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Nonsymmetric trapped surfaces in the Schwarzschild and Vaidya spacetimes

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Marginally trapped surfaces (MTSs) are commonly used in numerical relativity to locate black holes. For dynamical black holes, it is not known generally if this procedure is sufficiently reliable. Even for Schwarzschild black holes, Wald and Iyer constructed foliations which come arbitrarily close to the singularity but do not contain any MTSs. In this paper, we review the Wald-Iyer construction, discuss some implications for numerical relativity, and generalize to the well-known Vaidya spacetime describing spherically symmetric collapse of null dust. In the Vaidya spacetime, we numerically locate non spherically symmetric trapped surfaces which extend outside the standard spherically symmetric trapping horizon. This shows that MTSs are common in this spacetime and that the event horizon is the most likely candidate for the boundary of the trapped region.

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Introduction.—In stationary black hole spacetimes, there is a strong correspondence between marginally trapped surfaces (MTSs) and event horizons (EHs) because cross sections of stationary EHs are MTSs. MTSs are also featured prominently in the frameworks of isolated, dynamical and trapping horizons which have shed considerable light on the properties of classical and quantum black holes even in the dynamical regime. See e.g. [1–3] for reviews. Numerical simulations routinely look for MTSs to locate black holes on a Cauchy surface. This is because, while MTSs can be located on a Cauchy surface in real time, the EH can only be located *a posteriori* after the simulation has been successfully completed. MTSs can be useful for extracting physical information about a black hole in a numerical simulation [4].

However, in dynamical situations, the correspondence between MTSs and EHs is lost (beyond the fact that MTSs are enclosed by the EH); the event horizon is, in general, an expanding null surface, while outgoing light rays from a MTS have, by definition, zero expansion. An explicit example was constructed by Wald and Iyer [5] where they showed that even in the Schwarzschild spacetime, there exist perfectly regular Cauchy surfaces which come arbitrarily close to the singularity and foliate the spacetime, but which nevertheless do not contain any MTSs. While this has not been an issue in most numerical simulations to date, it raises the question of whether MTSs can be found generally in numerical simulations of black hole spacetimes.

However, there are other results which indicate that MTSs should be common in black hole spacetimes. For example, it was suggested by Eardley [6] that an MTS can be locally perturbed in an arbitrary spacelike direction to yield a 1-parameter family of MTSs. A precise formulation of this idea and its proof follows from recent results of Andersson *et al.* [7]. From this result, it seems plausible that for a "generic" Cauchy surface Σ passing through the black hole region, one should be able to perturb a nearby MTS to lie on Σ. Thus, Cauchy surfaces passing through the black hole should "generically" contain a MTS. A related issue is the question of the boundary of the trapped region of a black hole spacetime; clearly, a MTS cannot be perturbed to lie outside this boundary. Eardley suggests that the boundary should be the event horizon, while arguments by Hayward suggest that the boundary should be a trapping horizon [8]. For stationary black holes the two notions coincide, but not for dynamical black holes.

In this paper, we study these issues from a numerical relativity perspective. In particular, we study the spherically symmetric Schwarzschild and Vaidya spacetimes using foliations not adapted to the spherical symmetry. Very little is known so far about trapped or marginally trapped surfaces on such slices, either analytically or numerically. We find that for Vaidya, MTSs are indeed easy to locate, and we do not encounter the problem suggested by Wald and Iyer. This suggests that Cauchy surfaces of the type suggested by Wald and Iyer (which presumably exist also in the Vaidya spacetime) are exceptional. In Vaidya, we find that the nonsymmetric marginal surfaces lie partially outside the standard $r = 2M$ surface (which in this case is a trapping horizon, and lies inside the event horizon), thus indicating that the event horizon is a better candidate for the boundary of the trapped region as suggested by Eardley. In the remainder of this paper, we review basic concepts regarding trapped surfaces and horizons, outline Wald and Iyer's construction with an explicit example and remark on its implications for numerical relativity, and finally discuss trapped surfaces in the Vaidya spacetime.

Trapped surfaces and horizons.—Let ℓ^a and n^a be the future directed null normals of a closed 2-surface S. Let q_{ab} be the 2-metric on S induced by the spacetime metric.

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The expansion of ℓ^a is $\Theta_{(\ell)} = q^{ab}\nabla_a \ell_b$ with a similar definition for $\Theta_{(n)}$. The surface S is said to be trapped if both expansions are negative: $\Theta_{(\ell)} < 0$ and $\Theta_{(n)} < 0$. For a *marginally trapped surface*, one or both of these inequalities are replaced by an equality instead. Weakly trapped surfaces have $\Theta_{(n)} \leq 0$, $\Theta_{(\ell)} \leq 0$. All these definitions are invariant under arbitrary positive rescalings of ℓ^a and n^a.

The definition of a *marginally outer-trapped surface* (MOTS) requires a choice of an "outgoing" direction with respect to future null infinity or spatial infinity. This choice of outgoing direction breaks the symmetry between the two null normals ℓ^a and n^a. A MOTS is thus a MTS with $\Theta_{(\ell)} = 0$, where ℓ is the outgoing direction.

The *trapped region* is the region where trapped surfaces exist. This is defined either in the full spacetime or on a Cauchy surface Σ. A point is in the trapped region if there is a trapped surface which contains that point. Similarly, a point is in the trapped region of Σ iff there is a trapped surface on Σ that contains this point. The *apparent horizon* (AH) on Σ is the boundary of the trapped region of Σ. As such, its definition is so complicated that it is numerically not feasible to look for it directly. However, an AH is also a MTS [9], and these can be efficiently detected.

Finally, a *marginal surface* (MS) [8] is a surface where one of the null normal's expansion vanishes, i.e., $\Theta_{(\ell)} = 0$ where ℓ can be any of the two null directions, *with no restriction on* $\Theta_{(n)}$. Unlike the definition of a MOTS, marginal surfaces do not require globally defined outgoing/ingoing directions. It is called a *future marginal surface* if $\Theta_{(n)} < 0$ and a *past marginal surface* if $\Theta_{(n)} > 0$. A future marginal surface is the same as a MTS and usually arises in numerical simulations as the cross section of a *dynamical horizon* (DH) [10] or an *isolated horizon* (IH) [4], or more generally, a trapping horizon [8].

The Wald-Iyer construction.—Wald and Iyer [5] construct a foliation of the extended Schwarzschild spacetime in which the spacelike hypersurfaces come arbitrarily close to the singularity, but nevertheless do not contain any trapped surfaces. It should be noted that these foliations, while special, are not pathological in any sense and they can be readily constructed in a numerical code. Wald and Iyer prove that, if the intersection of the slice with the trapped region lies in the past of, roughly speaking, "a single event on the future singularity," no slice of such a foliation contains a trapped surface. This construction relies on the existence of angular horizons in the black hole region, just as in a cosmological spacetime near the initial singularity.

An explicit example of one such Cauchy surface is easy to construct. Consider the extended Schwarzschild spacetime in Kruskal coordinates (T, X, θ, ϕ) [see Eq. (6.4.29) of [11]]. The hypersurface $T = k\cos\theta$ can be easily shown to satisfy the Wald-Iyer condition for $k < 1/2$. Thus, even though this surface enters the black hole region, it does not contain any trapped surfaces. Such a slice is depicted schematically in Fig. 1. It intersects the black hole horizon for $T > 0$, and the white hole horizon for $T < 0$.

Even though it is a fact that the above Cauchy surface does not contain a MOTS, standard apparent horizon trackers employed in numerical simulations will happily find an "apparent horizon" on this slice. This apparent contradiction is an issue of terminology. What the apparent horizon tracker will locate is the intersection of the Cauchy surface with the surface $T = X$, which is the bold line in Fig. 1. The intersection is the 2-sphere given by $T = X = k\cos\theta$. In numerical relativity, one typically chooses that part of $\mathcal{I}^+$ which belongs to one specific asymptotically flat end of the spacetime. Thus, the outgoing null normal ℓ^a and the ingoing null normal n^a are the ones shown in Fig. 1. With this choice of ℓ^a, the surface given above satisfies $\Theta_{(\ell)} = 0$. However, this "apparent horizon" is not a MTS because $\Theta_{(n)} < 0$ on the black hole portion and $\Theta_{(n)} > 0$ on the white hole portion.

What is often loosely called an "apparent horizon" in numerical relativity, or almost as loosely, "marginally outer-trapped surface," is really only a marginal surface, or a future marginal surface if the condition $\Theta_{(n)} < 0$ is checked (which it is often not). Determining the globally outgoing direction is usually either unpractical or impossible in numerical simulations. If done, it requires some additional knowledge of the simulated spacetime that the code itself generally does not have. The easiest way to avoid such situations in numerical relativity is to explicitly make sure that the apparent horizon is future trapped by verifying that $\Theta_{(n)}$ is negative everywhere, both in the initial data and during evolution. Regarding black hole

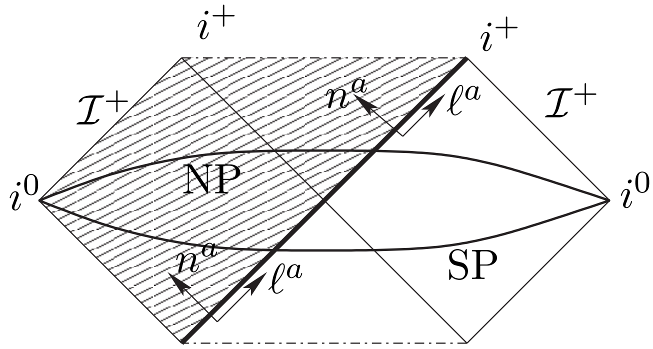


FIG. 1. Penrose diagram of the extended Schwarzschild spacetime, from a numerical relativity point of view. The bold line is the null world tube of marginal surfaces. The hatched area is the region of spacetime that is invisible to the "observer at infinity," which is located at the $\mathcal{I}^+$ to the right. Every point in this figure is a sphere as in conventional Penrose diagrams, *except* for the curves labeled NP and SP. If we represent the Cauchy surface as $f(T, X, \theta, \phi) = 0$, then NP and SP are, respectively, the projections of the north pole ($\theta = 0$) and south pole ($\theta = \pi$); the intermediate angles lie in between. NP enters the trapped region, but SP does not.

initial data, the construction presented in [12] will ensure that the marginal surfaces are future trapped and will therefore avoid any of the Wald-Iyer slices if the lapse function is kept nonnegative everywhere.

Trapped surfaces in the Vaidya spacetime.— Generalizing to dynamical situations, consider the Vaidya spacetime which describes a spherically symmetric collapse of null dust (radiation) [13]. This is an astrophysically unrealistic toy model, but it does serve as a very useful testing ground. It has been extensively used, for example, to study the formation of naked singularities. In ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ), the metric is

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right)dv^2 + 2dvdr + r^2d\Omega^2, \quad (1)$$

where the mass function $M(v)$ can be specified as a function of the null coordinate v. For constant $M(v)$, this is just the standard Schwarzschild metric in ingoing Eddington-Finkelstein coordinates. The stress energy tensor is determined by the derivative of $M(v)$:

$$T_{ab} = \frac{\dot{M}(v)}{4\pi r^2}\partial_a v \partial_b v \quad (2)$$

where $\dot{M} = \partial M/\partial v \geq 0$. We shall use a time coordinate defined as $t = v - r$ and we shall take the mass function to be nonzero only for $v > 0$. Thus, the spacetime is flat for $v \leq 0$.

It is easy to see that the only spherically symmetric MTSs are the spheres given by $r = 2M(v_0)$ for a specified v_0. These will be the apparent horizons on spherically symmetric Cauchy surfaces which intersect the $r = 2M(v)$ surface. Let us denote the $r = 2M(v)$ surface by H. It is easy to show that H is spacelike and is a trapping horizon. The EH lies outside H and is strictly separated from H when $\dot{M} > 0$; at late times, H asymptotes to the EH [10].

Let us now consider non spherically symmetric Cauchy surfaces. There is now an important qualitative difference from the Schwarzschild case. There, the analog of H was null and expansion free; the intersection of any spacelike surface with H was then a marginal surface, as long as this intersection was, topologically, a complete sphere. This is also true more generally when the black hole is isolated in an otherwise dynamical spacetime (if H is an isolated horizon). However, in genuinely dynamical situations, H is spacelike as in the Vaidya example. In this case, if the intersection of a Cauchy surface with H is not one of the spherically symmetric marginal surfaces, *then the intersection cannot be a marginal surface even if it is a complete 2-sphere*. This statement follows directly from Theorem 4.2 of [14]. Thus the question naturally arises: are there apparent horizons on nonsymmetric Cauchy surfaces in the Vaidya spacetime? One would expect there to be Wald-Iyer Cauchy surfaces which come arbitrarily close to the singularity but which do not contain any marginal surfaces, but we shall now see that apparent horizons do exist on a large class of nonsymmetric Cauchy surfaces.

We choose the mass function

$$M(v) = \begin{cases} 0 & \text{for } v \leq 0, \\ M_0 v^2/(v^2 + W^2) & \text{for } v > 0, \end{cases} \quad (3)$$

with the constants $M_0 = 1$ and $W = 1/10$. This is a short pulse of radiation that forms a black hole with the final mass M_0. This mass function is only C^1 at $v = 0$, but our results are unchanged qualitatively for other mass functions with higher differentiability. For this mass function the singular point $v = 0$, $r = 0$ is locally naked (see e.g. [15]), but this is not relevant for our purposes.

We examine the spacetime with a slicing that is only axially symmetric. We use a time coordinate $\bar{t}$ given by

$$\bar{t} = t - \alpha z = v - r(1 + \alpha\cos\theta), \quad (4)$$

where $t = v - r$ is the standard Vaidya time, and the constant α determines how much the slice is boosted in the z direction. We chose $\alpha = 10/11$. We have also examined other more complicated foliations, but the results presented below do not change qualitatively.

The results are shown in Figs. 2 and 3. Figure 2 shows the t-z section of the Cauchy surfaces for a range of $\bar{t}$ values, and the distorted MSs on these sections. This clearly shows that the MSs extend outside H and can also extend into the flat region. Figure 3 shows the MS on the $\bar{t} = -0.3$ and $\bar{t} = 0$ slices. The MS at $\bar{t} = 0$ is a

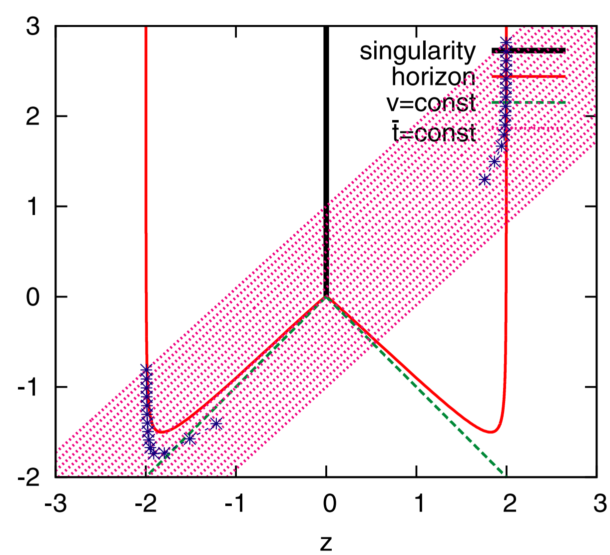
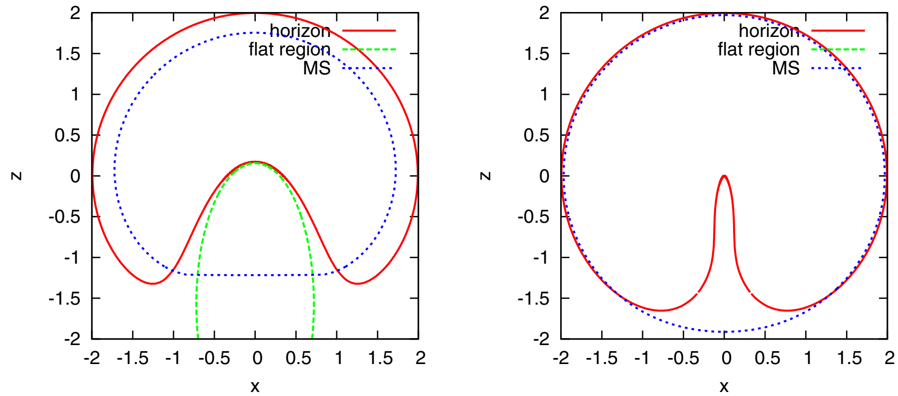


FIG. 2 (color online). The t-z section of the Vaidya spacetime where $t = v - r$. The tilted lines are (sections of) the axisymmetric surfaces given in Eq. (4) for a range of $\bar{t}$ values. The marginal surfaces on these sections are marked by a "$\star$." The dynamical horizon H [the $r = 2M(v)$ surface] are the pair of bold curves, and the two dashed straight lines are the $v = 0$ light cone which is the boundary of the flat portion of the spacetime. The singularity is the positive t axis ($z = 0$, $t \geq 0$).



(a) $\bar{t} = -0.3$ (b) $\bar{t} = 0.0$

FIG. 3 (color online). The location of the distorted marginal surface on the *x*-*z* section of the Cauchy surfaces at $\bar{t} = -0.3, 0.0$. The solid curve is the intersection with the dynamical horizon H, the dotted curve is the distorted MS, and the region inside the dashed curve is the $v < 0$ region, i.e., the flat portion of spacetime. Note that at $\bar{t} = 0.0$, the Cauchy surface does not intersect the flat portion at all, and the distorted MS coincides closely with the DH except near its south pole. In both cases, the north pole is inside the DH while the south pole is outside.

future marginally trapped surface, i.e., $\Theta_{(n)} < 0$. At $\bar{t} = -0.3$, the MS extends into the flat portion, and this part of S is planar, with $\Theta_{(n)} = \Theta_{(\ell)} = 0$. On the rest of the sphere, $\Theta_{(n)} < 0$ as expected; this is therefore a *weakly* marginally trapped surface. Furthermore, the 3-dimensional world tube obtained by stacking up all the MSs turns out to be spacelike; the ones with $\Theta_{(n)} < 0$ form a dynamical horizon.

We also look for surfaces with a small nonvanishing expansion $\Theta_{(\ell)} = \pm 10^{-3}$. These surfaces can be viewed as radial deformations of the MS; the outward deformation has $\Theta_{(\ell)} > 0$, and $\Theta_{(\ell)} < 0$ for the inward deformation. At $\bar{t} = 0$, the inward deformation is strictly trapped and the outward deformation is strictly untrapped. At $\bar{t} = -0.3$, the inward deformation has $\Theta_{(n)} > 0$ in the flat region and $\Theta_{(n)} < 0$ elsewhere. The outward deformation has $\Theta_{(n)} < 0$ everywhere and is thus strictly untrapped. We have not been able to find strictly trapped surfaces which extend into the flat portion of spacetime. Sufficiently far in the future, the MSs asymptote to the spherically symmetric DH and also come arbitrarily close to the EH. Finally, there are restrictions on the location of trapped surfaces in the presence of a dynamical horizon [14]. We have verified that these restrictions are satisfied. The existence of such distorted MSs was already suggested in [14], but with no restrictions on $\Theta_{(n)}$; here we have also shown $\Theta_{(n)} \leq 0$.

Conclusions.—We have numerically studied nonsymmetric trapped surfaces in simple spherically symmetric spacetimes. We have seen that the Wald-Iyer example illustrates the importance of verifying $\Theta_{(n)} \leq 0$ for apparent horizons located numerically. In Vaidya, we have found trapped surfaces which extend outside the usual $r = 2M$ surface H. This shows that H is not the boundary of the trapped region. We have also found marginal surfaces that extend into the flat region of the spacetime.

The boundary of the trapped region should be spherically symmetric, since it is an invariantly defined geometric quantity in a spherically symmetric spacetime. This lends support to Eardley's conjecture that the event horizon is the boundary of the trapped region, since the EH is the only natural candidate. However, we have not found strictly trapped surfaces that extend into the flat region of the spacetime, so that the boundary may be inside the EH.

We conclude with some open questions that need to be addressed: (i) Is it possible to push the marginal surfaces arbitrarily close to the event horizon, even in the flat region? This would verify that the EH is truly the boundary of the trapped region. (ii) For asymptotically flat spacetimes, the event horizon is the natural candidate for the boundary of the trapped region. What is this boundary for non asymptotically flat spacetimes, e.g. asymptotically de Sitter spacetimes where the event horizon is not strictly defined?

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