the consideration of the late-time behavior of the monopole massive vector field in the Schwarzschild background in [13] where the suggestion of [12] was supported by asymptotic treatment.

One of the earliest papers dealing with massive vector field perturbations was that by Galtsov, Pomerantseva, and Chizhov [14], where it was shown that massive particles around black holes have quasistationary states with hydrogenlike spectrum. That was different from behavior of the Proca field in Coulomb potential, where bounded states cannot be formed [15]. In the paper [14], the perturbations equations were deduced for the first time, yet, as the system of equations for general value of multipole number  $\ell$  cannot be decoupled, the solution was obtained in the region far from a black hole [14]. On the contrary in [13], the perturbation equations were reduced to a single wavelike equation, but only for the case of spherically symmetrical perturbations and zero cosmological constant.

We are interested now to know what will happen with massive vector perturbations in a black hole background. In this case the situation is qualitatively different from the known massless vector or massive scalar cases. First, the wave equation for monopole massive vector perturbations cannot be reduced to that one for the massless vector field, just because the massless vector field does not have radiative monopole. Another distinctive feature: the corresponding effective potential is not positive definite everywhere outside the black hole, so one must check the stability of perturbations.

The most unexpected feature we have found in the present paper is that when increasing the mass of the field, the lowest frequency and the higher overtones behavior is qualitatively different: the fundamental mode decreases its damping rate what results in appearance of infinitely long living modes, while, on the contrary, all higher modes decrease their oscillation frequencies, leading to appearance of almost nonoscillatory damping modes.

The paper is organized as follows: in Sec. I we deduce the wave equation for perturbations of the Proca field in the background of Schwarzschild, Schwarzschild-de Sitter, and Schwarzschild-anti-de Sitter (SAdS) black holes. In Sec. II the stability of monopole perturbations is proved. Section III deals with quasinormal spectrum for massive vector perturbations of Schwarzschild, and Schwarzschild-anti-de Sitter backgrounds, including obtaining of the asymptotically high overtone limit. In the Conclusion we give a summary of obtained results.

## II. PERTURBATIONS OF PROCA FIELD IN A BLACK HOLE BACKGROUND

We shall consider here the Schwarzschild black hole solution with a  $\Lambda$ -term, i.e. Schwarzschild, Schwarzschild-de Sitter, and Schwarzschild-anti-de Sitter

backgrounds in which the massive vector field propagates. The black hole metric is given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

where

$$f(r) = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right).$$

The vector field is described by a four-potential  $A_\mu$ , which is supposed to interact with gravitational field minimally, i.e. the field equations are generally covariant analogs of the vector field equations in Minkowskian space-time. Therefore, the Proca equations

$$F_{;\nu}^{\mu\nu} - m^2 A^\mu = 0, \quad F^{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \quad (4)$$

in curved space-time, read

$$\frac{1}{\sqrt{-g}}((A_{\sigma,\rho} - A_{\rho,\sigma})g^{\rho\mu}g^{\sigma\nu}\sqrt{-g})_{,\nu} - m^2 A^\mu = 0. \quad (5)$$

From here and on the coordinates  $t$ ,  $r$ ,  $\theta$ , and  $\phi$  will be designated as 0, 1, 2, and 3, respectively.

With respect to angular coordinates we imply adequate expansion into spherical harmonics. Then, the field perturbations can be described by four scalar functions of the radial coordinate and time  $f^{\ell m}(r, t)$ ,  $h^{\ell m}(r, t)$ ,  $k^{\ell m}(r, t)$ , and  $a^{\ell m}(r, t)$ :

$$A_0 = f^{\ell m}(r, t)Y_{\ell m}(\theta, \phi), \quad (6)$$

$$A_1 = h^{\ell m}(r, t)Y_{\ell m}(\theta, \phi), \quad (7)$$

$$A_2 = k^{\ell m}(r, t)Y_{\ell m,\theta}(\theta, \phi) + \frac{a^{\ell m}(r, t)Y_{\ell m}(\theta, \phi)}{\sin\theta}, \quad (8)$$

$$A_3 = k^{\ell m}(r, t)Y_{\ell m,\phi}(\theta, \phi) - a^{\ell m}(r, t)\sin\theta Y_{\ell m,\theta}. \quad (9)$$

Considering Eq. (5) with  $\mu = 0, 1$  and substituting Eqs. (6)–(9) we arrive at the following equations:

$$\lambda(k_{,t}^{\ell m} - f^{\ell m}) - ((h_{,t}^{\ell m} - f_{,r}^{\ell m})r^2)_{,r}f(r) + m^2 r^2 f^{\ell m} = 0, \quad (10)$$

$$\lambda(k_{,r}^{\ell m} - h^{\ell m}) - ((f_{,r}^{\ell m} - h_{,t}^{\ell m})r^2)_{,t}f(r)^{-1} + m^2 r^2 h^{\ell m} = 0, \quad (11)$$

where  $\lambda = \ell(\ell + 1)$ . Here we got rid of the function  $a^{\ell m}(r, t)$ , so that the final perturbation dynamic can be described by the three independent functions of  $r$  and  $t$ .

The other two equations of (5), corresponding to  $\mu = 2, 3$ , result in a pair of equations with both even and odd spherical harmonics. Let us differentiate (10) with respect to  $r$  and (11) with respect to  $t$ . Then, consider the particular case of spherically symmetrical perturbations. Thereby taking  $l = 0$ , i.e., implicitly, discarding all terms contain-

ing derivatives with respect to angular variables, and introducing the new function

$$B = A_{r,t} - A_{t,r}, \quad (12)$$

we obtain the following equation:

$$f(r)(B_{,rr} - B_{,tt}) + \left(\frac{2}{r} - \frac{2M}{r^2}\right)B_{,r} + \left(\frac{2M}{r^3} - \frac{2}{r^2} + m^2\right)B = 0. \quad (13)$$

Assuming  $B \sim e^{i\omega t}$ , after introducing of  $\Psi = B/r$  and using of the tortoise coordinate  $r_*$ :  $dr_* = dr/f(r)$ , we get the radial wavelike equation:

$$\frac{\partial^2 \Psi(r)}{\partial r_*^2} + \omega^2 \Psi - V(r) \Psi = 0, \quad (14)$$

with the effective potential

$$V(r) = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \left(\frac{2}{r^2} - \frac{6M}{r^3} + m^2\right). \quad (15)$$

When the  $\Lambda$ -term vanishes, the wave equation (14) and (15) reduces to that obtained recently in [13] with the help of the Newman-Penrose tetrad formalism.  $\Lambda > 0$  ( $\Lambda < 0$ ) corresponds to asymptotically de Sitter (anti-de Sitter) solutions.

Yet, for perturbations of general multipolarity, all four equations of (5) can be reduced to the matrix equation for three scalar functions  $\Psi_\alpha(r)$ ,  $\alpha = 0, 1, 2$ ,

$$\frac{\partial^2 \Psi_\alpha(r)}{\partial r_*^2} + M_{\alpha\beta}(r, \omega) \Psi_\beta = 0, \quad (16)$$

and the matrix  $M_{\alpha\beta}(r, \omega)$  cannot be diagonalized by the  $r$ -independent transformations of the vector  $\Psi_\alpha(r)$ , i.e. the set of Eqs. (16) cannot be reduced to the wavelike Eqs. (14).

### III. EFFECTIVE POTENTIAL AND STABILITY

Even though we are limited now by spherically symmetric perturbations, one can hope it is possible to judge about stability of the system against massive vector field perturbations, because usually, if a system is stable against monopole perturbations, it is stable also against higher multipole perturbations. The effective potential for different values of field mass  $m$  is given on Fig. 1–3 for Schwarzschild, Schwarzschild-de Sitter, and Schwarzschild-anti-de Sitter black holes, respectively. To prove the stability of perturbations governed by the wave equation (14) and (15) we need to show, that the corresponding differential operator

$$A = -\frac{\partial^2}{\partial r_*^2} + V(r) \quad (17)$$

is positive self-adjoint operator in the Hilbert space of square integrable functions of  $r_*$ , so that there is no normalizable growing solution. This provides that all found

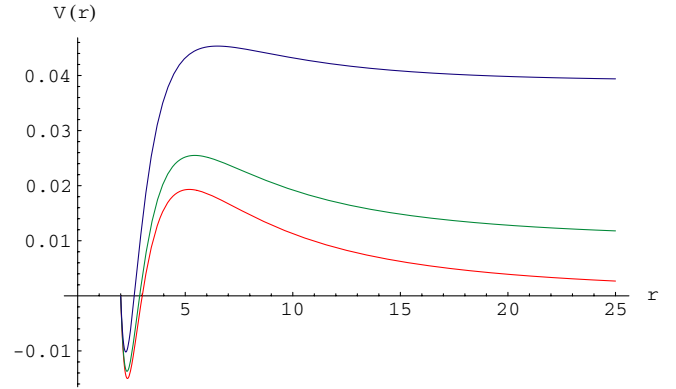


FIG. 1 (color online). Effective potential as a function of radial coordinate for Schwarzschild black hole for different values of mass of the field:  $m = 0.01$  (bottom),  $m = 0.1$ , and  $m = 0.2$  (top);  $M = 1$ .

quasinormal modes are damped. For massless scalar, vector, and gravitational perturbations (as well as for a massive scalar perturbations) of a four-dimensional Schwarzschild, Schwarzschild-de Sitter, and Schwarzschild-anti-de Sitter black holes, the effective potential is manifestly positive, and therefore the positivity of the self-adjoint operator is evident. As a result, the corresponding quasinormal modes for these cases are damped. Yet, for massive vector perturbations, as we see from Fig. 1–3, the effective potential has negative values near the event horizon. Nevertheless, the effective potential is bounded from below and we can apply here the method used in [16], which consists in extension of  $A$  to a semibounded self-adjoint operator in such a way, that the lower bound of the spectrum of the extension does not change. For this to perform, let us, following [16], introduce the operator

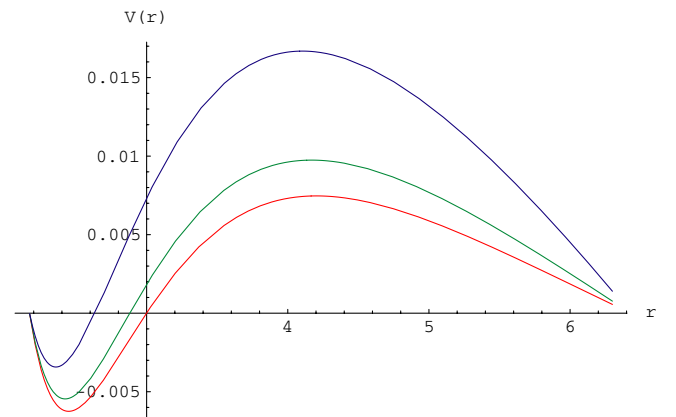


FIG. 2 (color online). Effective potential as a function of radial coordinate for Schwarzschild-de Sitter black hole for different values of mass of the field:  $m = 0.01$  (bottom),  $m = 0.1$ , and  $m = 0.2$  (top);  $M = 1$ ,  $\Lambda = 0.05$ .

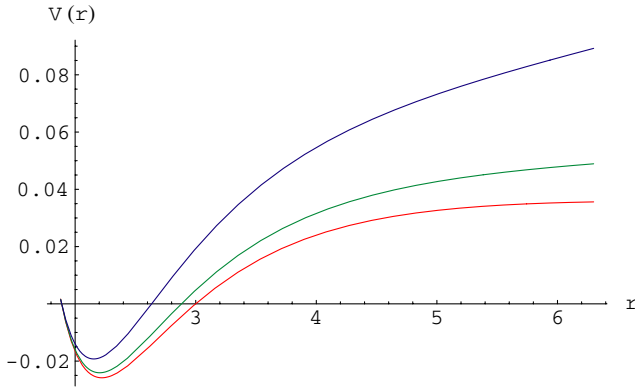


FIG. 3 (color online). Effective potential as a function of radial coordinate for Schwarzschild-anti-de Sitter black hole for different values of mass of the field:  $m = 0.01$  (bottom),  $m = 0.1$ , and  $m = 0.2$  (top);  $M = 1$ ,  $\Lambda = 0.05$ .

$$D = \frac{\partial^2}{\partial r_*^2} + S(r), \quad (18)$$

and we know that [16]:

$$(\Psi, A\Psi)_{L^2} = -(\Psi^* D\Psi)_{\text{boundary}} + \int dr_* (|D\Psi|^2 + W|\Psi|^2), \quad (19)$$

where

$$W = V + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) S'(r) - S^2(r). \quad (20)$$

Thus, we need to find the function  $S(r)$  which would make the effective potential  $W$  positive. After investigation of the form of the effective potential one can see that there is a set of functions  $S(r)$  satisfying this requirement. For instance, the function

$$S(r) = \frac{1}{r} \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \quad (21)$$

creates the following potential:

$$W = \frac{4(-3M + r)\Lambda - m^2(6M - 3r - r^3\Lambda)}{3r}. \quad (22)$$

Using Mathematica, one can show that this potential is positive outside the event horizon of a black hole. Thus a symmetric operator  $A$  is positive definite outside the black hole for positive and zero cosmological constant, and so is the self-adjoint extension. Yet, for the case of asymptotically anti-de Sitter black hole, the range of the tortoise coordinate is incomplete. At the same time, since the effective potential is divergent at spatial infinity the Dirichlet boundary conditions  $\Psi(r = \infty) = 0$  is physically motivated. Then, the boundary term in (19) does not contribute to the spectrum, and we obtain the positive self-adjoint extension of  $A$ . Thereby, we have proved that the

Schwarzschild, Schwarzschild-de Sitter, and Schwarzschild-anti-de Sitter space-times are stable against monopole massive vector field perturbations. It means that there are no growing quasinormal modes in the spectra of these perturbations. In the next section we shall compute the quasinormal modes for the asymptotically flat and AdS cases, and show, that all found modes are damped implying the stability.

#### IV. QUASINORMAL MODES

We shall be restricted here by consideration of quasinormal modes of asymptotically flat and AdS black holes as those which are most physically motivated. Quasinormal modes (QNMs) of asymptotically flat black holes may be observed by a future generation of gravitational antennas [17], while asymptotically AdS black holes have direct interpretation in the conformal field theory in the regime of strong coupling [18].

Let us start with asymptotically flat black holes. The effective potential (17) approaches constant values both at event horizon and spatial infinity in this case. Therefore the standard QN boundary conditions  $\Psi \sim e^{\pm ik_+ r_*}$ ,  $r_* \pm \infty$  ( $k_+ = \omega$ ,  $k_- = \sqrt{\omega^2 - m^2}$ ) are reasonable. More accurately, taking into consideration the subdominant asymptotic term at infinity, the QN boundary conditions are

$$\Psi(r^*) \sim C_+ e^{i\chi r^*} r^{iMm^2/\chi}, \quad (r, r^* \rightarrow +\infty), \quad (23)$$

$$\chi = \sqrt{\omega^2 - m^2}.$$

Note that the sign of  $\chi$  is to be chosen to remain in the same complex plane quadrant as  $\omega$ .

Following the Leaver method, one can eliminate the singular factor from  $\Psi$ , satisfying the ingoing wave boundary condition at the event horizon and (23) at infinity, and expand the remaining part into the Frobenius series that are convergent in the region between the event horizon and the infinity (see [11] for more details). The Frobenius series are:

$$\Psi(r) = e^{i\chi r} r^{2iM\chi + iMm^2/\chi} \left(1 - \frac{2M}{r}\right)^{-2iM\omega} \times \sum_n a_n \left(1 - \frac{2M}{r}\right)^n. \quad (24)$$

Substituting (24) into (14) we find a three-term recurrence relation:

$$\alpha_0 a_1 + \beta_0 a_0 = 0, \quad \alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \quad n > 0. \quad (25)$$

Then, using algebra of continued fractions we can find the quasinormal modes as those values of  $\omega$  for which

$$\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} - \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} - \alpha_{n-3}\gamma_{n-2}/\dots}} = \frac{\alpha_n\gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \alpha_{n+2}\gamma_{n+3}/\dots}}. \quad (26)$$

As this procedure is described in many papers (see for instance [19] and references therein), we shall not describe it here in details, and, go over directly to the obtained results. We shall write  $\omega = \text{Re}\omega + i\text{Im}\omega$ , and the damped modes should have  $\text{Im}\omega < 0$ .

First of all, let us look at the Table I where first ten QNMs are presented for three small (with respect to the black hole mass  $M$ ) values of field mass  $m$ . Note that as we consider a vector field minimally interacting with gravity, i.e. backreaction of the vector field on the metric is not considered we cannot consider large values of  $m/M$ . We see from Table I that as the overtone number is increasing the difference between QNMs for different values of  $m$  is decreasing and becomes small even at around the tenth overtone. Thus, one can conclude (and we check this by computing high overtones), that high overtone behavior does not depend on the mass term  $m$  coming into the effective potential (15), that is in agreement with the previous study of high overtones for massive scalar field in [11]. Let us remind that asymptotic limit of the QN spectrum for massive vector field does not reduce to that for the massless case, because the effective potential (15), does not have physical meaning in the limit  $m = 0$ . The most unexpected feature of the quasinormal spectrum we found (see Fig. 4) is that the fundamental mode shows correlation with mass of the field  $m$ , totally different from all the remaining higher overtones. Thus, as the mass  $m$  is increasing, the real part of the fundamental mode is increasing, while the imaginary part is falling off to tiny values, leading thereby to existence of the so-called quasi-resonant modes, i.e. of infinitely long living oscillating modes [10,11]. On contrary, the second, third (see Fig. 4), and higher overtones have their real part decreasing to tiny values, and, the imaginary part is growing when the mass  $m$  is growing. Thus higher overtones can lead to existence

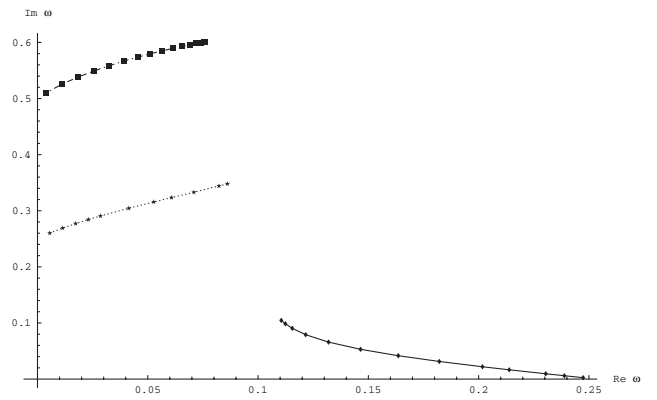


FIG. 4. Imaginary part of  $\omega$  as a function of real part of  $\omega$  for first three overtones for increasing mass:  $m \in (0.01, 0.28)$  for  $n = 0$  (diamond),  $m \in (0.01, 0.48)$  for  $n = 1$  (star),  $m \in (0.01, 0.75)$  for  $n = 2$  (box).

of almost pure imaginary modes which just damp without oscillations. We do not know examples of such a different behavior between the fundamental mode and higher overtones, for massless fields of any spin [20], or for massive scalar field [9–11], at least for asymptotically flat or de Sitter black holes. The infinitely long living modes can exist for massive scalar field perturbations [10], but for *all* modes [11], not only for the fundamental one. Note however, that for massless vector perturbations of asymptotically AdS black holes under Dirichlet boundary conditions, the fundamental mode is pure imaginary (see for instance [21] and references therein), what represents the hydrodynamic mode in the dual conformal field theory [22]. So, the qualitative difference between fundamental and higher overtones is not absolutely new phenomena, yet, completely unexpected for asymptotically flat space-times. Note also that despite the fact that we have two tendencies: approaching  $\text{Re}\omega$  the constant value  $\log 3/8\pi M$ , when  $n$  is growing, and at the same time approaching zero, when  $m$  is growing, there is no contradiction: to approach the limit

TABLE I. First ten quasinormal modes for Schwarzschild black hole:  $M = 1$ ,  $m = 0.01$ ,  $m = 0.1$ , and  $m = 0.25$ .

n	$m = 0.01$		$m = 0.1$		$m = 0.25$	
	$\text{Re}(\omega_0)$	$-\text{Im}(\omega_0)$	$\text{Re}(\omega_0)$	$-\text{Im}(\omega_0)$	$\text{Re}(\omega_0)$	$-\text{Im}(\omega_0)$
0	0.110 523	0.104 649	0.121 577	0.079 112	0.222 081	0.012 994
1	0.086 079	0.348 013	0.082 277	0.344 140	0.062 605	0.325 191
2	0.075 725	0.601 066	0.074 036	0.599 791	0.065 511	0.592 979
3	0.070 401	0.853 671	0.069 451	0.853 002	0.064 570	0.849 359
4	0.067 068	1.105 630	0.066 451	1.105 200	0.063 243	1.102 860
5	0.064 737	1.357 140	0.064 299	1.356 830	0.062 006	1.355 170
6	0.062 991	1.608 340	0.062 660	1.608 110	0.060 925	1.606 850
7	0.061 619	1.859 320	0.061 359	1.859 140	0.059 991	1.858 140
8	0.060 504	2.110 150	0.060 293	2.110 001	0.059 182	2.109 180
9	0.059 575	2.360 860	0.059 400	2.360 730	0.058 475	2.360 050



$\log 3/8\pi M$ , real part of  $\omega$  should increase again after some certain  $n = n_c$  [23]. We can observe it on the Table I (third column), where we can observe the “local maximum” of  $\text{Re}\omega$  at  $n = 2$ .

The modes in Table I and Fig. 4 were found with the help of the above described Frobenius technique. For lower overtones, one can use, alternatively, the WKB approach suggested in [24] and consequently developed to 3rd [25] and 6th [26] WKB orders beyond the eikonal approximation. The WKB formula has been used recently in a lot of papers [27] and comparison with accurate numerical data in many cases [28] shows good accuracy of the WKB formula up to the 6th WKB order. Here we can compare the results with WKB values, but only for the fundamental overtone, because for higher ones:  $n > \ell = 0$ , and the WKB method cannot be applied.

Note also that an effective potential takes negative values near the event horizon, and, the WKB formula does not take into consideration “sub” scattering by the local minimum of the potential and should not be so accurate as in the case of the ordinary positive definite potential. For example, for  $m = 0.01$  we get  $0.110523 - 0.104649i$  with the help of the Frobenius method, and  $0.1195 - 0.0871i$  by WKB formula [24]. The larger  $m$  is, the worse convergence of the WKB method. Generally we see that the accuracy of WKB approach is not satisfactory here because the WKB formula is actually good only for  $\ell > n$ .

Now let us go over to asymptotically high overtones. It is known [11] that the mass term does not change the infinitely high overtone asymptotic of the Schwarzschild black hole. Thus it is natural to expect that the same will take place for a massive vector field. Yet, as there are no monopole dynamical degrees of freedom for massless vector perturbations, we cannot formally take  $m = 0$  in the considered effective potential. Therefore, using the Nollert’s method [29], we computed numerically high overtones for nonvanishing values of  $m$  (see Fig. 5). From Fig. 5 one can learn that as  $n$  is growing, the real part approaches  $\ln 3/8\pi M$ , while the spacing in imaginary

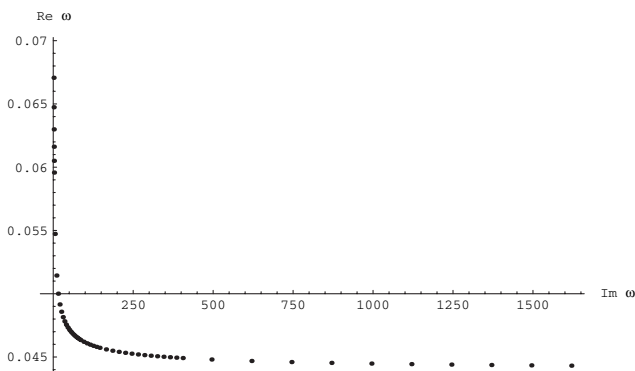


FIG. 5. Imaginary part of  $\omega$  as a function of real part of  $\omega$  for high overtone behavior:  $m = 0.01$ ,  $M = 1$ .

part approaches constant:

$$\text{Re } \omega_n \rightarrow \frac{\ln 3}{8\pi M}, \quad \text{Im } \omega_n \rightarrow \frac{(2n-1)}{8M}, \quad n \rightarrow \infty. \quad (27)$$

It is different from the asymptotic limit which takes place for higher multipole perturbations of massless vector field [30]: because for the latter case the real part of  $\omega$  asymptotically approaches zero [30]. To see better that in the obtained plot  $\text{Re}\omega$  approaches  $\ln 3/8\pi M$  let us make fit on values  $n = 1000, 1500, 2000, \dots, 6000$ . In a similar fashion with Nollert’s approach, we see that fit in powers of  $1/\sqrt{n}$  is better than in powers of  $1/n$ , and gives

$$\text{Re } \omega_n \approx 0.04372 + \frac{0.04780}{\sqrt{n}} + \frac{0.03353}{n}, \quad n \rightarrow \infty. \quad (28)$$

This is very close to  $\ln 3/(8\pi M) \approx 0.04373$ . When increasing the number of overtones, the obtained fit is closer to  $\ln 3/(8\pi M)$ .

Following the arguments of [30], it is straightforward to reproduce numerically obtained asymptotic (28) in an analytical way. For this it is enough to remember that the effective potential (15), has the following asymptotic behavior in the origin:

$$V(r) \rightarrow \frac{12M^2}{r^4}, \quad r \rightarrow 0, \quad (29)$$

and at the event horizon

$$V(r) \rightarrow \text{const}(r-2M) + O((r-2M)^2), \quad r \rightarrow 2M. \quad (30)$$

Therefore the general asymptotic solution near the origin is

$$\Psi(r_*) = c_1 \sqrt{\omega r_*} J_1(\omega r_*) + c_2 \sqrt{\omega r_*} J_{-1}(\omega r_*), \quad r \rightarrow 0. \quad (31)$$

Then, repeating all relevant steps of [30] and taking into account that near the event horizon the wave function has the following asymptotic

$$\Psi(r_*) \sim e^{2Mi\omega \ln(r_*(r)-r_*(r=2M))}, \quad r \rightarrow 2M, \quad (32)$$

and equating the two monodromy (which look similar to those in [30]) one gets (28).

We see that the high overtone asymptotic (28) is the same as for gravitational perturbations. This is easily understood, because the effective potential looks like that for gravitational perturbations with formally taken  $\ell = 1$  plus massive term times  $f(r)$ . Then, as we have shown here for vector and in [11] for scalar fields, the massive term does not contribute in high overtone asymptotic.

The quasinormal behavior of Schwarzschild-anti-de Sitter black holes is essentially dependent on the radius of a black hole: one can distinguish the three regimes of large ( $r_+ \gg R$ ), intermediate ( $r_+ \sim R$ ), and small ( $r_+ \ll R$ ) AdS black holes. From detailed previous study of

massless fields, one can learn that QNMs of large AdS black holes are proportional to the black hole radius, and therefore to the temperature [18]. QNMs of intermediate AdS black holes do not show simple linear dependence on radius [18]. Finally, QNMs of small AdS black holes approach normal modes of empty AdS space-time [31]. In the limit of asymptotically high damping, QNMs show equidistant spectrum with the same spacing between nearby modes for different massless fields (scalar, electromagnetic, and gravitational) [21].

Using the Horowitz-Hubeny method, we obtain the quasinormal frequencies for SAdS black hole numerically. As this method is described in a lot of recent works, we shall outline only the key points of it here. The Schwarzschild-AdS metric function can be written in the form

$$f(r) = 1 - \frac{r_0}{r} + \frac{r^2}{R^2}, \quad (33)$$

where  $R$  is the anti-de Sitter radius. The corresponding effective potential is divergent at infinity and is polynomial function of  $r$ . Therefore, one can expand the wave function  $\Psi$  near the event horizon in the form:

$$\Psi(x) = \sum_{n=0}^{\infty} a_n (x - x_+)^n, \quad x_+ = 1/r_+. \quad (34)$$

Here  $r_+$  is the largest of the zeros of the metric function  $f(r)$ . The Dirichlet boundary conditions we shall use here imply that

$$|\Psi(r = \infty)| = 0. \quad (35)$$

Then we need to truncate the sum (34) at some large  $n = N$ , in order to observe the convergence of the values of the root of the Eq. (35)  $\omega$  to some true quasinormal frequency.

The fundamental quasinormal frequencies are shown in Table II for large, intermediate, and small SAdS black holes for different values of  $m$ . From Table II one can see that both real and imaginary parts of the quasinormal frequency are increasing when the mass of the field is growing.

Finally, let us find the normal modes of pure AdS space-time for the case of massive vector field. The metric

function  $f(r)$  of pure AdS space-time has the form:

$$f(r) = 1 + \frac{r^2}{R^2}. \quad (36)$$

We can put the anti-de Sitter radius to be  $R = 1$  in further calculations. The tortoise coordinate is connected with the Schwarzschild radial coordinate by the relation:

$$r = \tan r^*. \quad (37)$$

Then, the effective potential has the form:

$$V = \frac{2}{\sin^2 r^*} + \frac{m^2}{\cos^2 r^*}. \quad (38)$$

Let us introduce a new variable

$$z = \sin^2 r^*. \quad (39)$$

Then the wave equation can be written in the form:

$$4z(1-z)\Psi_{,zz}(z) + 2(1-2z)\Psi_{,z}(z) + \left(\omega^2 - \frac{2}{z} - \frac{m^2}{1-z}\right)\Psi = 0. \quad (40)$$

After introducing a new function

$$\Psi = \Phi z^\alpha (1-z)^\beta,$$

the wave equation takes the form:

$$z(1-z)\Phi_{,zz}(z) + \left(\frac{1}{2} + 2\alpha - (2\alpha + 2\beta + 1)z\right)\Phi_{,z}(z) + \left(\frac{2\alpha(\alpha-1) + \alpha-1}{2z} - \frac{2\beta(\beta-1) + \beta - (m^2/2)}{2(1-z)} + \frac{\omega^2}{4} - (\alpha + \beta)^2\right)\Phi = 0. \quad (41)$$

We are in position now to choose the values of  $\alpha$  and  $\beta$ , so that the terms proportional to  $1/z$  and  $1/(1-z)$  vanish. The general solution is

TABLE II. Fundamental quasinormal modes for large ( $r_+ = 100R$ ), intermediate ( $r_+ = 1R$ ), and small ( $r_+ = 1/10R$ ) Schwarzschild-anti-de Sitter black hole.

m	$r_+ = 100R$		$r_+ = 1R$		$r_+ = 1/10R$	
	Re( $\omega_0$ )	-Im( $\omega_0$ )	Re( $\omega_0$ )	-Im( $\omega_0$ )	Re( $\omega_0$ )	-Im( $\omega_0$ )
0.01	184.959 733	266.385 59	2.798 314	2.671 325	2.6929	0.1010
0.05	185.109 096	266.671 681	2.800 496	2.674 197	2.6949	0.1012
0.1	185.571 219	267.521 739	2.807 245	2.683 084	2.700	0.103
0.15	186.325 972	268.912 194	2.818 276	2.697 634	2.709	0.1035
0.2	187.352 413	270.806 932	2.833 289	2.717 471	2.7247	0.1039
0.25	188.625 045	273.161 253	2.851 928	2.742 088	2.7416	0.1055

$$\begin{aligned} \Psi = & C_1 z^{(1/2)-\alpha} (1-z)_2^\beta F_1 \left( \frac{1}{2} - \alpha + \beta - \frac{\omega}{2}, \frac{1}{2} - \alpha + \beta \right. \\ & \left. + \frac{\omega}{2}, \frac{3}{2} - 2\alpha, z \right) + C_2 z^\alpha (1-z)_2^\beta F_1 \left( \alpha + \beta - \frac{\omega}{2}, \alpha \right. \\ & \left. + \beta + \frac{\omega}{2}, \frac{1}{2} + 2\alpha, z \right), \end{aligned} \quad (42)$$

where constants  $C_1, C_2$  may be complex. We require regularity of the solution at the origin  $z = 0$ ,  $C_1 = 0$ , and, the vanishing of the wave function at spatial infinity implies

$$2F_1 \left( \alpha + \beta - \frac{\omega}{2}, \alpha + \beta + \frac{\omega}{2}, \frac{1}{2} + 2\alpha, z \right) = 0. \quad (43)$$

Note also that the choice  $\alpha = 1$ ,  $\beta = \frac{1}{4}(1 - \frac{1}{4} \times \sqrt{1 + 4m^2})$  corresponds to the above boundary conditions both at infinity and at the event horizon.

Now it is not hard to see that  $\omega$  has the form:

$$\omega_n = \sqrt{|\Lambda|/3} \left( 2n + 3 + \frac{1}{4} \left( 1 - \frac{1}{4} \sqrt{1 + 4m^2} \right) \right). \quad (44)$$

This is different from the AdS normal modes for massless vector field  $\omega_n = \sqrt{\Lambda/3}(2n + 2 + \ell)$  [30], where  $\ell = 1, 2, \dots$ , i.e. monopole perturbations are not dynamical.

Note that it is expected that similar to scalar field behavior [31], the massive vector quasinormal modes of SAdS black holes should approach their pure AdS values (44) as the mass of the black hole goes to zero. For the fundamental mode, we see that according to formula (44), for  $m = 0.1$ , one has  $\omega = 3.004$ , and, according to the extrapolation of the data in Fig. 6 obtained numerically with the help of Horowitz-Hubeny method,  $\omega$ , indeed, approaches some constant value close to 3. Unfortunately we cannot check the accurate numerical correspondence to the formula (44), because the series (34) converges very slowly for small black hole radius, and therefore one needs

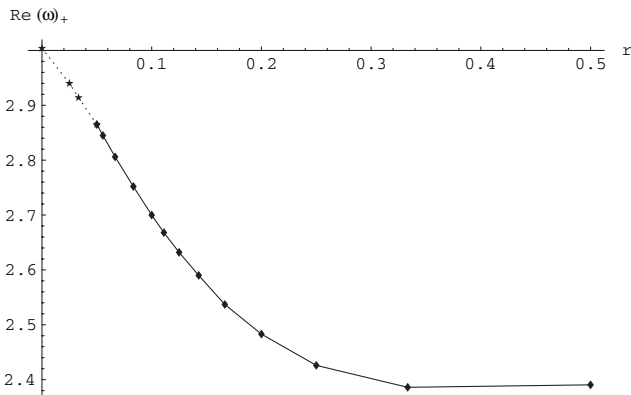


FIG. 6. Real part of  $\omega$  as a function of the black hole radius  $r_+$  for small AdS black holes:  $m = 0.1$ ,  $R = 1$ .

TABLE III. Higher overtones for large ( $r_+ = 100R$ ) Schwarzschild-anti-de Sitter black hole.

n	m = 0.1		m = 0.2		m = 0.3	
	Re( $\omega_0$ )	-Im( $\omega_0$ )	Re( $\omega_0$ )	-Im( $\omega_0$ )	Re( $\omega_0$ )	-Im( $\omega_0$ )
1	316.780	492.755	318.601	495.990	321.437	501.024
2	447.102	717.863	448.936	721.089	451.792	726.109
3	577.201	942.921	579.043	946.141	581.913	951.155
4	707.218	1167.952	709.064	1171.171	711.940	1176.185
5	837.194	1392.973	839.044	1396.191	841.925	1401.203
6	967.150	1617.987	969.002	1621.204	971.886	1626.215
7	1097.093	1842.997	1098.946	1846.214	1101.832	1851.226
8	1227.027	2068.006	1228.881	2071.222	1231.77	2022.23
9	1356.956	2293.009	1358.811	2296.230	1361.70	2301.24

enormous computer time to achieve the regime of very small black holes.

Numerical data for high overtones, in the regime of large black holes, is shown in Table III. There one can see that, indeed, in concordance with analytical formula (34), one has

$$\frac{\omega_{n+1} - \omega_n}{r_+} \approx 1.29 - 2.25i. \quad (45)$$

At a sufficiently high  $n$ , the above formula is valid independently of the value of the mass field  $m$ . It is also valid for any large  $r_+$  because in the regime of large black holes the QNMs are proportional to the black hole radius  $r_+$  for massive fields as well. One could say that a quasinormal mode at high damping consists of two contributions [32]. One is proportional to an overtone number  $n$  and thereby equals to a spacing between nearby modes; it is called “gap.” Another contribution does not depend on  $n$  in the limit  $n \rightarrow \infty$ , called “offset.” Thus, one has

$$\omega_n = [\text{offset}] + [\text{gap}]n, \quad [\text{gap}] = 2\sqrt{3}\pi R^2 e^{-i\pi/3}/9, \quad (46)$$

where  $R$  is the anti-de Sitter radius, i.e. at high overtones, the spectrum is equidistant with spacing which does not depend on  $m$ , and is the same as for gravitational or massless vector perturbations. This should be true also for intermediate and small AdS black holes, yet to check this numerically one needs considerable computer time, because of the slow convergence of the series for small black holes.

All numerical computations in this paper were made with the help of *Mathematica*.

## V. CONCLUSION

Fortunately the monopole perturbations of the Proca field in the Schwarzschild-(A)dS black hole background can be reduced to the wavelike equation with some effective potential. Even though the effective potential is not positive definite everywhere outside black hole, we have



proved that spherically symmetrical perturbations of massive vector field is stable, i.e. there are not growing modes in this case. This is confirmed by numerical computations of the QNMs spectrum, which is done for Schwarzschild and Schwarzschild-AdS black holes. A quite unexpected property, we found, is that the behavior of the fundamental mode and all higher overtones (for the asymptotically flat case) are qualitatively different: when increasing the field mass  $m$ , the damping rate of the fundamental mode goes gradually to zero, leading to appearing of infinitely long living mode, while all higher overtones, on the contrary, decrease their  $\text{Re}\omega$  that results in existence of almost pure imaginary modes, i.e. damping modes without oscillations.

Asymptotics of infinitely high overtones for Schwarzschild black holes are the same as for corresponding gravitational (massless) perturbations. In particular, for Schwarzschild black hole, real oscillation frequency approaches  $\ln 3/8\pi M$ , while damping rates become equidistant with spacing equal  $1/4M$ . This value of high damping

asymptotic, which coincides with that for massless scalar and gravitational fields for Schwarzschild black holes can be easily explained by two factors: 1) the mass term does not contribute to the limit of infinite damping of the quasinormal spectrum, and 2) when formally taking the limit  $m = 0$  in the effective potential which governs the evolution of massive vector perturbations, one has the potential which looks qualitatively like that for gravitational perturbations.

For asymptotically AdS black holes the quasinormal spectrum is equidistant at high overtones with spacing which does not depend on the mass of the field.

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