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Non-Abelian Einstein-Born-Infeld-dilaton cosmology

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The non-Abelian Einstein-Born-Infeld-dilaton theory, which rules the dynamics of tensor-scalar gravitation coupled to a $su(2)$-valued gauge field ruled by Born-Infeld Lagrangian, is studied in a cosmological framework. The microscopic energy exchange between the gauge field and the dilaton which results from a nonuniversality of the coupling to gravity modifies the usual behavior of tensor-scalar theories coupled to matter fluids. General cosmological evolutions are derived for different couplings to gravitation and a comparison to universal coupling is highlighted. Evidences of cosmic acceleration are presented when the evolution is interpreted in the Jordan physical frame of a matter respecting the weak equivalence principle. The importance for the mechanism of cosmic acceleration of the dynamics of the Born-Infeld gauge field, the attraction role of the matter fluid, and the nonuniversality of the gravitational couplings are briefly outlined.

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I. INTRODUCTION

In the description of the very beginning of the universe, well before the big bang nucleosynthesis, field theoretical models are to be considered instead of the usual hydrodynamical description of matter. Those kinds of models, inspired by high-energy physics, have led to numerous progress in modern cosmology, trying to solve various problems from cosmic acceleration or the flatness problem to the magnitude of the cosmological constant or the existence of topological defects. The motivation of this paper lies in two questions, among many others, that raise from the description of the very first moments by high-energy physics.

First is the question whether large-scale massless gauge fields can play any interesting role in cosmology. Indeed, such fields could have existed in the early universe before the phase transitions of spontaneous symmetry breaking but, if they were ruled by usual Yang-Mills (YM) conformally invariant dynamics, their primeval excitations have probably been swept away by inflation. This point motivated some authors to study the impact of the Born-Infeld (BI)-type modification of gauge dynamics, suggested by string theory, on cosmology (see [1]). The BI Lagrangian breaks down the scale invariance of the gauge fields beyond some critical energy, and therefore it is not obvious to conclude directly on the becoming of such gauge fields during and after an inflation period. Furthermore, it was proved in [1] that such gauge fields of BI-type cannot provide any cosmic acceleration on their own although they can mimic a fluid of negative pressure. Before going further on this question, it is therefore of first importance to study more deeply in a cosmological context the interaction between gauge and scalar fields as suggested in models inspired by high-energy physics. The second question is to see what happens to a possible scalar sector of gravitation during the cosmological evolution. Indeed, string theories predict the existence of the dilaton [2], a Lorentz scalar partner to the tensor Einstein graviton as a low-energy limit of bosonic actions. This large theoretical framework provides a physical background for tensor-scalar modification [3] of general relativity, in which gravitation is mediated by a long-range scalar field acting in complement of the usual spin 2 gravity fields. Although this question has been widely studied when dilaton—or more generally tensor-scalar theories—is coupled to matter during radiation and matter dominated era, the case of a microscopic field model which would not be coupled universally to gravity, as suggested in string theory, has been less considered. In particular, how the interaction between scalar and gauge fields modifies their respective dynamics and the resulting cosmological evolution will be the main subject of the present paper.

But before going any further, let us locate the present work in the existing literature. In this paper, we will focus on cosmological solutions of the Einstein-Born-Infeld-dilaton (EBID) equations for flat spacetimes. Cosmologies with large-scale massless homogeneous and isotropic gauge fields with gauge group SU(2) and ruled by usual YM dynamics have been studied for a long time [4–8]. The gravitational instability of flat spacetimes filled with such gauge fields was studied in [9]. Generalization to higher gauge groups have also been studied [10,11] in the case of flat and closed cosmologies. The Einstein-Born-Infeld cosmology with non-Abelian gauge fields deriving from gauge group SU(2) has been studied thoroughly in [1] for flat, closed, and open spacetimes for any value of the cosmological constant. The minimal coupling of large-scale cosmological gauge fields and scalar multiplets has been studied in [12,13]. The Einstein-Yang-Mills-dilaton (EYMD) equations for flat cosmologies and a special case of nonuniversal coupling to gravity have been derived in [14,15] where the authors highlighted the energy exchange

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between the dilaton and the gauge fields and briefly discussed its effects on inflation, entropy crisis, and the Polonyi problem (domination of a nearly massless dilaton at late times). However, they did not propose a complete solution to the EYMD field equations in the cosmological framework which could allow to address completely these issues. Moreover, they did not discuss the general influence of nonuniversal coupling to gravity as well. The EYMD system was also studied in [16] in the case of closed Friedmann-Lemaitre-Robertson-Walker (FLRW) with a static gauge field[1] and vanishing dilaton potential and cosmological constant.

As we mentioned before, tensor-scalar cosmologies have been widely studied, with a large spectrum of applications for physical cosmology: inflation, primordial nucleosynthesis, cosmic microwave background, …. The question of the convergence of tensor-scalar theories to general relativity during the cosmological evolution has been widely studied in [17–20] and references therein. For the so-called "*Einstein conformal frame*," where the gravitational and scalar fields have pure spin 2 and spin 0 dynamics, respectively, the scalar sector of gravitation disappears naturally during cosmic expansion due to its coupling to matter. The cosmological evolution of the dilaton emerging from string theory has been studied in [21].

In this paper, we will consider the cosmological evolution of the dilaton coupled directly to a large-scale non-Abelian gauge field ruled by BI dynamics which go beyond the scale invariance of YM theory. A nonuniversal coupling to gravity, as suggested in preceding works, will lead to quite different results to the usual coupling of tensor-scalar theories to a fluid. For example, when the gauge fields are governed by YM scale-invariant dynamics, the scalar sector of gravitation remains directly coupled to the gauge fields although they mimic a radiation fluid. However, it is well known that tensor-scalar theories decouple from radiation (except during phase transition). Through both numerical computations and analytical solutions, we will show how the dilaton evolution is modified by nonuniversal coupling to the metric. This will lead to remarkable consequences for cosmology.

The structure of this paper will be as follows. In Sec. II, we establish the general field equations for the EBID cosmology. In Sec. III, we first remind the reader about BI cosmology as was studied in [1]. The non-Abelian BI cosmology can be split in two extreme regimes depending on the energy density of the gauge field compared with a critical scale introduced in the BI theory. For large energies, the gauge field is shown to mimic a fluid of negative pressure with $p/\rho = -1/3$ while in the low-energy limit the scale-invariant YM dynamics is retrieved and the gauge field looks like a radiation fluid. On the other hand, we also remind the reader about dilaton cosmology, studied in [17,21]. As the equations are to be established and solved with pure spin degrees of freedom in the Einstein frame, the interpretation in terms of the Jordan physical frame is also recalled. The case of universal coupling is solved there with usual properties of tensor-scalar theories. In Sec. IV, we will focus on the strong-field limit of the EBID system with general coupling to gravity and in Sec. V the low-energy limit which consists of a generalized version of the EYMD system appearing in [14,15] is treated. In Sec. VI, we analyze a complete cosmological evolution of the EBID system and present evidences for possible cosmic acceleration in the physical Jordan frame. This frame is defined with respect to a pressureless matter fluid that has been added to the gauge sector. The acceleration is shown to resist to the attraction provided by matter and appears to be intrinsically related to the nonuniversality of the coupling to gravity. Finally, we conclude in Sec. VII by some perspectives to the present work.

II. FIELD EQUATIONS OF EINSTEIN-BORN-INFELD-DILATON COSMOLOGY

Most of the interest of field models, for example, deriving from string theory in the low-energy limit, comes from their nonuniversal coupling to the gravity fields $g_{\mu\nu}$ and its scalar counterpart ϕ (each type of matter field has in general its own coupling function to the dilaton, see [21]). This results in a violation of the weak equivalence principle and therefore the gravitational interaction of these microscopic field models is different from a usual tensor-scalar theory where the weak equivalence principle is usually assumed. Without imposing such a violation of the weak equivalence principle at a microscopic scale, field models would not be different than considering a tensor-scalar theory in the presence of a fluid with the equation of state of the considered fields. Therefore, we will make use of a general form of the action for the non-Abelian Einstein-Born-Infeld-dilaton system, which takes into account a possible violation of the weak equivalence principle. This action writes down

$$S = \int\left\{-\frac{1}{2\kappa}R - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) - A^4(\phi)\mathcal{L}_{\rm BI}(B^2(\phi)g_{\mu\nu}, A_\mu)\right\}\sqrt{-g}d^4x + S_m[\psi_m, C^2(\phi)g_{\mu\nu}]. \quad (1)$$

In this action, the gravitational interaction is described by the scalar curvature R and the dilaton ϕ, κ being the "*bare*" gravitational coupling constant and $A(\phi)$, $B(\phi)$, and $C(\phi)$ being three different coupling functions of the dilaton to matter. The first two illustrate the coupling of the gauge sector to the volume form and to the Einstein metric

[1]Because of this particular topology, the EYMD system does not reduce to pure Einstein-dilaton field equations when the gauge field is static.

$g_{\mu\nu}$ inside the Born-Infeld Lagrangian $\mathcal{L}_{\rm BI}$ (where A_μ are the non-Abelian gauge potentials) and the last coupling function $C(\phi)$ is related to another type of matter ruled by the action S_m. Other parametrizations of Einstein-Born-Infeld-dilaton action were considered in the literature, for example, with $A = 1$ and $B = \exp(k/2\phi)$ in [14–16,22] and with $B = A^2 = \exp(k/2\phi)$ in [23]. The non-Abelian gauge interaction is described by the Born-Infeld Lagrangian built upon the field strength tensor

$$F_{\mu\nu} = F^{\mathbf{a}}_{\mu\nu} T_{\mathbf{a}}$$

(where $T_{\mathbf{a}}$ are the generators of the gauge group under consideration[2]) and its dual tensor $\tilde{F}_{\mu\nu}$. Indeed, this Lagrangian, denoted by $\mathcal{L}_{\rm BI}$ in (1), is defined as

$$\mathcal{L}_{\rm BI} = \epsilon_c(\mathcal{R} - 1) = \epsilon_c\left(\sqrt{1 + \frac{B^{-4}(\phi)}{2\epsilon_c} F_{\mu\nu}F^{\mu\nu} - \frac{B^{-8}(\phi)}{16\epsilon_c^2}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2} - 1\right), \tag{2}$$

where ϵ_c is the Born-Infeld critical energy and $B(\phi)$ is the dilaton coupling function which is equal to $e^{k/2\phi}$ when nonperturbative effects are not taken into account (see [14,15,21]). In this case, k will be called the dilaton coupling constant. Throughout this paper, we will assume the Planck system of units, in which $\hbar = c = 1$ and $G = m_{\rm Pl}^{-2}$, with the Planck mass $m_{\rm Pl} = 1.2211 \times 10^{19}$ GeV and the gravitational coupling constant is $\kappa = 8\pi G$. We have also set the gauge coupling constant to unity, as it actually defines a system of units for the dilaton field ϕ, provided the dilaton is massless ($V = 0$). The Born-Infeld critical energy ϵ_c defines the scale above which nonlocal effects of string theory arise and where the scale invariance of the gauge fields is broken. For example, in the low-energy limit $\epsilon_c \to \infty$ of the Born-Infeld part of the action (1), we recover the usual, conformally invariant, Yang-Mills (YM) Lagrangian density for the non-Abelian gauge field:

$$\sqrt{-g}\mathcal{L}_{\rm YM} = -\sqrt{-g}A^4(\phi)B^{-4}(\phi)\tfrac{1}{4}F^{\mathbf{a}}_{\mu\nu}F^{\mu\nu}_{\mathbf{a}}. \tag{3}$$

In this paper, we will focus on cosmology and therefore we will adopt the prescriptions of the cosmological principle which states that the spatial sections of our Universe are homogeneous and isotropic. For the sake of simplicity, we will also restrict ourselves to the case of flat spacetimes, which constitutes however a very nice approximation of the present universe and its early stages as well. The metric describing such spacetimes is the one of FLRW:

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2), \tag{4}$$

where $a(t)$ is the scale factor and N is the so-called *lapse function* of the Hamiltonian Arnowitt-Deser-Misner approach to general relativity. This function can be fixed by a specific choice of time coordinate (gravitational gauge freedom). The symmetries prescribed by the cosmological principle we assume impose that the dilaton scalar field ϕ depends only on time. A remarkable fact about non-Abelian gauge fields is that they admit nontrivial homogeneous and isotropic configurations at the opposite of their Abelian U(1) counterparts [see [1,5] and references therein for a complete discussion of the SU(2) case]. The main reason for that is because only the gauge invariant quantities such as the field strength tensor have to exhibit the symmetries explicitly, while the gauge potentials can be symmetric up to a gauge transformation (see [4,24] for more general gauge groups). As a result, the energy can be distributed amongst the different gauge degrees of freedom while the stress-energy tensor remains compatible with the maximal symmetry of the spacetime background. In this paper, we will restrict ourselves to the case of $su(2)$-valued gauge potentials, for which the ansatz

$$\mathbf{A} = A^{\mathbf{a}}_\mu T_{\mathbf{a}} dx^\mu = \sigma(t) T_{\mathbf{m}} dx^{\mathbf{m}} \tag{5}$$

of the connection one-form $\mathbf{A}$ makes the gauge invariant quantities satisfying the required symmetry (see [4,5]). The remaining dynamical degrees of freedom of the gauge potential are now expressed by the field $\sigma(t)$. However, our results will not depend on this particular choice as the ansatz above can be generalized to higher gauge groups (see [10,11]). In the equation above, the generators $T_{\mathbf{m}}$ of the Lie algebra of the gauge group SU(2) are to be taken in the coordinate dependent basis of the gauge degrees of freedom space as follows:

$$T_{\mathbf{r}} = \sin\theta\cos\varphi T_{\mathbf{1}} + \sin\theta\sin\varphi T_{\mathbf{2}} + \cos\theta T_{\mathbf{3}}$$

$$T_\theta = \frac{\partial}{\partial\theta}T_{\mathbf{r}} \qquad T_\varphi = \frac{\partial}{\sin\theta\partial\varphi}T_{\mathbf{r}},$$

with $T_{\mathbf{i}} = \frac{1}{2}\sigma_i$ the usual basis of the Lie algebra $su(2)$ (σ_i being the Pauli matrices) with the following standard normalization conditions and commutation relations:

$$tr(T_{\mathbf{a}}T_{\mathbf{b}}) = \tfrac{1}{2}\delta_{\mathbf{ab}}, \qquad [T_{\mathbf{a}}, T_{\mathbf{b}}] = i\epsilon^{\mathbf{c}}_{\mathbf{ab}}T_{\mathbf{c}}.$$

The ansatz (5) is of course independent of the particular choice of the action for the gauge field, as was shown in [1].

The symmetries implied by the cosmological principle therefore allow us to write down (1) as an effective one-dimensional action, after integrating over R^3 and dividing by the infinite volume of its orbits:

$$S_{\rm eff} = \int dt\left\{-\frac{3}{\kappa}\frac{\dot{a}^2 a}{N} + \frac{\dot{\phi}^2}{2}\frac{a^3}{N} - V(\phi)Na^3 - Na^3\epsilon_c A^4(\phi)(\mathcal{R} - 1)\right\} + S_m, \tag{6}$$

where a dot denotes a derivative with respect to the time t and where $\mathcal{R}$ is given by

[2]Gauge indices will be noted as bold Latin letters.

$$\mathcal{R} = \sqrt{1 - 3\frac{B^{-4}(\phi)}{\epsilon_c}\left(\frac{\dot{\sigma}^2}{a^2N^2} - \frac{\sigma^4}{a^4}\right) - 9\frac{B^{-8}(\phi)}{\epsilon_c^2 a^6 N^2}\dot{\sigma}^2\sigma^4}, \tag{7}$$

where $\sigma(t)$ is the gauge potential as defined in (5). Following [1], it is also convenient to write $\mathcal{R}$ as

$$\mathcal{R} = \sqrt{1-\Gamma}\sqrt{1+\Delta} \tag{8}$$

with

$$\Gamma = \frac{3\dot{\sigma}^2 B^{-4}(\phi)}{\epsilon_c a^2 N^2}, \qquad \Delta = \frac{3\sigma^4 B^{-4}(\phi)}{\epsilon_c a^4}. \tag{9}$$

From the action (6) and relation (7), it is straightforward to write down the field equations for the Einstein-Born-Infeld-dilaton system by varying this action over the following degrees of freedom: N, a, ϕ and σ. First, the Euler-Lagrange equation for the variable N gives the Hamiltonian constraint

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\left[\frac{\dot{\phi}^2}{2} + V(\phi) + \epsilon_c A^4(\phi)(\mathcal{P}-1) + \rho_m\right], \tag{10}$$

which we will refer to as the *Friedmann* equation. In the previous equation, ρ_m stands for the energy density of the matter fluid ruled by S_m and the function $\mathcal{P}$ is defined in terms of Γ and Δ in (9) as

$$\mathcal{P} = \sqrt{\frac{1+\Delta}{1-\Gamma}}. \tag{11}$$

The careful reader should have noticed that, after varying over N, we set the gravitational gauge to $N = 1$, meaning that we work with the synchronous time coordinate (another convenient choice for the study of the gauge dynamics in the Yang-Mills regime is the conformal gauge $N = a$ as it naturally exhibits the conformal invariance). The Friedmann equation allows us to define the Born-Infeld effective energy density of the gauge field as a generalization of what was proposed in [1]

$$\rho_{\rm BI} = \epsilon_c A^4(\phi)(\mathcal{P}-1). \tag{12}$$

The Euler-Lagrange equation for the scale factor a gives the acceleration equation:

$$\frac{\ddot{a}}{a} = \frac{\kappa}{3}\left[(V(\phi) - \dot{\phi}^2) + \epsilon_c A^4(\phi)(\mathcal{P}^{-1}-1) - \frac{1}{2}(\rho_m + 3p_m)\right], \tag{13}$$

where p_m stands for the pressure of the additional matter fluid. This allow us to define the Born-Infeld effective pressure

$$p_{\rm BI} = \frac{\epsilon_c}{3}A^4(\phi)(3 - \mathcal{P} - 2\mathcal{P}^{-1}) \tag{14}$$

and the equation of state

$$\lambda_{\rm BI} = \frac{p_{\rm BI}}{\rho_{\rm BI}} = \frac{1}{3}\left(\frac{\epsilon_c A^4(\phi) - \rho_{\rm BI}}{\epsilon_c A^4(\phi) + \rho_{\rm BI}}\right) \tag{15}$$

for the gauge part of the EBID system as in [1]. Here we see that the $su(2)$-valued gauge fields ruled by Born-Infeld Lagrangian can be represented by a fluid with an equation of state that varies continuously from $-\frac{1}{3}$, when the BI energy density is much larger than the "critical field" $A^{-4}(\phi)\rho_{\rm BI} \gg \epsilon_c$, to $\frac{1}{3}$ at low energies $A^{-4}(\phi)\rho_{\rm BI} \ll \epsilon_c$. These two extreme regimes correspond to a gas of Nambu-Goto strings in three spatial dimensions on one hand (strong-field limit) and radiations on the other (weak-field limit). The transition between these regimes occurs at vanishing pressure when the BI energy density is of order of the BI critical energy scale ϵ_c. At low energies ($\epsilon_c \to \infty$), the gauge field behaves like radiation as expected because the BI Lagrangian reduces to the conformally invariant Yang-Mills one.

It is also important to notice that there is no cosmic acceleration with the metric $g_{\mu\nu}$ as long as the dilaton is massless. Indeed, the highest value of $\ddot{a}$ that can be achieved in this frame is identically zero [see (13)], in the limit of the pure Einstein-Born-Infeld system at high energies ($\dot{\phi} = 0$, $\lambda_{BI} = -\frac{1}{3}$). However, we will see that cosmic acceleration may appear once we examine the behavior in another frame.

By varying the action (6) with respect to the dilaton ϕ, we find the Klein-Gordon equation:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{dV(\phi)}{d\phi} - 2\epsilon_c A^4(\phi)[\beta(\phi)(\mathcal{P} + \mathcal{P}^{-1} - 2\mathcal{R}) + \alpha(\phi)(2\mathcal{R} - 2)] - \gamma(\phi)(\rho_m - 3p_m), \tag{16}$$

where $\alpha(\phi) = \frac{d\ln A(\phi)}{d\phi}$, $\beta(\phi) = \frac{d\ln B(\phi)}{d\phi}$, and $\gamma(\phi) = \frac{d\ln C(\phi)}{d\phi}$. The key point of the physics in the EBID system lies in the fact that the dilaton field couples differently to the gauge sector of the theory depending on the values of the coupling functions A and B. Although the cosmological dynamics of the gauge fields ruled by Born-Infeld Lagrangian can be regarded as a fluid with an equation of state given by (15), the coupled dynamics of the dilaton and the gauge field does not reduce in general to a scalar-tensor theory with this fluid as background. This is only the case when we have a universal coupling to the metric $g_{\mu\nu}$, i.e. when $A = B$. In the general case, there exists a nontrivial energy exchange between the dilaton and the gauge sectors of the theory that will dominate at late epochs as we shall see further. Finally, the Euler-Lagrange equation for the gauge field σ gives the Born-Infeld equation that rules the gauge potential dynamics:

$$\ddot{\sigma} + 2\frac{\sigma^3}{a^2}\mathcal{P}^{-2} - 2\frac{\dot{a}}{a}\dot{\sigma}\left(\frac{1}{2} - \mathcal{P}^{-2}\right) + 2\dot{\phi}\,\dot{\sigma}\left[2\alpha(\phi)\frac{\mathcal{R}}{\mathcal{P}} - \beta(\phi)\left(2\frac{\mathcal{R}}{\mathcal{P}} + 1 - \mathcal{P}^{-2}\right)\right] = 0\cdot \tag{17}$$

This equation is essentially the same as that in [1] except from the coupling term proportional to $\dot{\phi}$ which accounts for the direct energy exchange at the microscopic level between the fields. It is important to notice that we did not assume any direct coupling between the gauge field and the additional matter fluid, which will allow one to treat them separately. Indeed, from Eq. (17), and following [1], it is possible to derive an energy conservation equation for the BI density :

$$\dot{\rho_{\rm BI}} = -2\frac{\dot{a}}{a}\rho_{\rm BI}\frac{\rho_{\rm BI} + 2\epsilon_c A^4(\phi)}{\rho_{\rm BI} + \epsilon_c A^4(\phi)} + 4\alpha(\phi)\dot{\phi}\epsilon_c A^4(\phi)(\mathcal{R} - 1) - 2\beta(\phi)\dot{\phi}\epsilon_c A^4(\phi)\left(\frac{\Delta + \Gamma}{\mathcal{R}} - 2\mathcal{P} + 2\mathcal{R}\right)\cdot \tag{18}$$

Now that we have derived the complete set of the EBID field equations for cosmology, we propose the reader to briefly review some basic features of Born-Infeld cosmology on one hand and dilaton cosmology on the other. In the rest of this paper, we will only consider a massless dilaton [i.e., vanishing self-interaction potential $V(\phi) = 0$].

III. BORN-INFELD AND DILATON COSMOLOGIES

A. Non-Abelian Born-Infeld cosmology

The Non-Abelian Born-Infeld cosmology in various spacetimes with different values of the curvature and the cosmological constant was described in detail in [1]. The field equations governing these models are those of the previous section with a vanishing dilaton $\phi = \dot{\phi} = 0$, constant coupling functions $A = B = 1$, and no additional matter fluid $\rho_m = 0$. The equation (18) for BI energy conservation can be written as

$$\dot{\rho_{\rm BI}} = -2\frac{\dot{a}}{a}\rho_{\rm BI}\frac{\rho_{\rm BI} + 2\epsilon_c}{\rho_{\rm BI} + \epsilon_c} \tag{19}$$

which admits a first integral:

$$a^4\rho_{\rm BI}(\rho_{\rm BI} + 2\epsilon_c) = C, \tag{20}$$

where C is a positive constant. In the strong-field limit, $\rho_{\rm BI} \gg \epsilon_c$, the BI energy density redshifts as $\rho_{\rm BI} \approx a^{-2}$ while in the weak limit, $\rho_{\rm BI} \ll \epsilon_c$, we retrieve the radiation behavior $\rho_{\rm BI} \approx a^{-4}$ characteristic of the conformal invariance of the gauge field at such energies. This allows one to treat separately the spacetime evolution and the dynamics of the gauge field. Although the complete analytical solutions for both gravitational and gauge sector were derived in [1], let us illustrate simply the main features of this cosmological model. First, the strong-field limit $\rho_{\rm BI} \gg \epsilon_c$ corresponds to $\mathcal{P} \gg 1$. In this limit, the acceleration Eq. (13) reduces to

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{3}\epsilon_c$$

whose general solution is

$$a(t) = a^* \sin\left(\sqrt{\frac{\kappa\epsilon_c}{3}}t\right)$$

where we set $a(0) = 0$ and a^* is the value of the scale factor at the time $t^* = \sqrt{3/(\kappa\epsilon_c)}\frac{\pi}{2}$ (in Planck units). Therefore, the cosmic expansion starts with a zero acceleration at the singularity. Then, setting $\mathcal{P} \gg 1$ in Eq. (17) brings

$$\ddot{\sigma} - \frac{\dot{a}}{a}\dot{\sigma} = 0$$

which shows that $\dot{\sigma}$ scales as a. Near the singularity, the behavior of the gauge field is therefore

$$\sigma(t) = \mp\frac{\sqrt{3\epsilon_c}}{3}a^* \cos\left(\sqrt{\frac{\kappa\epsilon_c}{3}}t\right),$$

and the gauge potential σ starts at rest.

Then, in the weak-field regime, $\rho_{\rm BI} \ll \epsilon_c$ and $\mathcal{P} \approx 1$ ($\epsilon_c \to \infty$). This limit corresponds to the Einstein-Yang-Mills cosmological solution studied in [5]. The conformal invariance of the gauge field in that regime yields that the scale factor behaves like in the radiation-dominated era: $a(t) \approx \sqrt{t}$ (in synchronous time). On the other hand, the energy conservation equation (18) now reduces to

$$a^2\dot{\sigma}^2 + \sigma^4 = \frac{C}{3\epsilon_c}$$

which can be integrated in terms of the Jacobi elliptic function. Moving to the conformal time coordinate $dt = a d\eta$, we find

$$\sigma(\eta) = \mathcal{E}^{1/4} cn(\mathcal{E}^{1/4}\eta; -1),$$

where $cn(u, k)$ is the Jacobi elliptic function and $\mathcal{E} = C/(3\epsilon_c)$. In synchronous time, the gauge potential σ oscillates with a fixed amplitude and a growing period. More generally, it is possible to derive a general solution for the gauge potential. Let us rewrite the first integral (20) in terms of $\mathcal{P}$ as

$$\mathcal{P} = \sqrt{1 + \frac{C}{\epsilon_c^2 a^4}}. \tag{21}$$

Using the definitions (11) and (9), the previous equation may be integrated to give the gauge potential (in the conformal gauge $dt = a d\eta$):

$$\sigma(\eta) = a_0^2\sqrt{\epsilon_c}cn(\mathcal{P}^{-1}\eta; -1)\cdot$$

Figure 1 illustrates the evolution of the scale factor, the gauge potential, and the equation of state during the expansion of a non-Abelian Born-Infeld universe. The figures correspond to the numerical integration of Eqs. (13) and (17) with $\phi = \dot{\phi} = 0$ and $A = B = 1$. During numerical evolution, we monitor the violation of the Hamiltonian constraint (10) (see the appendix for more details on integrating the EBID system). In this case, this violation does not exceed a part on 10^{-12}. Initial conditions at $t_i = 0$ were assumed such as $\rho_{\rm BI}(t_i) = 100\epsilon_c$, $a(t_i) = a_i = 1$, and $\dot{\sigma}(t_i) = 0$ ($\epsilon_c = 10^{-4} \times m_{\rm Pl}^4$).

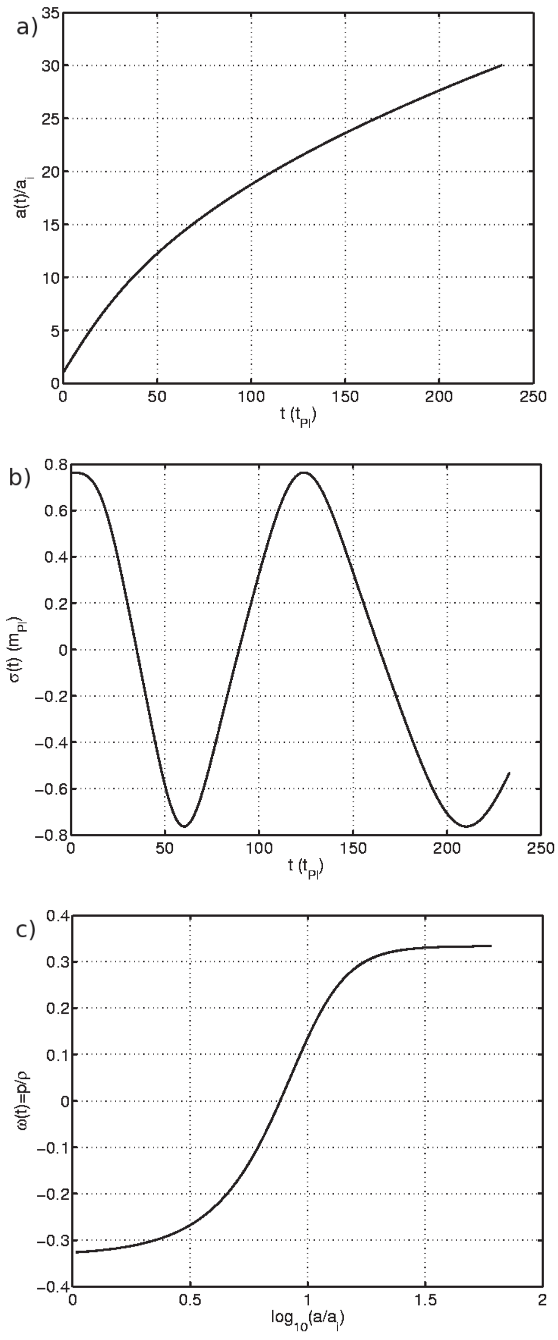

FIG. 1. Illustration of Non-Abelian Born-Infeld cosmology: (a) scale factor; (b) gauge potential; (c) equation of state of the Born-Infeld "*fluid*."

B. Dilaton cosmology

Dilaton cosmology can be retrieved from our fundamental equations by setting σ equal to 0. Indeed, tensor-scalar theories can be written in the so-called "*Einstein*" conformal frame:

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g_*}\{R_* - 2g_*^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\} + S_m[\psi_m, C^2(\varphi)g^*_{\mu\nu}], \tag{22}$$

where κ is therefore the "*bare*" gravitational coupling constant, S_m is the action for the matter fields ψ_m, $\varphi = \sqrt{\kappa/2}\phi$, and $g^*_{\mu\nu}$ the "*Einstein*" metric tensor which corresponds to basic gravitational variables with pure spin 2 propagation modes. This metric is measured by using purely gravitational rods and clocks and allows one to account for the dynamics in a simpler way[3] than an observable frame in which the metric tensor $\tilde{g}_{\mu\nu}$ is universally coupled to matter fields ψ_m. This frame is called the "*Jordan-Fierz*" frame in which the action (22) can be written

$$S = \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\left\{\Phi\tilde{R} - \frac{\omega(\Phi)}{\Phi}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\right\} + S_m[\psi_m, \tilde{g}_{\mu\nu}], \tag{23}$$

where $\tilde{R}$ is the curvature scalar built upon the physical metric $\tilde{g}_{\mu\nu}$ which is measured using nongravitational rods and clocks and where $\omega(\Phi)$ is called the coupling function. The scalar field Φ now gives the effective gravitational coupling constant. In this frame, matter fields evolve in the same way that they could do in general relativity because the action of matter does not depend explicitly on the scalar field ϕ, as matter couples universally to the physical metric $\tilde{g}_{\mu\nu}$. Einstein and Jordan frames can be linked together through the conformal transformation

$$\tilde{g}_{\mu\nu} = C^2(\varphi)g^*_{\mu\nu} \tag{24}$$

and the following relation between the scalar field ϕ in the Einstein frame (pure spin 0 dynamics) and its counterpart Φ in the Jordan physical frame:

$$\Phi^{-1} = \kappa C^2(\varphi), \tag{25}$$

with $\varphi = \sqrt{\kappa/2}\phi$. The energy density and pressure of the matter fluid in both frames are related by

[3]In particular, this frame represents gravitation with its pure scalar and tensor degrees of freedom and the limit of general relativity is not singular.

$$\rho_m = C^4 \tilde{\rho}_m \tag{26}$$

$$p_m = C^4 \tilde{p}_m, \tag{27}$$

where ρ_m and p_m represent these quantities expressed in the Einstein frame. At the opposite of scalar-tensor theories, we did not assume here an universal coupling to the metric tensor $g_{\mu\nu}$. Indeed, the action (1) reduces to a tensor-scalar theory in the Einstein frame (22) only when $A(\phi) = B(\phi) = C(\phi)$, i.e. when the weak equivalence principle applies. Let us now remind the reader about the behavior of such tensor-scalar theories in the presence of a background cosmological fluid. When the matter fields ψ_m are represented under the approximation of a perfect fluid, the dynamics of the scalar field is ruled by (see [17–20])

$$\frac{2}{(3-\varphi'^2)}\varphi'' + (1-\lambda)\varphi' + (1-3\lambda)\gamma(\varphi) = 0, \tag{28}$$

where $\gamma(\varphi) = \frac{d \ln C(\varphi)}{d\varphi}$ and $\lambda = \rho/p$ is the equation of state for the cosmological fluid. In the previous equation, a prime denotes the derivative with respect to the variable $p = \ln(a/a_i)$. The action of the cosmological fluid is thus to damp the dynamics of the scalar field while it is rolling down some effective potential depending on the coupling function. Furthermore, the scalar field now has an effective, velocity-dependent, mass of

$$m(\varphi) = \frac{2}{(3-\varphi'^2)}, \tag{29}$$

where the field has a limiting speed $\varphi' \leq \sqrt{3}$ for which its effective mass diverges. This relativistic limit corresponds to the case where the energy density of the background fluid is negligible compared to the kinetic energy of the scalar field (the universe is dominated by the kinetic energy of the scalar field).

C. The universal coupling for EBID cosmology

Let us now turn back to the EBID system we wrote in the previous section and focus on the gauge sector only by setting $S_m = 0$. If we now assume a universal coupling to the metric $g_{\mu\nu}$ by setting $A = B$, we now have for the dilaton equation (16)

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \alpha(\phi)(\rho_{\rm BI} - 3p_{\rm BI}) = 0, \tag{30}$$

where

$$\rho_{\rm BI} - 3p_{\rm BI} = 2\epsilon_c A^4(\phi)(\mathcal{P} + \mathcal{P}^{-1} - 2).$$

In this case of universal coupling, we recognize the equation for the scalar field in the presence of a background fluid for general tensor-scalar theories.[4] The equations (10), (13), and (16) can be simply solved in the Jordan frame by using the results on the non-Abelian Born-Infeld cosmology (previous paragraph). The equation (17) for the gauge field in the Einstein frame can also be written

$$\ddot{\sigma} + 2\frac{\sigma^3}{a^2}\mathcal{P}^{-2} - 2\frac{\dot{a}}{a}\dot{\sigma}\left(\frac{1}{2} - \mathcal{P}^{-2}\right) - 2\alpha(\phi)\dot{\phi}\,\dot{\sigma}(1 - \mathcal{P}^{-2}) = 0\cdot \tag{31}$$

Therefore, in the weak energy regime $\rho_{\rm BI} \ll \epsilon_c$ ($\mathcal{P} \approx 1$), the gauge field undergoes a conformally invariant dynamics (non-Abelian radiation, $\rho_{\rm BI} = 3p_{\rm BI}$) and decouples from the scalar field.

In a radiation-dominated universe, where the equation of state is $\lambda = p/\rho = 1/3$, the dynamics of the scalar field is given by the following solution to (28) (cf. [17,18,20]):

$$\varphi(p) = \varphi_\infty - \sqrt{3}\ln[Ke^{-p} + \sqrt{1 + K^2 e^{-2p}}], \tag{32}$$

where the integration constant K is determined from the initial velocity $\varphi'(p=0) = \varphi'_0$:

$$K = \frac{\varphi'_0}{3 - \varphi'^2_0}\cdot$$

This should correspond to the low-energy limit of the Born-Infeld field equations when a universal coupling is assumed: the scalar field velocity in p-time should be damped to zero by the cosmological expansion. It should be noticed that, when there is no universal coupling $A \neq B$, we do keep an energy exchange between the dilaton and the gauge fields and the usual dynamics of tensor-scalar theories will be modified. Moreover, when there is no universal coupling, the energy exchange between the gauge potentials and the dilaton field will prevent the dynamics to be purely dictated by the solution (32). As we shall see in Sec. V, this solution will accurately describe the early epochs of evolution when the field is almost relativistic. However, at late times, energy transfer between dilaton and gauge fields will substantially alter the dynamics.

Once again, let us assume a universal coupling and consider the strong-field limit of the Born-Infeld system where we have $\lambda = -1/3$. The dilaton equation (28) now becomes

$$\frac{\varphi''}{3 - \varphi'^2} + \frac{2}{3}\varphi' + \alpha(\varphi) = 0\cdot \tag{33}$$

Let us now write down for the dilaton coupling function

$$A^2(\phi) = e^{k\phi} \tag{34}$$

$$\omega(\Phi) = \frac{2\kappa - 3k^2}{2k^2} \tag{35}$$

$$\alpha(\varphi) = \frac{k}{\sqrt{2\kappa}} \tag{36}$$

[4] We also have $\rho_{\rm BI} = A^4 \tilde{\rho}_{\rm BI}$ [see (12)] for the relation between the energy density expressed in the Einstein and Jordan frames (quantities with a tilde).

$$|3 + 2\omega(\Phi)| = \alpha^{-2}(\varphi) \tag{37}$$

and face the simplest tensor-scalar theory of Brans-Dicke type ($\varphi = \sqrt{\kappa/2}\phi$). Using CONVODE [25], it is possible to find an analytic solution for φ' under the following implicit form when we use the coupling function (34):

$$(6\mathcal{A}^2 - 8)(p + p_0) = \sqrt{3}\mathcal{A}\ln\left(\left|\frac{\varphi' - \sqrt{3}}{\varphi' + \sqrt{3}}\right|\right) - 2\ln\left(\frac{\varphi'^2 - 3}{(3\mathcal{A} + 2\varphi')^2}\right), \tag{38}$$

where $\mathcal{A} = k/\sqrt{2\kappa}$ and p_0 some integration constant. When $p \to \infty$, there is an attractor for φ', namely,

$$\varphi'(p \to \infty) = -\frac{3}{2}\frac{k}{\sqrt{2\kappa}}. \tag{39}$$

Therefore, there is also a maximum value for the dilatonic coupling constant k for which the attractor corresponds to the relativistic limit for the dilaton ($|\varphi'_\infty| \to \sqrt{3}$):

$$k_{\max} = \sqrt{\frac{8\kappa}{3}}. \tag{40}$$

In the nonrelativistic limit $|\varphi'| \ll \sqrt{3}$, Eq. (33) now becomes

$$\varphi'' + 2\varphi' + 3\mathcal{A} = 0,$$

which can be solved easily to give

$$\varphi(p) = -\frac{3}{2}\frac{k}{\sqrt{2\kappa}}\left(p + \frac{1}{2}e^{-2p} - \frac{1}{2}\right), \tag{41}$$

where we assumed $\varphi_i = 0$. We can see that, due to the constant potential term in Eq. (33), the value of the dilaton field goes to $-\infty$ ($+\infty$ if $k < 0$) with the time variable p. However, the gauge energy density will also decrease with time and finally the assumption of strong field will be no longer true as ρ_{BI} becomes less than the critical energy ϵ_c. At the end of the evolution, we should retrieve the radiation case for which the solution in case of universal coupling was described above. Once again, a nonuniversal coupling to the metric yields modification of these behaviors as we shall see further. Now that we have recalled the main features of Born-Infeld and dilaton cosmologies as well as EBID system with universal coupling, let us now discuss how the nonuniversal coupling in the general EBID system will modify a tensor-scalar cosmological picture. This will be done in three steps: in the following section, we will focus on the strong-field limit where the gauge field mimics a Nambu-Goto string gas; then on the low-energy limit which corresponds to the Yang-Mills regime for the gauge fields and finally to the general cosmological evolution where transition between both regimes occurs.

IV. THE STRONG-FIELD REGIME

As we have seen earlier, the strong-field limit is reached when the BI critical energy ϵ_c can be neglected with regards to the gauge field energy density. Setting $\mathcal{P} \gg 1$ into the EBID field equations (10), (13), (16), and (17) with $\rho_m = p_m = 0$ and $V = 0$, we find

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\left[\frac{\dot{\phi}^2}{2} + \epsilon_c A^4(\phi)\mathcal{P}\right] \tag{42}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{3}[\dot{\phi}^2 + \epsilon_c A^4(\phi)] \tag{43}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + 2\epsilon_c A^4(\phi)\mathcal{P}[-\beta(\phi) + 2\alpha(\phi)] = 0 \tag{44}$$

$$\ddot{\sigma} - \frac{\dot{a}}{a}\dot{\sigma} + 2\dot{\phi}\,\dot{\sigma}[2\alpha(\phi) - 3\beta(\phi)] = 0\cdot \tag{45}$$

where we used $\frac{\mathcal{R}}{\mathcal{P}} = 1 - \Gamma$ and assumed $\Gamma \ll 1$ which will be verified afterwards by the agreement between analytical and numerical solutions. In the following, we will assume the exponential coupling function (34) for the sake of simplicity. However, the qualitative analysis will be valid for any coupling functions. In terms of the new time variable

$$p = \ln\left(\frac{a}{a_i}\right)$$

(where a_i defines the initial zero value for p), we can combine Eqs. (42)–(44) to obtain

$$\frac{\varphi''}{3 - \varphi'^2} + \frac{2}{3}\varphi' + (-\beta(\varphi) + 2\alpha(\varphi)) = 0, \tag{46}$$

where $\varphi = \sqrt{\kappa/2}\phi$. In order to particularize, we can set now $\alpha = 0$ ($A = 1$) and $B(\varphi) = \exp(k/\sqrt{2\kappa}\varphi)$ and find for the dilaton equation

$$\frac{\varphi''}{3 - \varphi'^2} + \frac{2}{3}\varphi' - \frac{k}{\sqrt{2\kappa}} = 0 \tag{47}$$

which admits a solution similar to the case of a tensor-scalar theory with Nambu-Goto string gas (33):

$$(6\mathcal{A}^2 - 8)(p + p_0) = -\sqrt{3}\mathcal{A}\ln\left(\left|\frac{\varphi' - \sqrt{3}}{\varphi' + \sqrt{3}}\right|\right) - 2\ln\left(\frac{\varphi'^2 - 3}{(3\mathcal{A} - 2\varphi')^2}\right), \tag{48}$$

where $\mathcal{A} = k/\sqrt{2\kappa}$. When $p \to \infty$, the attractor for φ' is now exactly the opposite of the universal coupling case (39):

$$\varphi'(p \to \infty) = \frac{3}{2}\frac{k}{\sqrt{2\kappa}}. \tag{49}$$

The maximum value for the dilatonic coupling constant k

is the same as before [Eq. (40)] and the nonrelativistic limit can be obtained from (41) with an opposite sign. The constant potential term in Eq. (46) is negative so that the value of the dilaton field is pushed toward $+\infty$ ($-\infty$ if $k < 0$) with the time variable p, as long as the gauge field remains in the strong-field limit $\rho_{\rm BI} \gg \epsilon_c$.

Figure 2(a) illustrates the evolution of the dilaton velocity with respect to the time variable p in the strong-field limit. The trajectory has been computed numerically [solid line in Fig. 2(a)] from the integration of the full EBID system with the initial conditions indicated in the caption [with $A = 1$ and $B(\varphi) = \exp(k/\sqrt{2\kappa}\varphi)$]. Also shown is the analytical approximation of the strong-field limit given by Eq. (48) represented by big dots.

Figure 2(b) gives the evolution of the dilaton field along the cosmic expansion for the case $A = 1$ and $B(\varphi) = \exp(k/\sqrt{2\kappa}\varphi)$. Starting with a negative velocity, the dilaton is damped to a minimum before being accelerated to infinite values (with $k > 0$). Fortunately, as the BI energy density will decrease with time, the strong-field limit $\rho_{\rm BI} \gg \epsilon_c$ will soon be no longer valid. We shall see further that, in the Yang-Mills limit, the same coupling to gauge fields will bring the dilaton to infinitely negative values. Of course, a similar conclusion can be found when k is negative. Figure 2(c) gives the behavior of the scale factor in the case discussed here. As the dilaton field is relativistic at infinitely low times $p \to -\infty$ and therefore dominates the energy content of the universe, the expansion starts with an infinite rate, breaking the "*renormalization*" that was done in simple BI cosmologies. Finally, let us focus on the gauge sector of the EBID system. In the strong-field limit, the features of Born-Infeld cosmologies for the gauge field are conserved: it starts at rest, damped by the cosmic expansion, before entering the oscillation regime of the weak energy Yang-Mills limit. Figure 2(d) illustrates the evolution of $\mathcal{P}$ for the numerical solution (solid line) with dilaton compared to the evolution in a simple BI universe with same initial BI energy density (dash-dotted line). This holds for the particular coupling $A = 1$ and $B(\varphi) = \exp(k/\sqrt{2\kappa}\varphi)$. If we now transpose A and B, it is easy to see from (46) that the attracting value for φ' will be twice the value of the universal coupling (39). Therefore, when

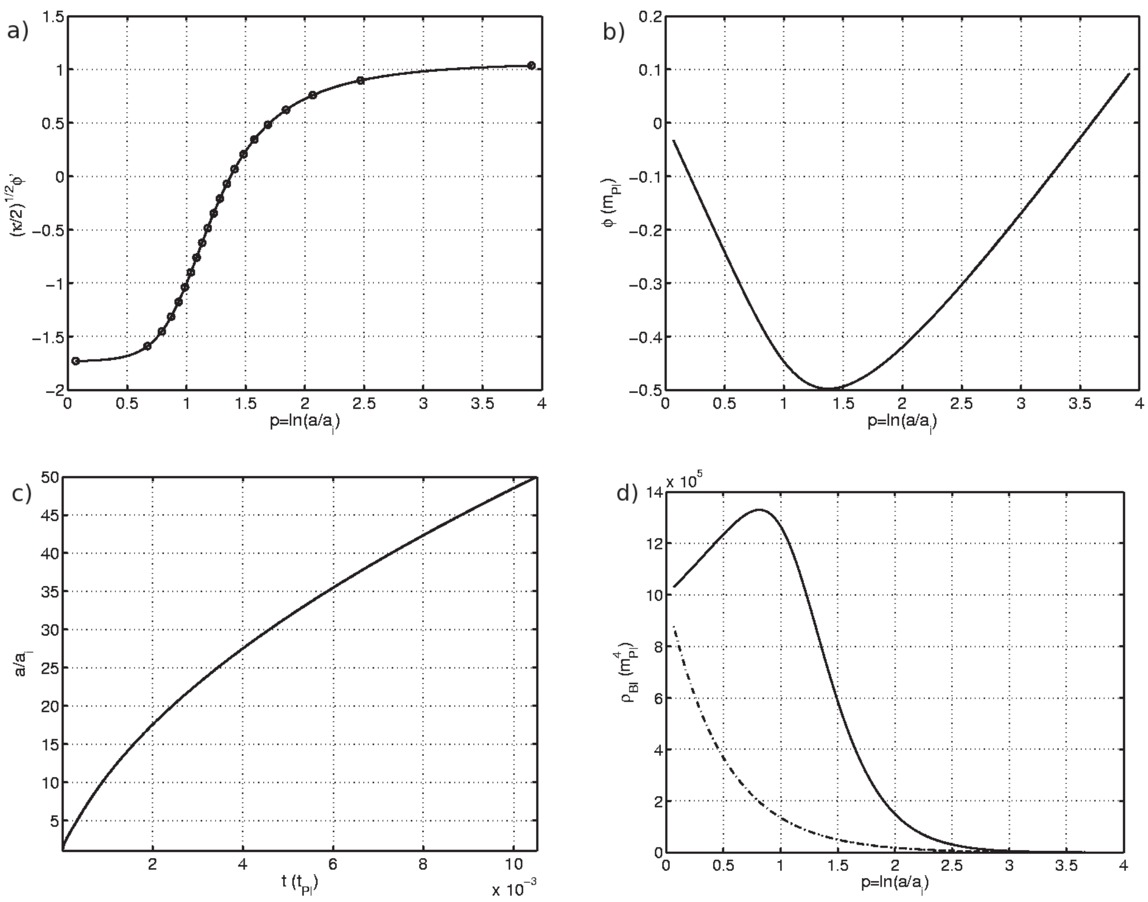

FIG. 2. Evolution of the EBID system in the strong-field regime with the p-time variable (a) φ', (b) dilaton field ϕ, and (c) scale factor $a(t)$. (d) Energy density of the BI gauge field ($a_i = 1$, $\rho_{\rm BI}(a_i)/\epsilon_c = 10^{10}$, $\epsilon_c = 10^{-4} \times m_{Pl}^4$, $k = 5$, $\phi_i' = -1.73$, $\delta H/H < 10^{-13}$).

the coupling to the volume form is weaker than the coupling to the metric,[5] the scalar field behaves just the opposite way than in the universal coupling (with $\lambda = -1/3$). Up to this point, we have obtained both analytical and numerical solutions for the strong-field limit, $\rho_{\rm BI} \gg \epsilon_c$, of the EBID system. We have also explained qualitatively the effects of nonuniversal coupling to the Einstein metric $g_{\mu\nu}$ on the dynamics of the scalar field. Let us now turn to the weak-field regime in which the gauge sector is ruled by Yang-Mills Lagrangian.

V. THE WEAK-FIELD REGIME: SOLUTIONS OF THE EINSTEIN-YANG-MILLS-DILATON SYSTEM

The weak-field regime of the EBID system is reached when the BI energy density $A^4(\varphi)\rho_{\rm BI}$ becomes much smaller than the critical energy ϵ_c. In this case, the Lagrangian ruling the gauge sector takes the usual Yang-Mills form (3), which gives for the spatially homogeneous and isotropic gauge potentials (5):

$$\mathcal{L}_{\rm YM} = -\frac{3}{2}\left(\frac{\sigma^4}{a^4} - \frac{\dot{\sigma}^2}{N^2a^2}\right). \qquad (50)$$

Taking into account the limit $\epsilon_c \to \infty$ (and $\mathcal{P} \approx 1$) into the EBID field equations (10), (13), (16), and (17) (with $V = 0$), we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\left[\frac{3}{2}A^4(\phi)B^{-4}(\phi)\left(\frac{\dot{\sigma}^2a^2 + \sigma^4}{a^4}\right) + \frac{\dot{\phi}^2}{2}\right] \qquad (51)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{3}\left[\frac{3}{2}A^4(\phi)B^{-4}(\phi)\left(\frac{\dot{\sigma}^2a^2 + \sigma^4}{a^4}\right) + \dot{\phi}^2\right] \qquad (52)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - 6A^4(\phi)B^{-4}(\phi)(\alpha(\phi) - \beta(\phi)) \times\left(\frac{\dot{\sigma}^2a^2 - \sigma^4}{a^4}\right) = 0 \qquad (53)$$

$$\ddot{\sigma} + 2\frac{\sigma^3}{a^2} + \frac{\dot{a}}{a}\dot{\sigma} + 4\dot{\phi}\,\dot{\sigma}(\alpha(\phi) - \beta(\phi)) = 0\cdot \qquad (54)$$

These equations constitute the Einstein-Yang-Mills-dilaton system, and the special case of $A = 1$ and $B(\phi) = \exp(k/2\phi)$ can be found in [14,15]. In this paper, the authors highlighted the importance of the energy exchange between the dilaton and the Yang-Mills field. Indeed, this coupling yields a new force term in the field equation for the dilaton (53) and the gauge field (54) which disappear in case of universal coupling ($A = B$ and $\alpha = \beta$). With nonuniversal coupling the gravitation is now sensitive to the force term when it is coupled to Yang-Mills radiation although its equation of state is those of radiation [see (28)]. Let us now describe the dynamics of the different fields in the EBID system for this low-energy regime. First, let us move to the p-time variable $p = \ln(a/a_i)$ and use the acceleration and Friedmann equations (51) and (52) to rewrite (53) as

$$\frac{\varphi''}{3 - \varphi'^2} + \frac{\varphi'}{3} - 2(\alpha(\phi) - \beta(\phi))\frac{\dot{\sigma}^2a^2 - \sigma^4}{\dot{\sigma}^2a^2 + \sigma^4} = 0, \quad (55)$$

with

$$\varphi = \sqrt{\frac{\kappa}{2}}\phi\cdot$$

The energy exchange term in $\dot{\sigma}^2a^2 - \sigma^4$ inside relations (53) and (55) is in general oscillating, due to the self-coupling of the non-Abelian gauge field [term in $2\sigma^3/a^2$ in (54)]. A way to handle this easily is to replace it by an effective source term which would account for the average effect of the gauge field oscillations. Let us therefore proceed to the following replacement:

$$\frac{\dot{\sigma}^2a^2 - \sigma^4}{\dot{\sigma}^2a^2 + \sigma^4} \approx \aleph \qquad (56)$$

with $\aleph$ some constant expressing the effectiveness of the energy exchange between dilaton and gauge fields in the weak-field regime. This constant $\aleph$ can for instance be estimated numerically by computing the average of the driving term in (55) over one period. Equation (55) is the same field equation as (28) for the tensor-scalar theory of the dilaton but now with a nonvanishing force term due to our averaging of the gauge oscillations. By averaging the gauge oscillations, we obtain a similar equation to the strong energy limit [Eq. (46)] seen in the previous section. Therefore, we can use the same procedure as before: if we set $A = 1$ and $B(\phi) = \exp(k/2\phi)$, we can propose the following implicit solution for φ':

$$(6\mathcal{A}^2 - 2)(p + p_0) = -\sqrt{3}\mathcal{A}\ln\left(\left|\frac{\varphi' + \sqrt{3}}{\varphi' - \sqrt{3}}\right|\right) - \ln\left(\frac{\varphi'^2 - 3}{(3\mathcal{A} + \varphi')^2}\right), \qquad (57)$$

where $\mathcal{A} = 2\aleph k/\sqrt{2\kappa}$. Once again, the p-time derivative of the dilaton field φ' evolves towards the following attractor:

$$\varphi'(p \to \infty) = -6\frac{\aleph k}{\sqrt{2\kappa}},$$

and the maximum value allowed for the dilatonic coupling constant k for which the dilaton remains relativistic ($\varphi'_\infty \to -\sqrt{3}$) is

$$k_{\rm max} = \sqrt{\frac{\kappa}{6\aleph^2}}\cdot$$

It is important to notice the opposite sign between the attractors of the strong and weak-field regimes which

[5]For example, in the case $A = 1$ and $B(\varphi) = \exp(k/\sqrt{2\kappa}\varphi)$ we just discussed.

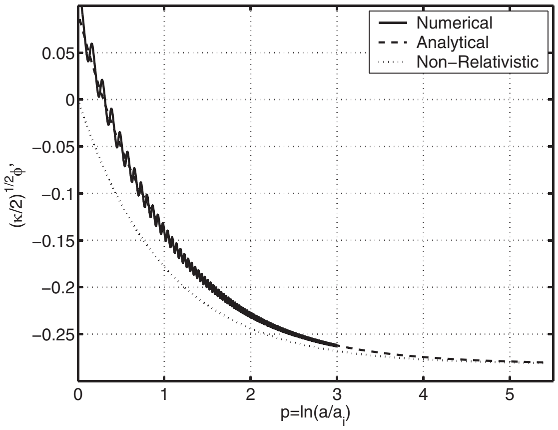

FIG. 3. Evolution of φ' as a function of the p-time variable in the weak-field limit ($a_i = 1$, $\rho_{\rm BI}(a_i)/\epsilon_c = 10^{-3}$, $\epsilon_c = 10^{-4} \times m_{\rm Pl}^4$, $k = 1$, $\phi_i' = 0.1$, $\delta H/H < 10^{-7}$).

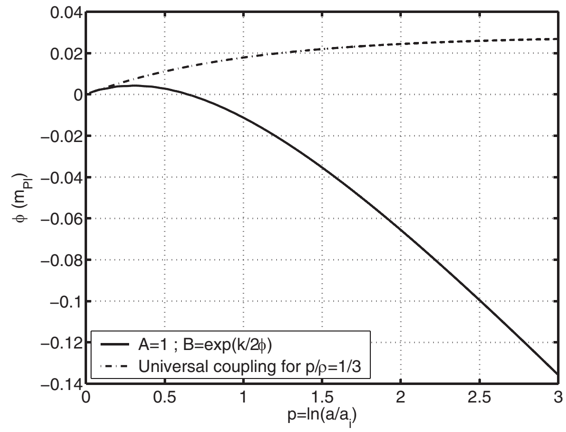

FIG. 4. Evolution of the dilaton field ϕ as a function of the p-time variable in the weak-field limit (same parameters as above).

will have important consequences on the cosmological evolution of the dilaton. In the nonrelativistic limit, $\phi'^2 \ll 3$, we find the following solution for the dilaton:

$$\varphi = -3\mathcal{A}(e^{-p} + p - 1) + \varphi_i, \tag{58}$$

and we see that the dilaton tends to $-\infty$, if the dilatonic coupling constant is positive ($\aleph > 0$). Figure 3 compares our analytical solution (57) coming from an averaging approximation (dashed line) to a numerical solution of the full EBID system (solid line). In the YM regime, the velocity of the dilaton φ' appears oscillating, around average values given by the approximation (57) with $\aleph = 1/3$. This value numerically appeared to account for the average behavior of the EBID system in the low-energy limit for a wide range of parameters (k, ϵ_c, φ_i, or φ_i'). Therefore, averaging the oscillations of the source term to about a third of their amplitude seems in very good agreement with numerical solutions. Therefore, the attractor for the p-time derivative of the dilaton is now

$$\varphi'(p \to \infty) = -2\frac{k}{\sqrt{2\kappa}}, \tag{59}$$

while the maximum value allowed for the dilatonic coupling constant is

$$k_{\rm max} = \sqrt{\frac{3\kappa}{2}}. \tag{60}$$

When $A = 1$ and $B(\phi) = \exp(k/2\phi)$, the p-time derivative of the dilaton appears to converge to a constant negative value which is directly given by the nonrelativistic approximation (58) which is valid when k is small compared with $\sqrt{\kappa}$ (dotted line). When $B = 1$ and $A(\phi) = \exp(k/2\phi)$, it is obvious from Eq. (55) that the attractor has exactly the opposite value. Therefore, the case of nonuniversal coupling $A \neq B$ is quite different to what happens in a usual tensor-scalar theory: in a radiation-dominated universe, the p-time velocity of the dilaton freezes to zero [see the solution (32)]. Indeed, this will make the dilaton field diverging after an infinite amount of time as can be seen in Fig. 4. In this figure, we represented the evolution of the dilaton in the numerical solution with $A = 1$ and $B(\phi) = \exp(k/2\phi)$ (solid line) to the solution (32) for the same field in a radiation-dominated universe. More precisely, the dilaton tends to $-\infty$. As a conclusion, although the gauge field looks like radiation at a large-scale level, the nonuniversal microscopic coupling between dilaton and gauge sectors finally dominates at large redshift and freezes the energy contribution of both sectors in such a way that none of these components completely dominates. Figure 5 presents the evolution of the gauge field velocity $\dot{\sigma}$ with the scale factor. The gauge field appears to be damped by its coupling to the dilaton and in fact the whole gauge sector loses energy at a rate fixed by (18).

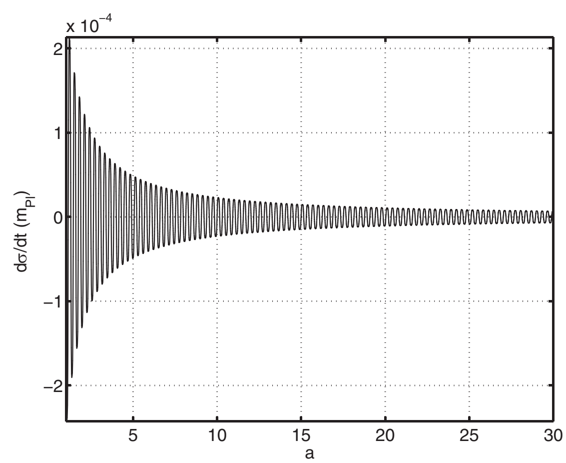

FIG. 5. Evolution of the gauge field velocity as a function of the scale factor in the weak-field limit (same parameters as above).

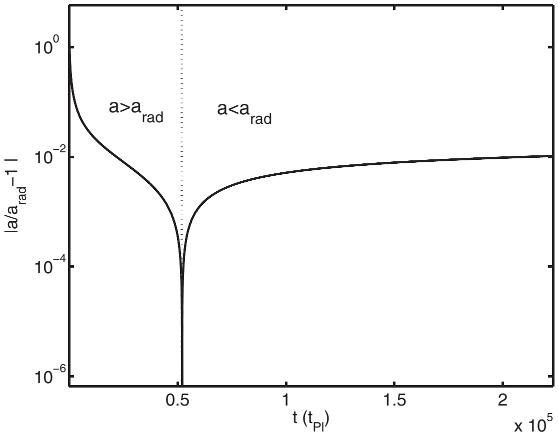

FIG. 6. Evolution of the departure from a radiation-dominated universe of the weak-field limit during expansion (same parameters as above).

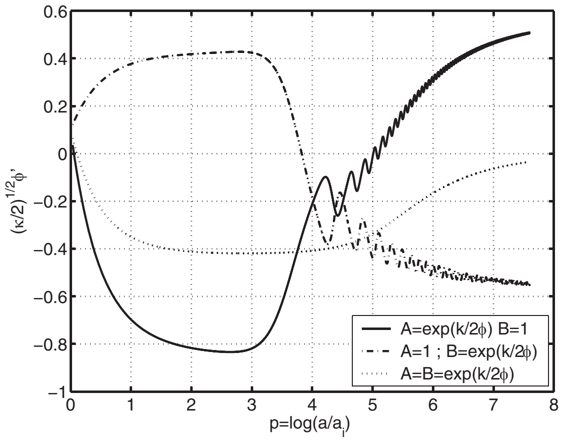

FIG. 7. Evolution of the dilaton velocity with p-time for three different couplings: $\beta = 0$ and $\alpha = k/2$ (solid line), $\alpha = 0$ and $\beta = k/2$ (dash-dotted line), and universal coupling ($A = B$, dotted line). [$a_i = 1$, $\rho_{\rm BI}(a_i)/\epsilon_c = 10^4$, $\epsilon_c = 10^{-4} \times m_{\rm Pl}^4$, $k = 2$, $\phi_i' = 0.1$, $\delta H/H < 10^{-10}$].

Figure 6 shows the departure of the scale factor from the radiation solution:

$$\left| \frac{a}{a_{\rm rad}} - 1 \right|$$

for the numerical solution presented in this section. We see that the departure is important when the scalar field is dominating at early times ($t < 5 \times 10^4 t_{\rm Pl}$) then finally converges to a slightly less strong expansion at late times ($a \approx 0.99 \times a_{\rm rad}$), when the equilibrium has been reached.

Before going further, let us summarize the cosmological evolution of the fields constituting the EBID system in the YM regime. First, the dilaton is damped until its velocity is attracted to a negative (respectively positive) value for the particular coupling $A = 1$ and $B(\phi) = \exp(k/2\phi)$ and $k > 0$ [respectively $B = 1$ and $A(\phi) = \exp(k/2\phi)$]. However, it should have been damped to rest if it would have been plunged in a radiation-dominated universe with universal coupling. Because its velocity has been attracted to a negative value, the dilaton field will eventually diverge linearly to $-\infty$ ($k > 0$). This is exactly the opposite situation of the strong-field limit that was presented before where the coupling term drives the dilaton to infinitely high values. In a general situation where the gauge field starts with an energy much higher than the BI critical energy and then cools down to YM dynamics, one should expect that the dilaton reaches some extremum value ($\varphi' = 0$) during the transition. This will be treated in the next section.

VI. GENERAL COSMOLOGICAL EVOLUTION

Let us follow in detail some typical cosmological evolutions of the EBID system for various couplings. Starting at singularity, the dilaton is in general relativistic ($|\varphi| \to \infty$ and $\varphi'^2 \to 3$). The expansion therefore begins at $a = 0$ with an infinite rate and the gauge field dynamics is dominated by the nonlocal effects induced by the BI nonlinearity ($\rho_{\rm BI} \gg \epsilon_c$). As energy is exchanged between the dilaton and the gauge field in the strong-field regime, the dilaton velocity is attracted to some value depending on the coupling functions, as illustrated in Fig. 7. During this phase, the velocity of the dilaton in p-time is indeed a positive (negative) constant when $A = 1$ and $B(\phi) = \exp(k/2\phi)$ [$B = 1$ and $A(\phi) = \exp(k/2\phi)$ or the universal coupling $A = B$]. When the gauge energy density has decreased to the BI critical energy ($\rho_{\rm BI} \approx \epsilon_c$), the dilaton velocity leaves the strong-field attractor to enter the YM low-energy regime. The epoch of this transition varies according to the coupling functions considered (see Fig. 7). It then moves to the low-energy attractor by accomplishing damped oscillations around the analytical solutions proposed in the previous section (with $A \neq B$) or is damped to vanishing velocities when there is universal coupling (see Sec. III). With nonuniversal coupling ($A \neq B$), the value of the dilaton field reaches some extremum ($\varphi' = 0$) on its way to the second attractor. This extremum is unavoidable as we have seen that the attractors of the dilaton velocity in the strong and weak-field regimes are of opposite signs and its velocity will therefore vanish at some time during the transition between these two attracting regimes. It is also interesting to examine the evolution of the gauge field energy density. Figure 8 represents the gauge field energy density ($\mathcal{P} - 1$) related to the curves in Fig. 7. With universal coupling $A = B$, we retrieve a cosmological evolution given by Eq. (19) and its first integral (20): $\rho_{\rm BI} \approx a^{-2}$ in the strong-field regime and $\rho_{\rm BI} \approx a^{-4}$ at low energies. The assumption of nonuniversal coupling now leads to different evolutions of the gauge field energy density, which in fact are given by the more general energy conservation equation (18). The differences between the evolutions come from the particular trajectories illustrated in Fig. 7.

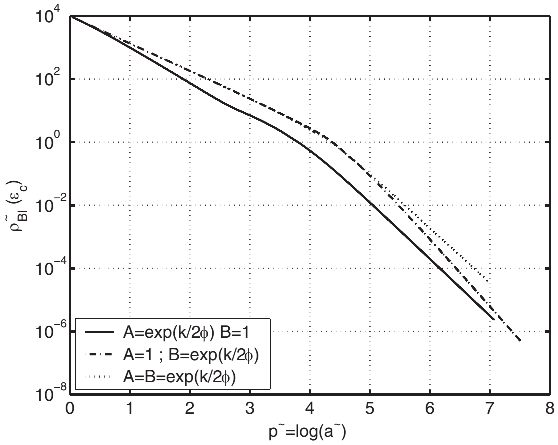

FIG. 8. Evolution of the physical energy density $\tilde{\rho}$ with $\tilde{p} = \ln(\exp(k/2\phi)a)$ for the curves in Fig. 7 (same parameters as above).

Now that we have reviewed the main features of a general full cosmological evolution from high to low-energy regimes, it is also important to discuss the evolution of the observed cosmological parameters like the scale factor, the Hubble expansion rate, or the accelerating parameter in the Jordan physical frame. In order to define such a frame, we will introduce an additional matter pressureless fluid which will verify the weak equivalence principle. The energy density ρ_m of this fluid in the Einstein frame is related to the physical energy density $\tilde{\rho}_m$ through the relation (26). The coupling function $C(\phi)$ to ordinary matter defines now our "*observable*" Jordan frame by

$$\tilde{g}_{\mu\nu} = C^2(\phi) g_{\mu\nu}. \tag{61}$$

In the Jordan frame obtained by the previous conformal transformation, the energy density $\tilde{\rho}_m$ of the matter fluid is ruled by the same conservation laws as in general relativity. This is true because there is no direct interaction between the gauge sector and the additional matter fluid and therefore they decouple from each other. The field equation for the gauge potential σ does not need to be modified as we do not assume any direct coupling with the pressureless fluid. We will now consider the field equations for the EBID system we have written in Sec. II for the Einstein metric $g_{\mu\nu}$ with a pressureless fluid $p_m = 0$. The observable scale factor in the Jordan frame will be given by

$$\tilde{a} = C(\phi) a, \tag{62}$$

while the synchronous time in the Jordan frame is denoted by

$$d\tilde{t} = C(\phi) dt. \tag{63}$$

Then, the Hubble expansion rate can be derived directly:

$$\tilde{H} = \frac{d\tilde{a}}{\tilde{a} d\tilde{t}} = C^{-1}(\phi)(H + \gamma(\phi)\dot{\phi}), \tag{64}$$

where $H = \dot{a}/a$ is the Hubble parameter in the Einstein frame, and $\gamma(\phi) = d\ln C(\phi)/d\phi$. The acceleration parameter $\tilde{q}$ in the Jordan frame can be written

$$\tilde{q} = \frac{\ddot{\tilde{a}}\,\tilde{a}}{\dot{\tilde{a}}^2} = a\left(\frac{d\gamma(\phi)}{d\phi}\dot{\phi}^2 a + \gamma(\phi)\ddot{\phi} a + \gamma(\phi)\dot{\phi}\,\dot{a} + \ddot{a}\right) \times (\dot{a} + a\gamma(\phi)\dot{\phi})^{-2}, \tag{65}$$

where a dot over a quantity expressed in the Jordan frame means a derivative with respect to the synchronous time $\tilde{t}$ in that frame. Although there is no possibility of a cosmic acceleration ($q > 0$) in the Einstein frame [unless one considers a nonvanishing potential, see relation (13)], this does not rule out a possible acceleration for the observable scale factor given by (62). Indeed, the existence of fluid (constituted by our Born-Infeld non-Abelian gauge field) that violates the weak equivalence principle will result in a possibility of cosmic acceleration. To show that this is actually the case even in the presence of a matter fluid which would make the tensor-scalar theory converging to general relativity if taken alone, we will use the coupling function $\gamma(\varphi) = \varphi$. Figure 9 illustrates evolutions of the acceleration parameter in the Jordan frame $\tilde{q}(\tilde{p})$, given by (65) as a function of the Jordan scale factor $\tilde{a}$. The solutions presented here are expanding universes ($\tilde{H} > 0$). Four different couplings have been considered, including the simple case of $A = B \neq C$ which correspond to different couplings of gravitation to the gauge and the matter sector. The pressureless fluid energy density at the start has been chosen to dominate the BI energy density by more than 1 order of magnitude. As the gauge field starts in the strong-field regime, its energy density will scale approximately with a^{-2}, depending on the coupling functions (see also Fig. 8). Therefore, the gauge sector rapidly dominates the

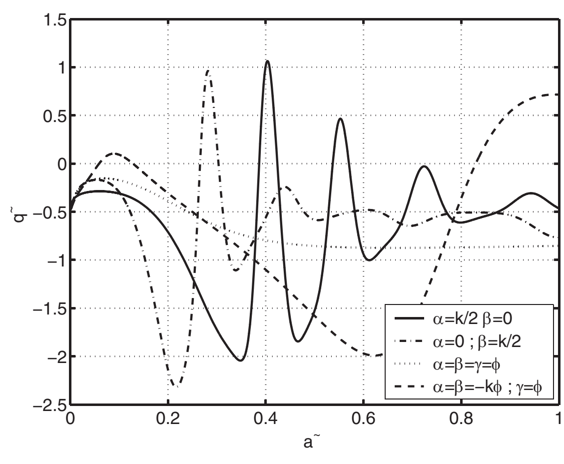

FIG. 9. Evolution of acceleration parameter in the Jordan frame for four different couplings: $\beta = 0$ and $\alpha = k/2$ (solid line); $\alpha = 0$ and $\beta = k/2$ (dash-dotted line); universal coupling ($\alpha = \beta = \gamma = \varphi$, dotted line) and $\alpha = \beta = k\phi$, $\gamma = \varphi$ (dashed line, $k < 0$). [$a_i = 1$, $\rho_{BI}(a_i)/\epsilon_c = 10^5$, $\epsilon_c = 10^{-4} \times m_{Pl}^4$, $k = 4$, $\delta H/H < 10^{-10}$, $\varphi_i' = 0.1$, $\rho_m(a_i)/\rho_{BI}(a_i) \approx 100$].

energy content of the universe. The dynamics of the dilaton is as described earlier: after having moved to the strong-field regime attractor, the transition to YM dynamics occurs and the dilaton quickly moves to the low-energy attractor. During this transition, acceleration appears in the Jordan frame defined by the pressureless fluid as indicated in Fig. 9. We see also that any violation of the weak equivalence principle (here by taking $C \neq A$, $C \neq B$ or $C \neq A = B$) leads to cosmic acceleration even with tensor-scalar theory that would alone converge to general relativity. An important condition for cosmic acceleration is to have a repulsive force term in the dilaton equation. Therefore, the EBID system with a nonuniversal coupling to gravitation offers the interesting possibility to build a scenario for dark energy or inflation. For the curves represented here, the acceleration periods are shorter than in a usual ΛCDM ($\Omega_\Lambda = 0.7$; $\Omega_m = 0.3$; $H_0 = 70$ km/s/Mpc) or quintessence model[6] and therefore a more complete study should be done to determine if it is possible to explain distance-redshifts measurements with EBID fields. It should also be noticed that an EBID dark energy scenario would predict a finite period of acceleration. Indeed, as the gauge field will recover a YM dynamics at the end of its evolution, its energy density will finally scale as a^{-4} and will finally be dominated by a pressureless fluid. Therefore, the questions whether cosmic acceleration ($\tilde{q} > 0$) can occur, with which intensity, and for how long, seem to depend on both initial energy distribution, the value of the dilaton coupling constant, and the critical BI energy scale ϵ_c. More work should focus on that point to see if it would be possible to build a physically relevant quintessence model with the EBID field equations. However, the perspectives of cosmic acceleration in the Jordan frame do exist and this looks particularly interesting for our view of modern cosmology.

VII. CONCLUSION

The non-Abelian Einstein-Born-Infeld-dilaton model provides an interesting framework, motivated by string theory, to study the impact of large-scale non-Abelian gauge fields on tensor-scalar theories of the gravitational interaction. In this paper, we focused on the cosmological evolution of an homogeneous and isotropic configuration of these fields in a flat background. The microscopic coupling between the dilaton and gauge fields induced by nonuniversal coupling to the metric leads to an energy exchange between both gravitational and gauge sectors that will alter the usual dynamics of tensor-scalar theories (for which the coupling is universal).

As in non-Abelian Born-Infeld cosmology, we considered two different regimes depending on the gauge energy density compared to the critical energy that parametrizes the Born-Infeld Lagrangian. We have derived both analytical and numerical solutions to describe the cosmological evolution of the whole system. In the case of nonuniversal coupling, the gravitational scalar field no more depends on the equation of state of the gauge field and the dynamics is altered as follows.

In the particular case of a Brans-Dicke theory, in which nonperturbative terms for the dilaton are not considered, we have shown that the energy exchange resulting from the particular couplings damps the dilaton to a frozen non-vanishing velocity. In the high-energy regime of the gauge dynamics, the attracting value for the velocity is positive when the gauge field couples more to the metric than to the volume form. The opposite situation happens in the low-energy regime where the gauge field is ruled by Yang-Mills Lagrangian. Therefore, in a general cosmological evolution where the gauge field cools down to low energies, there is a transition between the two attractors. Their values are directly proportional to the value of the dilatonic coupling constant.

However, it is well known from experimental tests of the gravitational theory, especially the determination of the post-Newtonian parameter $\bar{\gamma}$, that the value of the coupling ω_0 is at least of order 500, roughly 10^{-3} for α_0 (see [26] for a recent estimation of the post-Newtonian parameter γ). The influence of the dilaton potential is also important to consider. Furthermore, the constraints on the weak equivalence principle obtained by the tests on the universality of free fall exclude a violation of this principle that would exceed a part in 10^{-12}. One can therefore argue that the effect of such nonuniversal couplings should be neglected. But, if the violation of the weak equivalence principle only applies to large-scale fields which do not couple to ordinary matter and whose distribution is roughly homogeneous, their energy density on our scales is far beyond experimental reach and the violation could be hard to exhibit.

The interesting possibility introduced by such a violation is a cosmic acceleration in the physical frame associated to ordinary matter. In this work, we build a first simple model based on our treatment of the EBID field equations that exhibits periods of acceleration in the presence of ordinary matter verifying the weak equivalence principle. The acceleration has been shown to resist to the attracting property of the accompanying matter and seems to be a general feature of a nonuniversal coupling to gravitation. Furthermore, this model respects the weak energy condition $\rho + 3p > 0$ in the frame of the physical degrees of

[6]To give an idea on how these universes accelerate, we remind the reader about the following values of the acceleration parameter for various energy content:

$$q(\text{radiation}) = -1 \qquad q(\text{relativistic } \phi) = -2 \qquad q(\Lambda) = 1$$
$$q(\text{ghost}) = 2,$$

where Λ stands for the cosmological constant (the de Sitter solution to which inflation is usually matched as an exponential expansion).

freedom (the Einstein frame). The Born-Infeld dynamics of the gauge field plays a crucial role in this kind of dark energy model by ensuring a late arising of this mechanism (when the gauge field mimics a Nambu-Goto string gas) and even predicts an end to the dark energy domination (when the gauge field looks like radiation). An interesting perspective to this work would be to use this remarkable feature of non-Abelian Born-Infeld gauge fields to build physical models for quintessence and, maybe, inflation.

In conclusion, we can say that our study of non-Abelian Born-Infeld gauge fields coupled to tensor-scalar gravity opens new and interesting perspectives for the question of the attraction to general relativity as well as other crucial topics of modern cosmology such as inflation or dark energy.

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APPENDIX: NUMERICAL INTEGRATION OF THE EBID SYSTEM

Here we give some details of the numerical integration of the full EBID system we use to illustrate this paper. In order to integrate the system of equations (10), (13), (16), and (17), we choose the following procedure. First, we rewrite Eqs. (13), (16), and (17) as a system of six first-order ODE's and keep the Hamiltonian constraint (10) to check the consistency of the numerical computation. Then, let us redefine the fields in such a way that they will be approximately of the same order of magnitude (this will avoid stiffness problems in the integration):

$$a = \frac{A}{a_i} \qquad \phi = \Phi m_{\rm Pl} \qquad \sigma = \Sigma m_{\rm Pl}, \tag{A1}$$

where A, Φ and Σ will be the fields to integrate. It is also useful to set $k = K/m_{\rm Pl}$ and $\epsilon_c = \epsilon_c^l m_{\rm Pl}^4$. Once the equations have been rewritten under these considerations, we choose the initial conditions as follows: a_i is set to 1 and ϕ_i to zero (we therefore start with a "*bare*" gauge coupling constant equal to unity in the Einstein frame). We choose the ratio $\rho_{\rm BI}(a = a_i)/\epsilon_c = r$ so that we can control the type of gauge dynamics (BI, YM, or transition) we start with. Then, we choose the value of $\phi'(a_i) = \phi_i'$ so that the initial expansion rate will be given by $H_i^2 = \kappa/3\rho_{\rm BI}(a_i)/(1 - \kappa\phi_i'^2/6)$. This gives also $\dot{\phi}_i$ as it is equal to $H_i\phi_i'$. Then, without loss of generality, we can assume $\sigma_i = 0$ and determine $\dot{\sigma}_i$ from the postulated value of $\rho_{\rm BI}(a = a_i)$. Numerical integration of the system of six ODE's is performed using the standard method of Shampine-Gordon [27]. To monitor the accuracy of the numerical solution, we compute the absolute violation of the Hamiltonian constraint (10):

$$\frac{\delta H}{H} = \frac{|H - \dot{A}/A|}{H}$$

all along the integration. The numerical integration makes the violation of the Hamiltonian constraint diverging exponentially with time and we indicated in the previous figures the final absolute error reached for each of the numerical solutions that were presented.

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