PHYSICAL REVIEW D **73,** 023523 (2006)

Fluctuations of the luminosity distance

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(Received 7 November 2005; published 27 January 2006)

We derive an expression for the luminosity distance in a perturbed Friedmann universe. We define the correlation function and the power spectrum of the luminosity distance fluctuations and express them in terms of the initial spectrum of the Bardeen potential. We present semianalytical results for the case of a pure CDM (cold dark matter) universe. We argue that the luminosity distance power spectrum represents a new observational tool which can be used to determine cosmological parameters. In addition, our results shed some light into the debate whether second order small scale fluctuations can mimic an accelerating universe.

DOI: 10.1103/PhysRevD.73.023523 PACS numbers: 98.80.−k, 98.62.En, 98.62.Py, 98.80.Es

I. INTRODUCTION

Some years ago, to the biggest surprise for the physics community, measurements of luminosity distances to far away type Ia supernovae have indicated that the Universe presently undergoes a phase of accelerated expansion [1]. If the Universe is homogeneous and isotropic, i.e. a Friedmann-Lemaître universe, this means that the energy density is dominated by some exotic "dark energy" which obeys an equation of state of the form $P < -\rho/3$. The best known dark energy candidate is vacuum energy or, equivalently, a cosmological constant. This discovery has lately been supported by several other combined data sets, like the cosmic microwave background (CMB) anisotropies combined with either large scale structure or measurements of the Hubble parameter [2].

On the other hand, since quite some time, it is known that locally measured cosmological parameters like H_0 or the deceleration parameter q_0 might not be the ones of the underlying Friedmann universe, but they might be dressed by local fluctuations [3]. Therefore, it is of great importance to derive a general formula of the luminosity distance in a universe with perturbations. To some extent, this has been done in several papers before [4,5]. But the formula which we derive here is new. We shall comment on the relations later on.

Lately, it has even been argued that second order perturbations might be responsible for the observed acceleration and that no cosmological constant or dark energy is needed [6,7]. This claim is very surprising, as it seems to require that backreaction leads to big perturbations out to very large scales, contrary to what is observed in the CMB. This proposal has thus promptly initiated a heated debate [8].

On the one hand, the present work is a contribution in this context. We calculate the measurable luminosity distance in a perturbed Friedmann universe and determine its fluctuations (within linear perturbation theory). We show that these remain smaller than 1 and therefore higher order perturbations are probably not relevant. The main point of our procedure is that we use only *measurable* quantities and not some abstract averaged expansion rate to determine the deceleration parameter. We actually calculate the luminosity distance $d_L(\mathbf{n}, z)$ where $\mathbf{n}$ defines the direction of the observed supernova and z its redshift. We then determine the power spectrum $C_\ell(z, z')$ defined by

$$d_L(\mathbf{n}, z) = \sum_{\ell m} a_{\ell m}(z) Y_{\ell m}(\mathbf{n}), \qquad (1)$$

$$C_\ell(z, z') = \langle a_{\ell m}(z) a^*_{\ell m}(z') \rangle. \qquad (2)$$

Here the $\langle\cdot\rangle$ denotes a statistical average. Like for the cosmic microwave background, statistical isotropy implies that the C_ℓ's are independent of m.

We then analyze whether the deviations of the angular diameter distance from its background value can be sufficient to fake an accelerating universe.

Aside from this problem, the new variable which is defined and calculated in this paper might in principle present an interesting and novel observational tool to determine cosmological parameters. And this is actually the main point of our work. We hope to initiate a new observational effort, the measurement of the luminosity distance power spectrum, with this paper. A detailed numerical calculation of the d_L power spectrum and the implementation of a parameter search algorithm are postponed to future work. Here we simply show that for large redshifts, $z \geq 0.4$ and sufficiently high multipoles, $\ell > 10$ the lensing effect dominates. However, at smaller redshift and especially at low ℓ's other terms can become important, most notably the Doppler term due to the peculiar motion of the supernova.

The paper is organized as follows: In Sec. II we derive a general formula for the luminosity distance valid in (nearly) arbitrary geometries. In Sec. III we apply the formula to a perturbed Friedmann universe. In Sec. IV we derive general expressions for the d_L power spectrum

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in terms of the Bardeen potentials. We then evaluate our expressions in terms of relatively crude approximations and some numerical calculations for a simple $\Omega_M = 1$ CDM model in Sec. V. In Sec. VI we discuss our results and conclude.

Notation.—We denote 4-vectors by arbitrary letters, sometimes with and sometimes without Greek indices, $k = (k^\mu)$. Three-dimensional vectors are denoted bold face or with Latin indices, $\mathbf{y} = (y^i)$. We use the metric signature $(-, +, +, +)$. The covariant derivative of the 4-vector k in direction of the 4-vector n is often denoted by $\nabla_n k \equiv (n^\mu k^\alpha_{;\mu})$.

II. THE LUMINOSITY DISTANCE IN INHOMOGENEOUS GEOMETRIES

We consider an inhomogeneous and anisotropic universe with geometry $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. We place a standard candle emitting with total luminosity L (energy per unit proper time) at spacetime position S. Its four-velocity is u_S. An observer at spacetime position O with four-velocity u_O (see Fig. 1) receives the energy flux F (energy per unit proper time and per surface). The luminosity distance between the source at S and the observer at O is defined by

$$d_L(S, O) = \sqrt{\frac{L}{4\pi F}}. \tag{3}$$

The observer measures the flux F and "knows" the intrinsic luminosity L of the standard candle. Furthermore, she determines the source redshift z and direction $\mathbf{n}$ and

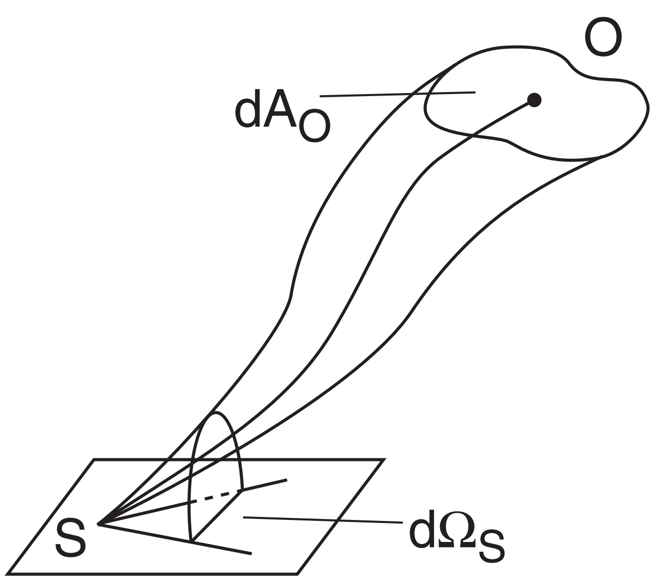

FIG. 1. A light beam emitted at the source event S ending on the observer O. At the source position, the plane normal to the source four-velocity is indicated.

thereby obtains the function $d_L(\mathbf{n}, z)$, which we now want to express in terms of the spacetime geometry.

Be $d\Omega_S$ the infinitesimal solid angle around the source and $dA(x)$ the infinitesimal surface element on the surface normal to the photon beam at the position x along the photon trajectory from S to O, then

$$d_L^2(S, O) = \frac{dA_O}{d\Omega_S}(1 + z)^2 = |\det J(O, S)|(1 + z)^2. \tag{4}$$

Here J is the so-called Jacobi map mapping initial directions $\delta\theta^\alpha_S$ around the source into vectors δx^μ_O transversal to the photon beam at the observer position [9],

$$\delta x^\mu_O = \mathcal{J}^\mu{}_\alpha(O, S)\delta\theta^\alpha_S. \tag{5}$$

The factor $1 + z = \omega_S/\omega_O$ is the redshift of the source. There is a factor $1 + z$ due to the redshift of the emitted energy and a second factor due to the time dilatation in $F \propto dE_O/d\tau_O$ with respect to $L = dE_S/d\tau_S$. If k denotes the 4-vector of the photon momentum and u_S and u_O are the source and observer 4-velocities, respectively, we have

$$-\omega_S \equiv (k \cdot u_S) = g_{\mu\nu}(S)k^\mu(S)u^\nu_S(S), \tag{6}$$

and

$$-\omega_O \equiv (k \cdot u_O) = g_{\mu\nu}(O)k^\mu(O)u^\nu_O(O). \tag{7}$$

If we have a standard candle source of which we know L and we measure F, we can therefore determine $|\det J(O, S)|^{1/2}\omega_S/\omega_O$, which contains information about the spacetime geometry. Of course it also depends on the source and observer velocities. The Jacobi map $\mathcal{J}^\mu{}_\alpha(O, S)$ maps direction vectors normal to the photons direction and normal to u_S at S into vectors normal to the photon direction and u_O at O. It depends on the source velocity u_S and on the curvature tensor along the photon geodesic from S to O. As we shall see, it does not depend on the observer velocity u_O.

Even though in the form (5), J is given by the 4×4 matrix $\mathcal{J}^\mu{}_\alpha(O, S)$, we have to take into account that the vectors δx^μ_O as well as $\delta\theta^\alpha_S$ live in the two-dimensional subspace normal to u_O, respectively, u_S and normal to the photon direction at O and S. The latter are given by

$$n_O = \frac{1}{\omega_O}(k(O) + (k(O) \cdot u_O)u_O), \tag{8}$$

and

$$n_S = \frac{1}{\omega_S}(k(S) + (k(S) \cdot u_S)u_S). \tag{9}$$

The photon direction vectors n_S and n_O are normalized spacelike vectors pointing into the photon direction in the reference frame of the source at S and of the observer at O, respectively. Denoting the projectors onto the subspaces normal to u_S, n_S and u_O, n_O by P_S and P_O we have

$$(P_S)^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu_S u_{S\nu} - n^\mu_S n_{S\nu}, \tag{10}$$

and

$$(P_O)^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu_O u_{O\nu} - n^\mu_O n_{O\nu}. \tag{11}$$

The true Jacobi map is $J(O, S) = P_O \mathcal{J} P_S$ understood as a two-dimensional linear map. For convenience we shall write it as a four-dimensional application and determine its determinant as the product of the two nonvanishing eigen values.

To determine the Jacobi map we now derive a differential equation for the evolution of the difference vector $\delta x^\mu(\lambda)$ in a given direction $\delta\theta^\alpha_S$ along the photon trajectory. The final value $\delta x^\mu(\lambda_O)$ then depends linearly on the initial conditions $\delta\theta^\alpha_S$. For this we denote the photon trajectory by $f^\alpha(\lambda, \mathbf{0})$ and parameterize neighboring lightlike geodesics by $f^\alpha(\lambda, \delta\mathbf{y})$. The 4-vector

$$k^\alpha(\delta\mathbf{y}) = \frac{\partial f^\alpha(\lambda, \delta\mathbf{y})}{\partial\lambda}$$

is the tangent of neighboring photons at $\delta\mathbf{y}$ and

$$\delta x^\alpha = \frac{\partial f^\alpha}{\partial y^i}\delta y^i$$

connects the geodesics $f^\alpha(\lambda, \mathbf{0})$ and $f^\alpha(\lambda, \delta\mathbf{y})$. Since the "beam" $f^\alpha(\lambda, \mathbf{y})$ describes photons which are all emitted at the same event S they have the same phase (eikonal) $\mathcal{S}$. With $k_\alpha = -\nabla_\alpha \mathcal{S}$ we therefore have

$$0 = \nabla_{\delta x}\mathcal{S} \equiv \delta x^\alpha \nabla_\alpha \mathcal{S} = -\delta x^\alpha k_\alpha. \tag{12}$$

In order for the 4-vectors $\delta x^\alpha(\mathbf{y})$ to sweep a surface normal to u_O at the observer event O at $\lambda = \lambda_O$, we also need $(\delta x(\lambda_O) \cdot u_O) = 0$. This is *a priori* not true. However, we can reparameterize f by

$$\lambda \to \bar\lambda = \lambda + h(\mathbf{y}) \quad \text{and} \quad \mathbf{y} \to \bar{\mathbf{y}} = \mathbf{g}(\mathbf{y}). \tag{13}$$

Under this reparameterization δx transforms as $\delta x^\alpha \to \overline{\delta x}^\alpha = \delta x^\alpha + k^\alpha \delta h$. It is easy to see that $g_{\alpha\beta}\delta x^\alpha \delta x^\beta = g_{\alpha\beta}\overline{\delta x}^\alpha \overline{\delta x}^\beta$, hence the length of the vector δx is invariant under this reparameterization. Since u_O is timelike, $(k(\lambda_O) \cdot u_O) \neq 0$ and we can hence choose a parameterization such that $(\delta x(\lambda_O) \cdot u_O) = 0$.

The directions $\delta\theta^\alpha$ are given by

$$\delta\theta^\alpha = \frac{1}{\omega_S}(\nabla_k \delta x)^\alpha = \frac{1}{\omega_S}(\nabla_{\delta x} k)^\alpha. \tag{14}$$

The last equality requires a brief calculation which can be found, e.g. in [9]. To convince oneself that the above definition of $\delta\theta^\alpha$ is suitable, one easily verifies (see [9]) that $\delta\theta^\alpha_S$ is normal to the source velocity u_S and the photon direction n_S and that it is normalized.

To find the differential equation for $\delta x(\lambda)$ we use the relations

$$\begin{aligned} R^\alpha{}_{\beta\mu\nu}k^\beta &= (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)k^\alpha, \\ (\nabla_k \delta x)^\beta &= k^\alpha \nabla_\alpha \delta x^\beta = \delta x^\alpha \nabla_\alpha k^\beta = (\nabla_{\delta x} k)^\beta. \end{aligned} \tag{15}$$

Furthermore,

$$\begin{aligned} R^\alpha_{\beta\mu\nu}k^\beta k^\mu \delta x^\nu &= k^\mu \delta x^\nu (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)k^\alpha \\ &= \delta x^\nu \nabla_k(\nabla_\nu k^\alpha) - k^\mu \nabla_{\delta x}(\nabla_\mu k^\alpha) \\ &= \nabla_k(\delta x^\nu \nabla_\nu k^\alpha) - (\nabla_k \delta x^\nu)(\nabla_\nu k^\alpha) \\ &\quad - \nabla_{\delta x}(k^\mu \nabla_\mu k^\alpha) + (\nabla_{\delta x} k^\mu)(\nabla_\mu k^\alpha) \\ &= \nabla_k(\delta x^\nu \nabla_\nu k^\alpha) \\ &= \nabla_k(k^\nu \nabla_\nu \delta x^\alpha) = \nabla_k(\omega_S \delta\theta^\alpha). \end{aligned} \tag{16}$$

From the third to the fourth line we have used that $\nabla_k \delta x = \nabla_{\delta x} k$ and $\nabla_k k = 0$. We therefore obtain the system of equations

$$\nabla_k(\omega_S \delta\theta^\alpha) = R^\alpha_{\beta\mu\nu}k^\beta k^\mu \delta x^\nu, \tag{17}$$

$$\nabla_k(\delta x^\alpha) = \omega_S \delta\theta^\alpha. \tag{18}$$

With the definition of the covariant derivative this finally gives

$$\begin{aligned} \frac{d(\delta x^\alpha)}{d\lambda} &= -\Gamma^\alpha_{\mu\nu}k^\mu \delta x^\nu + \omega_S \delta\theta^\alpha \\ &\equiv C^\alpha_\nu(\lambda)\delta x^\nu + \omega_S \delta\theta^\alpha, \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{d(\omega_S \delta\theta^\alpha)}{d\lambda} &= R^\alpha_{\beta\mu\nu}k^\beta k^\mu \delta x^\nu - \Gamma^\alpha_{\mu\nu}k^\mu \omega_S \delta\theta^\nu \\ &\equiv A^\alpha_\nu(\lambda)\delta x^\nu + C^\alpha_\nu(\lambda)\omega_S \delta\theta^\nu, \end{aligned} \tag{20}$$

where we have set

$$C^\alpha_\beta(\lambda) = -\Gamma^\alpha_{\mu\beta}k^\mu \quad \text{and} \quad A^\alpha_\beta(\lambda) = R^\alpha_{\rho\mu\beta}k^\rho k^\mu. \tag{21}$$

We now define

$$\vec{Z} = \begin{pmatrix} \delta x^\alpha \\ \omega_S \delta\theta^\alpha \end{pmatrix}. \tag{22}$$

This (8-component) vector then satisfies the equation

$$\frac{d\vec{Z}(\lambda)}{d\lambda} = B(\lambda)\vec{Z}(\lambda), \tag{23}$$

with

$$B(\lambda) = \begin{pmatrix} C^\alpha_\beta(\lambda) & \delta^\alpha_\beta \\ A^\alpha_\beta(\lambda) & C^\alpha_\beta(\lambda) \end{pmatrix}. \tag{24}$$

The initial conditions are $\delta x^\alpha(\lambda_S) = 0$ since all photons start from the same source event and $(k^\alpha \delta\theta_\alpha)(\lambda_S) = (u^\alpha_S \delta\theta_\alpha(\lambda_S)) = 0$ as we have seen above. The solution of Eq. (23) therefore provides a linear relation between the initial condition $\delta\theta^\alpha(\lambda_S)$ and $\delta x^\alpha(\lambda)$,

$$\delta x^\alpha(\lambda) = \mathcal{J}^\alpha_\beta(\lambda)\delta\theta^\beta(\lambda_S). \tag{25}$$

With $\mathcal{J}(\lambda_O)$ we can then easily determine the true Jacobi map $J(O, S) = P_O \mathcal{J}(\lambda_O) P_S$.

III. THE LUMINOSITY DISTANCE IN A PERTURBED FRIEDMANN UNIVERSE

A. Conformally related luminosity distances

We consider two geometries related by

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = a^2(x)g_{\mu\nu}dx^\mu dx^\nu = a^2(x)ds^2. \quad (26)$$

We want to relate the angular diameter distances of the two metrics. If $\tilde{k}$ is a lightlike geodesic for the metric $d\tilde{s}^2$ with affine parameter $\tilde{\lambda}$, then $k = a^2\tilde{k}$ is a lightlike geodesic for ds^2 with affine parameter λ determined by

$$\frac{d\tilde{\lambda}}{d\lambda} = a^2.$$

Furthermore, be $\tilde{u}^\mu = \frac{dx^\mu}{d\tilde{\tau}}$ the 4-velocity of an observer with metric $d\tilde{s}^2$ and be $\tilde{\tau}$ its proper time such that $\tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -1$, then $u^\mu = \frac{dx^\mu}{d\tau}$ is the corresponding 4-vector of the observer with respect to the metric ds^2 with proper time τ if $\frac{d\tilde{\tau}}{d\tau} = a$. In other words

$$\tilde{u}^\mu = \frac{dx^\mu}{d\tilde{\tau}} = \frac{dx^\mu}{d\tau}\frac{d\tau}{d\tilde{\tau}} = a^{-1}u^\mu. \quad (27)$$

The redshift of a photon emitted at S and observed at O determined with respect to the two metrics is therefore related by

$$1 + \tilde{z} = \frac{\tilde{\omega}_S}{\tilde{\omega}_O} = \frac{(\tilde{g}_{\mu\nu}\tilde{k}^\mu\tilde{u}^\nu)_S}{(\tilde{g}_{\mu\nu}\tilde{k}^\mu\tilde{u}^\nu)_O} = \frac{a_O(g_{\mu\nu}k^\mu u^\nu)_S}{a_S(g_{\mu\nu}k^\mu u^\nu)_O} = \frac{a_O}{a_S}(1+z). \quad (28)$$

To determine the relation between the Jacobi maps ${J^\alpha}_\beta = \frac{\delta x^\alpha_O}{\delta\theta^\beta_S}$ we just have to remember that angles are not affected by conformal transformations, but distances scale with the conformal factor a. Therefore

$$\tilde{J}(S,O) = \frac{\delta\tilde{x}^\alpha_O}{\delta\theta^\beta_S} = a_O\frac{\delta x^\alpha_O}{\delta\theta^\beta_S} = a_O J(S,O), \quad (29)$$

$$\det\tilde{J}(S,O) = a_O^2\det J(S,O). \quad (30)$$

For the angular distance relation we finally obtain

$$\tilde{d}_L = (1+\tilde{z})\sqrt{|\det\tilde{J}(S,O)|} = \frac{a_O^2}{a_S}(1+z)\sqrt{|\det J(S,O)|} = \frac{a_O^2}{a_S}d_L. \quad (31)$$

This relation is very useful in Friedmann cosmology. The Friedmann metric is given by

$$d\tilde{s}^2 = a^2(-d\eta^2 + \gamma_{ij}dx^i dx^j) = a^2 ds^2, \quad (32)$$

where γ is the metric of a 3-space with constant curvature K. The luminosity distance of a photon emitted at conformal time η_S and observed at η_O with respect to the metric ds^2 is simply $\eta_O - \eta_S = \int_{\eta_S}^{\eta_O} d\eta$. The Friedmann equation for a universe containing matter, radiation, curvature, and a cosmological constant reads

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2[\Omega_m a^{-1} + \Omega_{\rm rad}a^{-2} + \Omega_K + \Omega_\Lambda a^2], \quad (33)$$

where we have normalized $a_O = 1$ and we have introduced the density parameters $\Omega_m = \rho_m(\eta_O)/\rho_c(\eta_O)$, $\Omega_{\rm rad} = \rho_{\rm rad}(\eta_O)/\rho_c(\eta_O)$, $\Omega_K = -K/H_0^2$, and $\Omega_\Lambda = \Lambda/(3H_0^2)$.

After the variable transformation to $z + 1 = 1/a$, $dz = -da/a^2$ we obtain

$$d\eta = \frac{H_0^{-1}dz}{\sqrt{\Omega_{\rm rad}(1+z)^4 + \Omega_m(z+1)^3 + \Omega_K(z+1)^2 + \Omega_\Lambda}}.$$

This leads to the well-known expression for the luminosity distance to an object emitting at redshift z_S observed today at $z_O = 0$,

$$d_L(z_S)^{\rm Friedman} = \frac{\eta_0 - \eta_S}{a_S} = \frac{1+z_S}{H_0}\int_1^{z_S+1}\frac{dx}{\sqrt{\Omega_{\rm rad}x^4 + \Omega_m x^3 + \Omega_K x^2 + \Omega_\Lambda}}. \quad (34)$$

Comparing this expression with the measured luminosity distance from supernovae type Ia at different redshifts has led to the claim that the cosmological constant be non-vanishing [1].

B. The Jacobi map in a perturbed Friedmann universe

We now consider a Friedmann universe with scalar perturbations. In longitudinal (or Newtonian) gauge the metric is given by

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = a^2[-(1+2\Psi)d\eta^2 + (1-2\Phi)\gamma_{ij}dx^i dx^j]. \quad (35)$$

For perfect fluids the metric perturbations Ψ and Φ are equal. We assume in the sequel $\Phi = \Psi$. Furthermore, we consider a spatially flat universe ($K = 0$), so that $\gamma_{ij} = \delta_{ij}$.

We now determine the luminosity distance for the metric

$$ds^2 = -(1+2\Psi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j. \quad (36)$$

We then relate this to the physical luminosity distance via the relation (31).

We assume that the galaxy containing the supernova as well as the one containing the observer are moving with the cosmic fluid. To first order in the perturbations, the four-velocity of the cosmic fluid is given by

$$(u^\mu) = (1 - \Psi, v^i), \tag{37}$$

where v^i is the peculiar velocity field.

1. Redshift

The photon geodesic is obtained by integrating the geodesic equation to first order. Since the background is Minkowski, the background photon momentum is constant and we may normalize the affine parameter such that $\bar{k}^0 = 1$ and $\bar{k}^i = n^i$ with $\sum_1^3 n^i n^i = 1$. Here overbars denote background quantities. For the perturbed 4-velocity of the photon we may still assume $k_S^0 = 1$. The geodesic equation then gives (to first order)

$$k^0(\lambda_O) - k^0(\lambda_S) = k^0(\lambda_O) - 1 = -2\int_{\lambda_S}^{\lambda_O} d\lambda \nabla\Psi(\lambda)\cdot\mathbf{n} = -2\Psi|_S^O + 2\int_{\lambda_S}^{\lambda_O} d\lambda\dot{\Psi},$$

and

$$k^i(\lambda_O) - k^i(\lambda_S) = 2n^i(\Psi_O - \Psi_S) - 2\int_{\lambda_S}^{\lambda_O} d\lambda \partial_i\Psi(\lambda).$$

The redshift of a photon emitted at spacetime position S and observed at O then becomes

$$1 + z = \frac{(g_{\mu\nu}k^\mu u^\nu)_S}{(g_{\mu\nu}k^\mu u^\nu)_O} = 1 + [\Psi + \mathbf{v}\cdot\mathbf{n}]_S^O - 2\int_{\lambda_S}^{\lambda_O} d\lambda\dot{\Psi}. \tag{38}$$

2. The perturbed Jacobi map

To determine the Jacobi map we have to solve the system (23) to first order. We first determine the maps $C^\alpha{}_\beta$ and $A^\alpha{}_\beta$ which make up the matrix $B^N{}_M$. According to Eq. (21), $C^\alpha{}_\beta = -\Gamma^\alpha_{\beta\gamma}k^\gamma$. Since $\Gamma^\alpha_{\beta\gamma}$ is already first order, we may insert the zeroth order expression for k^γ leading to

$$C_0^0 = -\Psi', \qquad C_i^0 = -\partial_i\Psi + \dot{\Psi}n^i,$$
$$C_0^i = -\partial_i\Psi + \dot{\Psi}n^i, \qquad C_j^i = \Psi'\delta_j^i + \partial_j\Psi n^i - \partial_i\Psi n^j. \tag{39}$$

Here we denote the derivative along the geodesic with a prime and the derivative with respect to conformal time by an over-dot, $\frac{d}{d\lambda} \equiv {}'$ and $\frac{\partial}{\partial\eta} \equiv \dot{}$. The matrix A is given by $A^\alpha_\beta = R^\alpha_{\mu\nu\beta}k^\mu k^\nu$. Again, since $R^\alpha_{\mu\nu\beta}$ is of first order, we may insert the zeroth order expression for the photon velocity. Note that $A_{\alpha\beta}$, unlike $C_{\alpha\beta}$, is symmetric. Computing the Riemann tensor of our perturbed metric we obtain

$$A_0^0 = 2\ddot{\Psi} + \frac{d^2\Psi}{d\lambda^2} - 2\frac{d\dot{\Psi}}{d\lambda},$$
$$A_i^0 = 2\partial_i\dot{\Psi} - \frac{d\partial_i\Psi}{d\lambda} - \frac{d\dot{\Psi}}{d\lambda}n^i,$$
$$A_0^i = -A_i^0,$$
$$A_j^i = -\frac{d^2\Psi}{d\lambda^2}\delta_j^i - 2\partial_j\partial_i\Psi + \frac{d\partial_j\Psi}{d\lambda}n^i + \frac{d\partial_i\Psi}{d\lambda}n^j. \tag{40}$$

The Christoffel symbols and the Ricci tensor of the perturbed metric are given in Appendix A. Spatial indices i or j are raised and lowered with the flat metric δ_{ij}. Therefore, no special attention is paid to their position.

To solve it, we now split the system (23) into its zeroth and first order components,

$$\vec{Z} = \vec{Z}^{(0)} + \vec{Z}^{(1)} \quad \text{and} \quad B = \bar{B} + B^{(1)}. \tag{41}$$

To zeroth order, the photons move along straight lines and the energy is not redshifted so that we simply obtain $\delta\bar{\theta}^\alpha(\lambda) = \delta\theta_S^\alpha$, $\bar{\omega}(\lambda) = \omega_S$, and $\delta\bar{x}^\alpha(\lambda) = (\lambda - \lambda_S)\omega_S\delta\bar{\theta}_S^\alpha$. For the Jacobi map this implies $\bar{\mathcal{J}}^\alpha_\beta = (\lambda_O - \lambda_S)\omega_S\delta^\alpha_\beta$. The projector onto the tangent space normal to the observer velocity and the photon direction is simply $\bar{P}_S = \bar{P}_O = \bar{P}$, where

$$\bar{P}^{\,0}_{\,0} = \bar{P}^0_i = \bar{P}^i_0 = 0, \qquad \bar{P}^i_j = \delta^i_j - n^i n_j. \tag{42}$$

The zeroth order two-dimensional Jacobi map is therefore given by $\bar{J}^\alpha_\beta = (\bar{P}\bar{\mathcal{J}}\bar{P})^\alpha_\beta$

$$\bar{J}^{\,0}_{\,0} = \bar{J}^0_i = \bar{J}^i_0 = 0, \qquad \bar{J}^i_j = (\lambda_O - \lambda_S)\omega_S(\delta^i_j - n^i n_j). \tag{43}$$

The two-dimensional determinant of the Jacobi map is therefore $\det\bar{J} = (\lambda_O - \lambda_S)^2\omega_S^2$, leading to the flat space luminosity distance $d_L = \lambda_O - \lambda_S = \eta_O - \eta_S$. For the last equality we have used that $\bar{n}^0 = \frac{d\eta}{d\lambda} = 1$. In an unperturbed Friedmann universe this reproduces (34).

Since C and A are already first order, the first order differential equation becomes

$$\frac{d}{d\lambda}\delta x^{\alpha(1)}(\lambda) = C^{\alpha(1)}_\beta(\lambda)\delta\bar{x}^\beta(\lambda) + (\omega_S\delta\theta^\alpha)^{(1)}(\lambda),$$
$$\frac{d}{d\lambda}(\omega_S\delta\theta^\alpha)^{(1)}(\lambda) = A^{\alpha(1)}_\beta(\lambda)\delta\bar{x}^\beta(\lambda) + C^{\alpha(1)}_\beta(\lambda)\bar{\omega}_S\delta\bar{\theta}^\beta(\lambda). \tag{44}$$

Making use of the background solution we obtain

$$(\delta\theta^\alpha)^{(1)}(\lambda) = \int_{\lambda_1}^{\lambda} d\lambda'(A^\alpha_\beta(\lambda')(\lambda' - \lambda_S) + C^\alpha_\beta(\lambda'))\delta\bar{\theta}^\beta_S + (\delta\theta^\alpha_S)^{(1)}, \tag{45}$$

$$\delta x^{\alpha(1)}(\lambda) = \bigg[\int_{\lambda_S}^{\lambda} d\lambda' C^{\alpha}_{\beta}(\lambda')(\lambda'-\lambda_S) + \int_{\lambda_S}^{\lambda} d\lambda' \int_{\lambda_S}^{\lambda'} d\lambda''(A^{\alpha}_{\beta}(\lambda'')(\lambda''-\lambda_S) + C^{\alpha}_{\beta}(\lambda''))\bigg]\bar{\omega}_S \delta\bar{\theta}^{\beta}_S + (\lambda-\lambda_S)(\omega_S\delta\theta^{\alpha}_S)^{(1)}. \tag{46}$$

The first order contribution to the unprojected Jacobi map then becomes

$$\omega_S^{-1}\mathcal{J}^{\alpha(1)}_{\beta}(\lambda_O) = \int_{\lambda_S}^{\lambda_O} d\lambda C^{\alpha}_{\beta}(\lambda)(\lambda-\lambda_S) + \int_{\lambda_S}^{\lambda_O} d\lambda \int_{\lambda_S}^{\lambda} d\lambda'(A^{\alpha}_{\beta}(\lambda')(\lambda'-\lambda_S) + C^{\alpha}_{\beta}(\lambda')). \tag{47}$$

We want to calculate

$$J^{(1)} = (P_O\mathcal{J}P_S)^{(1)} = \bar{P}_O\mathcal{J}^{(1)}\bar{P}_S + P_O^{(1)}\bar{\mathcal{J}}\bar{P}_S + \bar{P}_O\bar{\mathcal{J}}P_S^{(1)}. \tag{48}$$

A short calculation, inserting our results for C and A gives

$$(\bar{P}_O\mathcal{J}^{(1)}\bar{P}_S)^i{}_j = U\cdot(\delta^i_j - n^i n_j) + W^i_j - n^i n^k W_{kj} - n_j n^k W^i_k + n^i n_j n^k n^l W_{kl}, \tag{49}$$

with

$$U = -2\Psi_S(\lambda_O-\lambda_S) + 2\int_{\lambda_S}^{\lambda_O} d\lambda\Psi(\lambda) \quad \text{and} \quad W_{ij} = -2\int_{\lambda_S}^{\lambda_O} d\lambda \int_{\lambda_S}^{\lambda} d\lambda' \partial_i\partial_j\Psi(\lambda')(\lambda'-\lambda_S). \tag{50}$$

Implicit summation over repeated (spatial) indices is assumed and $n^i = n_i$, $W^i_j = W_{ij} = W^{ij}$.

Calculating also the first order contributions to the projections we finally obtain

$$\begin{aligned} J^0_0 &= 0, \qquad J^0_i = \omega_S(\lambda_O-\lambda_S)(v^i_O - n^i n_k v^k_O), \qquad J^i_0 = \omega_S(\lambda_O-\lambda_S)(-v^i_S + n^i n^k v^k_S), \\ J^i_j &= \omega_S(\lambda_O-\lambda_S)\bigg\{\bigg(1 - 2\Psi_S + \frac{2}{\lambda_O-\lambda_S}\int_{\lambda_S}^{\lambda_O} d\lambda\Psi(\lambda)\bigg)\delta^i_j \\ &\quad + n^i n_j\bigg(-1 + 2\Psi_S - \frac{2}{\lambda_O-\lambda_S}\int_{\lambda_S}^{\lambda_0} d\lambda\Psi(\lambda) - \mathbf{n}(\mathbf{v}_O+\mathbf{v}_S) - 2\int_{\lambda_S}^{\lambda_O} d\lambda\nabla\Psi(\lambda)\mathbf{n} + 2\mathbf{n}\cdot\mathbf{k}^{(1)}_S\bigg) + n^i v_{O\,j} + n_j v^i_S \\ &\quad + 2\int_{\lambda_S}^{\lambda_O} d\lambda\partial_j\Psi(\lambda)n^i - n^i k^{(1)}_{Sj} - n_j k^{(1)i}_S - \frac{2}{\lambda_O-\lambda_S}\int_{\lambda_S}^{\lambda_0} d\lambda\int_{\lambda_S}^{\lambda} d\lambda'(\lambda'-\lambda_S)(\partial_i\partial_j\Psi - n^i n^k\partial_j\partial_k\Psi - n^j n^k\partial_i\partial_k\Psi \\ &\quad + n^i n^j n^k n^l\partial_k\partial_l\Psi)(\lambda')\bigg\}. \end{aligned} \tag{51}$$

Like in the unperturbed case, the two eigen values of the Jacobi map are equal. This is due to the fact that the shear contribution to the Jacobi map still vanishes in first order. A short computation gives the eigen values α,

$$\alpha = \omega_S(\lambda_O-\lambda_S)\bigg\{1 - 2\Psi_S + \frac{2}{\lambda_O-\lambda_S}\int_{\lambda_S}^{\lambda_O} d\lambda\Psi(\lambda) - \frac{1}{\lambda_O-\lambda_S}\int_{\lambda_S}^{\lambda_O} d\lambda\int_{\lambda_S}^{\lambda} d\lambda'(\lambda'-\lambda_S)(\nabla^2\Psi(\lambda') - \partial_i\partial_j\Psi(\lambda')n^i n^j)\bigg\}. \tag{52}$$

The luminosity distance of the perturbed Minkowski spacetime is given by $d_L = (\omega_S/\omega_O)\alpha$. Inserting the above expressions and taking into account the perturbation of the emission frequency, $\omega_S = -(g_{\mu\nu}k^\mu u^\nu)_S = \bar{\omega}_S + \omega^{(1)}_S$, we obtain

$$\begin{aligned} d_L &= (\eta_O-\eta_S)\bigg\{1 - \Psi_O + \mathbf{n}\cdot(\mathbf{v}_O - 2\mathbf{v}_S) + \frac{2}{\eta_O-\eta_S}\int_{\eta_S}^{\eta_O} d\eta\Psi + 2\int_{\eta_S}^{\eta_0} d\eta\nabla\Psi\cdot\mathbf{n} \\ &\quad + \frac{2}{\eta_O-\eta_S}\int_{\eta_S}^{\eta_0} d\eta\int_{\eta_S}^{\eta} d\eta'\nabla\Psi\cdot\mathbf{n} - \frac{1}{\eta_O-\eta_S}\int_{\eta_S}^{\eta_0} d\eta\int_{\eta_S}^{\eta} d\eta'(\eta'-\eta_S)(\nabla^2\Psi - n^i n^j\partial_i\partial_j\Psi)\bigg\}. \end{aligned} \tag{53}$$

Here we have also transformed the parameter λ into the conformal time η via the relation

$$\frac{d\eta}{d\lambda} = n^0(\lambda) = 1 - 2\int_{\lambda_S}^{\lambda} d\lambda' \nabla\Psi \cdot \mathbf{n}.$$

Now η is parametrizing the unperturbed photon geodesic and we interpret the potential as a function of η, $\Psi(\eta) = \Psi(\eta, \mathbf{x}(\eta))$. We use the notation $\dot{\Psi} \equiv \partial_\eta \Psi$, so that $\frac{d\Psi}{d\eta} = \dot{\Psi} + \mathbf{n} \cdot \nabla\Psi$. We now also take into account expansion, which gives $\tilde{d}_L = \frac{a_O^2}{a_S} d_L$.

Furthermore, we relate the peculiar velocities to the Bardeen potential via the first order perturbations of Einstein's equations. Setting $(\tilde{u}^\mu) = a^{-1}(1 - \Psi, v^i)$ gives [10],

$$v^i(\eta) = -\frac{1}{4\pi G a^2(\rho + p)}\left(\frac{\dot{a}}{a}\partial_i \Psi + \partial_i \dot{\Psi}\right). \tag{54}$$

With this we find the following result for the luminosity distance in an perturbed Friedmann universe

$$\begin{aligned}\tilde{d}_L(\eta_S, \mathbf{n}) = \frac{a_O^2}{a_S}(\eta_O - \eta_S)\Bigg\{&1 - \Psi_O + \mathbf{v}_O \cdot \mathbf{n} + \frac{2}{\eta_O - \eta_S}\int_{\eta_S}^{\eta_O} d\eta \Psi \\ &+ 2\mathbf{n} \cdot \left[\int_{\eta_S}^{\eta_O} d\eta \nabla\Psi + \frac{1}{\eta_O - \eta_S}\int_{\eta_S}^{\eta_O} d\eta \int_{\eta_S}^{\eta} d\eta' \nabla\Psi + \frac{1}{4\pi G a_S^2(\rho + p)(\eta_S)}(\mathcal{H}\nabla\Psi + \nabla\dot{\Psi})(\eta_S)\right] \\ &- \frac{1}{\eta_O - \eta_S}\int_{\eta_S}^{\eta_O} d\eta \int_{\eta_S}^{\eta} d\eta'(\eta' - \eta_S)(\nabla^2\Psi - n^i n^j \partial_i \partial_j \Psi)\Bigg\},\end{aligned} \tag{55}$$

where we have introduced $\mathcal{H} \equiv \dot{a}/a = a^{-1}\frac{da}{d\eta} \equiv Ha$. In what follows, we further simplify the formulas by normalizing the scale factor to $a_O \equiv 1$.

Here we have used the linear perturbation theory solution for the source velocity v_S. One might argue that the supernovae are highly nonlinear objects inside galaxies and do not move with the velocity obtained from linear perturbation theory. However, we shall be interested in distances and angles which are sufficiently large so that the nonlinear contributions to the supernova velocities are uncorrelated and therefore considering only the linear part of it in the correlation function is sufficient.

Equation (55) is the luminosity distance of a source in direction $-\mathbf{n}$ at conformal time η_S. However, this quantity is not directly measurable. What we do measure instead is the redshift of the source $z_S = \bar{z}_S + \delta z_S$, where $\bar{z}_S + 1 = 1/a(\eta_S)$. Now

$$\tilde{d}_L(\eta_S, \mathbf{n}) = \tilde{d}_L(\eta(\bar{z}_S), \mathbf{n}) \equiv \tilde{d}_L(\bar{z}_S, \mathbf{n}) = \tilde{d}_L(z_S, \mathbf{n}) - \frac{d}{d\bar{z}_S}\tilde{d}_L(z_S, \mathbf{n})\delta z_S. \tag{56}$$

Furthermore,

$$\frac{d}{d\bar{z}_S}\tilde{d}_L(z_S, \mathbf{n}) = (1 + z_S)^{-1}\tilde{d}_L + \mathcal{H}_S^{-1} + \text{ first order} \quad \text{and}$$

$$\delta\tilde{z}_S = (1 + z_S)\delta z_S = (1 + z_S)\left[\Psi_S - \Psi_O + 2\int_{\eta_S}^{\eta_O} d\eta \mathbf{n} \cdot \nabla\Psi + (\mathbf{v}_O - \mathbf{v}_S) \cdot \mathbf{n}\right]. \tag{57}$$

Inserting this in Eq. (55) leads to

$$\begin{aligned}\tilde{d}_L(z_S, \mathbf{n}) = (1 + z_S)\Bigg\{&(\eta_O - \eta_S) + \frac{1}{\mathcal{H}_S}(\Psi_O - \mathbf{v}_O \cdot \mathbf{n}) - (\eta_O - \eta_S + \mathcal{H}_S^{-1})\Psi_S \\ &+ 2\int_{\eta_S}^{\eta_O} d\eta \Psi + 2\mathbf{n} \cdot \left[-\frac{1}{\mathcal{H}_S}\int_{\eta_S}^{\eta_O} d\eta \nabla\Psi + \int_{\eta_S}^{\eta_O} d\eta \int_{\eta_S}^{\eta} d\eta' \nabla\Psi \right. \\ &\left. + \frac{\eta_O - \eta_S - \mathcal{H}_S^{-1}}{8\pi G a_S^2(\rho + p)(\eta_S)}(\mathcal{H}\nabla\Psi + \nabla\dot{\Psi})(\eta_S)\right] - \int_{\eta_S}^{\eta_O} d\eta \int_{\eta_S}^{\eta} d\eta'(\eta' - \eta_S)(\nabla^2\Psi - \partial_i \partial_j \Psi n^i n^j)\Bigg\}.\end{aligned} \tag{58}$$

After several integrations by part, one can also derive the following expression for the luminosity distance, which also can be found elsewhere [4,7], where it has been derived using the evolution equations of the expansion and the shear

$$\tilde{d}_L(z_S, \mathbf{n}) = (1+z_S)(\eta_O - \eta_S)\Bigg\{1 - \frac{1}{(\eta_O-\eta_S)\mathcal{H}_S}\mathbf{v}_O\cdot\mathbf{n} - \left(1 - \frac{1}{(\eta_O-\eta_S)\mathcal{H}_S}\right)\mathbf{v}_S\cdot\mathbf{n} - \left(2 - \frac{1}{(\eta_O-\eta_S)\mathcal{H}_S}\right)\Psi_S + \left(1 - \frac{1}{(\eta_O-\eta_S)\mathcal{H}_S}\right)\Psi_O + \frac{2}{(\eta_O-\eta_S)}\int_{\eta_S}^{\eta_O} d\eta\Psi + \frac{2}{(\eta_O-\eta_S)\mathcal{H}_S}\int_{\eta_S}^{\eta_O} d\eta\dot{\Psi} - 2\int_{\eta_S}^{\eta_O} d\eta\frac{(\eta-\eta_S)}{(\eta_O-\eta_S)}\dot{\Psi} + \int_{\eta_S}^{\eta_O} d\eta\frac{(\eta-\eta_S)(\eta_O-\eta)}{(\eta_O-\eta_S)}\ddot{\Psi} - \int_{\eta_S}^{\eta_O} d\eta\frac{(\eta-\eta_S)(\eta_O-\eta)}{(\eta_O-\eta_S)}\nabla^2\Psi\Bigg\}. \quad (59)$$

A detailed derivation of this result starting from Eq. (58) is given in Appendix B. In this equation the first line, apart from the background contribution, contains the terms due to peculiar motion of the observer and emitter (Doppler terms). The second line can be identified as "gravitational redshift." This is, however, not entirely correct since this term does not vanish even if $\Psi_S = \Psi_O$. The third line collects integrated effects proportional to line of sight integrals of Ψ and its time derivative, and the fourth and last line represent the lensing term with $\nabla^2\Psi \propto \delta\rho$. This term has been discussed in the literature before [11]. An equivalent of the above formula also can be found in [12].

Equations (58) and (59) are the final expressions for the luminosity distance in a perturbed Friedmann universe, as a function of the measured source redshift z_S and its direction $-\mathbf{n}$. In the next section we determine the luminosity distance power spectrum which is, in principle, an observable quantity.

IV. THE LUMINOSITY DISTANCE POWER SPECTRUM

We now want to determine the power spectrum of the perturbed luminosity distance, as defined in the introduction. For notational simplicity, we drop the ˜ and use d_L to denote the luminosity distance in a perturbed Friedman universe. From Eqs. (1) and (2) and the addition theorem for spherical harmonics, one obtains the correlation function

$$\bar{d}_L(z_S)^{-1}\bar{d}_L(z_{S'})^{-1}\langle d_L(z_S, \mathbf{n})d_L(z_{S'}\mathbf{n}')\rangle = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell(z_S, z_{S'})P_\ell(\mathbf{n}\cdot\mathbf{n}'), \quad (60)$$

where P_ℓ is the Legendre polynomial of order ℓ.

A. The dipole

Let us first briefly look at the dipole coming from the peculiar motion of the observer, the term containing the scalar product $\mathbf{n}\cdot\mathbf{v}_O$. The power spectrum of this term is given by

$$\langle d_L^{(v)}(z_S, \mathbf{n})d_L^{(v)}(z_{S'}, \mathbf{n}')\rangle = \frac{(z_S+1)(z_{S'}+1)}{3\mathcal{H}_S\mathcal{H}_{S'}}\langle v_O^2\rangle(\mathbf{n}\cdot\mathbf{n}'). \quad (61)$$

We assume that, like for the anisotropies in the cosmic microwave background, this term completely dominates the dipole. The luminosity distance dipole therefore has the same direction as the CMB dipole. To determine its amplitude we insert $\bar{d}_L(\eta_S) = (z_S+1)(\eta_O - \eta_S)$. We then obtain

$$C_1 = \left[\frac{4\pi}{9}\langle v_O^2\rangle\right]\frac{\mathcal{H}_S^{-1}\mathcal{H}_{S'}^{-1}}{(\eta_O-\eta_S)(\eta_O-\eta_{S'})}. \quad (62)$$

The CMB dipole is given by the expression in square brackets. In a pure CDM universe with $\mathcal{H} = 2/\eta$ and $\eta_O/\eta_S = \sqrt{z_S+1}$ we obtain for the amplitude of the luminosity distance dipole

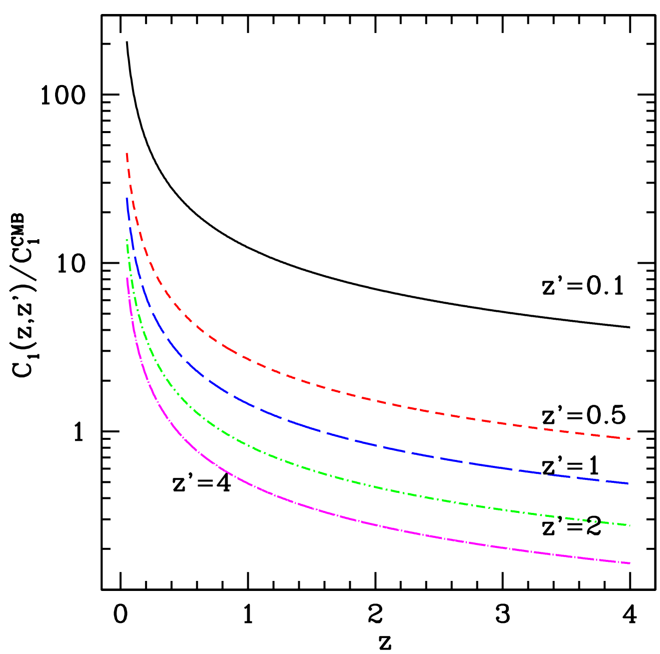

FIG. 2 (color online). We show the dipole amplitude in a pure CDM universe in units of the CMB dipole as a function of z for $z' = 0.1$, 0.5, 1, 2, and 4 from top to bottom.

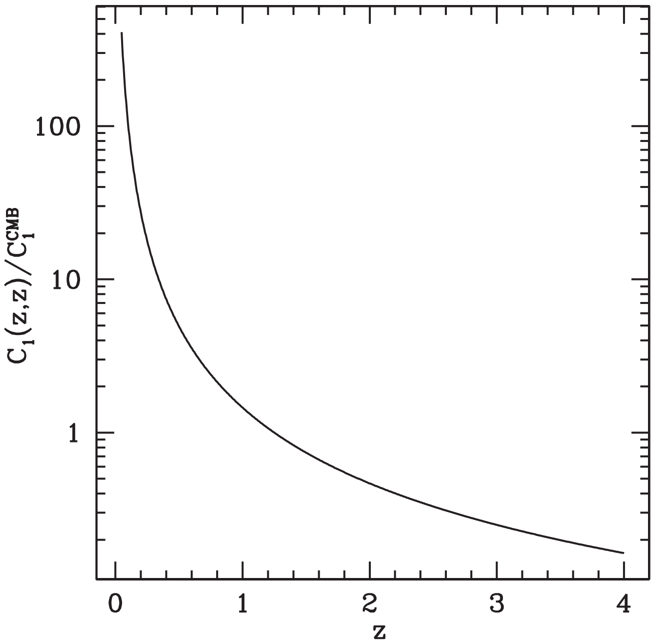

FIG. 3. We show the dipole amplitude in units of the CMB dipole as a function of $z = z'$ in a pure CDM universe.

$$C_1(z, z') = C_1^{\rm CMB} \frac{1}{4(\sqrt{z+1} - 1)(\sqrt{z'+1} - 1)}. \qquad (63)$$

In Fig. 2 the relative amplitude of C_1 as a function of z for different values of z' is shown. In Fig. 3 we plot $C_1(z, z)$. It seems to be most promising to measure the dipole at relatively low redshift. But, of course, the redshift must be sufficiently high such that the peculiar velocities of the supernovae themselves are not strongly correlated with each other or with our peculiar motion. Hence the distance of a supernova at z or z' should be sufficient for linear perturbation theory to apply. This is safely achieved for $z, z' \gtrsim 0.1$. At $z = z' = 0.1$ we have $C_1(0.1, 0.1) \simeq 105 \times C_1^{\rm CMB}$, hence an enhancement of about a factor 100 with respect to the CMB dipole. This factor is even somewhat larger, in a Λ-dominated cosmology. Through its dependence on $\mathcal{H}(z)$, measuring the amplitude of this dipole alone can already lead to new observational constraints on the expansion history of the Universe.

B. The higher multipoles

We now want to express the higher C_ℓ's in terms of the power spectrum for the Bardeen potential. We define the Fourier transform

$$\Psi(\eta, \mathbf{k}) = \int d^3x e^{-i\mathbf{k}\mathbf{x}} \Psi(\eta, \mathbf{x}). \qquad (64)$$

We split the deterministic time evolution into a "transfer function" $T_k(\eta)$, such that $\Psi(\eta, \mathbf{k}) = T_k(\eta)\Psi(\mathbf{k})$. We normalize the transfer function such that $\lim_{k\to 0} T_k(\eta_0) = 1$. The power spectrum P_Ψ of $\Psi(\mathbf{k})$ is defined by

$$k^3 \langle \Psi(\mathbf{k}) \Psi^*(\mathbf{k}') \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') P_\Psi(k). \qquad (65)$$

The δ^3 function is a consequence of statistical homogeneity. We need to determine the correlation function of Ψ for the positions $\mathbf{x} = \mathbf{x}_O - \mathbf{n}(\eta_O - \eta)$ and $\mathbf{x}' = \mathbf{x}_O - \mathbf{n}'(\eta_O - \eta')$. In terms of the power spectrum the correlation function of Ψ and of its derivatives as they enter in Eq. (58) can be written as (for details see Appendices C and D)

$$\langle \Psi(\eta, \mathbf{x}) \Psi(\eta', \mathbf{x}') \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{(\Psi)}(z, z') P_\ell(\mathbf{n} \cdot \mathbf{n}') \quad \text{with}$$
$$C_\ell^{(\Psi)}(z, z') = \frac{2}{\pi} \int \frac{dk}{k} T_k(\eta) T_k(\eta') P_\Psi(k) j_\ell(k(\eta_O - \eta)) j_\ell(k(\eta_O - \eta')), \qquad (66)$$

$$\langle \mathbf{n} \cdot \nabla \Psi(\eta, \mathbf{x}) \Psi(\eta', \mathbf{x}') \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{(nd\Psi)}(z, z') P_\ell(\mathbf{n} \cdot \mathbf{n}') \quad \text{with}$$
$$C_\ell^{(nd\Psi)}(z, z') = -\frac{2}{\pi} \int dk T_k(\eta) T_k(\eta') P_\Psi(k) j'_\ell(k(\eta_O - \eta)) j_\ell(k(\eta_O - \eta')), \qquad (67)$$

$$\langle n^i n^j \partial_i \partial_j \Psi(\eta, \mathbf{x}) \Psi(\eta', \mathbf{x}') \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{(nndd\Psi)}(z, z') P_\ell(\mathbf{n} \cdot \mathbf{n}') \quad \text{with}$$
$$C_\ell^{(nndd\Psi)}(z, z') = \frac{2}{\pi} \int dk k T_k(\eta) T_k(\eta') P_\Psi(k) j''_\ell(k(\eta_O - \eta)) j_\ell(k(\eta_O - \eta')), \qquad (68)$$

$$\langle \nabla^2 \Psi(\eta, \mathbf{x}) \Psi(\eta', \mathbf{x}') \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{(dd\Psi)}(z, z') P_\ell(\mathbf{n} \cdot \mathbf{n}') \quad \text{with}$$
$$C_\ell^{(dd\Psi)}(z, z') = \frac{-2}{\pi} \int dk k T_k(\eta) T_k(\eta') P_\Psi(k) j_\ell(k(\eta_O - \eta)) j_\ell(k(\eta_O - \eta')), \qquad (69)$$

$$\langle \mathbf{n}\cdot\nabla\Psi(\eta,\mathbf{x})\mathbf{n}'\cdot\nabla\Psi(\eta',\mathbf{x}')\rangle = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell^{(nd\Psi nd)}(z,z')P_\ell(\mathbf{n}\cdot\mathbf{n}') \quad \text{with}$$
$$C_\ell^{(nd\Psi nd)}(z,z') = \frac{2}{\pi}\int dkkT_k(\eta)T_k(\eta')P_\Psi(k)j'_\ell(k(\eta_O-\eta))j'_\ell(k(\eta_O-\eta')), \tag{70}$$

$$\langle \mathbf{n}\cdot\nabla\Psi(\eta,\mathbf{x})\nabla^2\Psi(\eta',\mathbf{x}')\rangle = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell^{(nd\Psi dd)}(z,z')P_\ell(\mathbf{n}\cdot\mathbf{n}') \quad \text{with}$$
$$C_\ell^{(nd\Psi dd)}(z,z') = \frac{2}{\pi}\int dkk^2T_k(\eta)T_k(\eta')P_\Psi(k)j'_\ell(k(\eta_O-\eta))j_\ell(k(\eta_O-\eta')), \tag{71}$$

$$\langle \mathbf{n}\cdot\nabla\Psi(\eta,\mathbf{x})n^in^j\partial_i\partial_j\Psi(\eta',\mathbf{x}')\rangle = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell^{(nd\Psi ndnd)}(z,z')P_\ell(\mathbf{n}\cdot\mathbf{n}') \quad \text{with}$$
$$C_\ell^{(nd\Psi ndnd)}(z,z') = -\frac{2}{\pi}\int dkk^2T_k(\eta)T_k(\eta')P_\Psi(k)j'_\ell(k(\eta_O-\eta))j''_\ell(k(\eta_O-\eta')), \tag{72}$$

$$\langle n^in^j\partial_i\partial_j\Psi(\eta,\mathbf{x})n'^in'^j\partial_i\partial_j\Psi(\eta',\mathbf{x}')\rangle = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell^{(nndd\Psi nndd)}(z,z')P_\ell(\mathbf{n}\cdot\mathbf{n}') \quad \text{with}$$
$$C_\ell^{(nndd\Psi nndd)}(z,z') = \frac{2}{\pi}\int dkk^3T_k(\eta)T_k(\eta')P_\Psi(k)j''_\ell(k(\eta_O-\eta))j''_\ell(k(\eta_O-\eta')), \tag{73}$$

$$\langle n^in^j\partial_i\partial_j\Psi(\eta,\mathbf{x})\nabla^2\Psi(\eta',\mathbf{x}')\rangle = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell^{(ndnd\Psi dd)}(z,z')P_\ell(\mathbf{n}\cdot\mathbf{n}') \quad \text{with}$$
$$C_\ell^{(ndnd\Psi dd)}(z,z') = -\frac{2}{\pi}\int dkk^3T_k(\eta)T_k(\eta')P_\Psi(k)j''_\ell(k(\eta_O-\eta))j_\ell(k(\eta_O-\eta')),$$
$$\langle \nabla^2\Psi(\eta,\mathbf{x})\nabla^2\Psi(\eta',\mathbf{x}')\rangle = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell^{(dd\Psi dd)}(z,z')P_\ell(\mathbf{n}\cdot\mathbf{n}') \quad \text{with}$$
$$C_\ell^{(dd\Psi dd)}(z,z') = \frac{2}{\pi}\int dkk^3T_k(\eta)T_k(\eta')P_\Psi(k)j_\ell(k(\eta_O-\eta))j_\ell(k(\eta_O-\eta')). \tag{74}$$

Using these definitions we can write the correlation function of the luminosity distance as

$$\frac{\langle d_L(z_S,\mathbf{n})d_L(z_{S'},\mathbf{n}')\rangle}{\bar{d}_L(z_S)\bar{d}_L(z_{S'})} = \sum_\ell \frac{2\ell+1}{4\pi}P_\ell(\mathbf{n}\mathbf{n}') \times (C_\ell^{(1)}+C_\ell^{(2)}+C_\ell^{(3)} + C_\ell^{(4)}+C_\ell^{(5)}), \tag{75}$$

where $C_\ell^{(i)}$ collects all the contributions to C_ℓ which contain integrals of the form $\int dkk^{i-2}\ldots$ The detailed expressions for the $C_\ell^{(i)}$'s are given in Appendix D. Here we just note that the term $C_\ell^{(5)}$ represents the lensing contribution. As we shall see, it dominates for sufficiently high redshift and sufficiently large ℓ. Another important contribution is $C_\ell^{(3)}$ which contains the peculiar velocity of the emitter, the Doppler term. (It also includes other contributions which are, however, always subdominant.)

The results of this section allow the determination of the luminosity distance for a given initial spectrum $P_\Psi(k)$ and given transfer function $T_k(\eta)$. The transfer function, the conformal time $\eta(z)$, as well as the conformal Hubble parameter $\mathcal{H}(z)$ depend crucially on the cosmological parameters. In a forthcoming paper [13] we will present a code to determine the luminosity distance power spectrum numerically and discuss its dependence on cosmological parameters. In this work, where we mainly want to present the method, we approximately calculate the power spectrum for a simple case to gain some intuition about the order of magnitude of the different terms.

V. RESULTS FOR A PURE CDM UNIVERSE

In this section we approximate the luminosity distance power spectrum semianalytically for the simple case of a cold dark matter (CDM) universe without cosmological constant, $\Omega_m = 1$, $\Omega_\Lambda = 0$. We assume a scale-invariant spectrum of initial fluctuations,

$$P_\Psi(k) = A(k\eta_0)^{n-1} = A, \qquad n = 1. \tag{76}$$

The amplitude A is known from the Wilkinson Microwave Anisotropy Probe (WMAP) experiment, $A \simeq 10^{-10}$ [2].

In the radiation-dominated past of the universe, the Bardeen potential is constant on super horizon scales, $k\eta < 1$, and oscillates and decays like $1/a^2 \propto 1/\eta^2$ on subhorizon scales. During matter domination, the

Bardeen potential is constant [10]. To take this gross behavior into account, we approximate the transfer function during the matter era by

$$T_k(\eta)T_k(\eta') = T_k^2 \simeq \frac{1}{1+\beta(k\eta_{\rm eq})^4}, \qquad (77)$$

where $\eta_{\rm eq}$ denotes the value of conformal time at matter and radiation equality. Comparing this rather crude approximation with the numerical one, which can be found e.g. in Dodelson's book [14], we find $\beta \simeq 3\times 10^{-4}$. In addition, there is a log correction which comes from the logarithmic growth of matter perturbations during the radiation era. We shall take it into account only for the dominant term $C_\ell^{(5)}$. Furthermore, we use that during the matter-dominated era $4\pi Ga^2(\rho+p) = \frac{3}{2}(\dot a/a)^2 = \frac{3}{2}\times(2/\eta)^2 = 6/\eta^2$.

To determine the power spectrum, we have to perform integrals over time of the form

$$I(f) = \int_{\eta_S}^{\eta} d\eta' f(\eta') j_\ell(k(\eta_0-\eta')) = \frac{1}{k}\int_x^{x_S} dx' f(\eta_0 - x'/k) j_\ell(x'), \qquad (78)$$

where we have introduced $x = k(\eta_0-\eta)$. The spherical Bessel function of order ℓ is peaked at $x\simeq\ell$. For values much smaller than ℓ it is suppressed like $(x/\ell)^\ell$ and for values much larger that ℓ it oscillates and decays like $1/x$. In our crude approximation, we neglect contributions to this integral from outside the first peak and approximate the integral over the first peak by the value of f at $x=\ell$ multiplied by the area under the peak. This gives

$$I(f) \simeq \frac{1}{k}I_\ell f\left(\eta_0-\frac{\ell}{k}\right)\theta\left(k-\frac{\ell}{\eta_0-\eta_S}\right)\theta\left(\frac{\ell}{\eta_0-\eta}-k\right), \qquad (79)$$

where I_ℓ is the area under the first peak of the Bessel function j_ℓ and θ denotes the Heaviside function, $\theta(x)=0$, if $x\le 0$ and $\theta(x)=1$, if $x>0$. Numerically we have found $I_\ell^2 \simeq 1.58/\ell$. Most of the resulting integrals over k can either be obtained analytically in terms of hypergeometric functions [15] or they can be approximated by the same method. Finally, one k integral contributing to the Doppler term $C_\ell^{(3)}$ has to be performed numerically. More details are given in Appendix E.

We have tested our approximations by comparing them with the numerical result and have found that we nearly always overestimate the numerical result, but never by more than a factor of 2. The approximations are quite bad at low $\ell \le 5$, but become reasonable later. A fully numerical evaluation as we shall perform it in [13] will probably give a somewhat smaller result but not by more that a factor of 2 to 4. Here, we are not so much interested in numerical accuracy as in qualitative features of the different contributions to the power spectrum.

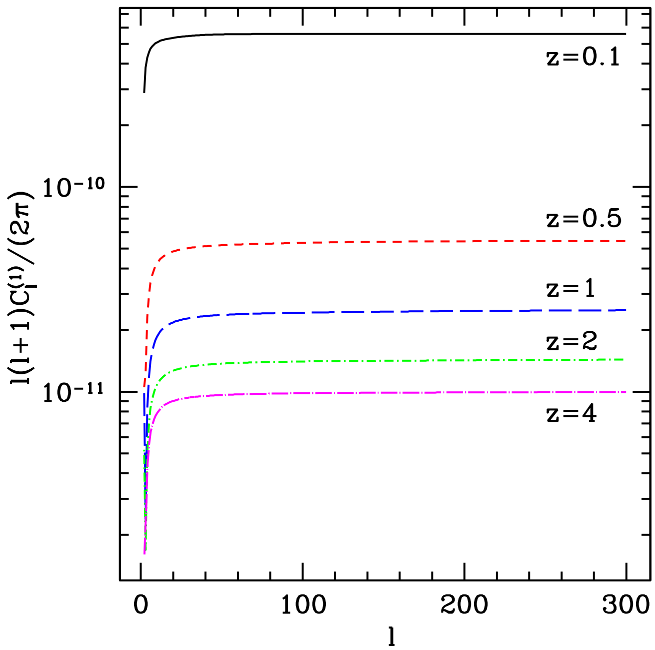

FIG. 4 (color online). The contribution of the redshift term $\ell(\ell+1)C_\ell^{(1)}(z,z)/(2\pi)$ for $z = 0.1$, 0.5, 1, 2, and 4 (from top to bottom).

In Figs. 4–9 we show $\ell(\ell+1)C_\ell^{(i)}(z,z)$ for different values of z. For $\ell \gtrsim 10$, the lensing contribution $C_\ell^{(5)}$ always dominates if $z>0.2$. It is interesting to note that the

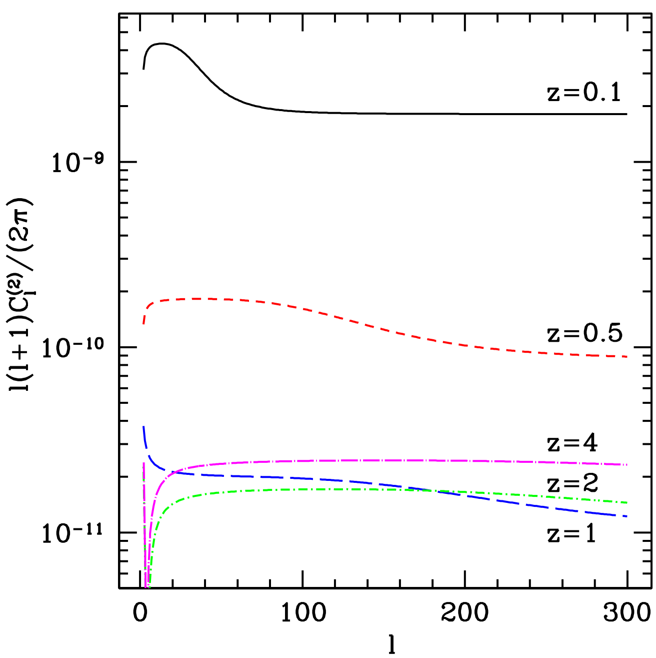

FIG. 5 (color online). The contribution $C_\ell^{(2)}$. We choose the same line styles like in Fig. 4. The contributions for $z = 0.1$, 0.5, and 1 are negative while those for $z = 2$, 4 pass through 0 at low ℓ, visible as a spike. This may well be due to our approximative treatment.

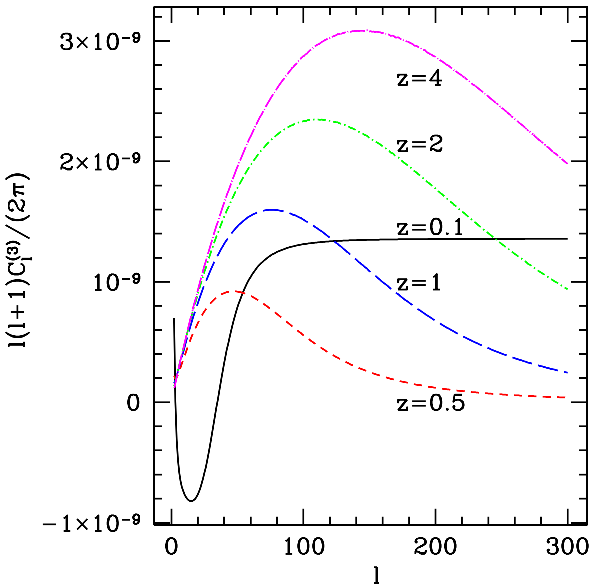

FIG. 6 (color online). The contribution $\ell(\ell + 1)C_\ell^{(3)}(z, z)/(2\pi)$, without the numerical part, for $z = 4$, 2, 1, 0.5, and 0.1 (from top to bottom). Note that here we have chosen linear as opposed to a log representation.

different contributions do not scale in the same way with ℓ. Only $C^{(1)}$ and $C^{(2)}$ are scale invariant with

$$\ell(\ell + 1)C_\ell^{(1)} \simeq 10^{-10}, \tag{80}$$

$$\ell(\ell + 1)C_\ell^{(2)} \simeq -10^{-10}. \tag{81}$$

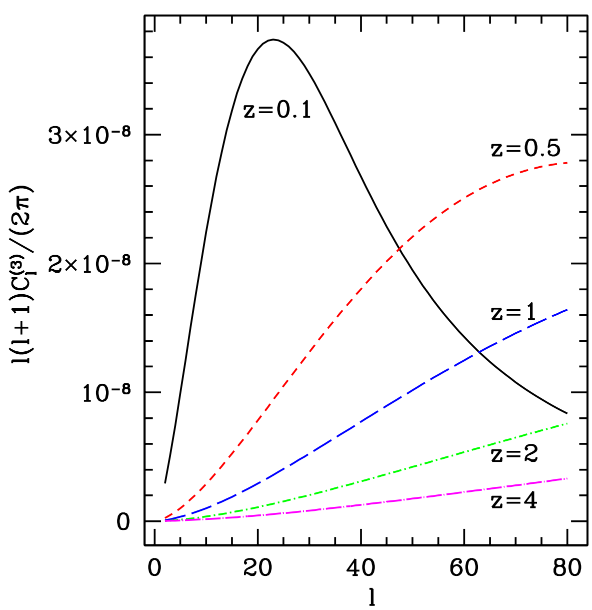

FIG. 7 (color online). The Doppler contribution of $\ell(\ell + 1)C_\ell^{(3)}(z, z)/(2\pi)$ which has been determined numerically for $z = 0.1$, 0.5, 1, 2, and 4 (from top to bottom). Our numerical code is stable only for $\ell \lesssim 80$ and we therefore plot only this part of the curve.

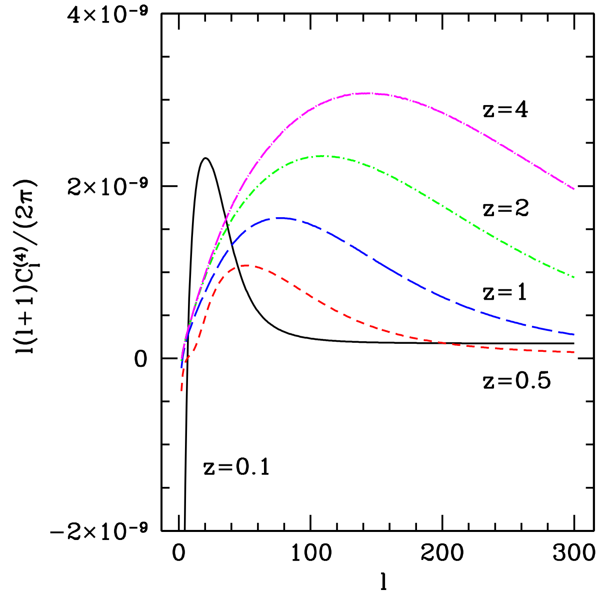

FIG. 8 (color online). The contribution $\ell(\ell + 1)C_\ell^{(4)}(z, z)/(2\pi)$ for $z = 4$, 2, 1, 0.5, and 0.1 (from top to bottom).

The other contributions grow up to a redshift dependent maximum (minimum) from where they decay. They may become scale invariant at higher ℓ, but until $\ell = 300$ the scale-invariant piece is only clearly visible for $z = 0.1$. Higher values of z have their maximum contribution at higher ℓ and have not decayed into a scale-invariant behavior until $\ell = 300$. The lensing contribution $C^{(5)}$ even just grows. For $z = 0.1$ it does seem to reach a scale-invariant plateau; for $z = 0.5$ it seems just to reach the

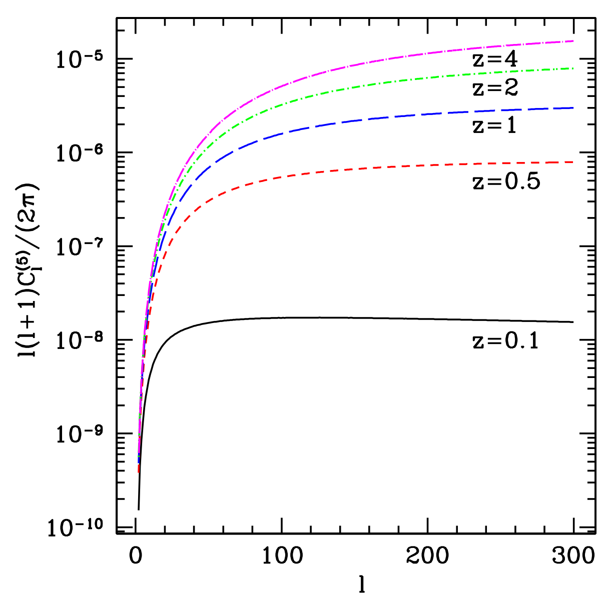

FIG. 9 (color online). The lensing contribution $\ell(\ell + 1)C_\ell^{(5)}(z, z)/(2\pi)$ for $z = 4$, 2, 1, 0.5, and 0.1 (from top to bottom). For clarity, we have again chosen a log representation in this graph.

turns over around $\ell = 300$. For values $z > 0.5$ shown in Fig. 9, the spectrum is simply growing and has not yet reached the turn over until $\ell = 300$.

The most surprising result is the high amplitude of the lensing term $C^{(5)}$. Let us discuss this term in more detail. After performing the time integrals as outlined above, an integral $\int dk k^{-1} T_k^2$ from $\ell/(\eta_0 - \eta_S)$ to infinity is left. If we neglect the log in the transfer function, this amounts to

$$\int_{\frac{l}{\eta_O - \eta_S}}^{\infty} \frac{dk}{k} T_k^2 \simeq \begin{cases} \log(\frac{\eta_0 - \eta_S}{\ell \beta^{1/4} \eta_{\rm eq}}) & \text{if } \frac{\ell}{\eta_0 - \eta_S} < \frac{1}{\beta^{1/4} \eta_{\rm eq}} \\ \frac{(\eta_0 - \eta_S)^4}{4\ell^4 \beta \eta_{\rm eq}^4} & \text{else.} \end{cases}$$

Together with the factor $I_\ell^2 \times \ell^2$ from the time integrations, we obtain a ℓ^{-1} behavior of $\ell(\ell+1)C_\ell^{(5)}$ at large ℓ, which is not seen in Fig. 9. However, when taking into account also the log correction, the correct amplitude and scaling with ℓ can be estimated in this way (for more details see Appendix D).

This dominant term comes actually from the second derivatives of Ψ, hence from the Riemann tensor which describes the tidal force field, i.e. geodesic deviations.

If the k integral would not be decaying, $\ell(\ell+1)C_\ell^{(5)}$ would be growing like $\sim \ell^3$. But the integrand becomes small for fluctuations with a wave number smaller than about $k_{\rm eq} \equiv 1/(\beta^{1/4}\eta_{\rm eq})$. Therefore $\ell(\ell+1)C_\ell^{(5)}$ has a (broad) maximum $\ell_{\max} \simeq k_{\rm eq}(\eta_0 - \eta_S)$. Hence $\ell_{\max}$ is increasing with the source redshift. For $z_S \simeq 1$, hence $\eta_0 - \eta_S \simeq 0.3\eta_0 \simeq 30\eta_{\rm eq}$ we find $\ell_{\max} \simeq 250$. The general expression for a matter-dominated universe is

$$\ell_{\max}(z_S) \simeq 760 \times \frac{\sqrt{z_S + 1} - 1}{\sqrt{z_S + 1}}. \tag{82}$$

Our first important finding is that the tidal force field, represented by $C_\ell^{(5)}$ totally dominates the final result for redshifts $z_S \gtrsim z_{S'} \gtrsim 0.2$. In a numerical treatment, where we want to reach a 1% level accuracy, it is sufficient to consider only $C_\ell^{(5)}$ for redshifts $z_S \gtrsim z_{S'} \gtrsim 0.5$. Secondly, naïvely one would expect a result of the order of $\langle \Psi^2 \rangle \simeq A \simeq 10^{-10}$, but we found nearly 10^{-5} for supernovae with redshift $z_S \sim 2$. This comes from the fact that in the time integral for $C^{(5)}$, the fluctuation is multiplied by the conformal distance $\eta - \eta_S$. A small angular deviation at η builds up to a large deviation at η_S if the distance is large. Furthermore, we deal with an integrated effect where even if the deviation from each fluctuation is similar, more small fluctuations pile up on the way from the supernovae into the telescope. Even if these are uncorrelated, we still gain a factor $\sqrt{N}$ by piling them up. These arguments are somewhat simplistic, but they explain why the term with most time integrals and with the factor $(\eta - \eta_S)$ dominates.

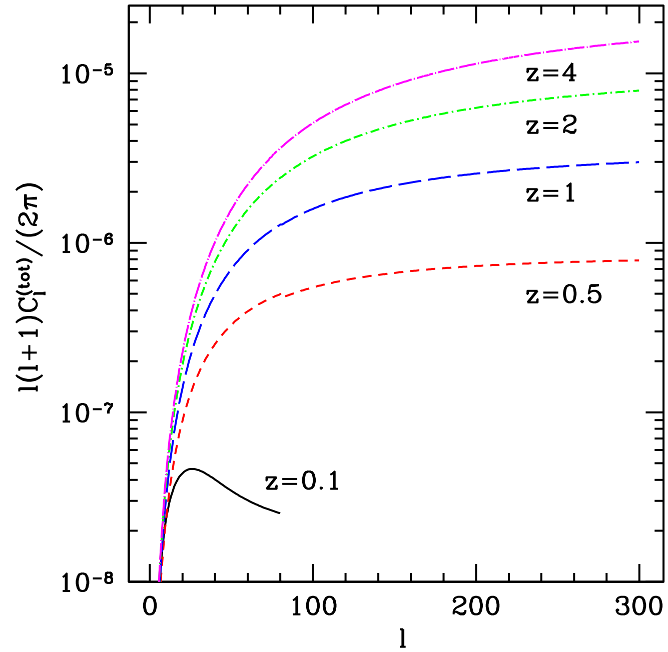

FIG. 10 (color online). The total $\ell(\ell+1)C_\ell(z,z)/(2\pi)$ is shown for $z = 4$, 2, 1, 0.5, and 0.1 (from top to bottom). Note that for $z > 0.1$ it reproduces simply $C_\ell^{(5)}$. For $z = 0.1$ the contribution of the Doppler part of $\ell(\ell+1)C_\ell^{(3)}(z,z)/(2\pi)$ is important, which we have computed only for $\ell \lesssim 80$. For clarity, we have again chosen a log representation in this graph.

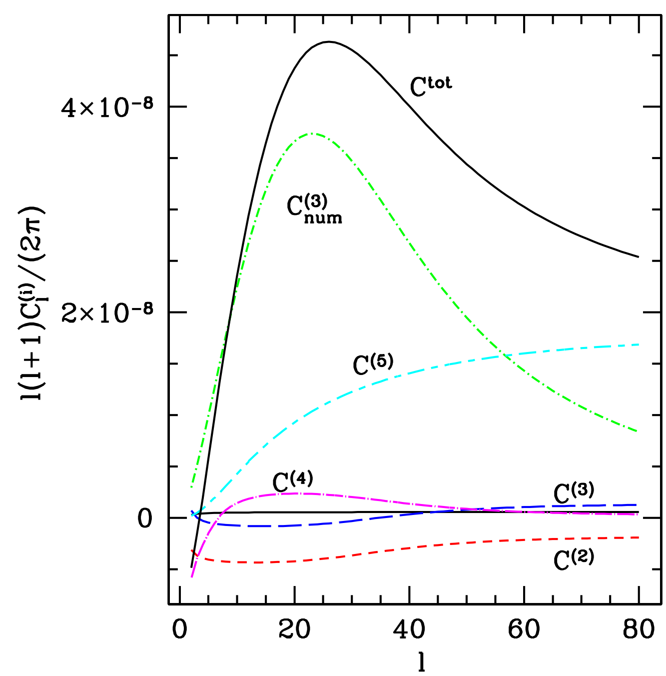

FIG. 11 (color online). The different contributions to $\ell(\ell+1)C_\ell(z,z)/(2\pi)$ for $z = 0.1$ are shown. For this low redshift they are all of the same order of magnitude. For low ℓ's our approximations are not trustable, they even lead to negative values for $C_\ell^{\rm tot}$ for $\ell \leq 3$.

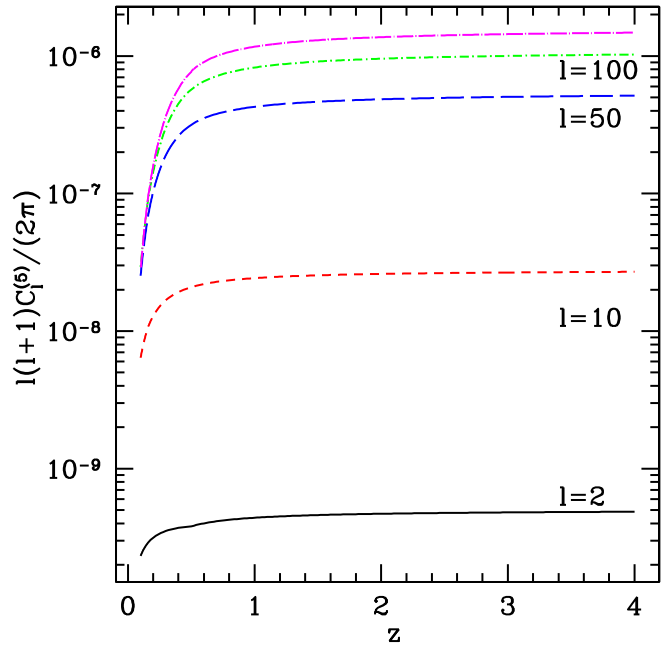

FIG. 12 (color online). The contribution $\ell(\ell+1)C_\ell^{(5)}(z, z')/(2\pi)$ is shown as a function of z with z' fixed to $z' = 0.5$ and $\ell = 200$, 100, 50, 10, and 2 (from top to bottom). Above $\ell \simeq 50$, the ℓ dependence of the result becomes weak as expected. For $z > 0.1$ this represents actually also the total contribution to C_ℓ.

In Fig. 10 we show the sum

$$\ell(\ell+1)\left[\sum_i C_\ell^{(i)}(z, z)\right]\frac{1}{2\pi}.$$

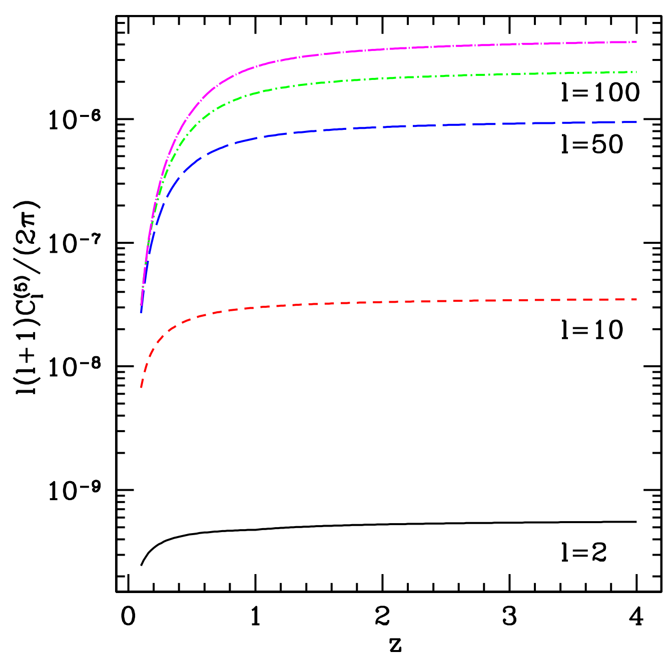

FIG. 13 (color online). Like Fig. 12, but for $z' = 1$.

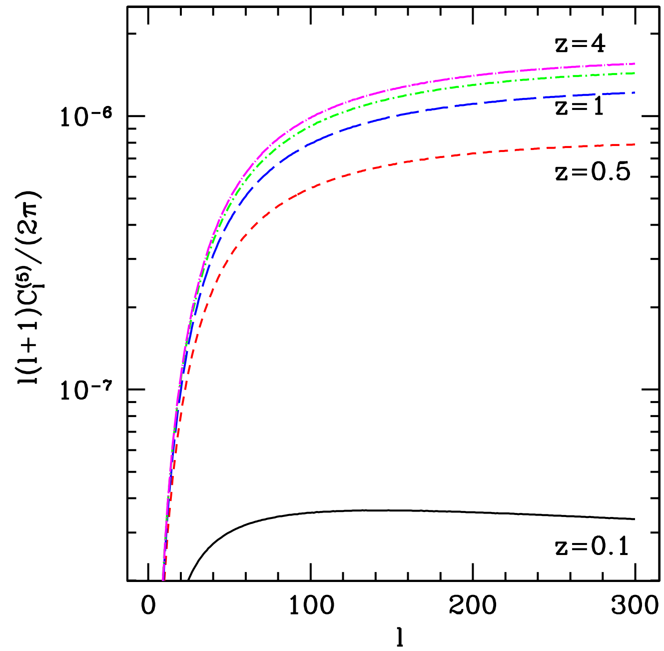

FIG. 14 (color online). The contribution $\ell(\ell+1)C_\ell^{(5)}(z, z')/(2\pi)$ is shown for $z' = 0.5$ and $z = 4$, 2, 1, 0.5, and 0.1 (from top to bottom). Again, for $z \neq 0.1$ this result is equivalent to the full C_ℓ.

For $z > 0.1$, the total results are indistinguishable from $C_\ell^{(5)}$ alone. Only for $z = 0.1$ all terms contribute, especially the numerical part of $C_\ell^{(3)}$ dominates. We plot this line only until $\ell = 80$ since we have no reliable results on the numerical contribution to $C_\ell^{(3)}$ for higher values of ℓ. The different contributions to C_ℓ for $z = z' = 0.1$ are shown in more detail in Fig. 11.

It is also interesting to study the behavior of $C_\ell(z, z')$ for fixed z' as a function of z and for fixed $z \neq z'$ as a function of ℓ. We show this behavior in Figs. 12–14. Somewhat surprisingly $C_\ell(z, z')$ shows no peak at $z = z'$. It is therefore not problematic to include relatively large bins Δz in a study of $C_\ell(z, z)$.

VI. CONCLUSIONS AND OUTLOOK

In this work we have determined the correlation function of the luminosity distance fluctuations. We have found that at redshifts $z \geq 0.2$, the result is dominated entirely by the "lensing term" $\langle|\Delta\Psi|^2\rangle$ which is proportional to the density fluctuation. Geometrically it comes from the term $A_i^j = R^j_{\mu\nu i}k^\mu k^\nu$ i.e. the Riemann tensor. Hence this contribution is due to the tidal force field. We have seen that it is dominated by fluctuations of the size $\lambda \simeq \eta_{\rm eq}$ which enter the horizon at matter radiation equality. These fluctuations have not been damped during the radiation era, but they are the smallest and therefore the most numerous which have not suffered damping. Their effect can therefore add up most along the path of the photon.

We have found that within linear perturbation theory, the d_L-power spectrum is nearly 5 orders of magnitude larger than the CMB anisotropy power spectrum. But nevertheless, the fluctuations obtained within linear theory are still much smaller than 1. We also have seen that small scale fluctuations do not significantly contribute to the C_ℓ's for low ℓ's. i.e. on large scales. This indicates that they cannot change the observed $d_L(z)$ by factors of order unity, which would be needed to mimic accelerated expansion in a matter-dominated universe. Also the variance, i.e., the typical deviation of a given luminosity distance $d_L(\mathbf{n}, z)$ from the mean, which is dominated by small scale fluctuations (the lensing contribution) is

$$\bar{d}_L(z)^{-2}\langle d_L(\mathbf{n}, z)^2\rangle = \frac{1}{4\pi}\sum_\ell (2\ell + 1)C_\ell \simeq 10^{-5} \ll 1.$$

Our findings thus indicate that the explanation of accelerated expansion put forward in [6] is probably not realized. Of course we have not taken into account the change of the transfer function due to nonlinearities. To determine this effect more precisely we would have to take into account the nonlinearities, especially in the integral for $C_\ell^{(5)}$.

We suggest that the newly derived luminosity distance power spectrum given by the $C_\ell(z_S, z_{S'})$ can be used as a new observational tool to determine cosmological parameters. For 1% accuracy of the fluctuations at $z_S \gtrsim 0.5$, only $C_\ell^{(5)}$ has to be taken into account and therefore the numerical complexity of the problem seems to be quite moderate. In a future paper [13] we shall investigate the possibilities to measure $C_\ell(z_S, z_{S'})$ with the supernovae searches which are presently under way or in planning.

ACKNOWLEDGMENTS

We thank Martin Kunz, Dominik Schwarz, and Lam Hui for useful and stimulating discussions. We thank Marc-Olivier Bettler for his help with a figure. We are grateful to the Swiss National Science foundation for financial support.

APPENDIX A: CHRISTOFFEL SYMBOLS AND THE RIEMANN TENSOR OF SCALAR PERTURBATIONS IN NONEXPANDING SPACETIME

Here we write down the Christoffel symbols and Riemann tensor for the metric

$$g_{\mu\nu}dx^\mu dx^\nu = -(1 + 2\Psi)d\eta^2 + (1 - 2\Psi)\gamma_{ij}dx^i dx^j$$

to first order in the gravitational potential Ψ:

$$\Gamma^0_{00} = \dot{\Psi}, \tag{A1}$$

$$\Gamma^0_{0i} = \partial_i\Psi, \tag{A2}$$

$$\Gamma^i_{00} = \partial^i\Psi, \tag{A3}$$

$$\Gamma^j_{0i} = -\delta^j_i\dot{\Psi}, \tag{A4}$$

$$\Gamma^0_{ij} = -\gamma_{ij}\dot{\Psi}, \tag{A5}$$

$$\Gamma^i_{jm} = -\delta^i_j\partial_m\Psi - \delta^i_m\partial_j\Psi + \gamma_{jm}\partial^i\Psi, \tag{A6}$$

$$R^0{}_{00j} = 0, \tag{A7}$$

$$R^0{}_{0ij} = 0, \tag{A8}$$

$$R^0{}_{i0j} = -\nabla_i\nabla_j\Psi - \gamma_{ij}\ddot{\Psi}, \tag{A9}$$

$$R^0{}_{ijm} = \gamma_{ij}\nabla_m\Psi - \gamma_{im}\nabla_j\dot{\Psi}, \tag{A10}$$

$$R^i{}_{00j} = -\nabla^i\nabla_j\Psi - \delta^i_j\ddot{\Psi}, \tag{A11}$$

$$R^i{}_{0jm} = \delta^i_j\nabla_m\dot{\Psi} - \delta^i_m\nabla_j\dot{\Psi}, \tag{A12}$$

$$R^i{}_{j0m} = -\delta^i_m\nabla_j\dot{\Psi} + \gamma_{jm}\nabla^i\dot{\Psi}, \tag{A13}$$

$$R^i{}_{jmn} = -[\delta^i_n\nabla_j\nabla_m - \delta^i_m\nabla_j\nabla_n + \gamma_{jm}\nabla^i\nabla_n - \gamma_{jn}\nabla^i\nabla_m]\Psi. \tag{A14}$$

Here ∇_i denotes the covariant derivative with respect to the metric γ_{ij}.

APPENDIX B: THE DERIVATION OF EQ. (59)

We first reintroduce the velocity of the source $\mathbf{v}_S$ and we collect all terms which contain spatial derivatives of the form $n^i\partial_i\Psi$ at the end. This brings (58) into the form (we dismiss the tilde in this appendix)

$$d_L(z_S, \mathbf{n}) = (1+z_S)\bigg\{(\eta_O - \eta_S) + \frac{1}{\mathcal{H}_S}(\Psi_O - \mathbf{v}_O\cdot\mathbf{n}) - (\eta_O - \eta_S + \mathcal{H}_S^{-1})\Psi_S - (\eta_O - \eta_S - \mathcal{H}_S^{-1})\mathbf{v}_S\cdot\mathbf{n} + 2\int_{\eta_S}^{\eta_O} d\eta\Psi - \int_{\eta_S}^{\eta_O} d\eta\int_{\eta_S}^{\eta} d\eta'(\eta'-\eta_S)\nabla^2\Psi - \frac{2}{\mathcal{H}_S}\int_{\eta_S}^{\eta_O} d\eta\mathbf{n}\cdot\nabla\Psi + 2\int_{\eta_S}^{\eta_O} d\eta\int_{\eta_S}^{\eta} d\eta'\mathbf{n}\cdot\nabla\Psi + \int_{\eta_S}^{\eta_O} d\eta\int_{\eta_S}^{\eta} d\eta'(\eta'-\eta_S)n^i n^j\partial_i\partial_j\Psi\bigg\}. \quad \text{(B1)}$$

Now we use

$$\frac{d\Psi}{d\eta} = \dot{\Psi} + \mathbf{n}\cdot\nabla\Psi$$

to convert all derivatives of the form $\mathbf{n}\cdot\nabla\Psi$ into time derivatives. This leads to

$$d_L(z_S, \mathbf{n}) = (1+z_S)\bigg\{(\eta_O - \eta_S) - \frac{1}{\mathcal{H}_S}(\Psi_O + \mathbf{v}_O\cdot\mathbf{n}) + (-2(\eta_O - \eta_S) + \mathcal{H}_S^{-1})\Psi_S - (\eta_O - \eta_S - \mathcal{H}_S^{-1})\mathbf{v}_S\cdot\mathbf{n} + (\eta_O - \eta_S)\Psi_O + 2\int_{\eta_S}^{\eta_O} d\eta\Psi - \int_{\eta_S}^{\eta_O} d\eta\int_{\eta_S}^{\eta} d\eta'(\eta'-\eta_S)\nabla^2\Psi + \frac{2}{\mathcal{H}_S}\int_{\eta_S}^{\eta_O} d\eta\dot{\Psi} - 2\int_{\eta_S}^{\eta_O} d\eta(\eta-\eta_S)\dot{\Psi} + \int_{\eta_S}^{\eta_O} d\eta\int_{\eta_S}^{\eta} d\eta'(\eta'-\eta_S)\ddot{\Psi}\bigg\}. \quad \text{(B2)}$$

Via integration by parts we can now convert the double integrals over time into single integrals. For this we use that for a regular function $f(\eta)$ integrating by parts $\int_{\eta_S}^{\eta_O} d\eta(\eta-\eta_S)^2 f(\eta)$ gives

$$\int_{\eta_S}^{\eta_O} d\eta\int_{\eta_S}^{\eta} d\eta'(\eta'-\eta_S)f(\eta') = \int_{\eta_S}^{\eta_O} d\eta(\eta-\eta_S)(\eta_O-\eta)f(\eta).$$

Using this in the two double integrals above we obtain Eq. (59).

APPENDIX C: THE POWER SPECTRUM

We use the Fourier transform convention

$$\Psi(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\Psi(\mathbf{x}), \quad \text{(C1)}$$

$$\Psi(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\Psi(\mathbf{k}). \quad \text{(C2)}$$

The time evolution of the Bardeen potential is given by the transfer function, $\Psi(\mathbf{k}, \eta) = T_k(\eta)\Psi(\mathbf{k})$, which is normalized such that $\Psi(\mathbf{k}, \eta_0) \to \Psi(\mathbf{k})$ for $k \to 0$. Since the Bardeen potential is constant on very large scales, this identifies $\Psi(\mathbf{k})$ also with the Bardeen potential right after inflation. The correlation function

$$\zeta_\Psi(|\mathbf{x}-\mathbf{y}|) \equiv \langle\Psi(\mathbf{x})\Psi(\mathbf{y})\rangle$$

depends only on the distance $|\mathbf{x}-\mathbf{y}|$, so that we obtain

$$\langle\Psi(\mathbf{k}, \eta)\Psi^*(\mathbf{k}', \eta')\rangle = T_k(\eta)T_{k'}(\eta')\int d^3x d^3y \zeta_\Psi(|\mathbf{x}-\mathbf{y}|) \times e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{y}} = T_k(\eta)T_{k'}(\eta')k^{-3}P_\Psi(k)(2\pi)^3 \times \delta^3(\mathbf{k}-\mathbf{k}'), \quad \text{(C3)}$$

where we have introduced the power spectrum

$$P_\Psi(k) = k^3\int d^3z\zeta_\Psi(\mathbf{z})e^{-i\mathbf{k}\cdot\mathbf{z}}. \quad \text{(C4)}$$

It is easy to verify that this definition is consistent with the one given in Eq. (65).

Standard inflationary scenarios give $P_\Psi \simeq A(k\eta_0)^{n-1}$ with $n \simeq 1$. From WMAP and other measurements of CMB anisotropies we have $A \sim 10^{-10}$. We first want to determine the correlation of the Bardeen potential at positions $\mathbf{x} = \mathbf{x}_O - \mathbf{n}(\eta_O - \eta)$ and $\mathbf{x}' = \mathbf{x}_O - \mathbf{n}'(\eta_O - \eta')$. With the above we have

$$\langle\Psi(\eta, \mathbf{x})\Psi(\eta', \mathbf{x}')\rangle = \frac{1}{(2\pi)^6}\int d^3k d^3k' T_k(\eta) \times T_{k'}(\eta')\langle\Psi(\mathbf{k})\Psi^*(\mathbf{k}')\rangle e^{-i\mathbf{k}\cdot\mathbf{n}(\eta_O-\eta)} \times e^{+i\mathbf{k}'\cdot\mathbf{n}'(\eta_O-\eta')}. \quad \text{(C5)}$$

Using the identity (see e.g. [15])

$$e^{i\mathbf{k}\cdot\mathbf{n}(\eta_O-\eta)} = \sum_\ell (2\ell+1)i^\ell j_\ell(k(\eta_O-\eta))P_\ell(\hat{\mathbf{k}}\cdot\mathbf{n}) \quad \text{(C6)}$$

and Eq. (C3) we obtain

$$\begin{aligned}\langle\Psi(\eta,\mathbf{x})\Psi(\eta',\mathbf{x}')\rangle &= \frac{1}{(2\pi)^3}\sum_{\ell\ell'}(2\ell+1)(2\ell'+1)i^{\ell-\ell'}\int\frac{dk}{k}T_k(\eta)T_k(\eta')\bigg[P_\Psi(k)j_\ell(k(\eta_O-\eta))j_{\ell'}(k(\eta_O-\eta'))\\ &\quad\times\int d\Omega_{\hat{\mathbf{k}}}P_\ell(\hat{\mathbf{k}}\cdot\mathbf{n})P_{\ell'}(\hat{\mathbf{k}}\cdot\mathbf{n}')\bigg]\\ &= \frac{1}{2\pi^2}\sum_\ell(2\ell+1)P_\ell(\mathbf{n}\cdot\mathbf{n}')\int\frac{dk}{k}T_k(\eta)T_k(\eta')P_\Psi(k)j_\ell(k(\eta_O-\eta))j_\ell(k(\eta_O-\eta'))\\ &= \sum_\ell\frac{2\ell+1}{4\pi}C_\ell^{(\Psi)}(z,z')P_\ell(\mathbf{n}\cdot\mathbf{n}'),\end{aligned} \tag{C7}$$

where we have used Eq. (66) for the last equals sign. Here $\hat{\mathbf{k}}$ is the unit vector in direction $\mathbf{k}$ and $d\Omega_{\hat{\mathbf{k}}}$ denotes the integral over the sphere of $\mathbf{k}$ directions.

In the same way one derives Eqs. (67) to (74). Each factor $i\mathbf{n}\cdot\mathbf{k}$ can be written as a derivative with respect to $\eta_O-\eta$ of the exponential and therefore replaces $j_\ell(k(\eta_O-\eta))$ by $-kj'_\ell(k(\eta_O-\eta))$. The Laplacian simply corresponds to a factor $-k^2$.

APPENDIX D: DETAILS FOR THE POWER SPECTRUM

In this appendix we write down in detail the expressions for the $C_\ell^{(i)}$'s used in this paper.

As mentioned in Sec. IV the power spectrum of the luminosity distance can be split in five different parts containing k integrals of different powers:

$C_\ell^{(1)}$ contains the integrals of the form $\int\frac{dk}{k}$, represents the redshift and parts of the integrated contributions.

$C_\ell^{(2)}$ contains the integrals of the form $\int dk$, represents the correlation of the Doppler term with the terms in $C^{(1)}$.

$C_\ell^{(3)}$ contains the integrals of the form $\int dk\cdot k$, represents the Doppler term and some (subdominant) integrated terms.

$C_\ell^{(4)}$ contains the integrals of the form $\int dk\cdot k^2$, is dominated by the correlation of the Doppler term with the lensing contribution.

$C_\ell^{(5)}$ contains the integrals of the form $\int dk\cdot k^3$, represents the lensing term.

From Eqs. (66) to (74) and the expression (58) for the luminosity distance we obtain the following expressions for the $C_\ell^{(i)}$'s

$$\begin{aligned}C_\ell^{(1)} &= \frac{2}{\pi}\int\frac{dk}{k}P_\Psi(k)\bigg[\frac{2}{\eta_O-\eta_S}\int_{\eta_S}^{\eta_0}d\eta T_k(\eta)j_\ell(k(\eta_0-\eta))-\bigg(1+\frac{1}{\mathcal{H}_S(\eta_O-\eta_S)}\bigg)T_k(\eta_S)j_\ell(k(\eta_O-\eta_S))\bigg]\\ &\quad\times\bigg[\frac{2}{\eta_O-\eta_{S'}}\int_{\eta_{S'}}^{\eta_0}d\eta T_k(\eta)j_\ell(k(\eta_0-\eta))-\bigg(1+\frac{1}{\mathcal{H}_{S'}(\eta_O-\eta_{S'})}\bigg)T_k(\eta_{S'})j_\ell(k(\eta_O-\eta_{S'}))\bigg],\end{aligned} \tag{D1}$$

$$\begin{aligned}C_\ell^{(2)} &= -\frac{4}{\pi}\int dkP_\Psi(k)\bigg[\frac{1}{3\mathcal{H}_S}\bigg(1-\frac{1}{\mathcal{H}_S(\eta_O-\eta_S)}\bigg)(T_k(\eta_S)+\mathcal{H}_S^{-1}\dot{T}_k(\eta_S))j'_\ell(k(\eta_0-\eta_S))\\ &\quad-\frac{1}{\mathcal{H}_S(\eta_O-\eta_S)}\int_{\eta_S}^{\eta_0}d\eta T_k(\eta)j'_\ell(k(\eta_0-\eta))+\frac{1}{\eta_0-\eta_S}\int_{\eta_S}^{\eta_0}d\eta\int_{\eta_S}^{\eta}d\eta' T_k(\eta')j'_\ell(k(\eta_0-\eta'))\bigg]\\ &\quad\times\bigg[\frac{2}{\eta_0-\eta_{S'}}\int_{\eta_{S'}}^{\eta_0}d\eta T_k(\eta)j_\ell(k(\eta_0-\eta))-\bigg(1+\frac{1}{\mathcal{H}_{S'}(\eta_O-\eta_{S'})}\bigg)T_k(\eta_{S'})j_\ell(k(\eta_0-\eta_{S'}))\bigg]+\eta_S\Leftrightarrow\eta_{S'},\end{aligned} \tag{D2}$$

$$C_\ell^{(3)} = \frac{8}{\pi}\int dkkP_\Psi(k)\bigg[\frac{1}{3\mathcal{H}_S}\Big(1-\frac{1}{\mathcal{H}_S(\eta_O-\eta_S)}\Big)(T_k(\eta_S)+\mathcal{H}_S^{-1}\dot{T}_k(\eta_S))j_\ell'(k(\eta_0-\eta_S)) - \frac{1}{\mathcal{H}_S(\eta_o-\eta_S)}\int_{\eta_S}^{\eta_0}d\eta T_k(\eta)j_\ell'(k(\eta_0-\eta)) + \frac{1}{\eta_0-\eta_S}\int_{\eta_S}^{\eta_0}d\eta\int_{\eta_S}^{\eta}d\eta' T_k(\eta')j_\ell'(k(\eta_0-\eta'))\bigg] \times\bigg[\frac{1}{3\mathcal{H}_{S'}}\Big(1-\frac{1}{\mathcal{H}_{S'}(\eta_O-\eta_{S'})}\Big)(T_k(\eta_{S'})+\mathcal{H}_{S'}^{-1}\dot{T}_k(\eta_{S'}))j_\ell'(k(\eta_0-\eta_{S'})) - \frac{1}{H_{S'}(\eta_O-\eta_{S'})}\int_{\eta_{S'}}^{\eta_0}d\eta T_k(\eta)j_\ell'(k(\eta_0-\eta)) + \frac{1}{\eta_0-\eta_{S'}}\int_{\eta_{S'}}^{\eta_0}d\eta\int_{\eta_{S'}}^{\eta}d\eta' T_k(\eta')j_\ell'(k(\eta_0-\eta'))\bigg] + \frac{2}{\pi(\eta_O-\eta_S)}\int dkkP_\Psi(k)\int_{\eta_S}^{\eta_0}d\eta\int_{\eta_S}^{\eta}d\eta'(\eta'-\eta_S)T_k(\eta')(j_\ell(k(\eta_0-\eta'))+j_\ell''(k(\eta_0-\eta'))) \times\bigg[\frac{2}{\eta_O-\eta_{S'}}\int_{\eta_{S'}}^{\eta_0}d\eta T_k(\eta)j_\ell(k(\eta_0-\eta)) - \Big(1+\frac{1}{H_{S'}(\eta_O-\eta_{S'})}\Big)T_k(\eta_{S'})j_\ell(k(\eta_O-\eta_{S'}))\bigg] + \eta_S \Leftrightarrow \eta_{S'}, \quad \text{(D3)}$$

$$C_\ell^{(4)} = -\frac{4}{\pi}\frac{1}{\eta_O-\eta_{S'}}\int dkk^2P_\Psi(k)\bigg[\frac{1}{3\mathcal{H}_S}\Big(1-\frac{1}{\mathcal{H}_S(\eta_O-\eta_S)}\Big)(T_k(\eta_S)+\mathcal{H}_S^{-1}\dot{T}_k(\eta_S))j_\ell'(k(\eta_0-\eta_S)) - \frac{1}{\mathcal{H}_S(\eta_O-\eta_S)}\int_{\eta_S}^{\eta_0}d\eta T_k(\eta)j_\ell'(k(\eta_0-\eta)) + \frac{1}{\eta_0-\eta_S}\int_{\eta_S}^{\eta_0}d\eta\int_{\eta_S}^{\eta}d\eta' T_k(\eta')j_\ell'(k(\eta_0-\eta'))\bigg] \times\int_{\eta_{S'}}^{\eta_0}d\eta\int_{\eta_{S'}}^{\eta}d\eta'(\eta'-\eta_{S'})T_k(\eta')(j_\ell(k(\eta_0-\eta'))+j_\ell''(k(\eta_0-\eta'))) + \eta_S\Leftrightarrow\eta_{S'}, \quad \text{(D4)}$$

$$C_\ell^{(5)} = \frac{2}{\pi}\frac{1}{(\eta_O-\eta_S)(\eta_O-\eta_{S'})}\int dkk^3P_\Psi(k)\bigg[\int_{\eta_S}^{\eta_0}d\eta\int_{\eta_S}^{\eta}d\eta'(\eta'-\eta_S)T_k(\eta')(j_\ell(k(\eta_0-\eta'))+j_\ell''(k(\eta_0-\eta')))\bigg] \times\bigg[\int_{\eta_{S'}}^{\eta_0}d\eta\int_{\eta_{S'}}^{\eta}d\eta'(\eta'-\eta_{S'})T_k(\eta')(j_\ell(k(\eta_0-\eta'))+j_\ell''(k(\eta_0-\eta')))\bigg]. \quad \text{(D5)}$$

APPENDIX E: INTEGRALS AND APPROXIMATIONS

Here we make full use of the relatively crude approximation (79)

$$\int_{x_1}^{x_2} dxf(x)j_\ell(x) \simeq I_\ell f(\ell)\theta(x_2-\ell)\theta(\ell-x_1), \quad \text{(E1)}$$

where θ denotes the Heaviside function, $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ else. Hence we neglect contributions to the integral which do not come from the region of the first peak of the Bessel function. This procedure is very useful to estimate the result, but cannot be trusted better than within a factor of about 2. We have tested it with numerical examples [16]. A more detailed numerical treatment will be presented elsewhere [13]. Furthermore, we assume a scale-invariant spectrum with $P_\Psi = A \simeq 10^{-10}$. We also use the fact that in a matter-dominated universe the transfer function does not depend on time and can be taken outside the time integrals.

We define $b_S = \frac{\eta_S}{\eta_O} = \frac{1}{\sqrt{1+z_S}}$, $x_S = k(\eta_0-\eta_S)$, and $\alpha_S = \beta(\frac{b_{\rm eq}}{1-b_S})^4$. Note that $x_{S'} = \frac{1-b_{S'}}{1-b_S}x_S$. In terms of these variables, the transfer function becomes

$$T^2(x_S) = \frac{1}{1+\alpha_S x_S^4}, \quad \text{(E2)}$$

except for the $C_\ell^{(5)}$, where we have to take into account the log correction.

1. $C_\ell^{(1)}$

$$C_\ell^{(1)}(z_S, z_{S'}) = \frac{2A}{\pi(1-b_{S'})}\left\{\frac{(2-b_S)(2-b_{S'})}{4(1-b_S)}\int_0^\infty \frac{dx_S}{x_S}T^2(x_S)j_\ell(x_S)j_\ell(x_S') + 4(1-b_S)\int_0^\infty \frac{dx_S}{x_S^3}T^2(x_S)\left(\int_0^{x_S} dx j_\ell(x)\right)\right.$$
$$\left.\times\left(\int_0^{x_{S'}} dx j_\ell(x)\right) - (2-b_{S'})\int_0^\infty \frac{dx_S}{x_S^2}T^2(x_S)\left(\int_0^{x_S} dx j_\ell(x)\right)\cdot j_\ell(x_{S'}) + b_S \Leftrightarrow b_{S'}\right\}. \tag{E3}$$

For the first term, the integral converges without the transfer function; we may therefore neglect it and perform the integral analytically. For the second and third terms, we use the approximation (E1). Assuming that $z_S < z_{S'}$ (if not, we reverse z_S and $z_{S'}$ in the formula), we obtain

$$C_\ell^{(1)}(z_S, z_{S'}) = \frac{2A}{\pi}\left\{4I_\ell^2\frac{1-b_S}{1-b_{S'}}\int_\ell^\infty \frac{dx_S}{x_S^3}\frac{1}{1+\alpha_S x_S^4} - \frac{I_\ell^2}{\ell^2}\frac{2-b_S}{1-b_{S'}}\frac{1}{1+\ell^4\alpha_S} + \frac{\sqrt{\pi}}{16}\frac{\Gamma(\ell)}{\Gamma(\ell+3/2)}(2-b_S)(2-b_{S'})\right.$$
$$\left.\times\frac{(1-b_S)^{\ell-1}(1-b_{S'})^{\ell-1}}{(2-b_S-b_{S'})^{2\ell}}F\left(\ell, \ell+1; 2\ell+2; \frac{4(1-b_S)(1-b_{S'})}{(2-b_S-b_{S'})^2}\right)\right\}. \tag{E4}$$

Here F denotes the hyper-geometric function and Γ is the Γ function. We use the notation and normalization of [15].

2. $C_\ell^{(2)}$

$$C_\ell^{(2)}(z_S, z_{S'}) = \frac{-2A}{\pi(1-b_{S'})}\left\{-\frac{(2-3b_S)(2-b_{S'})}{2(1-b_S)}\int_0^\infty \frac{dx_S}{x_S}T^2(x_S)j_\ell(x_S)j_\ell(x_S')\right.$$
$$-\frac{b_S(2-3b_S)(2-b_{S'})}{12(1-b_S)^2}\int_0^\infty dx_S T^2(x_S)j_l'(x_S)j_\ell(x_{S'}) - 4(1-b_S)\int_0^\infty \frac{dx_S}{x_S^3}T^2(x_S)\left(\int_0^{x_S} dx j_\ell(x)\right)$$
$$\times\left(\int_0^{x_{S'}} dx j_\ell(x)\right) + (6-7b_{S'})\int_0^\infty \frac{dx_S}{x_S^2}T^2(x_S)\left(\int_0^{x_S} dx j_\ell(x)\right)\cdot j_\ell(x_{S'})$$
$$\left.+\frac{b_{S'}(2-3b_{S'})}{3(1-b_S)}\int_0^\infty \frac{dx_S}{x_S}T^2(x_S)\left(\int_0^{x_S} dx j_\ell(x)\right)\cdot j_l'(x_{S'})\right\} + b_S \Leftrightarrow b_{S'}. \tag{E5}$$

Here again, the terms which contain only an integral over x_S can be calculated analytically when we neglect the decay of the transfer function. For the other terms we use the approximation (E1)

$$
\begin{aligned}
C_\ell^{(2)}(z_S, z_{S'}) = & \frac{-2A}{\pi(1-b_S)(1-b_{S'})}\bigg\{-8I_\ell^2(1-b_S)^2\int_\ell^\infty \frac{dx_S}{x_S^3}\frac{1}{1+\alpha_S x_S^4} + \frac{I_\ell^2}{\ell^2}\frac{1}{1+\alpha_S\ell^4}\bigg[(6-7b_S)(1-b_S) \\
& + \frac{4b_S(2-3b_S)}{3}\frac{\alpha_S\ell^4}{1+\alpha_S\ell^4}\bigg] - \frac{I_\ell}{\ell^2(2\ell+1)}\frac{b_S(2-3b_S)}{3}\frac{1}{1+\alpha_S\ell^4}\bigg[\ell(\ell-1)I_{\ell-1}\theta\Big(\ell-(\ell-1)\frac{1-b_S}{1-b_{S'}}\Big) \\
& - (\ell+1)^2 I_{\ell+1}\theta\Big(\ell-(\ell+1)\frac{1-b_S}{1-b_{S'}}\Big)\bigg] - \frac{I_\ell}{\ell^2(2\ell+1)}\frac{b_{S'}(2-3b_{S'})}{3}\frac{1}{1+\alpha_{S'}\ell^4} \\
& \times\bigg[\ell(\ell-1)I_{\ell-1}\theta\Big(\ell-(\ell-1)\frac{1-b_{S'}}{1-b_S}\Big) - (\ell+1)^2 I_{\ell+1}\theta\Big(\ell-(\ell+1)\frac{1-b_{S'}}{1-b_S}\Big)\bigg] \\
& - (4(1-b_S-b_{S'})+3b_Sb_{S'})\frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell(1-b_{S'})^\ell}{(2-b_S-b_{S'})^{2\ell}}F\Big(\ell,\ell+1;2\ell+2;\frac{4(1-b_S)(1-b_{S'})}{(2-b_S-b_{S'})^2}\Big) \\
& - \frac{b_S(2-3b_S)(2-b_{S'})}{12(2\ell+1)}\bigg[\frac{\sqrt{\pi}}{2}\frac{\Gamma(\ell+1)}{\Gamma(\ell+1/2)}\frac{(1-b_S)^{\ell-1}}{(1-b_{S'})^\ell}F\Big(\ell,-1/2;\ell+1/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\Big) \\
& - \frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell+2)}{\Gamma(\ell+5/2)}\frac{(1-b_S)^{\ell+1}}{(1-b_{S'})^{\ell+2}}F\Big(\ell+1,1/2;\ell+5/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\Big)\bigg] \\
& - \frac{b_{S'}(2-3b_{S'})(2-b_S)}{12(2\ell+1)}\bigg[\frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell}{(1-b_{S'})^{\ell+1}}F(\ell,1/2;\ell+3/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\Big) \\
& - \frac{\sqrt{\pi}}{2}\frac{\Gamma(\ell+2)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell}{(1-b_{S'})^{\ell+1}}F\Big(\ell+1,-1/2;\ell+3/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\Big)\bigg]\bigg\}. \qquad \text{(E6)}
\end{aligned}
$$

3. $C_\ell^{(3)}$

$$
\begin{aligned}
C_\ell^{(3)}(z_S, z_{S'}) = & \frac{2A}{\pi(1-b_S)(1-b_{S'})}\bigg\{\Big(2-\frac{9(b_S+b_{S'})}{2}+8b_Sb_{S'}\Big)\int_0^\infty\frac{dx_S}{x_S}T^2(x_S)j_\ell(x_S)j_\ell(x_{S'}) \\
& + \frac{b_S(2-3b_S)(2-3b_{S'})}{6(1-b_S)}\int_0^\infty dx_S T^2(x_S)j_l'(x_S)j_\ell(x_{S'}) + b_S\Leftrightarrow b_{S'} + \frac{b_Sb_{S'}(2-3b_S)(2-3b_{S'})}{36(1-b_S)^2} \\
& \times\int_0^\infty dx_S x_S T^2(x_S)j_l'(x_S)j_l'(x_{S'}) - 4(1-b_S)^2\int_0^\infty\frac{dx_S}{x_S^3}T^2(x_S)\Big(\int_0^{x_S}dx j_\ell(x)\Big)\Big(\int_0^{x_{S'}}dx j_\ell(x)\Big) \\
& + 3b_{S'}(1-b_S)\int_0^\infty\frac{dx_S}{x_S^2}T^2(x_S)\Big(\int_0^{x_S}dx j_\ell(x)\Big)\cdot j_\ell(x_{S'}) + b_S\Leftrightarrow b_{S'} - \frac{b_{S'}(2-3b_{S'})}{3} \\
& \times\int_0^\infty\frac{dx_S}{x_S}T^2(x_S)\Big(\int_0^{x_S}dx j_\ell(x)\Big)\cdot j_l'(x_{S'}) + b_S\Leftrightarrow b_{S'} + 2(1-b_S)^2\int_0^\infty\frac{dx_S}{x_S^3}T^2(x_S) \\
& \times\Big(\int_0^{x_S}dx\int_x^{x_S}dx'(x_S-x')j_\ell(x')\Big)\cdot\Big(\int_0^{x_{S'}}dx j_\ell(x)\Big) + b_S\Leftrightarrow b_{S'} - \frac{(2-b_{S'})(1-b_S)}{2}\int_0^\infty\frac{dx_S}{x_S^2}T^2(x_S) \\
& \times\Big(\int_0^{x_S}dx\int_x^{x_S}dx'(x_S-x')j_\ell(x')\Big)\cdot j_\ell(x_{S'}) + b_S\Leftrightarrow b_{S'}\bigg\}. \qquad \text{(E7)}
\end{aligned}
$$

Here, it is not possible to neglect the transfer function in the third integral, because for $z_S = z_{S'}$ the integral does not converge without $T^2(x_S)$. We therefore have to calculate the third (the Doppler term) term numerically:

$$
\begin{aligned}
C_\ell^{(3)}(z_S, z_{S'}) &= \frac{2A}{\pi(1-b_S)(1-b_{S'})}\Bigg\{-2I_\ell^2(1-b_S)^2\int_\ell^\infty \frac{dx_S}{x_S^3}\frac{1}{1+\alpha_S x_S^4}\bigg(2(1+\ell^2)+\frac{\ell x_S}{1-b_S}(b_S+b_{S'}-2)\bigg)\\
&+\frac{I_\ell^2}{\ell^2}\frac{1}{1+\alpha_S\ell^4}\bigg[3b_S(1-b_S)-\ell^2\bigg(1-\frac{b_S}{2}\bigg)(b_S-b_{S'})+b_S(b_S-2/3)\frac{4\alpha_S\ell^4}{1+\alpha_S\ell^4}\bigg]\\
&+\frac{I_\ell}{\ell^2(2\ell+1)}\frac{b_S(2-3b_S)}{3}\frac{1}{1+\alpha_S\ell^4}\bigg[\ell(\ell-1)I_{\ell-1}\theta\bigg(\ell-(\ell-1)\frac{1-b_S}{1-b_{S'}}\bigg)\\
&-(\ell+1)^2I_{\ell+1}\theta\bigg(\ell-(\ell+1)\frac{1-b_S}{1-b_{S'}}\bigg)\bigg]+\frac{I_\ell}{\ell^2(2\ell+1)}\frac{b_{S'}(2-3b_{S'})}{3}\frac{1}{1+\alpha_{S'}\ell^4}\\
&\times\bigg[\ell(\ell-1)I_{\ell-1}\theta\bigg(\ell-(\ell-1)\frac{1-b_{S'}}{1-b_S}\bigg)-(\ell+1)^2I_{\ell+1}\theta\bigg(\ell-(\ell+1)\frac{1-b_{S'}}{1-b_S}\bigg)\bigg]\\
&+\bigg(2-\frac{9}{2}(b_S+b_{S'})+8b_Sb_{S'}\bigg)\frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell(1-b_{S'})^\ell}{(2-b_S-b_{S'})^{2\ell}}\\
&\times F\bigg(\ell,\ell+1;2\ell+2;\frac{4(1-b_S)(1-b_{S'})}{(2-b_S-b_{S'})^2}\bigg)+\frac{b_S(2-3b_S)(2-3b_{S'})}{6(2\ell+1)}\bigg[\frac{\sqrt{\pi}}{2}\frac{\Gamma(\ell+1)}{\Gamma(\ell+1/2)}\frac{(1-b_S)^{\ell-1}}{(1-b_{S'})^\ell}\\
&\times F\bigg(\ell,-1/2;\ell+1/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\bigg)-\frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell+2)}{\Gamma(\ell+5/2)}\frac{(1-b_S)^{\ell+1}}{(1-b_{S'})^{\ell+2}}F\bigg(\ell+1,1/2;\ell+5/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\bigg)\bigg]\\
&+\frac{b_{S'}(2-3b_{S'})(2-3b_S)}{6(2\ell+1)}\bigg[\frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell}{(1-b_{S'})^{\ell+1}}F\bigg(\ell,1/2;\ell+3/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\bigg)\\
&-\frac{\sqrt{\pi}}{2}\frac{\Gamma(\ell+2)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell}{(1-b_{S'})^{\ell+1}}F\bigg(\ell+1,-1/2;\ell+3/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\bigg)\bigg]+\frac{b_Sb_{S'}(2-3b_S)(2-3b_{S'})}{36(1-b_S)^2}\\
&\times\int_0^\infty dx_S x_S T^2(x_S)j_\ell'(x_S)j_\ell'(x_{S'})\Bigg\}. \qquad \text{(E8)}
\end{aligned}
$$

The last term in this sum is determined by numerical integration over x_S.

4. $C_\ell^{(4)}$

$$
\begin{aligned}
C_\ell^{(4)}(z_S, z_{S'}) &= \frac{-2A}{\pi(1-b_S)(1-b_{S'})}\Bigg\{(2-3b_S)(1-b_{S'})\int_0^\infty\frac{dx_S}{x_S}T^2(x_S)j_\ell(x_S)j_\ell(x_{S'})\\
&+\frac{b_S(2-3b_S)(1-b_{S'})}{6(1-b_S)}\int_0^\infty dx_S T^2(x_S)j_l'(x_S)j_\ell(x_{S'})+4(1-b_S)^2\int_0^\infty\frac{dx_S}{x_S^3}T^2(x_S)\bigg(\int_0^{x_S}dx j_\ell(x)\bigg)\\
&\times\bigg(\int_0^{x_{S'}}dx j_\ell(x)\bigg)-2(3-4b_{S'})(1-b_S)\int_0^\infty\frac{dx_S}{x_S^2}T^2(x_S)\bigg(\int_0^{x_S}dx j_\ell(x)\bigg)\cdot j_\ell(x_{S'})\\
&-\frac{b_{S'}(2-3b_{S'})}{3}\int_0^\infty\frac{dx_S}{x_S}T^2(x_S)\bigg(\int_0^{x_S}dx j_\ell(x)\bigg)\cdot j_l'(x_{S'})-2(1-b_S)^2\int_0^\infty\frac{dx_S}{x_S^3}T^2(x_S)\bigg(\int_0^{x_S}dx j_\ell(x)\bigg)\\
&\cdot\bigg(\int_0^{x_{S'}}dx\int_x^{x_{S'}}dx'(x_{S'}-x')j_\ell(x')\bigg)+(2-3b_S)(1-b_S)\int_0^\infty\frac{dx_S}{x_S^2}T^2(x_S)\bigg(\int_0^{x_{S'}}dx\int_x^{x_{S'}}dx'(x_{S'}-x')j_\ell(x')\bigg)\\
&\cdot j_\ell(x_S)+\frac{b_S(2-3b_S)}{6}\int_0^\infty\frac{dx_S}{x_S}T^2(x_S)\bigg(\int_0^{x_{S'}}dx\int_x^{x_{S'}}dx'(x_{S'}-x')j_\ell(x')\bigg)\cdot j_l'(x_S)\Bigg\}+b_S\Leftrightarrow b_{S'}. \qquad \text{(E9)}
\end{aligned}
$$

$$C_\ell^{(4)}(z_S, z_{S'}) = \frac{-2A}{\pi(1-b_S)(1-b_{S'})}\bigg\{2I_\ell^2(1-b_S)^2\int_\ell^\infty \frac{dx_S}{x_S^3}\frac{1}{1+\alpha_S x_S^4}\bigg(2(2+\ell^2)+\frac{\ell x_S}{1-b_S}(b_S+b_{S'}-2)\bigg)$$
$$+\frac{I_\ell^2}{\ell^2}\frac{1}{1+\alpha_S\ell^4}\bigg[-2(1-b_S)(3-4b_S)+\ell^2(2-3b_S)(b_S-b_{S'})+b_S(b_S-2/3)\frac{4\alpha_S\ell^4}{1+\alpha_S\ell^4}\bigg]$$
$$+\frac{I_\ell}{\ell^2(2\ell+1)}\frac{b_S(2-3b_S)}{3}\frac{1}{1+\alpha_S\ell^4}\bigg[\ell(\ell-1)I_{\ell-1}\theta\bigg(\ell-(\ell-1)\frac{1-b_S}{1-b_{S'}}\bigg)$$
$$-(\ell+1)^2I_{\ell+1}\theta\bigg(\ell-(\ell+1)\frac{1-b_S}{1-b_{S'}}\bigg)\bigg]+\frac{I_\ell}{\ell^2(2\ell+1)}\frac{b_{S'}(2-3b_{S'})}{3}\frac{1}{1+\alpha_{S'}\ell^4}$$
$$\times\bigg[\ell(\ell-1)I_{\ell-1}\theta\bigg(\ell-(\ell-1)\frac{1-b_{S'}}{1-b_S}\bigg)-(\ell+1)^2I_{\ell+1}\theta\bigg(\ell-(\ell+1)\frac{1-b_{S'}}{1-b_S}\bigg)\bigg]$$
$$+\frac{I_\ell\ell}{2\ell+1}\frac{b_S(1-b_S)^3(2-3b_S)}{6}\bigg[\frac{\ell I_{\ell-1}}{\ell-1}\frac{b_{S'}-1+\ell(b_S-b_{S'})}{(1-b_S)^4+(\ell-1)^4b_{\rm eq}^4}\theta\bigg(\ell-1-\ell\frac{1-b_S}{1-b_{S'}}\bigg)$$
$$-I_{\ell+1}\frac{1-b_{S'}+\ell(b_S-b_{S'})}{(1-b_S)^4+(\ell+1)^4b_{\rm eq}^4}\theta\bigg(\ell+1-\ell\frac{1-b_S}{1-b_{S'}}\bigg)\bigg]+b_S\Leftrightarrow b_{S'}+(4-5(b_S+b_{S'})+6b_Sb_{S'})$$
$$\times\frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell(1-b_{S'})^\ell}{(2-b_S-b_{S'})^{2\ell}}F\bigg(\ell,\ell+1;2\ell+2;\frac{4(1-b_S)(1-b_{S'})}{(2-b_S-b_{S'})^2}\bigg)+\frac{b_S(2-3b_S)(1-b_{S'})}{6(2\ell+1)}$$
$$\times\bigg[\frac{\sqrt{\pi}}{2}\frac{\Gamma(\ell+1)}{\Gamma(\ell+1/2)}\frac{(1-b_S)^{\ell-1}}{(1-b_{S'})^\ell}F\bigg(\ell,-1/2;\ell+1/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\bigg)$$
$$-\frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell+2)}{\Gamma(\ell+5/2)}\frac{(1-b_S)^{\ell+1}}{(1-b_{S'})^{\ell+2}}F\bigg(\ell+1,1/2;\ell+5/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\bigg)\bigg]+\frac{b_{S'}(2-3b_{S'})(1-b_S)}{6(2\ell+1)}$$
$$\times\bigg[\frac{\sqrt{\pi}}{4}\frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell}{(1-b_{S'})^{\ell+1}}F\bigg(\ell,1/2;\ell+3/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\bigg)$$
$$-\frac{\sqrt{\pi}}{2}\frac{\Gamma(\ell+2)}{\Gamma(\ell+3/2)}\frac{(1-b_S)^\ell}{(1-b_{S'})^{\ell+1}}F\bigg(\ell+1,-1/2;\ell+3/2;\frac{(1-b_S)^2}{(1-b_{S'})^2}\bigg)\bigg]\bigg\}. \tag{E10}$$

5. $C_\ell^{(5)}$

$$C_\ell^{(5)}(z_S,z_{S'}) = \frac{2A}{\pi(1-b_S)(1-b_{S'})}\bigg\{(1-b_S)(1-b_{S'})\int_0^\infty\frac{dx_S}{x_S}T^2(x_S)j_\ell(x_S)j_\ell(x_{S'})+4(1-b_S)^2\int_0^\infty\frac{dx_S}{x_S^3}T^2(x_S)$$
$$\times\bigg(\int_0^{x_S}dxj_\ell(x)\bigg)\bigg(\int_0^{x_{S'}}dxj_\ell(x)\bigg)-2(1-b_S)^2\int_0^\infty\frac{dx_S}{x_S^2}T^2(x_S)\bigg(\int_0^{x_{S'}}dxj_\ell(x)\bigg)\cdot j_\ell(x_S)+b_S\Leftrightarrow b_{S'}$$
$$-2(1-b_S)^2\int_0^\infty\frac{dx_S}{x_S^3}T^2(x_S)\bigg(\int_0^{x_{S'}}dxj_\ell(x)\bigg)\bigg(\int_0^{x_S}dx\int_x^{x_S}dx'(x_S-x')j_\ell(x')\bigg)+b_S\Leftrightarrow b_{S'}$$
$$+(1-b_S)(1-b_{S'})\int_0^\infty\frac{dx_S}{x_S^2}T^2(x_S)\bigg(\int_0^{x_S}dx\int_x^{x_S}dx'(x_S-x')j_\ell(x')\bigg)j_\ell(x_{S'})+b_S\Leftrightarrow b_{S'}$$
$$+(1-b_S)^2\int_0^\infty\frac{dx_S}{x_S^3}T^2(x_S)\bigg(\int_0^{x_S}dx\int_x^{x_S}dx'(x_S-x')j_\ell(x')\bigg)\bigg(\int_0^{x_{S'}}dx\int_x^{x_{S'}}dx'(x_{S'}-x')j_\ell(x')\bigg)\bigg\}. \tag{E11}$$

The first term is dominated on a large scale and we may thus set $T\equiv 1$ so that it can be integrated analytically. For the other terms we use again the approximation (E1) for the integrals dx or dx'. The biggest contribution then comes from the last term where we have to perform two double integrals $dxdx'$, which result in $I_\ell^2(2+\ell^2-\ell x_S)(2+\ell^2-\ell x_{S'})\propto \ell^3$. In this term, which becomes large for large ℓ or large x_S, we take into account the log correction to the transfer function for better accuracy. From the expression in Ref. [14] and our definitions we find:

$$T^2(x_S) = \frac{1}{1 + [(\alpha_S x_S^4)/(\ln^2(1 + \frac{7.8\cdot 10^{-4}}{1-b_S} x_S))]}. \tag{E12}$$

Using our approximation (E1), we obtain

$$\begin{aligned} C_\ell^{(5)}(z_S, z_{S'}) = \frac{2A}{\pi(1-b_S)(1-b_{S'})} \bigg\{ & I_\ell^2 (1-b_S)^2 \int_\ell^\infty \frac{dx_S}{x_S^3} \frac{1}{1 + [(\alpha_S x_S^4)/(\ln^2(1 + \frac{7.8\cdot 10^{-4}}{1-b_S} x_S))]} (2 + \ell^2 - \ell x_S)(2 + \ell^2 - \ell x_{S'}) \\ & - \frac{I_\ell^2}{\ell^2} \frac{1-b_S}{1 + \hat{\alpha}_S \ell^4} (2 - (2+\ell^2) b_S + \ell^2 b_{S'}) + \frac{\sqrt{\pi}}{4} \frac{\Gamma(\ell)}{\Gamma(\ell + 3/2)} \frac{(1-b_S)^{\ell+1}(1-b_{S'})^{\ell+1}}{(2 - b_S - b_{S'})^{2\ell}} \\ & \times F\bigg(\ell, \ell+1; 2\ell+2; \frac{4(1-b_S)(1-b_{S'})}{(2-b_S-b_{S'})^2}\bigg)\bigg\}, \end{aligned} \tag{E13}$$

where

$$\hat{\alpha}_S = \beta \left(\frac{b_{\rm eq}}{1-b_S}\right)^4 \frac{1}{\ln^2(1 + \frac{7.8\cdot 10^{-4}\ell}{1-b_S})}. \tag{E14}$$

The remaining integral represents by far the largest contribution to $C_\ell^{(5)}$. For sources with equal redshifts $z_S = z_{S'} = z$, the spectrum $C_\ell^{(5)}(z, z)$ grows until $\hat{\alpha}_S \ell^4 \sim 1$ and decays for larger ℓ. Neglecting the log correction we have $\alpha_S = (\beta^{1/4} \frac{b_{\rm eq}}{1-b_S})^4 \equiv \ell_{\rm max}^{-4}$. Hence $C_\ell^{(5)}$ grows roughly until $\ell_{\rm max}$ and decays afterwards. With $b_{\rm eq} = (\eta_{\rm eq}/\eta_O) \simeq 0.01$ we obtain

$$\ell_{\rm max} \simeq 760 \frac{\sqrt{z_S + 1} - 1}{\sqrt{1 + z_S}}.$$

For a crude order of magnitude estimate, we first neglect the log corrections. For $\ell \ll \ell_{\rm max}$ the integral is dominated by the region $x_S < \ell_{\rm max}$ and we may simply integrate until $x_S \simeq \ell_{\rm max}$, neglecting the x_S^4 decay of the transfer function. In the opposite region, if $\ell \gg \ell_{\rm max}$, we may neglect the 1 in the denominator of the integral. An interpolation between this two asymptotic regimes gives

$$C_\ell^{(5)}(z_S, z_S) \simeq \frac{2A I_\ell^2 \ell^2}{\pi} \begin{cases} \ln(\frac{\ell_{\rm max}}{\ell}) + \frac{1}{4} & \text{if } \ell < \ell_{\rm max} \\ \frac{1}{4}(\frac{\ell_{\rm max}}{\ell})^4 & \text{if } \ell > \ell_{\rm max}. \end{cases} \tag{E15}$$

Since $I_\ell^2 \propto 1/\ell$ we see that $\ell(\ell+1)C_\ell^{(5)}$ grows like ℓ^3 for small ℓ's and it decays like $1/\ell$ for large ℓ's. The broad maximum is reached roughly at $\ell_{\rm max} \simeq 760 \frac{\sqrt{1+z_S}-1}{\sqrt{z_S+1}} = 760(1-b_S)$ and is of the order of $(A/\pi)\ell_{\rm max}^3$. This approximation is, however, surprisingly bad. We therefore take into account the log in the transfer function by simply replacing α_S by $\hat{\alpha}_S$, where ℓ in the expression for $\hat{\alpha}_S$ denotes the lower boundary of the integral. The expression for $\ell_{\rm max}$ then becomes ℓ dependent,

$$\ell_{\rm max} \simeq \frac{\sqrt{\ln(1 + 7.8 \times 10^{-4}\ell/(1-b_S))}}{\beta^{1/4} b_{\rm eq}} (1 - b_S). \tag{E16}$$

For $\ell < 1.3 \times 10^3 (1 - b_S) \equiv \ell_S$ the log can be expanded and $\ell_{\rm max}/\ell$ behaves like $\ell^{-1/2}$ leading to a linear growth of $\ell(\ell+1)C_\ell^{(5)}$. Only above ℓ_S it levels off. For $z_S = 2$, the asymptotic regime, where $\ell(\ell+1)C_\ell^{(5)}$ decays like $1/\ell$ is actually only reached at $\ell \sim 2000$, where our approximations (and linear perturbation theory) no longer hold.

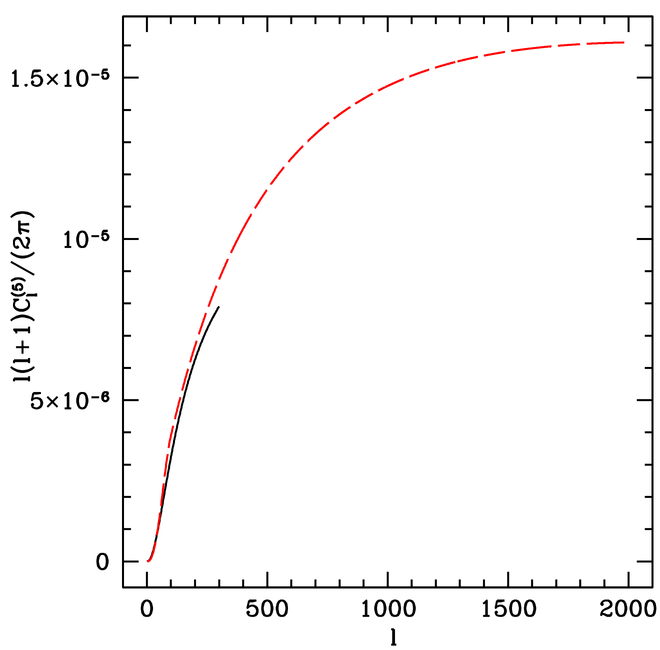

FIG. 15 (color online). The approximation for $\ell(\ell+1)C_\ell^{(5)}(z, z')/(2\pi)$ given in Eq. (E15) (dashed line) is compared with our numerical result (solid line) for $z = 2$.

In Fig. 15 we plot the approximation given in Eq. (E15) with $\ell_{\rm max}$ given in (E16) for $z_S = z_{S'} = 2$ and hence $\ell_S \simeq 540$. Actually, to have a better fit with the numerical integral we choose a slightly modified value, namely, $\tilde{\ell}_{\rm max} = 0.75\ell_{\rm max}$.

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[16] We must of course make sure that the above integral converges for large x_2. Otherwise the approximation is not meaningful.