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Anisotropically inflating universes

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We show that in theories of gravity that add quadratic curvature invariants to the Einstein-Hilbert action there exist expanding vacuum cosmologies with positive cosmological constant which do not approach the de Sitter universe. Exact solutions are found which inflate anisotropically. This behavior is driven by the Ricci curvature invariant and has no counterpart in the general-relativistic limit. These examples show that the cosmic no-hair theorem does not hold in these higher-order extensions of general relativity and raises new questions about the ubiquity of inflation in the very early universe and the thermodynamics of gravitational fields.

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I. INTRODUCTION

The inflationary universe is the central cosmological paradigm which astronomical observations aim to test, and by which we seek to understand how the universe might have evolved from a general initial condition into its present state of large-scale isotropy and homogeneity together with an almost flat spectrum of near-Gaussian fluctuations. The essential feature of this inflationary picture is a period of accelerated expansion during the early stages of the universe [1]. The simplest physically motivated inflationary scenario drives the acceleration by a scalar field with a constant potential, and the latter can also be described by adding a positive cosmological constant to the Einstein equations. In order to understand the generality of this scenario it is important to determine whether universal acceleration and asymptotic approach to the de Sitter metric always occurs. A series of cosmic no-hair theorems of varying strengths and degrees of applicability have been proved to demonstrate some necessary and sufficient conditions for its occurrence [2–6]. Similar deductions are possible for power-law [7,8] and intermediate inflationary behavior, [9], where accelerated expansion is driven by scalar-field potentials that have slow exponential or power-law falloffs, but we will confine our discussion to the situation that occurs when there is a positive cosmological constant, $\Lambda > 0$. So far, investigations have not revealed any strong reason to doubt that, when $\Lambda > 0$ and other matter is gravitationally attractive, any stable, ever-expanding general-relativistic cosmological model will approach isotropic de Sitter inflation exponentially rapidly within the event horizon of any geodesically moving observer. Similar conclusions result when we consider inflation in those generalizations of general relativity in which the Lagrangian is a function only of the scalar curvature, R, of spacetime. This similarity is a consequence of the conformal equivalence between these higher-order theories in vacuum and general relativity in the presence of a scalar field [10–12]. In this paper we will show that when quadratic terms formed from the Ricci curvature scalar, $R_{\mu\nu}R^{\mu\nu}$ are added to the Lagrangian of general relativity, then new types of cosmological solution arise when $\Lambda > 0$ which have no counterparts in general relativity. They inflate anisotropically and do not approach the de Sitter spacetime at large times. We give two new exact solutions for spatially homogeneous anisotropic universes with $\Lambda > 0$ which possess this new behavior. They provide counterexamples to the expectation that a cosmic no-hair theorem will continue to hold in simple higher-order extensions of general relativity. Other consequences of such higher-order theories have been studied in [13–15]. The presence of such quadratic terms as classical or quantum corrections to the description of the gravitational field of the very early universe will therefore produce very different outcomes following expansion from general initial conditions to those usually assumed to arise from inflation. This adds new considerations to the application of the chaotic and eternal inflationary theories [16] in conjunction with anthropic selection [17].

We will consider a theory of gravity derived from an action quadratic in the scalar curvature and the Ricci tensor. More specifically, ignoring the boundary term, we will consider the D-dimensional gravitational action

$$S_G = \frac{1}{2\kappa}\int_M d^D x\sqrt{|g|}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda). \quad (1)$$

Variation of this action leads to the following generalized Einstein equations (see, e.g., [18]):

$$G_{\mu\nu} + \Phi_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the ordinary matter sources, which we in this paper will assume to be zero, and

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$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}, \tag{3}$$

$$\Phi_{\mu\nu} \equiv 2\alpha R(R_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu}) + (2\alpha+\beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R + \beta\Box(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) + 2\beta(R_{\mu\sigma\nu\rho} - \tfrac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho}, \tag{4}$$

with $\Box \equiv \nabla^\mu\nabla_\mu$. The tensor $\Phi_{\mu\nu}$ incorporates the deviation from regular Einstein gravity, and we see that $\alpha = \beta = 0$ implies $\Phi_{\mu\nu} = 0$.

First, consider an Einstein metric, so that $R_{\mu\nu} = \lambda g_{\mu\nu}$. This is a solution of Eq. (2) with $T_{\mu\nu} = 0$ provided that

$$\Lambda = \frac{\lambda}{2}[(D-4)(D\alpha+\beta)\lambda + (D-2)]. \tag{5}$$

Hence, when $D = 4$ any Einstein space is a solution to Eq. (2) provided that $\Lambda = (D-2)\lambda/2$. In particular, if $\Lambda > 0$, de Sitter spacetime is a solution to Eq. (2). If $\Lambda = 0$, we need $\lambda = 0$ and de Sitter spacetime cannot be a solution.

Now consider solutions to Eq. (2) which are nonperturbative and α and β are not small. We know that solutions with $\beta = 0$, $\alpha \neq 0$ are conformally related to Einstein gravity with a scalar field $\phi = \ln(1 + 2\alpha R)$ that possesses a self-interaction potential of the form $V(\phi) = (e^\phi - 1)^2/4\alpha$, [10–12], and their inflationary behaviors for small and large $|\phi|$, along with that of theories derived from actions that are arbitrary functions of R, are well understood. However, there is no such conformal equivalence with general relativity when $\beta \neq 0$ and cosmologies with $\Lambda > 0$ can then exhibit quite different behavior.

II. THE FLAT DE SITTER SOLUTION

First consider the spatially flat de Sitter universe with metric

$$ds^2_{\text{dS}} = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2), \qquad H = \sqrt{\frac{\Lambda}{3}}. \tag{6}$$

The stability of this solution in terms of perturbations of the scale factor depends on the sign of $(3\alpha + \beta)$. In 4D, we can use the Weyl invariant and the Euler density, E, defined by [19]

$$\begin{aligned} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2, \\ E &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \end{aligned} \tag{7}$$

to eliminate the quadratic Ricci invariant in the action, since

$$\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} = \frac{1}{3}(3\alpha+\beta)R^2 + \frac{\beta}{2}(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - E).$$

Since integration over the Euler density is a topological invariant, the variation of E will not contribute to the equations of motion. The Friedmann-Robertson-Walker (FRW) universes are conformally flat so, for a small variation, the invariant $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ will not contribute either. Hence, sufficiently close to a FRW metric only the R^2 term will contribute. The stability of the FRW universe is therefore determined by the sign of $(3\alpha + \beta)$ [20]. One can check this explicitly using Eq. (2). We start with the metric ansatz:

$$ds^2 = -dt^2 + e^{2b(t)}(dx^2 + dy^2 + dz^2), \qquad H = \sqrt{\frac{\Lambda}{3}},$$

and note that in 4D the trace of Eq. (2) reduces to

$$-R + 2(3\alpha+\beta)\Box R + 4\Lambda = 0, \tag{8}$$

which can be used to determine the stability of the Ricci scalar. We can perturb the Ricci scalar by assuming a small deviation from the flat de Sitter metric of the form:

$$b(t) = Ht + b_1 e^{\lambda_1 t} + b_2 e^{\lambda_2 t} + \mathcal{O}(e^{2\lambda_i t}),$$

where b_1 and b_2 are arbitrary constants. Equation (8) implies

$$\lambda_{1,2} = -\frac{3H}{2}\left(1 \pm \sqrt{1 - \frac{2}{9H^2(3\alpha+\beta)}}\right), \tag{9}$$

if $(3\alpha + \beta) \neq 0$. For $(3\alpha + \beta) = 0$, we must have $b_1 = b_2 = 0$. From this expression we see that if $(3\alpha + \beta) > 0$ then the solution will asymptotically approach the flat de Sitter spacetime as $t \to \infty$; however, for $(3\alpha + \beta) < 0$ the solution is unstable. For the special case of $\beta = 0$, this result agrees with the stability analysis of [20]. A construction of an asymptotic series approximation around the de Sitter metric for the case $\beta = 0$ has also been performed [21–25]. In the case of general relativity ($\alpha = \beta = 0$) a number of results for the inhomogeneous case of small perturbations from isotropy and homogeneity when $\Lambda > 0$ have also been obtained [2–5,26–29].

We see that, as long as $(3\alpha + \beta) > 0$, any FRW model sufficiently close to the flat de Sitter model will asymptotically approach de Sitter spacetime and consequently obeys the cosmological no-hair theorem. We should emphasize that only FRW perturbation modes have been considered here. The question of whether the flat de Sitter universe is stable against general anisotropic or large inhomogeneous perturbations when $\alpha \neq 0$ and $\beta \neq 0$ is still unsettled. In the case of universes that are not "close" to isotropic and homogeneous FRW models we shall now show that the cosmic no-hair theorem for $\Lambda > 0$ vacuum cosmologies is not true: there exist ever-expanding vacuum universes with $\Lambda > 0$ that do not approach the de Sitter spacetime.

III. EXACT ANISOTROPIC SOLUTIONS

We now present two new classes of exact vacuum anisotropic and spatially homogeneous universes of Bianchi types II and VI_h with $\Lambda > 0$. These are new exact solutions of Eq. (2) with $(\alpha, \beta) \neq (0, 0)$.

A. Bianchi type *II* solutions

$$ds^2_{II} = -dt^2 + e^{2bt}\left[dx + \frac{a}{2}(zdy - ydz)\right]^2 + e^{bt}(dy^2 + dz^2), \tag{10}$$

where

$$a^2 = \frac{11 + 8\Lambda(11\alpha + 3\beta)}{30\beta}, \qquad b^2 = \frac{8\Lambda(\alpha + 3\beta) + 1}{30\beta}. \tag{11}$$

These solutions are spacetime homogeneous with a 5-dimensional isotropy group. They have a one-parameter family of 4-dimensional Lie groups, as well as an isolated one (with Lie algebras $A^q_{4,11}$ and $A^1_{4,9}$, respectively, in Patera *et al.*'s scheme [30]) acting transitively on the spacetime. An interesting feature of this family of solutions is that there is a lower bound on the cosmological constant, given by $\Lambda_{\min} = -1/[8(\alpha + 3\beta)] = -a^2/8$ for which the spacetime is static. For $\Lambda > \Lambda_{\min}$ the spacetime is inflating and shearing. The inflation does not result in an approach to isotropy or to asymptotic evolution close to the de Sitter metric. Interestingly, even in the case of a vanishing Λ the universe inflates exponentially but anisotropically. We also note from the solutions that the essential term in the action causing this solution to exist is the $\beta R_{\mu\nu}R^{\mu\nu}$ term and the distinctive behavior occurs when $\alpha = 0$. The solutions have no well-defined $\beta \rightarrow 0$ limit, and do not have a general-relativistic counterpart. They are nonperturbative. Similar solutions exist also in higher dimensions. Their existence seems to be related to so-called Ricci nilsolitons [31,32].

B. Bianchi type VI_h solutions

$$ds^2_{VI_h} = -dt^2 + dx^2 + e^{2(rt+ax)}[e^{-2(st+a\tilde{h}x)}dy^2 + e^{+2(st+a\tilde{h}x)}dz^2], \tag{12}$$

where

$$r^2 = \frac{8\beta s^2 + (3 + \tilde{h}^2)(1 + 8\Lambda\alpha) + 8\Lambda\beta(1 + \tilde{h}^2)}{8\beta\tilde{h}^2}, \qquad a^2 = \frac{8\beta s^2 + 8\Lambda(3\alpha + \beta) + 3}{8\beta\tilde{h}^2}, \tag{13}$$

and r, s, a, and $\tilde{h}$ are all constants. These are also homogeneous universes with a 4-dimensional group acting transitively on the spacetime. Both the mean Hubble expansion rate and the shear are constant. Again, we see that the solution inflates anisotropically and is supported by the existence of $\beta \neq 0$. It exists when $\alpha = 0$ and $\Lambda = 0$ but not in the limit $\beta \rightarrow 0$.

IV. AVOIDANCE OF THE NO-HAIR THEOREM

The no-hair theorem for Einstein gravity states that for Bianchi types *I–VIII* the presence of a positive cosmological constant drives the late-time evolution towards the de Sitter spacetime. An exact statement of the theorem can be found in the original paper by Wald [6]. It requires the matter sources (other than Λ) to obey the strong-energy condition. It has been shown that if this condition is relaxed then the cosmic no-hair theorem cannot be proved and counterexamples exist [7,33–35]. In [36], the cosmic no-hair conjecture was discussed for Bianchi cosmologies with an axion field with a Lorentz Chern-Simons term. Interestingly, exact Bianchi type *II* solutions, similar to the ones found here, were found which avoided the cosmic no-hair theorem. However, unlike for our solutions, these violations were driven by an axion field whose energy-momentum tensor violated the strong and dominant energy condition. The no-hair theorem for spatially homogeneous solutions of Einstein gravity also requires the spatial 3-curvature to be nonpositive. This condition ensures that universes do not recollapse before the Λ term dominates the dynamics but it also excludes examples like that of the Kantowski-Sachs $S^2 \times S^1$ universe which has an exact solution with $\Lambda > 0$ which inflates in some directions but is static in others. These solutions, found by Weber [37], were used by Linde and Zelnikov [38] to model a higher-dimensional universe in which different numbers of dimensions inflate in different patches of the universe. However, it was subsequently shown that this behavior, like the Weber solution, is unstable [39,40]. We note that our new solutions to gravity theories with $\beta \neq 0$ possess anisotropic inflationary behavior without requiring that the spatial curvature is positive and are distinct from the Kantowski-Sachs phenomenon.

The Bianchi type solutions given above inflate in the presence of a positive cosmological constant Λ. However, they are neither de Sitter, nor asymptotically de Sitter; nor do they have initial singularities. Let us examine how these models evade the conclusions of the cosmic no-hair theorem. Specifically, consider the type *II* solution, Eq. (10). We define the timelike vector $\mathbf{n} = \partial/\partial t$ orthogonal to the Bianchi type *II* group orbits, and introduce an orthonormal frame. We define the expansion tensor $\theta_{\mu\nu} = n_{\mu;\nu}$ and decompose it into the expansion scalar, $\theta \equiv \theta^\mu_\mu$ and the shear, $\sigma_{\mu\nu} \equiv \theta_{\mu\nu} - (1/3)(g_{\mu\nu} + n_\mu n_\nu)$, in the standard way. The Hubble scalar is given by $H = \theta/3$. For the type *II* metric, we find (in the orthonormal frame)

$$\theta = 2b, \qquad \sigma_{\mu\nu} = \tfrac{1}{6}\mathrm{diag}(0, 2b, -b, -b).$$

As a measure of the anisotropy, we introduce dimensionless variables by normalizing with the expansion scalar:

$$\Sigma_{\mu\nu} = \frac{3\sigma_{\mu\nu}}{\theta} = \mathrm{diag}\left(0, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}\right).$$

Interestingly, the expansion-normalized shear components are constants (and independent of the parameters α, β, and Λ) and this shows that these solutions violate the cosmological no-hair theorem (which requires $\sigma_{\mu\nu}/\theta \to 0$ as $t \to \infty$). To understand how this solution avoids the no-hair theorem of, say, Ref. [6], rewrite Eq. (2) as follows:

$$G_{\mu\nu} = \tilde{T}_{\mu\nu}, \qquad \tilde{T}_{\mu\nu} \equiv -\Lambda g_{\mu\nu} - \Phi_{\mu\nu} + \kappa T_{\mu\nu}.$$

In this form the higher-order curvature terms can be interpreted as matter terms contributing a fictitious energy-momentum tensor $\tilde{T}_{\mu\nu}$. For the Bianchi *II* solution we find

$$\begin{aligned}\tilde{T}_{\mu\nu} &= \tfrac{1}{4}\,\mathrm{diag}(5b^2 - a^2, -3b^2 + 3a^2,\\ &\quad -7b^2 - a^2, -7b^2 - a^2)\\ &= \mathrm{diag}(\tilde{\rho}, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3),\end{aligned} \tag{14}$$

where $\tilde{\rho}$ and $\tilde{p}_i$ are the energy density and the principal pressures, respectively. The no-hair theorems require the dominant energy condition (DEC) and the strong-energy condition (SEC) to hold. However, since $\tilde{\rho} + \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = -3b^2 < 0$ the SEC is always violated when $b \neq 0$. The DEC is violated when $\tilde{\rho} < 0$ and the weak energy condition is also violated because $\tilde{\rho} + \tilde{p}_2 = \tilde{\rho} + \tilde{p}_3 = -(a^2 + b^2)/2 < 0$. These violations also ensure that the singularity theorems will not hold for these universes and they have no initial or final singularities.

Are these solutions stable? Because of the complexity of the equations of motion it is difficult to extract information about the stability of these nonperturbative solutions in general. In the class of spatially homogeneous cosmologies the dynamical systems approach has been extremely powerful for determining asymptotic states of Bianchi models. A similar approach can be adopted to the class of models considered here; however, the complexity of the phase space increases dramatically due to the higher-derivative terms. Nonetheless, some stability results can be easily extracted. Consider, for example, a perfect fluid with a barotropic equation of state, $p = w\rho$, where w is constant. Because of the exponential expansion, the value of the deceleration parameter is $q \equiv -(1 + \dot{H}/H^2) = -1$ for the type *II* and VI_h solutions given. Hence, these vacuum solutions will be stable against the introduction of a perfect fluid with $w > -1$. This includes the important cases of dust ($w = 0$), radiation ($w = 1/3$) and inflationary stresses ($-1 < w < -1/3$).

For perturbations of the shear and the curvature, the situation is far more complicated. Even within the class of Bianchi models in general relativity a full stability analysis is lacking. However, in some cases, some of the modes can be extracted. Consider again the Bianchi type *II* solution, Eq. (10). Using the trace of the evolution equations, Eq. (8), we consider a perturbation of the Ricci scalar:

$$R \approx 4\Lambda + r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t}.$$

Using $\Box R = -(\ddot{R} + \theta\dot{R})$, which is valid for spatially homogeneous universes, Eq. (8) again implies, to lowest order:

$$\lambda_{1,2} = -\frac{3H}{2}\left(1 \pm \sqrt{1 - \frac{2}{9H^2(3\alpha + \beta)}}\right),$$

for $(3\alpha + \beta) \neq 0$. This shows that the perturbation of the Ricci scalar gives the same eigenmodes for the anisotropic solutions of types *II* and VI_h as it did for perturbations of de Sitter spacetime in Eq. (9). In order to determine the stability of other modes, like shear and anisotropic curvature modes, further analysis is required.

V. DISCUSSION

The solutions that we have found raise new questions about the thermodynamic interpretation of spacetimes. We are accustomed to attaching an entropy to the geometric structure created by the presence of a cosmological constant, for example, the event horizon of de Sitter spacetime. Do these anisotropically inflating solutions have a thermodynamic interpretation? If they are stable they may be related to dissipative structures that appear in nonequilibrium thermodynamics and which have appeared to have been identified in situations where de Sitter metrics appear in the presence of stresses which violate the strong-energy condition [7,33–35]. They also provide a new perspective on the physical interpretation of higher-order gravity terms in the gravitational Lagrangian.

In summary: we have found exact cosmological solutions of a gravitational theory that generalizes Einstein's by the addition of quadratic curvature terms to the action. These solutions display the new phenomenon of anisotropic inflation when $\Lambda > 0$. They do not approach the de Sitter spacetime asymptotically and provide examples of new outcomes for inflation that is driven by a $p = -\rho$ stress and begin from "general" initial conditions.

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