

Pseudoscalar meson masses in Wilson chiral perturbation theory for 2 + 1 flavorsS. Aoki,^{1,2} O. Bär,¹ T. Ishikawa,³ and S. Takeda¹¹*Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan*²*Riken BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA*³*Center for Computational Physics, University of Tsukuba, Tsukuba 305-8577, Japan*

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We consider 2 + 1 flavor Wilson chiral perturbation theory including the lattice spacing contributions of $O(a^2)$. We adopt a power counting appropriate for the unquenched lattice simulations carried out by the CP-PACS/JLQCD Collaboration and compute the pseudoscalar meson masses to one loop. These expressions are required to perform the chiral extrapolation of the CP-PACS/JLQCD lattice data.

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I. INTRODUCTION

The limitations of the quenched approximation in numerical lattice QCD simulations is by now well established. For example, the light hadron mass spectrum calculated by the CP-PACS Collaboration [1] deviates from the experimentally measured values by about 10%. Even though the quenching error is different for different observables, one must assume the quenching error to be of the same order for other quantities as well. Once the effect of dynamical up and down quarks is included, the quenching error is significantly reduced and the discrepancy between the numerically calculated and the experimentally measured values is much smaller compared with the quenched results [2]. Still, ignoring the effect of a dynamical strange quark in these unquenched 2 flavor simulations leads to an uncertainty, which is expected to be non-negligible. Only simulations with a dynamical strange quark will provide numerical results, which can be compared with experiment with confidence.

In order to eliminate the remaining source of quenching error the CP-PACS and JLQCD Collaborations have been carrying out unquenched 2 + 1 flavor simulations. A RG-improved gauge action and an $O(a)$ improved Wilson quark action have been adopted. The size of the lattice is modest ($L \simeq 2$ fm) and simulations at three different lattice spacings ($a \simeq 0.7$ fm, 1.0 fm, 1.22 fm) are planned so that the continuum limit can be taken. Five different masses for the degenerate up and down type quarks are simulated leading to pseudoscalar meson masses in the range $m_{PS}/m_V \simeq 0.62 - 0.78$. The physical strange quark mass lies between the two simulated strange quark masses and can therefore be reached by an interpolation. More details and the status of these simulations have been recently summarized in Ref. [3].

The masses for the up and down quarks are rather heavy and an extrapolation to their physical values is required. The functional forms for the extrapolation are usually given by Chiral Perturbation Theory (ChPT). In its stan-

dard form [4,5] the expressions derived in ChPT can be used after the continuum limit of the lattice data has been taken. However, for various reasons it is advantageous to perform the chiral extrapolation before the continuum limit. In this case ChPT needs to be formulated for lattice QCD at nonzero lattice spacing a . The main idea how this can be done was proposed in Ref. [6,7]. Since then many observables have been calculated at one loop order (for an overview see Ref. [8]). For lattice theories with Wilson fermions, however, all results were derived for unquenched 2 flavor Wilson ChPT (WChPT) and the chiral expressions are therefore not applicable for the CP-PACS/JLQCD simulations.

This is the first paper in a series where we provide the one loop expressions of 2 + 1 flavor WChPT for a variety of observables, which will be measured by the CP-PACS/JLQCD Collaboration. Here we present the results for the simplest observables, the pseudoscalar meson masses. The second paper is devoted to the vector meson masses and the third to the pseudoscalar decay constants and axial vector Ward identity quark mass [9,10]. In our calculations we include the lattice spacing contributions through $O(a^2)$ and adopt various power countings. Even though we have primarily the CP-PACS/JLQCD simulations in mind, our expressions are equally useful for other unquenched 2 + 1 flavor simulations employing Wilson fermions.

There is no fundamental difficulty in applying the framework of WChPT to 2 + 1 flavors, the main difference to the 2 flavor results is just the increased complexity of the final results. Since we follow the standard strategy of WChPT we will be brief in presenting our results. In Sec. II we explain the power counting which we assume and present the chiral Lagrangian up to next-to-leading order. The calculation of the pseudoscalar masses from this Lagrangian is straightforward and we summarize our final results in Sec. III. Many technical details and some intermediate results are collected in Appendices A and B.

II. THE CHIRAL EFFECTIVE LAGRANGIAN

A. The order counting

In continuum chiral perturbation theory (ChPT), $M = m$ or p^2 is the expansion parameter, where m is the quark mass and p is the momentum of the pseudoscalar meson. Since chiral symmetry is explicitly broken in lattice QCD with Wilson-type quarks, corrections due to the nonzero lattice spacing a are non-negligible. The construction of the chiral effective theory for Lattice QCD with Wilson fermions, so-called Wilson ChPT (WChPT), has become standard by now. From a conceptual point of view there is nothing new in applying the familiar techniques [6,11–13] to $2 + 1$ flavor lattice QCD. The only nontrivial choice one has to make is the order counting one adopts in the chiral expansion, which we are going to explain in this section.¹

The leading order (LO) chiral Lagrangian of the WChPT is constructed from $O(M)$ terms and the $O(a)$ term. Since the $O(a)$ term has the same chiral structure as the mass term, the LO Lagrangian of the WChPT assumes the same form as the one of continuum ChPT, provided one performs the replacement $m \rightarrow \tilde{m} = m + c_1 a$ with c_1 being a combination of two low-energy constants [6,11]. Based on this LO Lagrangian, however, the pion becomes a tachyon for $\tilde{m} = m + c_1 a < 0$ since the pion mass squared m_π^2 is given by $m_\pi^2 = 2B(m + c_1 a)$ at tree level. This is in contrast to continuum ChPT where chiral symmetry dictates that the pion mass is given by $m_\pi^2 = B|m|$. This already indicates that the first nontrivial modification to continuum ChPT starts at $O(a^2)$. Indeed, including the $O(a^2)$ term in the LO chiral Lagrangian removes the unphysical tachyon from the theory, as has been shown in Ref. [12] for the 2 flavor theory. We therefore conclude that, for the consistency of the theory, the $O(a^2)$ term should be included in the LO Lagrangian in the WChPT.

Although the $O(a)$ term is superficially larger than the $O(a^2)$ term, the former term is irrelevant since it is absorbed in the definition of the shifted mass \tilde{m} . The $O(a^2)$ correction, on the other hand, becomes important in the regime where $\tilde{m}/\Lambda_{\text{QCD}} \approx \Lambda_{\text{QCD}}^2 a^2$, even though the $\Lambda_{\text{QCD}}^2 a^2$ correction is much smaller than 1 in general.

Suppose we consider the $O(\tilde{M}) = O(\tilde{m}, p^2)$ and the $O(a^2)$ term as LO terms. Then the terms of $O(\tilde{M}^2, \tilde{M}a^2, a^4)$ can be regarded as next-to-leading order (NLO), since the loop expansion in WChPT increases in units of \tilde{M} . We remark that the $O(a^4)$ term is not as relevant as the $O(\tilde{M}^2, \tilde{M}a^2)$ terms for our final results for the pseudoscalar masses. We will need the tree level contribution of the $O(a^4)$ term only to satisfy $m_{\text{PS,NLO}}^2 = 0$ at

$m_{\text{PS,NLO}}^2 = 0$, where $m_{\text{PS,NLO}}$ is the pseudoscalar meson mass at NLO. This means that there is essentially no unknown low-energy constant associated with the $O(a^4)$ correction.²

The proper order counting of the $O(\tilde{M}a)$ term is more subtle than for the previously discussed terms.³ Depending on the size of the $O(\tilde{M}a)$ contributions we may include it at leading order where it subsequently enters the chiral logs, or we treat it as a NLO term and include it at tree level only. The size of this term is indeed expected to vary significantly, depending on details of the action of the underlying lattice theory. The $O(\tilde{M}a)$ term contains one power of a and stems from the Pauli term in the Symanzik's effective action [15–17], which is used in an intermediate step to match the lattice to the chiral effective theory. Consequently, the $O(\tilde{M}a)$ contribution in the chiral Lagrangian is directly proportional to the coefficient of the Pauli term [11], and the size of this coefficient is much smaller for improved theories with a clover term in the action than for standard Wilson fermions. For fully nonperturbatively improved theories the coefficient is equal to zero and no $O(\tilde{M}a)$ term appears in the chiral Lagrangian.⁴

Since we have no *a priori* knowledge about the size of the $O(\tilde{M}a)$ term we will be as general as possible and present our results for three different order countings of this term. We first keep the $O(\tilde{M}a)$ term at LO where it gives a contribution to the chiral log corrections. If the $O(\tilde{M}a)$ term is assumed to be much smaller than the other LO terms we can easily expand our results. Performing this expansion is equivalent to doing the calculation with keeping the $O(\tilde{M}a)$ term at NLO. Finally we can set this term to zero in order to obtain the results for nonperturbatively improved theories.

B. Leading order Lagrangian

According to the discussion in the previous section, we use the following LO Lagrangian, which consists of terms of $O(\tilde{M}, a^2, \tilde{M}a)$:

²This may seem odd within WChPT, but it is a simple consequence of the fact that the pseudoscalar meson becomes massless in lattice QCD at the critical quark mass. In other words, the $O(a^4)$ term merely results in an additional shift in the critical quark mass. An implicit assumption we make here is the presence of a phase where flavor and parity is spontaneously broken [14].

³The $O(a^3)$ term is unproblematic since the arguments we gave for the $O(a^4)$ contribution also apply for the $O(a^3)$ term.

⁴Strictly speaking this only holds if the chiral Lagrangian is parameterized in terms of renormalized quark masses which absorb some of the $O(a)$ cutoff effects. Particular terms linear in a will appear if other choices are used (see Sec. III C).

¹For notational simplicity we assume N degenerate quark masses in the following discussion.

$$\begin{aligned}
 L_{\text{LO}} = & \frac{f^2}{4} \langle \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \rangle - \frac{f^2 B}{2} \langle M_q \Sigma + \Sigma^\dagger M_q \rangle + \frac{f^2}{16} [c_2 \langle \Sigma + \Sigma^\dagger \rangle^2 + \tilde{c}_2 \langle (\Sigma + \Sigma^\dagger)^2 \rangle + c_4 \langle \Sigma - \Sigma^\dagger \rangle^2] \\
 & + \frac{f^2}{4} [c_0 \langle \Sigma + \Sigma^\dagger - 2 \rangle \langle \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \rangle + \tilde{c}_0 \langle (\Sigma + \Sigma^\dagger - 2) \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \rangle] + \frac{f^2 B}{8} [2c_3 \langle \Sigma + \Sigma^\dagger \rangle \langle M_q \Sigma + \Sigma^\dagger M_q \rangle \\
 & + \tilde{c}_3 \langle (\Sigma + \Sigma^\dagger) (M_q \Sigma + \Sigma^\dagger M_q) \rangle] + \frac{f^2 B}{8} [2c_5 \langle \Sigma - \Sigma^\dagger \rangle \langle M_q \Sigma - \Sigma^\dagger M_q \rangle + \tilde{c}_5 \langle (\Sigma - \Sigma^\dagger) (M_q \Sigma - \Sigma^\dagger M_q) \rangle], \quad (1)
 \end{aligned}$$

where $\langle X \rangle = \text{tr} X$, f is the pseudoscalar meson decay constant,

$$\Sigma = \exp \left[i \frac{1}{f} \sum_a \pi_a T^a \right], \quad (2)$$

is an element of SU(3) with π_a being the pseudoscalar meson fields. The SU(3) generators T^a are normalized according to $\text{tr} T^a T^b = \frac{1}{2} \delta_{ab}$. The first and the second term in the first line are the standard $O(p^2)$ and $O(\tilde{m})$ terms [5], respectively. The second line comprises the $O(a^2)$ terms [12,13]. The last three lines contain the $O(p^2 a)$ and $O(\tilde{m} a)$ contributions [11].

For notational simplicity only we use a different notation for the low-energy constants associated with the nonzero lattice spacing (the c and \tilde{c} 's) compared to the notation in Refs. [11–13]. In particular, we have chosen to absorb the explicit powers of the lattice spacing a into the coefficients c , \tilde{c} . Consequently, as a function of a these coefficients scale according to

$$c_0, \tilde{c}_0, c_3, \tilde{c}_3, c_5, \tilde{c}_5 = O(a), \quad c_2, \tilde{c}_2, c_4 = O(a^2). \quad (3)$$

The quark mass matrix is given by

$$\begin{aligned}
 M_q &= \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m_s \end{pmatrix} = M_0 \mathbf{1} + M_8 T^8, \quad (4) \\
 M_0 &= \frac{2m + m_s}{3}, \quad M_8 = \frac{2(m - m_s)}{\sqrt{3}},
 \end{aligned}$$

where isospin symmetry ($m_u = m_d = m$) is assumed. Note that $O(a)$ contribution is already absorbed in the definition of M_q , so there is no $O(a)$ term in the chiral Lagrangian. In the $a \rightarrow 0$ limit, the pseudoscalar meson masses are related to m and m_s according to

$$m_a^2 = \begin{cases} m_\pi^2 = 2Bm, & a = 1, 2, 3, \\ m_K^2 = B(m + m_s), & a = 4, 5, 6, 7, \\ m_\eta^2 = \frac{2B}{3}(m + 2m_s), & a = 8, \end{cases} \quad (5)$$

which, of course, agree with the continuum ChPT result. We also define the average mass

$$m_{\text{av}}^2 = \frac{1}{N^2 - 1} \sum_a m_a^2 = 2B \frac{2m + m_s}{3}, \quad (6)$$

which will be a useful short hand notation in the following.

Note that, except for a π_a independent constant, the term proportional to \tilde{c}_5 is identical to the term with \tilde{c}_3 . Therefore we can set $\tilde{c}_5 = 0$ without loss of generality.

C. Shifted quark mass at leading order

By expanding the LO Lagrangian to second order in π_a , we obtain

$$L^{(2)} = \frac{1}{2} \sum_a [(\partial_\mu \pi_a)^2 + \tilde{m}_a^2 \pi_a^2]. \quad (7)$$

The pseudoscalar meson masses at LO are therefore given by

$$\tilde{m}_a^2 = m_a^2 (1 - Nc_3 - \tilde{c}_3) - m_{\text{av}}^2 Nc_3 - Nc_2 - \tilde{c}_2. \quad (8)$$

In the following we keep the number of flavors N arbitrary, but put $N = 3$ in the final expressions.

We now define *shifted quark masses*, which satisfy

$$\tilde{m}_a^2 = \begin{cases} \tilde{m}_\pi^2 = 2B\tilde{m}, & a = 1, 2, 3, \\ \tilde{m}_K^2 = B(\tilde{m} + \tilde{m}_s), & a = 4, 5, 6, 7, \\ \tilde{m}_\eta^2 = \frac{2B}{3}(\tilde{m} + 2\tilde{m}_s), & a = 8. \end{cases} \quad (9)$$

Explicitly they are given by

$$\tilde{m} = m(1 - Nc_3 - \tilde{c}_3) - \frac{2m + m_s}{3} Nc_3 - \frac{Nc_2 + \tilde{c}_2}{2B}, \quad (10)$$

$$\tilde{m}_s = m_s(1 - Nc_3 - \tilde{c}_3) - \frac{2m + m_s}{3} Nc_3 - \frac{Nc_2 + \tilde{c}_2}{2B}. \quad (11)$$

From these expressions the LO critical mass for $N = 3$ is defined by the condition

$$\tilde{m}_{\text{av}}^2 = m_{\text{av}}^2 (1 - 6c_3 - \tilde{c}_3) - 3c_2 - \tilde{c}_2 = 0, \quad (12)$$

leading to

$$2Bm_{\text{critical}} = \frac{3c_2 + \tilde{c}_2}{1 - 6c_3 - \tilde{c}_3} \equiv -\delta m_{\text{av}}^2. \quad (13)$$

This definition for the critical quark mass assumes that all three quark masses are extrapolated to the massless point, including the strange quark mass. In numerical spectroscopy calculations, however, a different definition is sometimes employed where the strange quark mass is kept fixed at (approximately) its physical value. For this procedure the condition for the critical quark mass reads

$$\tilde{m}_\pi^2 = m_\pi^2(1 - 3c_3 - \tilde{c}_3) - m_{\text{av}}^2 3c_3 - 3c_2 - \tilde{c}_2 = 0, \quad (14)$$

which results in

$$2Bm_{\text{critical}}(m_s) = \frac{2Bm_s c_3 + 3c_2 + \tilde{c}_2}{1 - 5c_3 - \tilde{c}_3}. \quad (15)$$

The difference between these two values is therefore

$$m_{\text{critical}} - m_{\text{critical}}(m_s) = \frac{c_3}{1 - 5c_3 - \tilde{c}_3} \left[\frac{3c_2 + \tilde{c}_2}{1 - 6c_3 - \tilde{c}_3} - 2Bm_s \right]. \quad (16)$$

Indeed, numerical 2 + 1 flavor simulations [18] suggest a nonvanishing value for this difference with

$$m_{\text{critical}} - m_{\text{critical}}(m_s) > 0. \quad (17)$$

D. NLO Lagrangian

The NLO Lagrangian provides the necessary counter terms in order to remove the UV divergences in the 1-

loop integrals. The contribution of $O(\tilde{M}^2)$ is given by

$$\begin{aligned} L_{\text{NLO}, \tilde{M}^2} = & L_4 \langle \Sigma_{\mu\mu} \rangle \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle \\ & + L_5 \langle \Sigma_{\mu\mu} (\hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q) \rangle \\ & + L_6 \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle^2 + L_7 \langle \hat{M}_q \Sigma - \Sigma^\dagger \hat{M}_q \rangle^2 \\ & + L_8 \langle \hat{M}_q \Sigma \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \Sigma^\dagger \hat{M}_q \rangle, \end{aligned} \quad (18)$$

where we introduced $\Sigma_{\mu\mu} = \partial_\mu \Sigma \partial_\mu \Sigma^\dagger$ and $\hat{M}_q = B\tilde{M}_q$, where \tilde{M}_q is the shifted quark mass matrix constructed from \tilde{m} and \tilde{m}_s (cf. previous section). The NLO constants L_i are related with standard Gasser-Leutwyler coefficients L_i^{GL} [5] as $L_{4,5} = 2L_{4,5}^{GL}$ and $L_{6,7,8} = -4L_{6,7,8}^{GL}$.

The complete Lagrangian at $O(a^2\tilde{M})$ and $O(a\tilde{M}^2)$ is cumbersome and has not been computed so far. Here we only list the terms that contribute to the meson propagators, which we need for the calculation of the pseudoscalar masses. These terms are straightforwardly found by a spurion analysis with spurion fields proportional to the lattice spacing [6,11,13]. Our result for the $O(a^2\tilde{M})$ Lagrangian reads

$$\begin{aligned} L_{\text{NLO}, a^2\tilde{M}} = & \langle \Sigma_{\mu\mu} \rangle \left(W_0 + \frac{W_1}{4N^2} \langle \Sigma + \Sigma^\dagger \rangle^2 + \frac{W_2}{2N} \langle \Sigma^2 + (\Sigma^\dagger)^2 \rangle \right) + \frac{W_3}{4N} \langle \Sigma_{\mu\mu} (\Sigma + \Sigma^\dagger) \rangle \langle \Sigma + \Sigma^\dagger \rangle \\ & + \frac{W_4}{2} \langle \Sigma_{\mu\mu} (\Sigma^2 + (\Sigma^\dagger)^2) \rangle + W_5 \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle + W_6 \langle \Sigma + \Sigma^\dagger \rangle^2 \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle + W_7 \langle \Sigma^2 + (\Sigma^\dagger)^2 \rangle \\ & \times \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle + W_8 \langle \Sigma + \Sigma^\dagger \rangle \langle \hat{M}_q \Sigma^2 + (\Sigma^\dagger)^2 \hat{M}_q \rangle + W_9 \langle \hat{M}_q \Sigma^3 + (\Sigma^\dagger)^3 \hat{M}_q \rangle + W_{10} \langle \Sigma + \Sigma^\dagger \rangle \langle \hat{M}_q \rangle \\ & + W_{11} \langle (\partial_\mu \Sigma)^2 + (\partial_\mu \Sigma^\dagger)^2 \rangle + W_{12} \langle (\partial_\mu \Sigma)^2 (\Sigma^\dagger)^2 + \Sigma^2 (\partial_\mu \Sigma^\dagger)^2 \rangle. \end{aligned} \quad (19)$$

while the terms of $O(a\tilde{M}^2)$ are given by

$$\begin{aligned} L_{\text{NLO}, a\tilde{M}^2} = & \langle \Sigma_{\mu\mu} \rangle \left[2V_0 \langle \hat{M}_q \rangle + \frac{V_1}{2N} \langle \Sigma + \Sigma^\dagger \rangle \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle + V_2 \langle \hat{M}_q \Sigma^2 + (\Sigma^\dagger)^2 \hat{M}_q \rangle \right] + \frac{V_3}{2} \langle \Sigma_{\mu\mu} (\Sigma + \Sigma^\dagger) \rangle \\ & \times \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle + \frac{V_4}{2N} \langle \Sigma_{\mu\mu} (\hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q) \rangle \langle \Sigma + \Sigma^\dagger \rangle + V_5 \langle \Sigma_{\mu\mu} (\hat{M}_q \Sigma^2 + (\Sigma^\dagger)^2 \hat{M}_q) \rangle \\ & + 2V_6 \langle \Sigma_{\mu\mu} \hat{M}_q \rangle + V_{16} \langle (\partial_\mu \Sigma)^2 \hat{M}_q + \hat{M}_q (\partial_\mu \Sigma^\dagger)^2 \rangle + V_{17} \langle (\partial_\mu \Sigma)^2 \Sigma^\dagger \hat{M}_q \Sigma^\dagger + \Sigma \hat{M}_q \Sigma (\partial_\mu \Sigma^\dagger)^2 \rangle \\ & + \langle \Sigma + \Sigma^\dagger \rangle [V_7 \langle \hat{M}_q^2 \rangle + V_8 \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle^2 + V_9 \langle \hat{M}_q \Sigma \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \Sigma^\dagger \hat{M}_q \rangle] + \langle \hat{M}_q \Sigma + \Sigma^\dagger \hat{M}_q \rangle \\ & \times [V_{10} \langle \hat{M}_q \rangle + V_{11} \langle \hat{M}_q \Sigma^2 + (\Sigma^\dagger)^2 \hat{M}_q \rangle] + V_{12} \langle \hat{M}_q^2 \Sigma + \Sigma^\dagger \hat{M}_q^2 \rangle + V_{13} \langle \hat{M}_q \Sigma \hat{M}_q \Sigma^2 + (\Sigma^\dagger)^2 \hat{M}_q \Sigma^\dagger \hat{M}_q \rangle \\ & + V_{14} \langle \Sigma + \Sigma^\dagger \rangle \langle \hat{M}_q \Sigma - \Sigma^\dagger \hat{M}_q \rangle^2 + V_{15} \langle \hat{M}_q \Sigma^2 - (\Sigma^\dagger)^2 \hat{M}_q \rangle \langle \hat{M}_q \Sigma - \Sigma^\dagger \hat{M}_q \rangle. \end{aligned} \quad (20)$$

As in the leading order Lagrangian we have absorbed the explicit powers of the lattice spacing in the low-energy constants, and their scaling behavior is therefore

$$W_i = O(a^2), \quad i = 0 \dots 12, \quad (21)$$

$$V_i = O(a), \quad i = 0 \dots 17. \quad (22)$$

III. RESULTS

A. Quark mass dependence of meson masses

The calculation of the pseudoscalar masses from the chiral Lagrangian in the previous section is straightforward. We collect some details and intermediate results of our calculation in the appendix. Here we simply quote the final result for the quark mass and lattice spacing depen-

dence of the pseudoscalar meson masses. The total contribution from LO tree, LO 1-loop plus NLO tree for m_π and m_K and m_η are given as follows:

$$m_{\pi,\text{total}}^2 = x + 2y + \frac{1}{f^2} [L_\pi^r \{A_\pi^\pi x + B_\pi^\pi y + 5C\} + L_K^r \{A_K^\pi x + B_K^\pi y + 4C\} + L_\eta^r \{A_\eta^\pi x + B_\eta^\pi y + C\} - \{(C_0 + D_0)x + 2C_0 y + (C_{\text{av}} + D_{\text{av}})x^2 + D_{yy}^\pi y^2 + 2Dxy\}], \quad (23)$$

$$m_{K,\text{total}}^2 = x - y + \frac{1}{f^2} [L_K^r \{A_K^K x + B_K^K y + 6C\} + L_\pi^r \{A_\pi^K x + B_\pi^K y + 3C\} + L_\eta^r \{A_\eta^K x + B_\eta^K y + C\} - \{(C_0 + D_0)x - C_0 y + (C_{\text{av}} + D_{\text{av}})x^2 + D_{yy}^K y^2 - Dxy\}], \quad (24)$$

$$m_{\eta,\text{total}}^2 = x - 2y + \frac{1}{f^2} [L_\eta^r \{A_\eta^\eta x + B_\eta^\eta y + 3C\} + L_\pi^r \{A_\pi^\eta x + B_\pi^\eta y + 3C\} + L_K^r \{A_K^\eta x + B_K^\eta y + 4C\} - \{(C_0 + D_0)x - 2C_0 y + (C_{\text{av}} + D_{\text{av}})x^2 + D_{yy}^\eta y^2 - 2Dxy\}]. \quad (25)$$

The parameters

$$x = \tilde{m}_{\text{av}}^2 = \frac{2B}{3}(2\tilde{m} + \tilde{m}_s), \quad (26)$$

$$y = \frac{1}{4\sqrt{3}} 2B\tilde{M}_8 = \frac{2B}{6}(\tilde{m} - \tilde{m}_s),$$

represent the quark mass dependence. The chiral log is denoted by

$$L_a^r = \frac{\tilde{m}_a^2}{16\pi^2} \log(\tilde{m}_a^2), \quad (27)$$

whose coefficients contain

$$C = \frac{1}{6}Z, \quad A_\pi^\pi = \frac{1}{3}\left(2C_\pi^\pi - \frac{5}{2}X\right), \quad A_K^\pi = \frac{1}{3}(2C_K^\pi - 2X), \quad A_\eta^\pi = \frac{1}{3}\left(2C_\eta^\pi - \frac{1}{2}X\right), \quad B_\pi^\pi = \frac{1}{3}(4C_\pi^\pi - 5Y),$$

$$B_K^\pi = \frac{1}{3}(C_K^\pi - Y), \quad B_\eta^\pi = \frac{1}{3}(12c_5 - Y), \quad A_K^K = \frac{1}{3}(2C_K^K - 3X), \quad A_\pi^K = \frac{1}{3}\left(2C_\pi^K - \frac{3}{2}X\right),$$

$$A_\eta^K = \frac{1}{3}\left(2C_\eta^K - \frac{1}{2}X\right), \quad B_K^K = \frac{1}{3}(-2C_K^K + 3Y), \quad B_\pi^K = \frac{1}{3}\left(C_\pi^K - \frac{3}{4}Y\right), \quad B_\eta^K = \frac{1}{3}\left(-3C_\eta^K + \frac{5}{4}Y - 6c_5\right),$$

$$A_\eta^\eta = \frac{1}{3}\left(2C_\eta^\eta - \frac{3}{2}X\right), \quad A_\pi^\eta = \frac{1}{3}\left(2C_\pi^\eta - \frac{3}{2}X\right), \quad A_K^\eta = \frac{1}{3}(2C_K^\eta - 2X), \quad B_\eta^\eta = \frac{1}{3}(-4C_\eta^\eta + 5Y - 24c_5),$$

$$B_\pi^\eta = \frac{1}{3}(-3Y + 36c_5), \quad B_K^\eta = \frac{1}{3}(-3C_K^\eta + 5Y - 24c_5).$$

These terms are parametrized by

$$X = \tilde{A}[1 - 6c_3 - 4\tilde{c}_3 - 36c_3^2\tilde{B}],$$

$$Y = (1 - 3c_3 - 4\tilde{c}_3)\tilde{B}, \quad (28)$$

$$Z = [9c_2 + 4\tilde{c}_2 + \delta m_{\text{av}}^2(1 - 18c_3 - 4\tilde{c}_3)],$$

$$C_\pi^\pi = 2 + 18c_0 + 9\tilde{c}_0 + 15c_3\tilde{B},$$

$$C_K^\pi = 1 + 24c_0 + 6\tilde{c}_0 + 12c_3\tilde{B}, \quad (29)$$

$$C_\eta^\pi = 6c_0 + \tilde{c}_0 + 3c_3\tilde{B},$$

$$C_K^K = \frac{3}{2} + 24c_0 + 9\tilde{c}_0 + 18c_3\tilde{B},$$

$$C_\pi^K = \frac{3}{4} + 18c_0 + \frac{9}{2}\tilde{c}_0 + 9c_3\tilde{B}, \quad (30)$$

$$C_\eta^K = \frac{3}{4} + 6c_0 + \frac{5}{2}\tilde{c}_0 + 3c_3\tilde{B},$$

$$C_\eta^\eta = 6c_0 + 3\tilde{c}_0 + 9c_3\tilde{B}, \quad (31)$$

$$C_\pi^\eta = 18c_0 + 3\tilde{c}_0 + 9c_3\tilde{B},$$

$$C_K^\eta = 3 + 24c_0 + 10\tilde{c}_0 + 12c_3\tilde{B},$$

with

$$\tilde{A} = \frac{1}{1 - 2Nc_3 - \tilde{c}_3},$$

$$\tilde{B} = \frac{1}{1 - Nc_3 - \tilde{c}_3},$$

$$\delta m_{\text{av}}^2 = -\frac{Nc_2 + \tilde{c}_2}{1 - 2Nc_3 - \tilde{c}_3}.$$

The polynomial (nonlogarithmic) terms contain

$$\begin{aligned}
D_{yy}^{\pi} &= 16\tilde{L}_5 + 8(\tilde{L}_8 + \tilde{L}'_8) + 48V_{\Delta}, \\
D_{yy}^K &= 4\tilde{L}_5 + 20\tilde{L}_8 - 16\tilde{L}'_8 + 48V_{\Delta}, \\
D_{yy}^{\eta} &= 16\tilde{L}_5 + 24\tilde{L}_8 + 24\tilde{L}'_8 + 96\tilde{L}_7 + 48V_{\Delta}, \\
D &= C_{\text{av}} + 4\tilde{L}_5 + 4(\tilde{L}_8 + \tilde{L}'_8), \\
C_{\text{av}} &= 4(N\tilde{L}_6 + N\tilde{L}_4 + \tilde{L}_5), \\
D_{\text{av}} &= 2(\tilde{L}_8 + \tilde{L}'_8) + V_{\text{av}}, \\
C_0 &= 4(W_0 + W_1 + W_2 + W_3 + W_4) + 2W_5 + 8N^2W_6 \\
&\quad + 4NW_7 + 16NW_8 + 18W_9 - 8(W_{11} + W_{12}) \\
&= a^2W_C, \\
D_0 &= N[16(NW_6 + W_7) + 4W_8 + 2W_{10}] = a^2W_D,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{L}_4 &= L_4 + V_0 + V_1 + V_2 + V_3 = L_4 + aL_4^1, \\
\tilde{L}_5 &= L_5 + V_4 + V_5 + V_6 - V_{16} - V_{17} = L_5 + aL_5^1, \\
\tilde{L}_6 &= L_6 + 2NV_8 + \frac{1}{4}V_{10} + \frac{5}{2}V_{11} = L_6 + aL_6^1, \\
\tilde{L}_7 &= L_7 + 2NV_{14} + 2V_{15} = L_7 + aL_7^1, \\
\tilde{L}_8 &= L_8 + 2NV_9 + \frac{1}{2}V_{12} + \frac{5}{2}V_{13} = L_8 + aL_8^{1a}, \\
\tilde{L}'_8 &= L_8 + 2NV_9 + 2V_{13} = L_8 + aL_8^{1b}, \\
V_{\text{av}} &= N(V_7 + 2V_9 + 4NV_8), \\
V_{\Delta} &= \frac{1}{2}(V_7 + 2V_9).
\end{aligned}$$

Even though the NLO parameters have been used to remove the divergent terms from the loop integrals we use the same notation for these coefficients. We finally note that there exists the following constraint among some of the coefficients:

$$\sum_b A_b^a = \frac{1}{3}[6 + 96c_0 + 32\tilde{c}_0 + 60c_3B - 5X]. \quad (32)$$

Consequently, in the limit $y \rightarrow 0$ we obtain identical results for m_{π} , m_K or m_{η} , as it should be for three degenerate quark masses.

Obviously the final results for the pseudoscalar masses are fairly lengthy. From a practical point of view the number of independent unknown parameters in these expressions is crucial for their usefulness. Unknown parameters are the critical quark mass m_{critical} , the constant $2B$ and the decay constant f . The coefficients of the chiral log terms, given in Eq. (29)–(32), are defined through five independent $O(a)$ parameters, c_0 , \tilde{c}_0 , c_3 , \tilde{c}_3 , and c_5 , and the coefficient C , which is an independent $O(a^2)$ parameter. In the analytic NLO correction we find the independent combinations $C_{\text{av}} + D_{\text{av}}$, D_{yy}^{π} , D_{yy}^K , D_{yy}^{η} , and D , which start

at $O(1)$, and the two coefficients C_0, D_0 are of $O(a^2)$. Therefore, the total number of independent parameters is 13 besides m_{critical} , $2B$, and f .

B. $O(\tilde{M}a)$ term at NLO

In Eqs. (24)–(26), $O(\tilde{M}a)$ terms are considered at LO. In this subsection we present the results for the case that these terms are treated as NLO corrections. First of all, the expressions for the shifted quark masses simplify to

$$\tilde{m} = m - \frac{Nc_2 + \tilde{c}_2}{2B} = m - m_{\text{critical}}, \quad (33)$$

$$\tilde{m}_s = m_s - \frac{Nc_2 + \tilde{c}_2}{2B} = m_s - m_{\text{critical}}. \quad (34)$$

The quark mass dependence of the pseudoscalar meson masses becomes

$$\begin{aligned}
m_{\pi, \text{total}}^2 &= (x + 2y)[1 - Nc_3 - \tilde{c}_3] - xNc_3 \\
&\quad + \frac{1}{f^2} \left[L'_{\pi} \left\{ \frac{1}{2}x + y + 5C \right\} + L'_K 4C \right. \\
&\quad + L'_{\eta} \left\{ \frac{1}{2}x - \frac{1}{3}y + C \right\} - \{(C_0 + D_0)x + 2C_0y \\
&\quad \left. + (C_{\text{av}} + D_{\text{av}})x^2 + D_{yy}^{\pi}y^2 + 2Dxy \} \right], \quad (35)
\end{aligned}$$

$$\begin{aligned}
m_{K, \text{total}}^2 &= (x - y)[1 - Nc_3 - \tilde{c}_3] - xNc_3 \\
&\quad + \frac{1}{f^2} \left[L'_K 6C + L'_{\pi} \left\{ -\frac{1}{4}x + 3C \right\} \right. \\
&\quad + L'_{\eta} \left\{ \frac{1}{3}(x - y) + C \right\} - \{(C_0 + D_0)x - C_0y \\
&\quad \left. + (C_{\text{av}} + D_{\text{av}})x^2 + D_{yy}^K y^2 - Dxy \} \right], \quad (36)
\end{aligned}$$

$$\begin{aligned}
m_{\eta, \text{total}}^2 &= (x - 2y)[1 - Nc_3 - \tilde{c}_3] - xNc_3 \\
&\quad + \frac{1}{f^2} \left[L'_{\eta} \left\{ -\frac{1}{2}x + \frac{5}{3}y + 3C \right\} \right. \\
&\quad + L'_{\pi} \left\{ -\frac{1}{2}x - y + 3C \right\} + L'_K \left\{ \frac{4}{3}(x - y) + 4C \right\} \\
&\quad - \{(C_0 + D_0)x - 2C_0y + (C_{\text{av}} + D_{\text{av}})x^2 \\
&\quad \left. + D_{yy}^{\eta} y^2 - 2Dxy \} \right], \quad (37)
\end{aligned}$$

where

$$C = \frac{Z}{6} = \frac{(9-N)c_2 + 3\tilde{c}_2}{6},$$

$$D_{yy}^\pi = 16L_5 + 16L_8 = 4D_{yy}^K,$$

$$D_{yy}^\eta = 16L_5 + 48L_8 + 96L_7,$$

$$D = C_{\text{av}} + 4L_5 + 8L_8,$$

$$C_{\text{av}} = 4(NL_6 + NL_4 + L_5),$$

$$D_{\text{av}} = 4L_8,$$

$$\begin{aligned} C_0 &= 4(W_0 + W_1 + W_2 + W_3 + W_4) + 2W_5 + 8N^2W_6 \\ &\quad + 4NW_7 + 16NW_8 + 18W_9 - 8(W_{11} + W_{12}) \\ &= a^2W_C, \end{aligned}$$

$$D_0 = N[16(NW_6 + W_7) + 4W_8 + 2W_{10}] = a^2W_D.$$

The number of independent parameters is reduced compared to the result given in the previous section. Besides m_{critical} , $2B$ and f there are c_3 , \tilde{c}_3 , C , $L_4 + L_6$, $L_5 + L_8$ (note that $C_{\text{av}} + D_{\text{av}}$, D_{yy}^π , D_{yy}^K , and D can be expressed by $L_4 + L_6$ and $L_5 + L_8$.) and D_{yy}^η . The total number of independent parameters besides m_{critical} , $2B$, and f is reduced from 13 to 6.

However, for improved theories there is some lattice spacing dependence implicit in the definition of the renormalized quark mass. This results in 3 parameters ($b_B + b_m^{(2)}$, $b_m^{(1)}$, $b_m^{(3)}$), as we will show in the next subsection, where we discuss $O(a)$ improved theories.

C. Formula in $O(a)$ improved theories

We finally consider the case that a nonperturbatively $O(a)$ improved quark action (i.e. the clover quark action) is used in the lattice simulations [15–17,19,20]. In this case there are no on-shell $O(a)$ terms in the Symanzik's effective theory provided that the relevant improvement coefficients are tuned nonperturbatively to an appropriate value. In particular, $O(a)$ improvement requires that some a dependence is absorbed in the definition of renormalized masses and the gauge coupling:

$$m \rightarrow m + b_m^{(1)}m^2a + b_m^{(2)}(2m + m_s)ma + b_m^{(3)}(2m^2 + m_s^2)a, \quad (38)$$

$$m_s \rightarrow m_s + b_m^{(1)}m_s^2a + b_m^{(2)}(2m + m_s)m_s a + b_m^{(3)}(2m^2 + m_s^2)a, \quad (39)$$

$$g_0^2 \rightarrow g_0^2 \left(1 + b_g \frac{(2m + m_s)a}{3} \right), \quad (40)$$

where b_g and $b_m = b_m^{(1)} + 3(b_m^{(2)} + b_m^{(3)})$ are improvement coefficients defined in Ref. [19,20]. Therefore, as long as on-shell quantities are considered, there are no terms of $O(a)$, $O(\bar{M}a)$, $O(\bar{M}^2a)$ etc. in the WChPT Lagrangian, if we replace

$$\begin{aligned} \tilde{m} \rightarrow \bar{m} &= \tilde{m} + b_m^{(1)}\tilde{m}^2a + b_m^{(2)}(2\tilde{m} + \tilde{m}_s)\tilde{m}a \\ &\quad + b_m^{(3)}(2\tilde{m}^2 + \tilde{m}_s^2)a, \end{aligned} \quad (41)$$

$$\begin{aligned} \tilde{m}_s \rightarrow \bar{m}_s &= \tilde{m}_s + b_m^{(1)}\tilde{m}_s^2a + b_m^{(2)}(2\tilde{m} + \tilde{m}_s)\tilde{m}_s a \\ &\quad + b_m^{(3)}(2\tilde{m}^2 + \tilde{m}_s^2)a, \end{aligned} \quad (42)$$

$$B \rightarrow \bar{B} = B[1 + b_B(2\tilde{m} + \tilde{m}_s)a]. \quad (43)$$

Here the last modification comes from the mass dependence of g_0^2 in the Symanzik's effective theory.

We emphasize that there are no terms linear in a in the chiral Lagrangian and in the results for the pseudoscalar masses as long as one parameterizes it in terms of \bar{m} , which absorbs some $O(a)$ dependence through proper renormalization. Using \tilde{m} instead, which is simpler in practice since this mass is directly proportional to the difference between the bare and the critical quark mass, there is some $O(a)$ dependence left explicit.

Having made these remarks we can write down the WChPT expressions for nonperturbatively $O(a)$ improved theories:

$$\begin{aligned} m_{\pi,\text{total}}^2 &= \bar{x} + 2\bar{y} + \frac{1}{f^2} \left[L_\pi^r \left\{ \frac{1}{2}x + y + 5C \right\} + L_K^r 4C \right. \\ &\quad \left. + L_\eta^r \left\{ \frac{1}{2}x - \frac{1}{3}y + C \right\} - \left\{ (C_0 + D_0)x + 2C_0y \right. \right. \\ &\quad \left. \left. + (C_{\text{av}} + D_{\text{av}})x^2 + D_{yy}^\pi y^2 + 2D_{xy} \right\} \right], \end{aligned} \quad (44)$$

$$\begin{aligned} m_{K,\text{total}}^2 &= \bar{x} - \bar{y} + \frac{1}{f^2} \left[L_K^r 6C + L_\pi^r \left\{ -\frac{1}{4}x + 3C \right\} \right. \\ &\quad \left. + L_\eta^r \left\{ \frac{1}{3}(x - y) + C \right\} - \left\{ (C_0 + D_0)x - C_0y \right. \right. \\ &\quad \left. \left. + (C_{\text{av}} + D_{\text{av}})x^2 + D_{yy}^K y^2 - D_{xy} \right\} \right], \end{aligned} \quad (45)$$

$$\begin{aligned} m_{\eta,\text{total}}^2 &= \bar{x} - 2\bar{y} + \frac{1}{f^2} \left[L_\eta^r \left\{ -\frac{1}{2}x + \frac{5}{3}y + 3C \right\} \right. \\ &\quad \left. + L_\pi^r \left\{ -\frac{1}{2}x - y + 3C \right\} + L_K^r \left\{ \frac{4}{3}(x - y) + 4C \right\} \right. \\ &\quad \left. - \left\{ (C_0 + D_0)x - 2C_0y + (C_{\text{av}} + D_{\text{av}})x^2 \right. \right. \\ &\quad \left. \left. + D_{yy}^\eta y^2 - 2D_{xy} \right\} \right], \end{aligned} \quad (46)$$

where

$$\begin{aligned} \bar{x} &= \frac{2\bar{B}}{3}(2\bar{m} + \bar{m}_s) = x[1 + (b_B + b_m^{(2)})(2\tilde{m} + \tilde{m}_s)a \\ &\quad + (b_m^{(1)} + 3b_m^{(3)})(2\tilde{m}^2 + \tilde{m}_s^2)a], \end{aligned} \quad (47)$$

$$\bar{y} = \frac{2\bar{B}}{6}(\bar{m} - \bar{m}_s) = y[1 + (b_B + b_m^{(2)})(2\bar{m} + \bar{m}_s)a + b_m^{(1)}(\bar{m} + \bar{m}_s)a], \quad (48)$$

and

$$C = \frac{Z}{6} = \frac{(9-N)c_2 + 3\tilde{c}_2}{6},$$

$$D_{yy}^\pi = 16L_5 + 16L_8 = 4D_{yy}^K,$$

$$D_{yy}^\eta = 16L_5 + 48L_8 + 96L_7,$$

$$D = C_{\text{av}} + 4L_5 + 8L_8,$$

$$C_{\text{av}} = 4(NL_6 + NL_4 + L_5),$$

$$D_{\text{av}} = 4L_8,$$

$$C_0 = 4(W_0 + W_1 + W_2 + W_3 + W_4) + 2W_5 + 8N^2W_6 + 4NW_7 + 16NW_8 + 18W_9 - 8(W_{11} + W_{12}) = a^2W_C,$$

$$D_0 = N[16(NW_6 + W_7) + 4W_8 + 2W_{10}] = a^2W_D.$$

Independent parameters (besides m_{critical} , $2B$, and f) are $b_B + b_m^{(2)}$, $b_m^{(1)}$, $b_m^{(3)}$, C , $L_4 + L_6$, $L_5 + L_8$, and D_{yy}^η . The number of independent parameters besides m_{critical} , $2B$ and f is therefore 7, reduced from previously found 13.

IV. CONCLUDING REMARKS

In this paper we computed the pseudoscalar masses in $2 + 1$ flavor WChPT. We presented results for three different order countings, appropriate for various sizes of the $O(a\tilde{M})$ term in the chiral Lagrangian. Depending on the lattice action used in the numerical simulation (unimproved, perturbatively improved, nonperturbatively improved) one has to choose one result for the chiral

extrapolation. Since we have no prior knowledge about the size of the $O(a\tilde{M})$ contribution we suggest to perform fits to the data with all three forms and let the data decide which form is most appropriate.

The number of unknown fit parameters is significantly larger than in 2 flavor WChPT. Using our results requires sufficiently enough data points in order to perform the chiral fits. The CP-PACS/JLQCD Collaboration is currently performing $2 + 1$ flavor simulations at three lattice spacings with five values for the light up and down quark mass and two different strange quark masses. At least for these simulations the number of data points exceeds the number of unknown fit parameters. Performing the chiral extrapolation of the CP-PACS/JLQCD data using our results is work in progress.

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APPENDIX A: USEFUL FORMULAE

1. Expansions in terms of π

In this subsection we collect some useful formulae necessary for the expansion of the chiral Lagrangian in terms of the π fields.

a. LO terms

At LO we have to expand terms at $O(\pi^4)$.

$$\begin{aligned} \langle \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \rangle &= \frac{4}{f^2} \langle \partial_\mu \pi \partial_\mu \pi \rangle + \frac{8}{3f^4} \langle \partial_\mu \pi [\pi, \partial_\mu \pi] \pi \rangle, & \langle M\Sigma + \Sigma^\dagger M \rangle &= \langle 2M \rangle - \frac{4}{f^2} \langle M\pi^2 \rangle + \frac{4}{3f^4} \langle M\pi^4 \rangle, \\ \langle \Sigma + \Sigma^\dagger \rangle &= \langle 2 \rangle - \frac{4}{f^2} \langle \pi^2 \rangle + \frac{4}{3f^4} \langle \pi^4 \rangle, & \langle \Sigma + \Sigma^\dagger \rangle^2 &= \langle 2 \rangle^2 - \frac{8}{f^2} \langle 2 \rangle \langle \pi^2 \rangle + \frac{8}{3f^4} \langle 2 \rangle \langle \pi^4 \rangle + \frac{16}{f^4} \langle \pi^2 \rangle^2, \\ \langle (\Sigma + \Sigma^\dagger)^2 \rangle &= \langle 4 \rangle - \frac{16}{f^2} \langle \pi^2 \rangle + \frac{64}{3f^4} \langle \pi^4 \rangle, & \langle \Sigma + \Sigma^\dagger - 2 \rangle \langle \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \rangle &= -\frac{16}{f^4} \langle \pi^2 \rangle \langle \partial_\mu \pi \partial_\mu \pi \rangle, \\ & & \langle (\Sigma + \Sigma^\dagger - 2) \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \rangle &= -\frac{16}{f^4} \langle \pi^2 \partial_\mu \pi \partial_\mu \pi \rangle, \\ \langle \Sigma + \Sigma^\dagger \rangle \langle M\Sigma + \Sigma^\dagger M \rangle &= \langle 2 \rangle \langle 2M \rangle - \frac{8}{f^2} \langle M \rangle \langle \pi^2 \rangle - \frac{8}{f^2} \langle 1 \rangle \langle M\pi^2 \rangle + \frac{16}{f^4} \langle \pi^2 \rangle \langle M\pi^2 \rangle + \frac{8}{3f^4} \langle M \rangle \langle \pi^4 \rangle + \frac{8}{3f^4} \langle 1 \rangle \langle M\pi^4 \rangle, \\ \langle (\Sigma + \Sigma^\dagger) (M\Sigma + \Sigma^\dagger M) \rangle &= \langle 4M \rangle - \frac{16}{f^2} \langle M\pi^2 \rangle + \frac{64}{3f^4} \langle M\pi^4 \rangle, & \langle \Sigma - \Sigma^\dagger \rangle^2 &= 0, \\ \langle (\Sigma - \Sigma^\dagger)^2 \rangle &= -\frac{16}{f^2} \langle \pi^2 \rangle + \frac{64}{3f^4} \langle \pi^4 \rangle, & \langle \Sigma - \Sigma^\dagger \rangle \langle M\Sigma - \Sigma^\dagger M \rangle &= \frac{32}{3f^4} \langle \pi^3 \rangle \langle M\pi \rangle, \\ \langle (\Sigma - \Sigma^\dagger) (M\Sigma - \Sigma^\dagger M) \rangle &= -\frac{16}{f^2} \langle M\pi^2 \rangle + \frac{64}{3f^4} \langle M\pi^4 \rangle. \end{aligned}$$

b. NLO terms

We have to expand the NLO terms to $O(\pi^2)$.

$$\begin{aligned}
\langle \Sigma_{\mu\mu} \rangle \langle M\Sigma + \Sigma^\dagger M \rangle &= \frac{4}{f^2} \langle \partial_\mu \pi \partial_\mu \pi \rangle \langle 2M \rangle, & \langle \Sigma_{\mu\mu} (M\Sigma + \Sigma^\dagger M) \rangle &= \frac{4}{f^2} \langle \partial_\mu \pi \partial_\mu \pi 2M \rangle, \\
\langle M\Sigma + \Sigma^\dagger M \rangle^2 &= \langle 2M \rangle^2 - \frac{8}{f^2} \langle 2M \rangle \langle M\pi^2 \rangle, & \langle M\Sigma - \Sigma^\dagger M \rangle^2 &= -\frac{16}{f^2} \langle M\pi \rangle^2, \\
\langle M\Sigma M\Sigma + \Sigma^\dagger M\Sigma^\dagger M \rangle &= -\frac{8}{f^2} \langle M\pi M\pi + M^2 \pi^2 \rangle, & \langle \Sigma^2 + (\Sigma^\dagger)^2 \rangle &= \langle 2 \rangle - \frac{16}{f^2} \langle \pi^2 \rangle, \\
\langle M\Sigma^2 + (\Sigma^\dagger)^2 M \rangle &= \langle 2M \rangle - \frac{16}{f^2} \langle M\pi^2 \rangle, & \langle M\Sigma^3 + (\Sigma^\dagger)^3 M \rangle &= \langle 2M \rangle - \frac{36}{f^2} \langle M\pi^2 \rangle, \\
\langle M^2 \Sigma + \Sigma^\dagger M^2 \rangle &= \langle 2M^2 \rangle - \frac{4}{f^2} \langle M^2 \pi^2 \rangle, & \langle M\Sigma M\Sigma^2 + (\Sigma^\dagger)^2 M\Sigma^\dagger M \rangle &= \langle 2M^2 \rangle - \frac{4}{f^2} \langle 4M\pi M\pi + 5M^2 \pi^2 \rangle, \\
\langle M\Sigma - \Sigma^\dagger M \rangle &= \frac{4i}{f} \langle M\pi \rangle, & \langle M\Sigma^2 - (\Sigma^\dagger)^2 M \rangle &= \frac{8i}{f} \langle M\pi \rangle, \\
\langle (\partial_\mu \Sigma)^2 + (\partial_\mu \Sigma^\dagger)^2 \rangle &= \langle (\partial_\mu \Sigma)^2 (\Sigma^\dagger)^2 + \Sigma^2 (\partial_\mu \Sigma^\dagger)^2 \rangle &= -\frac{8}{f^2} \langle \partial_\mu \pi \partial_\mu \pi \rangle.
\end{aligned}$$

2. Formula for traces

After expanding the Lagrangian in terms of the π fields, we have to take the trace in the flavor space.

a. LO terms

$$\begin{aligned}
\langle 1 \rangle &= N, & \langle M_q \rangle &= NM_0, & \langle \pi^2 \rangle &= \frac{1}{2} \sum_a \pi_a^2, & 2B \langle M_q \pi^2 \rangle &= \frac{1}{2} \sum_a m_a^2 \pi_a^2, \\
m_a^2 &= \begin{cases} m_\pi^2 = 2Bm, & a = 1, 2, 3, \\ m_K^2 = B(m + m_s), & a = 4, 5, 6, 7, \\ m_\eta^2 = \frac{2B}{3}(m + 2m_s), & a = 8, \end{cases} & 2B \langle M_q \pi^4 \rangle &= \sum_{a,b,c,d} F^{abcd} \pi_a \pi_b \pi_c \pi_d, \\
\langle \pi^4 \rangle &= \frac{1}{4N} \sum_{a,b} \pi_a^2 \pi_b^2 + \frac{1}{8} \sum_{a \sim e} d^{abe} d^{cde} \pi_a \pi_b \pi_c \pi_d, \\
4F^{abcd} &= \frac{2BM_0}{2} \left\{ \frac{2}{N} \delta^{ab} \delta^{cd} + \sum_e d^{abe} d^{cde} \right\} + \frac{2BM_8}{4} \left\{ \frac{2}{N} (\delta^{ab} d^{cd8} + d^{ab8} \delta^{cd}) + \sum_e d^{ee8} d^{abe} d^{cde} \right\}, \\
\langle \partial_\mu \pi \partial_\mu \pi \rangle &= \frac{1}{2} \sum_a \partial_\mu \pi_a \partial_\mu \pi_a, & \langle \pi^2 \partial_\mu \pi \partial_\mu \pi \rangle &= \frac{1}{4N} \sum_{a,b} \pi_a^2 \partial_\mu \pi_b \partial_\mu \pi_b + \frac{1}{8} \sum_{a \sim e} d^{abe} d^{cde} \pi_a \pi_b \partial_\mu \pi_c \partial_\mu \pi_d, \\
\langle \partial_\mu \pi [\pi, \partial_\mu \pi] \pi \rangle &= -\frac{1}{4} \sum_{a \sim e} f^{abe} f^{cde} \partial_\mu \pi_a \pi_b \partial_\mu \pi_c \pi_d, & \langle \pi^3 \rangle &= \frac{1}{4} \sum_{a,b,c} d^{abc} \pi_a \pi_b \pi_c, & 2B \langle M_q \pi \rangle &= \frac{1}{2} 2BM_8 \pi_8.
\end{aligned}$$

b. NLO terms

$$\begin{aligned}
\langle 2B\tilde{M}_q \rangle &= N\tilde{m}_{\text{av}}^2, & \langle \pi^2 \rangle &= \frac{1}{2} \sum_a \pi_a^2, & \langle (2B\tilde{M}_q)^2 \rangle &= N^2(\tilde{m}_{\text{av}}^2)^2, \\
\langle (\partial_\mu \pi)^2 2B\tilde{M}_q \rangle &= \tilde{m}_{\text{av}}^2 \frac{1}{2} \sum_a (\partial_\mu \pi_a)^2 + \Delta\tilde{m}^2 \frac{1}{4} \sum_a d^{aa8} (\partial_\mu \pi_a)^2, & \langle (2B\tilde{M}_q)^2 \rangle &= N(\tilde{m}_{\text{av}}^2)^2 + \frac{1}{2} (\Delta\tilde{m}^2)^2, \\
\langle 2B\tilde{M}_q \pi^2 \rangle &= \frac{1}{2} \sum_a \tilde{m}_a^2 \pi_a^2, & \langle 2B\tilde{M}_q \pi \rangle^2 &= \frac{1}{4} (\Delta\tilde{m}^2)^2 \pi_8^2, \\
2\langle (2B\tilde{M}_q \pi)^2 \rangle &= (\tilde{m}_{\text{av}}^2)^2 \sum_a \pi_a^2 + \tilde{m}_{\text{av}}^2 \Delta\tilde{m}^2 \sum_a d^{aa8} \pi_a^2 + \frac{(\Delta\tilde{m}^2)^2}{4} \left[\frac{2}{N} \pi_8^2 + \sum_a \left\{ (d^{aa8})^2 - \sum_b (f^{ab8})^2 \right\} \pi_a^2 \right], \\
\langle (2B\tilde{M}_q)^2 \pi^2 \rangle &= \frac{1}{2} (\tilde{m}_{\text{av}}^2)^2 \sum_a \pi_a^2 + \frac{1}{2} \tilde{m}_{\text{av}}^2 \Delta\tilde{m}^2 \sum_a d^{aa8} \pi_a^2 + (\Delta\tilde{m}^2)^2 \sum_a \left\{ \frac{1}{4N} - \frac{1}{8\sqrt{3}} d^{aa8} \right\} \pi_a^2.
\end{aligned}$$

3. Group factors

We have to calculate some Lie group factors.

$$\begin{aligned}
C^{ab} &\equiv \sum_c f^{abc} f^{abc} = (1 - \delta^{ab}) C^{AB} = C^{ba}, & C^{\pi\pi} &= C^{K_1 K_1} = C^{K_2 K_2} = 1, \\
C^{\pi K} &= C^{K_1 K_2} = 1/4, & C^{\pi\eta} &= 0, & C^{K\eta} &= 3/4,
\end{aligned}$$

where π represents $a = 1, 2, 3$, K represents $a = 4, 5$ (K_1) and $a = 6, 7$ (K_2) and η represents $a = 8$.

$$\begin{aligned}
D^{ab} &\equiv \sum_c d^{aac} d^{bbc} = d^{aa3} d^{bb3} + d^{aa8} d^{bb8} = D^{ba}, & D^{\pi\pi} &= D^{K_1 K_1} = D^{K_2 K_2} = D^{\eta\eta} = 1/3, \\
D^{\pi K} &= D^{K_1 K_2} = -1/6, & D^{\pi\eta} &= -1/3, & D^{K\eta} &= 1/6.
\end{aligned}$$

$$\begin{aligned}
E^{ab} &\equiv \sum_c d^{cc8} d^{aac} d^{bbc} = \frac{1}{\sqrt{3}} [d^{aa3} d^{bb3} - d^{aa8} d^{bb8}] = E^{ba}, & E^{\pi\pi} &= E^{K_1 K_2} = E^{\eta\eta} = -\frac{1}{3\sqrt{3}}, \\
E^{\pi K} &= E^{K_1 K_1} = E^{K_2 K_2} = \frac{1}{6\sqrt{3}}, & E^{\pi\eta} &= \frac{1}{3\sqrt{3}}, & E^{K\eta} &= -\frac{1}{6\sqrt{3}}.
\end{aligned}$$

$$\begin{aligned}
\tilde{E}^{ab} &\equiv \sum_c d^{cc8} d^{abc} d^{abc} = \tilde{E}^{ba}, & \tilde{E}^{aa} &= \frac{1}{\sqrt{3}} \begin{cases} -1/3 & \text{for } \pi, \\ 1/6 & \text{for } K, \\ -1/3 & \text{for } \eta, \end{cases} & \tilde{E}^{\pi K} &= -\frac{1}{8\sqrt{3}}, & \tilde{E}^{\pi\eta} &= \frac{1}{3\sqrt{3}}, \\
\tilde{E}^{K_1 K_2} &= \frac{1}{4\sqrt{3}}, & \tilde{E}^{K\eta} &= -\frac{1}{24\sqrt{3}}, & \text{others} &= 0.
\end{aligned}$$

With these definition we have

$$\begin{aligned}
4F^{aabb} &= 2BM_0 \left\{ \frac{1}{N} + \frac{1}{2} D^{ab} \right\} + 2BM_8 \left\{ \frac{1}{2N} (d^{aa8} + d^{bb8}) + \frac{1}{4} E^{ab} \right\}, \\
4F^{abab} &= 2BM_0 \left\{ \frac{1}{N} \delta^{ab} + \frac{1}{2} \tilde{D}^{ab} \right\} + 2BM_8 \left\{ \frac{1}{N} \delta^{ab} d^{aa8} + \frac{1}{4} \tilde{E}^{ab} \right\}.
\end{aligned}$$

The following formulae for $N = 3$ are useful.

$$\frac{1}{N} + \frac{1}{2}D^{ab} + \tilde{D}^{ab} = \begin{pmatrix} 5/6 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 5/6 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 5/6 \end{pmatrix},$$

$$\frac{1}{2N}(d^{aa8} + d^{bb8}) + \frac{1}{4}E^{ab} + \frac{1}{2}\tilde{E}^{ab} = \frac{1}{4\sqrt{3}} \begin{pmatrix} 1/3 & 1 & 1 & 1/4 & 1/4 & 1/4 & 1/4 & 1 \\ 1/4 & 1/4 & 1/4 & -1/6 & -1/2 & -1/2 & -1/2 & -5/4 \\ 1 & 1 & 1 & -5/4 & -5/4 & -5/4 & -5/4 & -7/3 \end{pmatrix},$$

where $a = 1, 4, 8$.

4. 1-loop contractions

The following contraction formula is useful for the calculation of the meson propagators at 1-loop.

$$\langle \partial_\mu \pi [\pi, \partial_\mu \pi] \pi \rangle \rightarrow -\frac{1}{4} \sum_{a,b} C^{ab} [(\partial_\mu \pi_a)^2 I_0(\tilde{m}_b^2) + \pi_a^2 I_1(\tilde{m}_b^2)], \quad C^{ab} = \sum_c f^{abc} f^{abc},$$

$$I_0(m^2) = -\frac{m^2}{16\pi^2} [\Delta + 1 - \log m^2], \quad \Delta = \frac{2}{\epsilon} - \gamma + \log(4\pi), \quad I_1(m^2) = -m^2 I_0(m^2),$$

$$\langle 2BM \pi^4 \rangle \rightarrow \sum_{a,b} \pi_a^2 2I_0(\tilde{m}_b^2) [F^{aabb} + 2F^{abab}], \quad \langle \pi^4 \rangle \rightarrow \sum_{a,b} \pi_a^2 I_0(\tilde{m}_b^2) \left[\frac{1}{2N} (1 + 2\delta_{ab}) + \frac{1}{4} \sum_e (d^{aae} d^{bbe} + 2d^{abe} d^{abe}) \right],$$

$$\langle \pi^2 \rangle^2 \rightarrow \frac{1}{2} \sum_{a,b} \pi_a^2 I_0(\tilde{m}_b^2) (1 + 2\delta_{ab}), \quad \langle \pi^2 \rangle \langle 2BM \pi^2 \rangle \rightarrow \frac{1}{4} \sum_{a,b} \pi_a^2 I_0(\tilde{m}_b^2) (m_a^2 + m_b^2) (1 + 2\delta_{ab}),$$

$$\langle \pi^2 \rangle \langle (\partial_\mu \pi)^2 \rangle \rightarrow \frac{1}{4} \sum_{a,b} [(\partial_\mu \pi_a)^2 I_0(\tilde{m}_b^2) + \pi_a^2 I_1(\tilde{m}_b^2)],$$

$$\langle \pi^2 (\partial_\mu \pi)^2 \rangle \rightarrow \sum_{a,b} [(\partial_\mu \pi_a)^2 I_0(\tilde{m}_b^2) + \pi_a^2 I_1(\tilde{m}_b^2)] \left(\frac{1}{4N} + \frac{1}{8} \sum_e d^{aac} d^{bbc} \right),$$

$$\langle \pi^3 \rangle \langle 2BM \pi \rangle \rightarrow \frac{3}{8} 2BM_8 d^{aa8} [I_0(\tilde{m}_a^2) \pi_8^2 + I_0(\tilde{m}_8^2) \pi_a^2].$$

APPENDIX B: CALCULATION OF THE PSEUDOSCALAR MESON PROPAGATOR AT 1-LOOP

In this appendix we give some details of the calculation of the pseudoscalar meson propagator at 1-loop in the WChPT.

1. Effective action for BG fields

In order to calculate meson masses at 1-loop, we use the background (BG) field method. We first split the π field as

$$\pi = \pi_Q + \pi_G, \quad (\text{B1})$$

where π_Q represents quantum field while π_G is a BG field which satisfies the equation of motion. Inserting this into the chiral effective Lagrangian, we have

$$L_{\text{LO}}(\pi) = L_{\text{LO}}(\pi_G) + L^{(2)}(\pi_Q) + L^{(4)}(\pi_Q + \pi_G) - L^{(4)}(\pi_G). \quad (\text{B2})$$

Integrating out π_Q we obtain the following formula

$$e^{-S_{\text{eff}}(\pi_G)} = \int \mathcal{D}\pi_Q e^{-\int d^4x L_{\text{LO}}(\pi)} = e^{-\int d^4x L_{\text{LO}}(\pi_G)} \int \mathcal{D}\pi_Q e^{-\int d^4x L^{(2)}(\pi_Q)} \times e^{-\int d^4x (L^{(4)}(\pi_Q + \pi_G) - L^{(4)}(\pi_G))}. \quad (\text{B3})$$

This leads to

$$S_{\text{eff}}(\pi_G) = \int d^4x L_{\text{LO}}(\pi_G) - \log \langle e^{-\int d^4x (L^{(4)}(\pi_Q + \pi_G) - L^{(4)}(\pi_G))} \rangle, \quad (\text{B4})$$

where

$$\langle f(\pi_Q, \pi_G) \rangle = \int \mathcal{D}\pi_Q e^{-\int d^4x L^{(2)}(\pi_Q)} f(\pi_Q, \pi_G). \quad (\text{B5})$$

Expanding S_{eff} in terms of π_G , we obtain

$$S_{\text{eff}}(\pi_G) = \text{const.} + S_{\text{eff}}^{(2)}(\pi_G) + \sum_{n=3}^{\infty} S_{\text{eff}}^{(n)}(\pi_G), \quad (\text{B6})$$

where $S_{\text{eff}}^{(n)}(\pi_G)$ contains the n -th power of the field π_G . In the calculation of the pseudoscalar masses we are interested in the $n = 2$ case. We write

$$S_{\text{eff}}^{(2)}(\pi_G) = \int d^4x L_{\text{LO}}(\pi_G) + S_{1\text{-loop}}^{(2)}(\pi_G) + \dots, \quad (\text{B7})$$

where \dots represent the higher loop contributions. We call $S_{1\text{-loop}}^{(2)}$ the 1-loop contribution to the meson propagator and write

$$S_{1\text{-loop}}^{(2)}(\pi_G) = \int d^4x L_{1\text{-loop}}^{(2)}(\pi_G). \quad (\text{B8})$$

2. Expansion of the LO Lagrangian

We now expand the LO Lagrangian (1) in terms of the pseudoscalar field π_a . Using the expansion and trace formulae given in Appendix A, we obtain

$$\begin{aligned} L^{(2)} &= \langle (\partial_\mu \pi)^2 \rangle + 2B \langle M_q \pi^2 \rangle - \frac{c_2}{2} \langle 2 \rangle \langle \pi^2 \rangle - \tilde{c}_2 \langle \pi^2 \rangle \\ &\quad - c_3 \langle 2BM_q \rangle \langle \pi^2 \rangle - c_3 \langle 1 \rangle 2B \langle M_q \pi^2 \rangle - \tilde{c}_3 2B \langle M_q \pi^2 \rangle \\ &= \frac{1}{2} \sum_a [(\partial_\mu \pi_a)^2 + \tilde{m}_a^2 \pi_a^2], \end{aligned} \quad (\text{B9})$$

at second order in π_a , where the pseudoscalar meson masses at LO are given by

$$\tilde{m}_a^2 = m_a^2(1 - Nc_3 - \tilde{c}_3) - m_{\text{av}}^2 Nc_3 - Nc_2 - \tilde{c}_2, \quad (\text{B10})$$

$$m_{\text{av}}^2 = \frac{1}{N^2 - 1} \sum_a m_a^2. \quad (\text{B11})$$

For $N = 3$ flavors we have

$$m_a^2 = \begin{cases} m_\pi^2 = 2Bm, & a = 1, 2, 3, \\ m_K^2 = B(m + m_s), & a = 4, 5, 6, 7, \\ m_\eta^2 = \frac{2B}{3}(m + 2m_s), & a = 8. \end{cases} \quad (\text{B12})$$

$$m_{\text{av}}^2 = 2B \frac{2m + m_s}{3},$$

Equation (B9) gives the pseudoscalar meson propagator at LO.

The 4-th order terms in the LO Lagrangian become

$$\begin{aligned} L^{(4)} &= \frac{2}{3f^2} \langle \partial_\mu \pi [\pi, \partial_\mu \pi] \pi \rangle - \frac{1 - Nc_3 - 4\tilde{c}_3}{3f^2} 2B \langle M \pi^4 \rangle \\ &\quad + \frac{Nc_2 + 4\tilde{c}_2 + Nc_3 m_{\text{av}}^2}{3f^2} \langle \pi^4 \rangle + \frac{c_2}{f^2} \langle \pi^2 \rangle^2 \\ &\quad + \frac{2c_3}{f^2} \langle \pi^2 \rangle \langle 2BM \pi^2 \rangle - \frac{4c_0}{f^2} \langle \pi^2 \rangle \langle (\partial_\mu \pi)^2 \rangle \\ &\quad - \frac{4\tilde{c}_0}{f^2} \langle \pi^2 \rangle \langle \partial_\mu \pi \rangle^2 + \frac{4c_5}{3f^2} \langle \pi^3 \rangle \langle 2BM \pi \rangle. \end{aligned} \quad (\text{B13})$$

There terms give the 4-point interaction vertices of the pseudoscalar mesons.

3. 1-loop contribution to the propagator

Using the formulae in Appendix A, it is now easy to calculate $L_{1\text{-loop}}^{(2)}$. Including the tree level contribution we obtain

$$S_{\text{eff}}^{(2)}(\pi) = \int d^4x \frac{1}{2} \sum_a Z_a [(\partial_\mu \pi_a)^2 + m_{a,R}^2 \pi_a^2]. \quad (\text{B14})$$

For the π ($a = 1, 2, 3$), K ($a = 4, 5, 6, 7$), and η ($a = 8$) we find the wave function renormalization as

$$\begin{aligned} Z_\pi &= 1 - \frac{1}{3f^2} [L_\pi \{2 + 9(2c_0 + \tilde{c}_0)\} \\ &\quad + L_K \{1 + 6(4c_0 + \tilde{c}_0)\} + L_\eta \{6c_0 + \tilde{c}_0\}], \\ Z_K &= 1 - \frac{1}{3f^2} [L_\pi \{3/4 + 9(2c_0 + \tilde{c}_0/2)\} \\ &\quad + L_K \{3/2 + 3(8c_0 + 3\tilde{c}_0)\} \\ &\quad + L_\eta \{3/4 + 6c_0 + 5\tilde{c}_0/2\}], \\ Z_\eta &= 1 - \frac{1}{3f^2} [L_\pi 3(6c_0 + \tilde{c}_0) + L_K \{3 + 2(12c_0 + 5\tilde{c}_0)\} \\ &\quad + L_\eta 3(2c_0 + \tilde{c}_0)], \end{aligned} \quad (\text{B15})$$

where

$$L_\pi = I_0(\tilde{m}_\pi^2), \quad L_K = I_0(\tilde{m}_K^2), \quad L_\eta = I_0(\tilde{m}_\eta^2). \quad (\text{B16})$$

Similarly we have

$$\begin{aligned} m_{\pi,R}^2 &= \tilde{m}_\pi^2 + \frac{1}{3f^2} \left[L_\pi \left\{ C_\pi^\pi 2\tilde{m}_\pi^2 - \frac{5}{2} X \tilde{m}_{\text{av}}^2 - 5Y \frac{\Delta \tilde{m}^2}{4\sqrt{3}} + \frac{5}{2} Z \right\} + L_K \left\{ C_K^\pi (\tilde{m}_K^2 + \tilde{m}_\pi^2) - 2X \tilde{m}_{\text{av}}^2 - Y \frac{\Delta \tilde{m}^2}{4\sqrt{3}} + 2Z \right\} \right. \\ &\quad \left. + L_\eta \left\{ C_\eta^\pi (\tilde{m}_\eta^2 + \tilde{m}_\pi^2) - \frac{1}{2} X \tilde{m}_{\text{av}}^2 - (Y - 12c_5) \frac{\Delta \tilde{m}^2}{4\sqrt{3}} + \frac{1}{2} Z \right\} \right], \end{aligned} \quad (\text{B17})$$

$$m_{K,R}^2 = \tilde{m}_K^2 + \frac{1}{3f^2} \left[L_K \left\{ C_K^K 2\tilde{m}_K^2 - 3X\tilde{m}_{\text{av}}^2 + 3Y \frac{\Delta\tilde{m}^2}{4\sqrt{3}} + 3Z \right\} + L_\pi \left\{ C_\pi^K (\tilde{m}_\pi^2 + \tilde{m}_K^2) - \frac{3}{2} X\tilde{m}_{\text{av}}^2 - \frac{3}{4} Y \frac{\Delta\tilde{m}^2}{4\sqrt{3}} + \frac{3}{2} Z \right\} \right. \\ \left. + L_\eta \left\{ C_\eta^K (\tilde{m}_\eta^2 + \tilde{m}_K^2) - \frac{1}{2} X\tilde{m}_{\text{av}}^2 + \frac{5}{4} \left(Y - \frac{24}{5} c_5 \right) \frac{\Delta\tilde{m}^2}{4\sqrt{3}} + \frac{1}{2} Z \right\} \right], \quad (\text{B18})$$

$$m_{\eta,R}^2 = \tilde{m}_\eta^2 + \frac{1}{3f^2} \left[L_\eta \left\{ C_\eta^\eta 2\tilde{m}_\eta^2 - \frac{3}{2} X\tilde{m}_{\text{av}}^2 + 5 \left(Y - \frac{24}{5} c_5 \right) \frac{\Delta\tilde{m}^2}{4\sqrt{3}} + \frac{3}{2} Z \right\} \right. \\ \left. + L_\pi \left\{ C_\pi^\eta (\tilde{m}_\pi^2 + \tilde{m}_\eta^2) - \frac{3}{2} X\tilde{m}_{\text{av}}^2 - 3(Y - 12c_5) \frac{\Delta\tilde{m}^2}{4\sqrt{3}} + \frac{3}{2} Z \right\} \right. \\ \left. + L_K \left\{ C_K^\eta (\tilde{m}_K^2 + \tilde{m}_\eta^2) - 2X\tilde{m}_{\text{av}}^2 + 5 \left(Y - \frac{24}{5} c_5 \right) \frac{\Delta\tilde{m}^2}{4\sqrt{3}} + 2Z \right\} \right]. \quad (\text{B19})$$

for the pseudoscalar meson mass. The parameters in these expressions have already been given in subsection III A.

4. NLO contribution to meson propagators

Using the formulae for the expansion in powers of the pion field and the trace formulae in the Appendix A, we have

$$L_{\text{NLO}} = \frac{1}{2} Z_a^{\text{NLO}} [(\partial_\mu \pi_a)^2 + m_{a,\text{NLO}}^2 \pi_a^2], \quad (\text{B20})$$

where the wave function renormalization factor is given by

$$Z_a^{\text{NLO}} = \frac{1}{f^2} [\tilde{m}_{\text{av}}^2 z_{\text{av}} + \Delta\tilde{m}^2 z_\Delta^a + z_0],$$

$$z_{\text{av}} = 4(N\tilde{L}_4 + \tilde{L}_5), \quad z_\Delta^a = 2d^{aa8}\tilde{L}_5,$$

$$z_0 = 4(W_0 + W_1 + W_2 + W_3 + W_4) - 8(W_{11} + W_{12}),$$

and the mass term is defined as

$$m_{a,\text{NLO}}^2 = -\frac{1}{f^2} [\tilde{m}_a^2 \{\tilde{m}_{\text{av}}^2 C_{\text{av}} + \Delta\tilde{m}^2 C_\Delta^a + C_0\} \\ + \tilde{m}_{\text{av}}^2 \{\tilde{m}_{\text{av}}^2 D_{\text{av}} + \Delta\tilde{m}^2 D_\Delta^a + D_0\} + (\Delta\tilde{m}^2)^2 E_\Delta^a],$$

$$C_{\text{av}} = 4(N\tilde{L}_6 + N\tilde{L}_4 + \tilde{L}_5),$$

$$D_{\text{av}} = 2(\tilde{L}_8 + \tilde{L}'_8) + V_{\text{av}}, \quad C_\Delta^a = 2d^{aa8}\tilde{L}_5,$$

$$D_\Delta^a = 2d^{aa8}(\tilde{L}_8 + \tilde{L}'_8),$$

$$E_\Delta^a = e_a \tilde{L}_8 + e'_a \tilde{L}'_8 + V_\Delta + \delta_{a8} \left(2\tilde{L}_7 + \frac{1}{N} \tilde{L}'_8 \right),$$

$$C_0 = 4(W_0 + W_1 + W_2 + W_3 + W_4) + 2W_5 \\ + 8N^2W_6 + 4NW_7 + 16NW_8 + 18W_9 \\ - 8(W_{11} + W_{12}) = a^2 W_C,$$

$$D_0 = N[16(NW_6 + W_7) + 4W_8 + 2W_{10}] = a^2 W_D. \quad (\text{B21})$$

The constants here are given by

$$\tilde{L}_4 = L_4 + V_0 + V_1 + V_2 + V_3 = L_4 + aL_4^1, \quad (\text{B22})$$

$$\tilde{L}_5 = L_5 + V_4 + V_5 + V_6 - V_{16} - V_{17} = L_5 + aL_5^1, \quad (\text{B23})$$

$$\tilde{L}_6 = L_6 + 2NV_8 + \frac{1}{4}V_{10} + \frac{5}{2}V_{11} = L_6 + aL_6^1, \quad (\text{B24})$$

$$\tilde{L}_7 = L_7 + 2NV_{14} + 2V_{15} = L_7 + aL_7^1, \quad (\text{B25})$$

$$\tilde{L}_8 = L_8 + 2NV_9 + \frac{1}{2}V_{12} + \frac{5}{2}V_{13} = L_8 + aL_8^{1a}, \quad (\text{B26})$$

$$\tilde{L}'_8 = L_8 + 2NV_9 + 2V_{13} = L_8 + aL_8^{1b}, \quad (\text{B27})$$

and

$$V_{\text{av}} = N(V_7 + 2V_9 + 4NV_8), \quad V_\Delta = \frac{1}{2}(V_7 + 2V_9), \quad (\text{B28})$$

$$e_a = \frac{1}{N} - \frac{1}{2\sqrt{3}} d^{aa8}, \quad (e_\pi, e_K, e_\eta) = \left(\frac{1}{6}, \frac{5}{12}, \frac{1}{2} \right), \quad (\text{B29})$$

$$e'_a = \frac{1}{2} [(d^{aa8})^2 - \sum_b (f^{ab8})^2], \quad (\text{B30})$$

$$(e'_\pi, e'_K, e'_\eta) = \left(\frac{1}{6}, -\frac{1}{3}, \frac{1}{6} \right).$$

5. Cancellation of UV divergence

In order to perform the renormalization, we consider divergent parts of meson masses in Eq. (B17)–(B19), which are given by

$$[m_a^2]_{\text{div.}} = -\frac{\Delta}{48\pi^2 f^2} [C_x^a x + C_y^a y + C_{xx}^a x^2 + C_{yy}^a y^2 \\ + C_{xy}^a xy], \quad (\text{B31})$$

where $x = \tilde{m}_{\text{av}}^2$ and $y = \frac{1}{4\sqrt{3}} \Delta\tilde{m}^2$, in terms of which, $\tilde{m}_\pi^2 = x + 2y$, $\tilde{m}_K^2 = x - y$, and $\tilde{m}_\eta^2 = x - 2y$. Constants are

given by

$$\begin{aligned}
C_x^\pi &= C_x^K = C_x^\eta = 5Z, & C_y^\pi &= 2Z, & C_y^K &= -Z, & C_y^\eta &= -2Z, \\
C_{xx}^a &= 2\sum_b C_{ba}^a - 5X = 2(3 + 48c_0 + 16\tilde{c}_0 + 30c_3\tilde{B}) - 5X, \\
C_{yy}^\pi &= 8C_\pi^\pi - C_K^\pi - 7Y - 24c_5\tilde{B} = 15 + 120c_0 + 66\tilde{c}_0 + \tilde{B}[-7 + 129c_3 + 28\tilde{c}_3 - 24c_5], \\
C_{yy}^K &= 2C_K^K + 2C_\pi^K + 6C_\eta^K - 7Y + 12c_5\tilde{B} = 9 + 120c_0 + 42\tilde{c}_0 + \tilde{B}[-7 + 93c_3 + 28\tilde{c}_3 + 12c_5], \\
C_{yy}^\eta &= 8C_\eta^\eta + 3C_K^\eta - 21Y + 144c_5\tilde{B} = 9 + 120c_0 + 54\tilde{c}_0 + \tilde{B}[-21 + 171c_3 + 84\tilde{c}_3 + 144c_5], \\
C_{xy}^\pi &= 8C_\pi^\pi - C_K^\pi - 4C_\eta^\pi - 2X - 7Y + 12c_5\tilde{B} = 15 + 96c_0 + 62\tilde{c}_0 + 96c_3\tilde{B} - 2X - 7Y + 12c_5\tilde{B}, \\
C_{xy}^K &= -4C_K^K + 5C_\pi^K - 7C_\eta^K + X + \frac{7}{2}Y - 6c_5\tilde{B} = -\frac{1}{2}C_{xy}^\pi, \\
C_{xy}^\eta &= -8C_\eta^\eta + 4C_\pi^\eta - 5C_K^\eta + 2X + 7Y - 12c_5\tilde{B} = -C_{xy}^\pi.
\end{aligned}$$

On the other hand, the NLO contributions lead to

$$[m_a^2]_{\text{NLO}} = -\frac{1}{f^2}[D_{xx}^a + D_{yy}^a + D_{xx}^a x^2 + D_{yy}^a y^2 + D_{xy}^a xy],$$

where

$$\begin{aligned}
D_x^a &= C_0 + D_0, & D_y^\pi &= 2C_0, & D_y^K &= -C_0, & D_y^\eta &= -2C_0, & D_{xx}^a &= C_{\text{av}} + D_{\text{av}}, \\
D_{yy}^\pi &= 16\tilde{L}_5 + 8(\tilde{L}_8 + \tilde{L}'_8) + 48V_\Delta, & D_{yy}^K &= 4\tilde{L}_5 + 20\tilde{L}_8 - 16\tilde{L}'_8 + 48V_\Delta, \\
D_{yy}^\eta &= 16\tilde{L}_5 + 24\tilde{L}_8 + 24\tilde{L}'_8 + 96\tilde{L}_7 + 48V_\Delta, & D_{xy}^\pi &= 2C_{\text{av}} + 8\tilde{L}_5 + 8(\tilde{L}_8 + \tilde{L}'_8) = -2D_{xy}^K = -D_{xy}^\eta.
\end{aligned}$$

In order to cancel the UV divergences, the divergent part in the NLO terms must be chosen according to

$$\begin{aligned}
[C_0]_{\text{div.}} &= -\frac{\Delta}{48\pi^2}Z, & [D_0]_{\text{div.}} &= -\frac{\Delta}{48\pi^2}4Z, \\
[C_{\text{av}} + D_{\text{av}}]_{\text{div.}} &= -\frac{\Delta}{48\pi^2}[2(3 + 48c_0 + 16\tilde{c}_0 + 30c_3\tilde{B}) - 5X], \\
[16\tilde{L}_5 + 8(\tilde{L}_8 + \tilde{L}'_8) + 48V_\Delta]_{\text{div.}} &= -\frac{\tilde{\Delta}}{48\pi^2}[15 + 120c_0 + 66\tilde{c}_0 + \tilde{B}(-7 + 129c_3 + 28\tilde{c}_3 - 24c_5)], \\
[4\tilde{L}_5 + 20\tilde{L}_8 - 16\tilde{L}'_8 + 48V_\Delta]_{\text{div.}} &= -\frac{\tilde{\Delta}}{48\pi^2}[9 + 120c_0 + 42\tilde{c}_0 + \tilde{B}(-7 + 93c_3 + 28\tilde{c}_3 + 12c_5)], \\
[16\tilde{L}_5 + 96\tilde{L}_7 + 24(\tilde{L}_8 + \tilde{L}'_8) + 48V_\Delta]_{\text{div.}} &= -\frac{\tilde{\Delta}}{48\pi^2}[9 + 120c_0 + 54\tilde{c}_0 + \tilde{B}(-21 + 171c_3 + 84\tilde{c}_3 + 144c_5)], \\
[2C_{\text{av}} + 8\tilde{L}_5 + 8\tilde{L}_8 + 8\tilde{L}'_8]_{\text{div.}} &= -\frac{\Delta}{48\pi^2}[15 + 96c_0 + 62\tilde{c}_0 + 96c_3\tilde{B} - 2X - 7Y + 12c_5\tilde{B}].
\end{aligned}$$

Notice that we can remove all divergences $[m_a^2]_{\text{div.}}$ consistently by these parameters, as it should be.

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