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Matrix model maps in AdS/CFT correspondence

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We discuss an extension of a map between BPS states and free fermions. The extension involves states associated with a full two matrix problem which are constructed using a sequence of integral equations. A two parameter set of matrix model eigenstates is then related to states in SUGRA. Their wave functions are characterized by nontrivial dependence on the radial coordinate of AdS and of the Sphere, respectively. A kernel defining a one to one map between these states is then constructed.

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I. INTRODUCTION

Studies [1–8] of giant gravitons in AdS Supergravity (and dual $N = 4$ SYM theory) have lead to a simple (matrix model) picture for $1/2$ BPS states. In particular a free fermion model [8–10] of harmonic oscillators was identified and shown to simulate fully the dynamics of $1/2$ BPS states and their interactions. In [10] (referred to as LLM) a classical Ansatz for AdS (bubbling) configurations was constructed whose energy and flux were demonstrated to be in a one to one correspondence with those of a general fermionic droplet configuration. Further, relevant studies of this free fermion map have recently been carried out [11–33].

It is clear that it would be desirable to extend the map to more general states and go beyond the simple case of free fermions. This would require an investigation (and solution) of more complex two (or multi) matrix models—a formidable task. In the present work we present a step in this direction. We will attempt to extend the correspondence from the fermionic family of states (representing a single diagonal matrix) to a more general set associated with states of a two matrix quantum mechanics. As was already seen in [8] which concerned itself with the case of $1/2$ BPS states one can start with a system of two matrices, or a complex matrix and perform a truncation to a single Hermitian matrix (in the manner analogous to a similar phenomena in the quantum Hall effect). The reduction was explained in [8] to be the Hilbert space equivalent of a holomorphic projection where the set of observables are given by traces of the complex matrix Z only. The introduction of mixed traces, involving the second (conjugate) matrix immediately leads to a nontrivial dynamical problem whose eigenstates were never constructed.

We will first address this problem of constructing invariant eigenstates of the two matrix quantum system. For this we develop in some detail a hybrid formalism, treating one of the matrices fully in the standard collective field theory manner, while the other is treated in the coherent state representation. This second matrix behaves then as an ‘‘impurity’’. The corresponding collective field theory of combined, mixed traces is then worked out and is shown to lead to a sequence of eigenvalue equations. These equations are seen to generalize an eigenvalue equation first found in [34], and first solved for its eigenstates in [35], describing angular degrees of freedom of the single matrix model. The sequence of eigenvalue equations can be solved for the present case of the oscillator potential. It provides a two parameter set of energy (dilatation operator) eigenvalues and a corresponding 2 dimensional space of eigenfunctions.

The central issue then becomes that of providing a correspondence between the eigenstates of the matrix model and states and eigenvalues in Supergravity. Here we work in a linearized approximation specifying a class of fluctuations with matching quantum numbers. The wave functions, in the AdSxS background are nontrivial, being given by hypergeometric functions or corresponding special functions. Nevertheless, we describe a $1-1$ map between a two dimensional subset and the two dimensional set of wave functions given by the matrix model. This map involves a transformation introduced originaly in the context of the 2d black hole and the corresponding matrix model [36]. This transform, appropriately interpreted, then provides a one to one map between the gravity and matrix model wave functions. We emphasize that being one to one this map is different from the well known holographic projection. It is expected that further studies of the map will be of relevance for reconstructing AdS quantum mechanics.

The outline of the paper is as follows. In Sec. II we give a review of the simple matrix model and of the fermion map. In Sec. III we address the two matrix problem describing its collective field formulation. We derive a sequence of eigenvalue equations and solve for eigenvalues and eigenfunctions. In Sec. IV we consider the wave functions of the AdSxS SUGRA and specify the transform to the matrix model eigenstates. Several Appendices contain further details.

II. REVIEW

We begin by reviewing and clarifying the existing map between the $1/2$ BPS SUGRA configurations and the states of the harmonic oscillator matrix model.

The matrix model degrees of freedom originate from a reduction of $N = 4$ Super Yang-Mills theory on $R \times S^3$. The Hamiltonian is therefore the dilatation operator and the Higgs fields become quantum mechanical matrix coordinates $\Phi_a(t)$, $a = 1\dots 6$. For the study performed in the present paper one can concentrate on the dynamics of two matrices

$$S = \frac{1}{2g_{\rm YM}^2}\int dt\,\mathrm{Tr}\Big(\dot\Phi_1^2 + \dot\Phi_2^2 - \Phi_1^2 - \Phi_2^2 - \frac{1}{2}[\Phi_1, \Phi_2]^2\Big).$$

The commutator interaction did not play a role in the 1/2 BPS correspondence and in what follows we will mainly concern ourselves with the simple quadratic harmonic oscillator model of two matrices

$$H = \frac{1}{2}\mathrm{Tr}(P_1^2 + P_2^2 + \Phi_1^2 + \Phi_2^2).$$

The symmetries of this reduced theory are given by the $U(1)$ charge

$$J = \mathrm{Tr}(P_1\Phi_2 - P_2\Phi_1),$$

and an $SL(2, R)$ symmetry algebra (allternatively $SU(2)$).

One has the complex matrices

$$Z = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2) \qquad Z^\dagger = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2),$$

and the conjugates

$$\Pi = \frac{1}{\sqrt{2}}(P_1 + iP_2) = -i\frac{\partial}{\partial Z^\dagger}$$
$$\Pi^\dagger = \frac{1}{\sqrt{2}}(P_1 - iP_2) = -i\frac{\partial}{\partial Z}.$$

Restriction to 1/2 BPS configurations corresponds in the matrix model to considering a subset of correlators involving only the chiral primary operators of the general form

$$\mathrm{Tr}\, Z^{k_1}\mathrm{Tr}Z^{k_2}\cdots\mathrm{Tr}Z^{k_n}.$$

For the corresponding reduction in Hilbert space one proceeds as follows (see [8,9]). It is useful to introduce the operators

$$A = \tfrac{1}{2}(Z + i\Pi),$$

and

$$B = \tfrac{1}{2}(Z - i\Pi).$$

In terms of these, the Hamiltonian and the $U(1)$ charge read

$$H = \mathrm{Tr}(A^\dagger A + B^\dagger B) \qquad J = \mathrm{Tr}(A^\dagger A - B^\dagger B).$$

One now has a sequence of eigenstates given by

$$\mathrm{Tr}\,((A^\dagger)^n)|0\rangle E = J = n \qquad \mathrm{Tr}((B^\dagger)^n)|0\rangle E = -J = n$$
$$\mathrm{Tr}((A^\dagger)^n)\mathrm{Tr}((B^\dagger)^m)|0\rangle E = n + m, \quad J = n - m.$$

Restriction to 1/2 BPS configurations corresponds in the matrix model Hilbert space to a reduction to a subsector given by A oscillators. It is useful to diagonalize $A, A^\dagger$ by using the unitary symmetry

$$A_{ij} = \lambda_i\delta_{ij} \qquad A_{ij}^\dagger = \lambda_i^\dagger\delta_{ij}.$$

The measure in these variables shows that we can treat the λ_i's as fermionic variables. The Hamiltonian for these fermionic oscillators is

$$H = \sum_i \lambda_i^\dagger\lambda_i.$$

The fermionic wave functions are

$$\psi_F(\lambda_1, \lambda_2, \dots, \lambda_n) = e^{-\sum_i \bar\lambda_i\lambda_i}\det\begin{pmatrix} \lambda_1^{l_1} & \lambda_1^{l_2} & \cdots & \lambda_1^{l_N} \\ \lambda_2^{l_1} & \lambda_2^{l_2} & \cdots & \lambda_2^{l_N} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_N^{l_1} & \lambda_N^{l_2} & \cdots & \lambda_N^{l_N}\end{pmatrix}.$$

After dividing the wave function by the Vandermonde determinant, we have that

$$\psi_{B;l_1,l_2,\dots,l_N}(\lambda_1, \lambda_2, \dots, \lambda_n) = e^{-\sum_i \bar\lambda_i\lambda_i}\chi_{l_1,l_2,\dots,l_N}(\lambda_1, \lambda_2, \dots, \lambda_N),$$

where $\chi_{l_1,l_2,\dots,l_N}$ denotes the character of a representation of $SU(N)$ that corresponds to a Young tableaux with l_1 boxes in the first row, l_2 boxes in the second one etc. Of special interest is the sequence of states corresponding to representations that contains 1 row of l boxes

$$\psi_{B;l_1,l_2,\dots,l_N}(\lambda_1, \lambda_2, \dots, \lambda_n) = e^{-\sum_i \bar\lambda_i\lambda_i}\chi_{l,0,\dots,0}$$
$$= e^{-\sum_i \bar\lambda_i\lambda_i}\chi_{l,0,\dots,0}(\lambda_1, \lambda_2, \dots, \lambda_N),$$

and another sequence that corresponds to a representation that contains 1 column of l boxes

$$\psi_{B;l_1,l_2,\dots,l_N}(\lambda_1, \lambda_2, \dots, \lambda_n) = e^{-\sum_i \bar\lambda_i\lambda_i}\chi_{l,0,\dots,0}$$
$$= e^{-\sum_i \bar\lambda_i\lambda_i}\chi_{1,1,\dots,1,0,\dots,0}$$
$$\times (\lambda_1, \lambda_2, \dots, \lambda_N).$$

In the fermionic picture [37] the first set of states represents particles and the second holes. These were explained in [6,8,9] to correspond to a giant graviton in AdS and to a giant graviton on the sphere, respectively. In terms of the moments

$$\phi_i = \sum_{j=0}^N \lambda_j^i,$$

one obtains Schur polynomials representing these states

$$\chi_{l,0,\dots,0}(\lambda_1, \lambda_2, \dots, \lambda_N) = P_l(\phi_1, \phi_2, \dots, \phi_N)$$
$$\chi_{1,1,\dots,1,0,\dots,0}(\lambda_1, \lambda_2, \dots, \lambda_N) = (-)^l P_l(-\phi_1, -\phi_2, \dots, -\phi_N).$$

They are exact eigenstates of a cubic collective field theory representing the bosonized version of 1d fermions. In terms of a two dimensional density field $\rho(x, y, t)$ the Hamiltonian is simply

$$H = \frac{1}{2}\int dx \int dy(x^2 + y^2)\rho(x, y, t).$$

Together with the nontrivial symplectic form

$$L_0 = 2\pi \int dx \int dy \rho(x) G(x - x')\dot{\tilde{\rho}}(x'),$$

one has a topological 2 + 1 dimensional scalar field theory [38] which can be reduced to a 1 + 1 dimensional collective field theory describing the dynamics of the boundary (of the droplet) $y_\pm(x, t)$ by

$$S = \frac{1}{2\pi}\int dt \int dx[y_+ \partial_x^{-1}\dot{y}_+ - y_- \partial_x^{-1}\dot{y}_- - ((y_+^3 - y_-^3) + x^2(y_+ - y_-))].$$

One can parametrize the boundary in terms of radial coordinates, in which the Lagrangian becomes quadratic. This is a simple manifestation of the integrability of this theory. This goes as follows:

Consider a closed curve $\vec{r}(s, t)$ in R^2 with parameter s which in our case describes the boundary of the fermi sea in the phase space. In general the equation of motion can be written in the form

$$\partial_t \vec{r} \times \partial_s \vec{r} = \partial_s A(\vec{r}),$$

with $A(\vec{r})$ defining the model that we are studying. For the case of free fermions in an oscillator potential one has

$$A(\vec{r}) = \tfrac{1}{2}\vec{r}^2.$$

If we parametrize the curve as

$$\vec{r}(x, t) = x\hat{x} + y_\pm(x, t)\hat{y},$$

we recover

$$\partial_t \vec{r} \times \partial_x \vec{r} = \partial_t y_\pm \qquad \partial_t y_\pm = -\tfrac{1}{2}\partial_x(y_\pm^2 + x^2).$$

If instead one uses polar coordinates to parametrize the boundary

$$\vec{r}(\phi, t) = \rho(\phi, t)\cos(\phi)\hat{x} + \rho(\phi, t)\sin(\phi)\hat{y},$$

and in this case we have

$$\partial_t \vec{r} \times \partial_\phi \vec{r} = \tfrac{1}{2}\partial_t \rho^2(\phi, t) \qquad \partial_t \rho^2 = \partial_\phi \rho^2.$$

It is instructive to derive the above linear equation of motion from the nonlinear one by using the field dependent coordinate transformation. It is simply given by

$$x = \rho[\phi(x, t), t]\cos(\phi(x, t))$$
$$y_+ = \rho[\phi(x, t), t]\sin(\phi(x, t)).$$

These are then the action-angle coordinates for the dynamics of the boundary.

In their work Lin, Lunin and Maldacena [10] have identified a nonlinear Ansatz for 10d Sugra which exactly reduces to the above, bosonic Hamiltonian of 1d fermions. To summarize the main features of the Ansatz, one has first the 10 dimensional metric

$$\begin{aligned} ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) \\ &\quad + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 \\ h^{-2} &= 2y\cosh G, \\ y\partial_y V_i &= \epsilon_{ij}\partial_j z, \\ y(\partial_i V_j - \partial_j V_i) &= \epsilon_{ij}\partial_y z, \\ z &= \frac{1}{2}\tanh G \end{aligned}$$

and a corresponding Ansatz for the gauge fields.

The only unknown function z is shown to obey the Laplace equation:

$$\partial_i \partial_i z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0,$$

which is solved as a boundary value problem:

$$z(x_1, x_2, y) = \frac{y^2}{\pi}\int_{\mathcal{D}} \frac{z(x_1', x_2', 0)dx_1' dx_2'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]^2}$$

$$V_i(x_1, x_2, y) = \frac{\epsilon_{ij}}{\pi}\int_{\mathcal{D}} \frac{z(x_1', x_2', 0)(x_j - x_j')dx_1' dx_2'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]^2}.$$

Remarkably, the flux and the energy of this general configuration were shown by LLM to take the form of the bosonized free fermion droplet

$$N = \frac{1}{4\pi^2 l_P^2}\int dx_1 \int dx_2\left(u(t, x_1, x_2) + \frac{1}{2}\right)$$

$$\Delta = \frac{1}{4\pi\hbar^2}\int dx_1 \int dx_2(x_1^2 + x_2^2)\left(u(t, x_1, x_2) + \frac{1}{2}\right) - \frac{1}{8\pi^2\hbar^2}\left(\int dx_1 \int dx_2 x_1^2 + x_2^2\left(u(t, x_1, x_2) + \frac{1}{2}\right)\right)^2.$$

It should be stressed that even though these expressions look two dimensional, effectively this is still only a 1 dimensional correspondence (it is described explicitly by the 1 + 1 dimensional bosonic scalar field theory). In addition to the formulas for the flux and the energy one also needs the symplectic form (which should coincide with the symplectic form established by Iso, Karabali and Sakita [38]) for the 2d fermion droplet. Another, simple way to see the one dimensionality is by an analysis of linearized fluctuations (we give this in Appendix A). One has

$$S = \sum_{n>0}\frac{1}{2}\int dt\left[\frac{1}{n^2}\dot{p}_n^2 + \dot{q}_n^2 - n^2 q_n^2 - p_n^2\right],$$

in agreement with the well known quadratic action for chiral primaries in AdS:

$$S = \sum_n \frac{8R_{\mathrm{AdS}}^8 n(n-1)}{(n+1)^2} \int_{\mathrm{AdS}^5} dx^5 \sqrt{g_{\mathrm{AdS}^5}} \times [\sigma^{-n}\Box\sigma^{+n} - n(n-4)\sigma^{-n}\sigma^{+n}].$$

It is supersymmetry which requires that $(\partial_t - \partial_\phi)\sigma = 0$ which for the 0 + 1 dimensional variables means that the "angular momentum" is equal to the energy. Choosing an opposite chirality for the fermions we would have had the condition $(\partial_t + \partial_\phi)\sigma = 0$ which would flip the sign in the relation between energy and "angular momentum".

III. MATRIX MODEL EIGENPROBLEM

We have seen in the discussion that the treatment of 1/2 BPS states corresponds to a reduction, namely, to one matrix quantum mechanics given by the canonical set A and $A^\dagger$. It is our interest to extend this correspondence to a larger set of states. In the matrix model they will be states involving the two matrices (A and B) of a two matrix model. This can be stated as a two matrix problem, with two hermitian matrices M and N in a quadratic potential, i.e., with Hamiltonian

$$H \equiv -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) + \frac{1}{2}\mathrm{Tr}(M^2) - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial N}\frac{\partial}{\partial N}\right) + \frac{1}{2}\mathrm{Tr}(N^2). \quad (1)$$

Using creation-annihilation operators for the matrix N_{ij} in a coherent basis, the Hamiltonian takes the form considered in this article:

$$\hat{H} \equiv -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) + \frac{1}{2}Tr(M^2) + \mathrm{Tr}\left(B\frac{\partial}{\partial B}\right). \quad (2)$$

We consider the action of this Hamiltonian on functionals of invariant variables (loops)

$$\Phi[\psi(k, s = 0, 1, 2, \ldots)],$$

where the $\psi(k, s = 0, 1, 2, \ldots)$ are states with s "B impurities":

$$\begin{aligned} \psi(k,0) &= \mathrm{Tr}(e^{ikM}) \\ \psi(k,1) &= \mathrm{Tr}(Be^{ikM}) \\ \psi(k,2) &= \int_0^k dk' \mathrm{Tr}(Be^{ik'M}Be^{i(k-k')M}) \\ &\ldots \end{aligned} \quad (3)$$

In terms of the eigenvalues λ_i and the angular variables V of the matrix $M = V\Lambda V^+$, we have

$$\begin{aligned} \psi(k,0) &= \Sigma_i e^{ik\lambda_i} \\ \psi(k,1) &= \Sigma_i (V^+BV)_{ii} e^{ik\lambda_i} \\ \psi(k,2) &= -2i\Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji}\frac{e^{ik\lambda_j}}{(\lambda_j - \lambda_i)} \\ &\ldots \end{aligned} \quad (4)$$

Using the chain rule, we obtain for the matrix M kinetic energy operator on the wave functional:

$$\begin{aligned} -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) = &-\frac{1}{2}\Sigma_s \int dk \quad \mathrm{Tr}\left(\frac{\partial^2\psi(k,s)}{\partial M\partial M}\right)\frac{\partial}{\partial\psi(k,s)} \\ &-\frac{1}{2}\Sigma_{s,s'}\int dk\int dk' \\ &\times \mathrm{Tr}\left(\frac{\partial\psi(k,s)}{\partial M}\frac{\partial\psi(k',s')}{\partial M}\right) \\ &\times \frac{\partial^2}{\partial\psi(k,s)\partial\psi(k',s')}. \end{aligned}$$

As it is traditional [39], we introduce the notation:

$$\begin{aligned} -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) = &-\frac{1}{2}\Sigma_s \int dk \quad \omega(k,s)\frac{\partial}{\partial\psi(k,s)} \\ &-\frac{1}{2}\Sigma_{s,s'}\int dk\int dk' \\ &\times \Omega(k,s{:}k',s')\frac{\partial^2}{\partial\psi(k,s)\partial\psi(k',s')}, \end{aligned} \quad (5)$$

$\omega(k,s)$ splits the loop $\psi(k,s)$ and $\Omega(k,s{:}k',s')$ joins the two loops $\psi(k,s)$ and $\psi(k',s')$.

We will find it useful to introduce a density description, or x representation:

$$\begin{aligned} \psi(x,s) &= \int \frac{dk}{2\pi} e^{-ikx}\psi(k,s), \\ \psi(k,s) &= \int dx e^{ikx}\psi(x,s). \end{aligned}$$

Any function of k (or x) transforms accordingly. Namely

$$\begin{aligned} \omega(x,s) &= \int \frac{dk}{2\pi} e^{-ikx}\omega(k,s) \\ \Omega(x,s;y,s') &= \int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}e^{-ikx}e^{-ik'y}\Omega(k,s;k',s'). \end{aligned}$$

For conjugates, we have

$$\begin{aligned} \frac{\partial}{\partial\psi(x,s)} &= \int dk e^{ikx}\frac{\partial}{\partial\psi(k,s)}; \\ \frac{\partial}{\partial\psi(k,s)} &= \int\frac{dx}{2\pi}e^{-ikx}\frac{\partial}{\partial\psi(x,s)}. \end{aligned}$$

In the density description, the kinetic operator then becomes

$$-\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) = -\frac{1}{2}\Sigma_s \int dx \omega(x,s)\frac{\partial}{\partial\psi(x,s)} - \frac{1}{2}\Sigma_{s,s'}\int dx \int dy \Omega(x,s:y,s') \times \frac{\partial^2}{\partial\psi(x,s)\partial\phi(y,s')}. \quad (6)$$

A. Spectrum and fluctuations in the zero impurity sector

Consider first the analysis for the spectrum of the zero impurity problem. This sector corresponds to the Quantum Mechanics of a single Hermitian matrix, and it has by now a standard solution [39–41], which is briefly reviewed in Appendix B. In this case, one has the standard cubic Hamiltonian

$$H^0_{\rm eff} = \frac{1}{2N^2}\int dx \partial_x \Pi(x)\psi(x,0)\partial_x\Pi(x) + N^2\left(\int dx \frac{\pi^2}{6}\psi^3(x,0) + \psi(x,0)\left(\frac{x^2}{2}-\mu\right)\right), \quad (7)$$

giving the well known Wigner distribution background in the limit as $N \to \infty$

$$\pi\psi(x,0) \equiv \pi\phi_0 = \sqrt{2\mu - x^2} = \sqrt{2-x^2}.$$

For the small fluctuation spectrum, one shifts the background

$$\psi(x,0) = \phi_0 + \frac{1}{\sqrt{\pi}N}\partial_x\eta; \qquad \partial_x\Pi(x) = -\sqrt{\pi}NP(x),$$

to find the quadratic operator

$$H^0_2 = \frac{1}{2}\int dx \pi\phi_0 P^2(x) + \frac{1}{2}\int dx\pi\phi_0(\partial_x\eta)^2.$$

The way to diagonalize is by now well known: one changes to the classical "time of flight" q

$$\frac{dx}{dq} = \pi\phi_0; \qquad x(q) = -\sqrt{2}\cos(q);$$

$$\pi\phi_0 = \sqrt{2}\sin(q); \qquad 0 \le q \le \pi.$$

One obtains the equation for a $2d$ massless boson:

$$H^0_2 = \frac{1}{2}\int dq P^2(q) + \frac{1}{2}\int dq(\partial_q\eta)^2. \quad (8)$$

In addition one needs to impose Dirichelet boundary conditions at the classical turning points, for a consistent time evolution of the constraint (B3). Therefore the spectrum in the zero impurity sector is

$$w_j = j; \qquad \phi_j = \sin(jq). \quad (9)$$

The following comment is in order: the harmonic oscillator potential is special, in that the effective Hamiltonian (7) can be equivalently written as (for discussions on the relationship between the two rewritings in the context of supersymmetric or stochastic stabilizations, see for instance [42–45])

$$H^0_{\rm eff} = \frac{1}{2N^2}\int dx\partial_x\Pi(x)\psi(x,0)\partial_x\Pi(x) + \frac{N^2}{2} \times \int dx\psi(x,0)\left(\int dy\frac{\psi(y,0)}{x-y} - x\right)^2.$$

It is then seen that the Wigner distribution background also satisfies the well known Brezin, Itzykson, Parisi, Zuber (BIPZ) [46] equation

$$\int dz\frac{\phi_0(z)}{(x-z)} = x. \quad (10)$$

Shifting about the background as above, we obtain for the quadratic Hamiltonian

$$H^0_2 = \frac{1}{2}\int dx\pi\phi_0 P^2(x) + \frac{1}{2}\int dx\pi\phi_0\left(\int\frac{dy}{\pi}\frac{\partial_y\eta(y)}{x-y}\right)^2.$$

This nonlocal Hamiltonian can be easily shown to be equivalent to (8). Let us examine this in slightly more detail: by changing to the classical time of flight q, we obtain

$$H^0_2 = \frac{1}{2}\int dqP^2(q) + \frac{1}{2}\int dq\left(\partial_q\int\frac{dq'}{\pi}\frac{\pi\phi_0(q')\eta(q')}{x(q)-x(q')}\right)^2.$$

The above nonlocal integral operator plays a prominent role in what follows and is discussed in Appendix C. Let us denote it by

$$\partial_q\int\frac{dq'}{\pi}\frac{\pi\phi_0(q')f(q')}{x(q)-x(q')} \equiv -i|\partial_q|f(q),$$

and by abuse of language (it does not satisfy a Leibnitz rule) refer to it as the "absolute derivative", for ease of notation. We note that $(-i|\partial_q|)^2 = \partial_q^2$ and that the appropriate eigenfunctions of this operator are $\phi_n = \sin(nq)$ with eigenvalue n as shown in Appendix C. Therefore the eigenfunctions (9) are also the solutions of

$$(i\partial_t + i|\partial_q|)\phi(q) = 0.$$

B. Quadratic Hamiltonian for states with impurities

We return now to the (pre-Hermitian) kinetic energy operator (6) (or (5)). We note that

$$\langle\psi(x,s)\rangle = \langle\psi(k,s)\rangle = 0; \qquad s = 1,2,3,\ldots.$$

This observation implies that for the multi-impurity spectrum it is sufficient to consider the zero impurity sector Jacobian already discussed [47], i.e.,

$$\frac{\partial}{\partial\psi(x,0)} \to J^{1/2}\frac{\partial}{\partial\psi(x,0)}J^{-1/2} = \frac{\partial}{\partial\psi(x,0)} - \frac{1}{2}\frac{\partial}{\partial\psi(x,0)}\ln J\frac{\partial}{\partial\psi(x,s)} \to \frac{\partial}{\partial\psi(x,s)}, \qquad s = 1,2,3,\ldots$$

where, to leading order in N

$$\partial_x\frac{\partial \ln J}{\partial\psi(x,0)} = \partial_x\int dy\Omega^{-1}(x,0;y,0)\omega(y,0) = 2\int dy\frac{\psi(y,0)}{(x-y)}. \qquad (11)$$

Let us now identify the terms in (6) which determine the quadratic operator in the multi-impurity sector.

We look for terms of the form $\psi(x,s)\partial/\partial\psi(x,s)$, $s>0$ when $\psi(x,0)\to\phi_0(x)$. Contributions of this form contained in the first term of (6) result from splittings of the loop $\psi(x,s)$ into a zero impurity loop and another with s impurities. We will denote this amplitude by $\bar{\omega}(x,s)$.

Contributions contained in the second term of (6) are obtained as a result of the similarity transformation described above, when we replace $\partial/\partial\psi(x,0)\to -(1/2)\partial/\partial\psi(x,0)\ln J$. We therefore obtain:

$$H_2^s = -\frac{1}{2}\int dx\bar{\omega}(x,s)\frac{\partial}{\partial\psi(x,s)} + \frac{1}{2}\times\int dx\int dy\Omega(x,0{:}y,s)\frac{\partial\ln J}{\partial\psi(x,0)}\frac{\partial}{\partial\psi(y,s)}. \qquad (12)$$

In a problem involving joining and splitting of loop states, the issue of closure of loop space is an important one. The first term in (12) always closes. This is because

$$\bar{\omega}(k,s) = -2\int_0^k dk'k'\psi(k',s)\psi(k-k',0).$$

This result is a straightforward application of a result established in [48]. In the x representation,

$$\bar{\omega}(z,s) = \int\frac{dk}{2\pi}e^{-ikz}\bar{\omega}(k,s) = -2\bigg[\psi(z,s)\int dx\frac{\phi_0(x)}{(x-z)^2} - \phi_0(z)\times\int dx\frac{\psi(x,s)}{(x-z)^2} + \int dx\phi_0(x)\partial_z\bigg(\frac{\psi(z,s)}{(z-x)}\bigg)\bigg].$$

Substituting this expression into (12) we obtain

$$H_2^s = \int dx\int dz\frac{\phi_0(z)\psi(x,s)-\psi(z,s)\phi_0(x)}{(x-z)^2}\frac{\partial}{\partial\psi(x,s)} - \int dx\int dz\frac{\phi_0(z)\psi(x,s)}{(x-z)}\partial_x\frac{\partial}{\partial\psi(x,s)} + \frac{1}{2}\int dx\int dy\Omega(x,0{:}y,s)\frac{\partial\ln J}{\partial\psi(x,0)}\frac{\partial}{\partial\psi(y,s)}. \qquad (13)$$

In general, for an arbitrary potential, the last term in (13) involving $\Omega(x,0{:}y,s)$ will not close. We will argue in the following that for the harmonic oscillator potential this term closes, by considering explicitly $s=1,2,3$ and then arguing for the general case.

C. The one impurity sector

It is straightforward to show that in this case

$$\Omega(k,0{:}k',1) = -kk'\psi(k+k',1),$$

from which it follows

$$\Omega(x,0;y,1) = \partial_x\partial_y(\psi(x,1)\delta(x-y)).$$

The term involving $\Omega(x,0{:}y,1)$ in (13) becomes

$$\frac{1}{2}\int dx\int dy\Omega(x,0{:}y,1)\frac{\partial\ln J}{\partial\psi(x,0)}\frac{\partial}{\partial\psi(y,1)} = \frac{1}{2}\int dx\partial_x\frac{\partial\ln J}{\partial\psi(x,0)}\psi(x,1)\partial_x\frac{\partial}{\partial\psi(x,1)} = \int dx\int dz\frac{\phi_0(z)}{(x-z)}\psi(x,1)\partial_x\frac{\partial}{\partial\psi(x,1)},$$

where we have used (11). We observe that this term cancels exactly a similar term in (13), and we obtain the final form for the quadratic Hamiltonian in the 1 impurity sector:

$$H_2^{s=1} = \int dx\int dz\frac{\phi_0(z)\psi(x,1)-\psi(z,1)\phi_0(x)}{(x-z)^2}\frac{\partial}{\partial\psi(x,1)}. \qquad (14)$$

The rescaling (B6) leaves the above Hamiltonian invariant, or equivalently the above Hamiltonian is of order 1 (N^0) in N, as was the case in the zero impurity sector. Writing the operator as

$$\int dx\int dy\psi(x,1)K(x,y)\frac{\partial}{\partial\psi(y,1)},$$

we obtain

$$\int dyK(x,y)\frac{\partial}{\partial\psi(y,1)} = \int dy\frac{\phi_0(y)}{(x-y)^2}\bigg(\frac{\partial}{\partial\psi(x,1)} - \frac{\partial}{\partial\psi(y,1)}\bigg).$$

Acting on a wave functional

$$\Phi = \int dzf(z)\psi(z,1),$$

we obtain the Marchesini-Onofri kernel [34,35,49,50]

$$\int dy\frac{\phi_0(y)}{(x-y)^2}(f(x)-f(y)) = \bigg(-\frac{d}{dx}\int dy\frac{\phi_0(y)}{(x-y)}\bigg)f(x) + \frac{d}{dx}\int dy\frac{\phi_0(y)f(y)}{(x-y)}.$$

Using (10), the first term yields $-f(x)$, and by changing to the time of flight coordinates the kernel can be written as

$$-f(q) - \frac{i}{\pi\phi_0}|\partial_q|(\pi\phi_0(q)f(q)),$$

or, for the spectrum equation

$$(-1 - i|\partial_q|)(\pi\phi_0(q)f(q)) = w(\pi\phi_0(q)f(q)).$$

As described in Appendix C the spectrum and eigenfunctions of this operator are

$$w_n = n - 1; \qquad \phi_n^{s=1} = \frac{\sin(nq)}{\sqrt{2}\sin(q)}; \qquad n = 1, 2, \ldots.$$

For the harmonic oscillator potential, these are the well known Tchebychev polynomials of the second kind. Adding the contribution from the $\mathrm{Tr}(B\partial/\partial B)$ term of the Hamiltonian we obtain

$$w_n = n; \qquad \phi_n^{s=1} = \frac{\sin(nq)}{\sqrt{2}\sin(q)}; \qquad n = 1, 2, \ldots. \quad (15)$$

D. The two impurities sector

For two impurities, we have

$$\begin{aligned}\Omega(k_0, 0{:}k, 2) &= -2k_0 \int dk' k' \mathrm{Tr}(Be^{i(k-k')M}Be^{i(k'+k_0)M}) \\ &= -2k_0 \Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji} \\ &\quad \times \left[-ik\frac{e^{i(k+k_0)\lambda_i}}{(\lambda_i - \lambda_j)} + e^{ik_0\lambda_i}\frac{e^{ik\lambda_i} - e^{ik\lambda_j}}{(\lambda_i - \lambda_j)^2}\right],\end{aligned}$$

and

$$\begin{aligned}\Omega(x, 0{:}y, 2) &= -2i\partial_x\partial_y\Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji}\delta(x - y) \\ &\quad \times \frac{\delta(y - \lambda_i)}{(\lambda_i - \lambda_j)} - 2i\partial_x\Sigma_{i,j}(V^+BV)_{ij} \\ &\quad \times (V^+BV)_{ji}\delta(x - \lambda_i) \\ &\quad \times \frac{\delta(y - \lambda_i) - \delta(y - \lambda_j)}{(\lambda_i - \lambda_j)^2}.\end{aligned}$$

The $\Omega(x, 0{:}y, 2)$ term in (13) takes the form

$$\begin{aligned}\frac{1}{2}\int dx \int dy \Omega(x, 0{:}y, 2)\frac{\partial \ln J}{\partial\psi(x, 0)}\frac{\partial}{\partial\psi(y, 2)} &= -i\int dx \int dy\Big[\Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji}\delta(x - y) \\ &\quad \times \frac{\delta(y - \lambda_i)}{(\lambda_i - \lambda_j)}\partial_x\frac{\partial \ln J}{\partial\psi(x, 0)}\partial_y\frac{\partial}{\partial\psi(y, 2)} - \Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji}\delta(x - \lambda_i) \\ &\quad \times \frac{\delta(y - \lambda_i) - \delta(y - \lambda_j)}{(\lambda_i - \lambda_j)^2}\partial_x\frac{\partial \ln J}{\partial\psi(x, 0)}\frac{\partial}{\partial\psi(y, 2)}\Big] \\ &= -2i\int dx\Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji}\frac{\delta(x - \lambda_i)}{(x - \lambda_j)}\int dz\frac{\phi_0(z)}{(x - z)}\partial_x\frac{\partial}{\partial\psi(x, 2)} \\ &\quad + 2i\int dy\Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji}\frac{\delta(y - \lambda_i)}{(\lambda_i - \lambda_j)^2}\Big[\int dz\frac{\phi_0(z)}{(\lambda_i - z)} \\ &\quad - \int dz\frac{\phi_0(z)}{(\lambda_j - z)}\Big]\frac{\partial}{\partial\psi(y, 2)}.\end{aligned}$$

For the harmonic oscillator potential, we can use the result (10), so that

$$\begin{aligned}&\frac{1}{2}\int dx \int dy\Omega(x, 0{:}y, 2)\frac{\partial \ln J}{\partial\psi(x, 0)}\frac{\partial}{\partial\psi(y, 2)} \\ &\quad = -2i\int dx\Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji}\frac{\delta(x - \lambda_i)}{(x - \lambda_j)} \\ &\quad\quad \times \int dz\frac{\phi_0(z)}{(x - z)}\partial_x\frac{\partial}{\partial\psi(x, 2)} + 2i\int dy\Sigma_{i,j}(V^+BV)_{ij} \\ &\quad\quad \times (V^+BV)_{ji}\frac{\delta(y - \lambda_i)}{(y - \lambda_j)}\frac{\partial}{\partial\psi(y, 2)}. \qquad (16)\end{aligned}$$

But from (4),

$$\begin{aligned}\psi(x, 2) &= \int\frac{dk}{2\pi}e^{-ikx}\psi(k, 2) \\ &= -2i\int\frac{dk}{2\pi}e^{-ikx}\Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji} \\ &\quad \times \frac{e^{ik\lambda_j}}{(\lambda_j - \lambda_i)} \\ &= -2i\Sigma_{i,j}(V^+BV)_{ij}(V^+BV)_{ji}\frac{\delta(x - \lambda_j)}{(x - \lambda_i)}.\end{aligned}$$

This allows us to express (16) entirely in terms of the density $\psi(x, 2)$ as

$$\frac{1}{2}\int dx \int dy \Omega(x, 0:y, 2)\frac{\partial \ln J}{\partial \psi(x, 0)}\frac{\partial}{\partial \psi(y, 2)} = \int dx \int dz \frac{\phi_0(z)}{(x-z)}\psi(x, 2)\partial_x \frac{\partial}{\partial \psi(x, 2)} - \int dx \psi(x, 2) \times \frac{\partial}{\partial \psi(x, 2)}.$$

As was the case for the one impurity sector, the first term above cancels the similar term in (13), and we obtain for the quadratic Hamiltonian in the 2 impurity sector:

$$H_2^{s=2} = \int dx \int dz \frac{\phi_0(z)\psi(x, 2) - \psi(z, 2)\phi_0(x)}{(x-z)^2}\frac{\partial}{\partial \psi(x, 2)} - \int dx \psi(x, 2)\frac{\partial}{\partial \psi(x, 2)}. \quad (17)$$

This is a shifted Marchesini-Onofri operator. It can be recast in the form:

$$(-2 - i|\partial_q|)(\pi \phi_0(q) f(q)) = w(\pi \phi_0(q) f(q)).$$

The spectrum and eigenfunctions of this operator are

$$w_n = n - 2; \qquad \phi_n^{s=2} = \frac{\sin(nq)}{\sqrt{2}\sin(q)}; \qquad n = 1, 2, \ldots.$$

Adding the contribution from the $\mathrm{Tr}(B\partial/\partial B)$ term of the Hamiltonian we obtain

$$w_n = n; \qquad \phi_n^{s=2} = \frac{\sin(nq)}{\sqrt{2}\sin(q)}; \qquad n = 1, 2, \ldots. \quad (18)$$

E. Multi-impurity spectrum

The pattern that emerges from the above discussion is clear: for s impurities and the harmonic oscillator potential, one obtains a shifted Marchesini-Onofri operator with spectrum and eigenfunctions

$$w_n = n - s; \qquad \phi_n^{s} = \frac{\sin(nq)}{\sqrt{2}\sin(q)}; \qquad n = 1, 2, \ldots.$$

When the contribution from the $\mathrm{Tr}(B\partial/\partial B)$ is added, we have for the full Hamiltonian

$$w_n = n; \qquad \phi_n^{s} = \frac{\sin(nq)}{\sqrt{2}\sin(q)}; \qquad n = 1, 2, \ldots. \quad (19)$$

To provide further evidence of this pattern, the 3 impurity case is treated explicitly in Appendix D. We also checked that by introducing multi local densities and then projecting to the 2 and 3 impurity states discussed here, we obtain the spectrum described above.

To summarize, as the $U(1)$ charge operator $\hat{J}$ it is represented by

$$\hat{J} = -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) + \frac{1}{2}\mathrm{Tr}(M^2) - \mathrm{Tr}\left(B\frac{\partial}{\partial B}\right),$$

and consequently $j = n - 2s$. Together with the energy eigenvalues $w = n$, these specify a two parameter family of states and a two dimensional complete set of eigenfunctions.

IV. SUGRA MAP

In this section we would like to identify the states of Sugra fluctuations and establish a one to one map with the eigenstates of the matrix problem found in the previous section. With the two matrices we hope to explore the extra coordinate which will be related to the radial coordinate of AdS and S. Since the other angular coordinates are ignored, it is sufficient to concentrate on the small fluctuation equations associated with $\mathrm{AdS}_3 \times S_3$ (the analysis for $\mathrm{AdS}_5 \times S_5$ reaches an identical conclusion). We have obtained in the matrix model solution a two parameter sequence of states with the eigenvalues $J = j$ and $w = j + 2n$. It is easy to find a corresponding sequence of states, which have the same eigenvalues. Actually there are two sequences, one with nontrivial functional dependence in the radial variable of AdS and the other in S. This situation is familiar from giant gravitons.

It will be clear that while the integer valued eigenvalues easily agree (between the matrix model and supergravity), the comparison of their wave functions is much less trivial and also much more interesting. In Sugra the wave functions are given as nontrivial special functions, while in the solution of the matrix eigenvalue problem they take the form of ordinary plane waves . The later obviously happens after the change from eigenvalue coordinate to the ''time of flight'' coordinate. We will establish a relationship between the two pictures in terms of a kernel describing a (canonical) change of variables.

A. The LLM kernel

It is useful first to work out the form of the kernel for the case of 1/2 BPS states given by the LLM map. For this one has to consider the LLM construction and perform the small fluctuation analysis. We do this in Appendix A where we also give the details of a transformation to the Lorentz-De Donder gauge. Furthermore, from now on the time of flight q will be denoted by τ.

Let us concentrate on the fluctuations associated with the metric $g_{\tilde{\Omega}\tilde{\Omega}}$. In the gauge of LLM the perturbation $\delta g_{\tilde{\Omega}\tilde{\Omega}}$ reads

$$\delta g_{\tilde{\Omega}\tilde{\Omega}} = -2\sinh\rho \sin\theta \sqrt{\frac{1+2u_{\mathrm{AdS}}}{1-2u_{\mathrm{AdS}}}}\frac{1}{(1+2u_{\mathrm{AdS}})^2}\tilde{u} d\tilde{\Omega}_3^2 = \sin^2\theta \frac{1}{2\pi}\int_0^{2\pi} d\tau \frac{(1-a^2)^2}{[1+a^2-2a\cos(\tau-\phi)]^2}\sum_j a_j e^{ij\tau} \quad (20)$$

$$a = \frac{\cos\theta}{\cosh\rho}$$

The a_j's are harmonic oscillator amplitudes. The relevant gauge transformation can be written in integral form as

$$\delta\theta = -\frac{\sin\theta\cos\theta}{\cosh^2\rho - \cos^2\theta}\sum_j a_j e^{ij\phi} = -\tan\theta\frac{1}{2\pi}\frac{a^2}{1-a^2} \times \int_0^{2\pi} d\tau \frac{1-a^2}{1+a^2-2a\cos(\tau-\phi)}\sum_j a_j e^{ij\tau}. \quad (21)$$

Performing the gauge transformation we have

$$\delta g_{\tilde{\Omega}\tilde{\Omega}} = \sin^2\theta\frac{1}{2\pi} \times \int_0^{2\pi} d\tau \frac{1-4a^2-a^4+4a^3\cos(\tau-\phi)}{[1+a^2-2a\cos(\tau-\phi)]^2}\sum_j a_j e^{ij\tau}. \quad (22)$$

In this form we see the relation

$$2|j|\sigma_j(t,\rho,\phi,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\tau \frac{1-4a^2-a^4+4a^3\cos(\tau-\phi)}{[1+a^2-2a\cos(\tau-\phi)]^2} a_j e^{ij\tau}. \quad (23)$$

After performing the field dependent gauge transformation in order to recognize the primary field coming from the metric and the three form one has the relation

$$|j|\sigma_j(t,\rho,\phi,\theta) = \frac{1}{2\pi}e^{ijt}\int_0^{2\pi} d\tau K^{\mathrm{LLM}}(\rho,\phi,\theta|\tau)e^{ij\tau}, \quad (24)$$

where the kernel is given by

$$K^{\mathrm{LLM}}(\rho,\phi,\theta|\tau) = \frac{1-4a^2-a^4+4a^3\cos(\tau-\phi)}{[1+a^2-2a\cos(\tau-\phi)]^2}, \qquad a = \frac{\cos\theta}{\cosh\rho}. \quad (25)$$

At this point we notice that $a<1$ at points where the measure of $AdS_3\times S^3$ is nonzero. In this region, we may consider introducing a cutoff L limiting the angular momentum j. We then have the kernel

$$|j|\sigma_j(t,\rho,\phi,\theta) = \frac{1}{2\pi}e^{ijt}\int_0^{2\pi} d\tau K_L^{\mathrm{LLM}}(\rho,\phi,\theta|\tau)e^{ij\tau}, \quad |j|\leq L \qquad 0 = \frac{1}{2\pi}e^{ijt}\int_0^{2\pi} d\tau K_L^{\mathrm{LLM}}(\rho,\phi,\theta|\tau)e^{ij\tau}, \quad |j|>L. \quad (26)$$

The kernel with the cutoff is given by

$$K_L^{\mathrm{LLM}}(\rho,\phi,\theta|\tau) = K^{\mathrm{LLM}}(\rho,\phi,\theta|\tau) + \frac{-\cos[L(\tau-\phi)] + a\cos[(L-1)(\tau-\phi)]}{1+a^2-2a\cos(\tau-\phi)}a^L + \frac{aL\cos[(L+2)(\tau-\phi)] - [1+L+2La^2]\cos[(L+1)(\tau-\phi)]}{[1+a^2-2a\cos(\tau-\phi)]^2}a^L + \frac{-a^2(L+1)\cos[(L-1)(\tau-\phi)] + a[2+2L+La^2]\cos[L(\tau-\phi)]}{[1+a^2-2a\cos(\tau-\phi)]^2}a^L. \quad (27)$$

We see that we have a strong convergence

$$\lim_{L\to\infty} K_L^{\mathrm{LLM}}(\rho,\phi,\theta|\tau) = K^{\mathrm{LLM}}(\rho,\phi,\theta|\tau). \quad (28)$$

B. Correspondence with the 2d black hole

To proceed with the construction of the kernel in our more general two dimensional case, it is also useful to take note of a correspondence with an equivalent problem that was considered in the case of a 2d black hole. We show in what follows that there is a simple connection between "off-shell" black hole wave functions and on-shell AdS wave functions that we have identified.

The wave functions that we consider correspond to highest weight states on $SO(4)$ but with a nontrivial dependence on the radial coordinate of AdS

$$f = \cos^l(\theta)e^{il\phi}\psi(t,\sigma).$$

We have the following eigenequation for ψ

$$-\cos^2(\sigma)\partial_t^2\psi + \cos^2(\sigma)\partial_\sigma^2 + \cot(\sigma)\partial_\sigma\psi = l(l-2)\psi$$
$$\Rightarrow -\partial_t^2\psi + \partial_\sigma^2\psi + \frac{1}{\cos(\sigma)\sin(\sigma)}\partial_\sigma\psi = \frac{l(l-2)}{\cos^2(\sigma)}\psi,$$

with the integration measure

$$dm = \sqrt{-g}g^{00}dtd\sigma = \tan(\sigma)d\sigma.$$

A change to a new function

$$R = \frac{1}{\cos(\sigma)}\psi,$$

with the new measure

$$dm = \frac{1}{2}\sin(2\sigma)dtd\sigma,$$

leads to the equation

$$-\partial_t^2 R + \frac{1}{\cos(\sigma)}\left[\partial_\sigma^2 + \frac{1}{\cos(\sigma)\sin(\sigma)}\partial_\sigma\right]\cos(\sigma)R = \frac{l(l-2)}{\cos^2(\sigma)}R$$

or

$$\partial_\sigma^2 R + 2\cot(2\sigma)\partial_\sigma R = \frac{l(l-2)+1}{\cos^2(\sigma)}R - \omega^2 R + R.$$

This can be compared with the 2d black hole equation[36] defined as a coset $\widetilde{SL}(2,R)/U(1)$. For the case of the Lorentzian black hole they are specified by the eigenvalue equation

$$\begin{aligned}\Delta_0 T_\nu^\lambda &= \left(-\frac{1}{4}-\lambda^2\right)T_\nu^\lambda \\ &\Rightarrow -\frac{1}{4\sinh(\frac{r}{2})}\partial_\tau^2 T_\nu^\lambda + \partial_r^2 T_\nu^\lambda + \coth(r)\partial_r T_\nu^\lambda \\ &= \left(-\frac{1}{4}-\lambda^2\right)T_\nu^\lambda \\ &\Rightarrow \frac{1}{\sinh(\frac{r}{2})}\nu^2 T_\nu^\lambda + \partial_r^2 T_\nu^\lambda + \coth(r)\partial_r T_\nu^\lambda \\ &= \left(-\frac{1}{4}-\lambda^2\right)T_\nu^\lambda,\end{aligned}$$

and the inner product is defined through the integration measure

$$\langle T_\nu^\lambda | T_{\nu'}^{\lambda'}\rangle = \delta(\nu-\nu')\int_0^\infty dr\sinh(r)(T_\nu^\lambda(r))^* T_{\nu'}^{\lambda'}(r).$$

We see that the two problems are related, through the following transformation transformations

$$l \to 1-2i\nu \qquad \omega \to i2\lambda \qquad \sigma \to \frac{i}{2}(r+\pi)$$

In Ref. [36], a transformation was constructed relating the wave functions in the black hole case to those of a c = 1 matrix model. The transformation reads

$$T_\nu^\lambda = \int_{-\infty}^{+\infty} dt_0 \int_0^{+\infty} d\sigma\delta\left[\sinh\left(\frac{r}{2}\right)\sinh\left(\frac{2t_0}{3}-\tau\right) - \cosh(2\sigma)\right]e^{-4i(t_0/3)}\cos(4\lambda\sigma).$$

and it involves a nontrivial kernel which specifies a canonical transformation from one problem to another. In the present case we will follow the construction of [36] and construct an analogous kernel which will relate AdS (and S) wave functions to those of the matrix eigenvalue problem.

C. The AdS kernel

We first give the main formulas defining the AdS kernel. The wave functions obey the equations

$$\begin{aligned}\nabla_{S^3}^2\sigma_{j,n}(t,\rho,\phi,\theta) &= -|j|(|j|+2)\sigma_{j,n}(t,\rho,\phi,\theta) \\ \nabla_{\mathrm{AdS}^3}^2\sigma_{j,n}(t,\rho,\phi,\theta) &= |j|(|j|-2)\sigma_{j,n}(t,\rho,\phi,\theta) - i\frac{\partial}{\partial\phi}\sigma_{j,n}(t,\rho,\phi,\theta) \\ &= j\sigma_{j,n}(t,\rho,\phi,\theta),\end{aligned} \tag{29}$$

and have an explicit solution in terms of hypergeometric functions

$$\begin{aligned}\sigma_{j,n}(t,\rho,\phi,\theta) &= e^{(j/|j|)i\omega_{j,n}t}\cos^{|j|}\theta e^{(j/|j|)ij\phi}\cosh^{-(|j|+2n)}\rho F(1-j-n,-n;1;-\sinh^2\rho) \\ \omega_{j,n} &= |j|+2n.\end{aligned} \tag{30}$$

We now use the integral representation

$$\begin{aligned}\sigma_{j,n}(t,\rho,\phi,\theta) &= e^{(j/|j|)i\omega_{j,n}t}\oint_C dz\frac{1}{i2\pi z}[\cosh\rho e^{-(j/|j|)i\phi}\times(\cosh\rho+z\sinh\rho)]^{-|j|-2n} \\ &\quad\times\left[\left(\frac{\cosh\rho}{z}+\sinh\rho\right)\times\left(\cosh\rho+z\sinh\rho\right)e^{-2(j/|j|)i\phi}\right]^n \\ &= e^{(j/|j|)i\omega_{j,n}t}\oint_C dz\frac{1}{i2\pi z}[e^{(j/|j|)i\phi}w]^{|j|+2n}\times[e^{-2(j/|j|)i\phi}v]^n,\end{aligned} \tag{31}$$

where

$$\begin{aligned}w &= \frac{1}{\cosh\rho(\cosh\rho+z\sinh\rho)} \\ v &= \left(\frac{\cosh\rho}{z}+\sinh\rho\right)(\cosh\rho+z\sinh\rho)\end{aligned} \tag{32}$$

$$|j|+2n \le L \tag{33}$$

to derive the kernel defined through

$$\begin{aligned}|j|\sigma_{j,n} &= e^{(j/|j|)i\omega_{j,n}t}\frac{1}{4\pi^2}\int_0^{2\pi}d\sigma\int_0^{2\pi}d\tau\times K_L(\rho,\phi,\theta|\sigma,\tau)e^{i(j/|j|)[(|j|+2n)\tau+n\sigma]}, \\ K_L(\rho,\phi,\theta|\sigma,\tau) &= \oint_C\frac{dz}{i2\pi z}(F_L(w|\tau)G_L(v|\sigma) - 2\tilde{F}_L(w|\tau)\tilde{G}_L(v|\sigma)).\end{aligned} \tag{34}$$

Explicitly, the functions involved in the definition of the kernel can be worked out after an introduction of a cutoff L for convergence. They take the slightly long forms:

$$F_L(w|\tau) = \frac{1 - 4w^2 - w^4 + 4w^3\cos(\tau-\phi)}{[1 + w^2 - 2w\cos(\tau-\phi)]^2} + \frac{-\cos[L(\tau-\phi)] + w\cos[(L-1)(\tau-\phi)]}{1 + w^2 - 2w\cos(\tau-\phi)} w^L + \frac{wL\cos[(L+2)(\tau-\phi)] - [1 + L + 2Lw^2]\cos[(L+1)(\tau-\phi)]}{[1 + w^2 - 2w\cos(\tau-\phi)]^2} w^L + \frac{-w^2(L+1)\cos[(L-1)(\tau-\phi)] + w[2 + 2L + Lw^2]\cos[L(\tau-\phi)]}{[1 + w^2 - 2w\cos(\tau-\phi)]^2} w^L \quad (35)$$

$$G_L(v|\sigma) = \frac{1 - v^2}{1 + v^2 - 2v\cos(\sigma - 2\phi)} + \frac{-\cos[L(\sigma - 2\phi)] + v\cos[(L-1)(\sigma - 2\phi)]}{1 + v^2 - 2v\cos(\sigma - 2\phi)} v^L \quad (36)$$

$$\tilde{F}_L(w|\tau) = \frac{1 - w^2}{1 + w^2 - 2w\cos(\tau-\phi)} + \frac{-\cos[L(\tau-\phi)] + w\cos[(L-1)(\tau-\phi)]}{1 + w^2 - 2w\cos(\tau-\phi)} w^L \quad (37)$$

$$\tilde{G}_L(v|\sigma) = \frac{v(v^2+1)\cos(\tau - 2\phi) - 2v^2}{[1 + v^2 - 2v\cos(\tau - 2\phi)]^2} + \frac{vL\cos[(L+2)(\tau-2\phi)] - [1 + L + 2Lv^2]\cos[(L+1)(\tau - 2\phi)]}{[1 + v^2 - 2v\cos(\tau-2\phi)]^2} v^L + \frac{-v^2(L+1)\cos[(L-1)(\tau-2\phi)] + v[2 + 2L + Lv^2]\cos[L(\tau-2\phi)]}{[1 + v^2 - 2v\cos(\tau - 2\phi)]^2} v^L. \quad (38)$$

D. The sphere kernel

We now consider the second sequence of wave functions, which are characterized by a nontrivial dependence on the radial coordinate of the sphere. The wave equations read

$$\nabla^2_{S^3}\sigma_{j,n}(t,\rho,\phi,\theta) = -(|j| + 2n)(|j| + 2n + 2) \times \sigma_{j,n}(t,\rho,\phi,\theta)\nabla^2_{AdS^3}\sigma_{j,n}(t,\rho,\phi,\theta) = (|j| + 2n)(|j| + 2n - 2)\sigma_{j,n}(t,\rho,\phi,\theta) - i\frac{\partial}{\partial\phi}\sigma_{j,n}(t,\rho,\phi,\theta) = j\sigma_{j,n}(t,\rho,\phi,\theta). \quad (39)$$

In the coordinate system where the metric is

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\psi^2 + d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\tilde{\psi}^2,$$

the normalizable solutions are given by

$$\sigma_{j,n} = e^{(j/|j|)i\omega_{j,n}t} e^{ij\phi}\cosh^{-|j|-2n}\rho\cos^{|j|}\theta F(1 + |j| + n, -n; 1; \sin^2\theta), \quad \omega_{j,n} = |j| + 2n. \quad (40)$$

We use the integral form of the wave functions

$$\sigma_{j,n} = e^{(j/|j|)i\omega_{j,n}t}e^{ij\phi}\oint_C \frac{dz}{i2\pi z}\left(\frac{\cos\theta - z\sin\theta}{\cosh\rho}\right)^{|j|+2n} \times \left(\frac{\cos\theta + \frac{\sin\theta}{z}}{\cos\theta - z\sin\theta}\right)^n = e^{(j/|j|)i\omega_{j,n}t}\oint_C dz\frac{dz}{i2\pi z}\left(\frac{\cos\theta - z\sin\theta}{\cosh\rho}e^{i(j/|j|)\phi}\right)^{|j|+2n} \times \left(\frac{\cos\theta + \frac{\sin\theta}{z}}{\cos\theta - z\sin\theta}e^{-i2(j/|j|)\phi}\right)^n = e^{(j/|j|)i\omega_{j,n}t}\oint_C dz\frac{dz}{i2\pi z}[e^{(j/|j|)i\phi}w]^{|j|+2n}[e^{-2(j/|j|)i\phi}v]^n, \quad (41)$$

where C is the unit circle on the complex plane and we defined

$$w = \frac{\cos\theta - z\sin\theta}{\cosh\rho} \qquad v = \frac{\cos\theta + \frac{\sin\theta}{z}}{\cos\theta - z\sin\theta}. \quad (42)$$

Introducing a cutoff on the angular momentum as we had for the LLM case

$$|j| + 2n \leq L, \quad (43)$$

we rewrite the wave function as

$$(|j| + 2n)\sigma_{j,n} = e^{(j/|j|)i\omega_{j,n}t} \frac{1}{4\pi^2} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\tau K_L(\rho, \phi, \theta|\sigma, \tau) e^{i(j/|j|)[(|j|+2n)\tau + n\sigma]},$$
$$K_L(\rho, \phi, \theta|\sigma, \tau) = \oint_C \frac{dz}{i2\pi z} F_L(w|\tau) G_L(v|\sigma). \tag{44}$$

The functions $F_L(w|\tau)$ and $G_L(v|\sigma)$ specifying the kernel in this case are found to be given by

$$\begin{aligned} F_L(w|\tau) = {}& \frac{1 - 4w^2 - w^4 + 4w^3\cos(\tau - \phi)}{[1 + w^2 - 2w\cos(\tau - \phi)]^2} + \frac{-\cos[L(\tau - \phi)] + w\cos[(L-1)(\tau - \phi)]}{1 + w^2 - 2w\cos(\tau - \phi)} w^L \\ &+ \frac{wL\cos[(L+2)(\tau - \phi)] - [1 + L + 2Lw^2]\cos[(L+1)(\tau - \phi)]}{[1 + w^2 - 2w\cos(\tau - \phi)]^2} w^L \\ &+ \frac{-w^2(L+1)\cos[(L-1)(\tau - \phi)] + w[2 + 2L + Lw^2]\cos[L(\tau - \phi)]}{[1 + w^2 - 2w\cos(\tau - \phi)]^2} w^L, \end{aligned} \tag{45}$$

and

$$G_L(v|\sigma) = \frac{1 - v^2}{1 + v^2 - 2v\cos(\sigma - 2\phi)} + \frac{-\cos[L(\sigma - 2\phi)] + v\cos[(L-1)(\sigma - 2\phi)]}{1 + v^2 - 2v\cos(\sigma - 2\phi)} v^L. \tag{46}$$

Let us now make the following comment regarding the cutoff that we have used. Since it imposes an upper limit on angular momenta it clearly plays a role of the 'exclusion principle'. Its removal seems to lead to singularities both in the sphere and the AdS case. One should remember then that this analysis is done at the linearized level, so there is no essential difference between the two cases. We can also show that if we restrict our attention to $1/2$ BPS wave functions (which would correspond to $n = 0$), the above kernel reduces to the kernel that we have found from the LLM construction. We notice that the function $F_L(w|\tau)$ is analytic in the unit circle C of the z-plane for every value of the remaining variables.

$$|j|\sigma_{j,0} = e^{ijt} \frac{1}{4\pi^2} \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma K_L(\rho, \phi, \theta|\sigma, \tau) e^{ij\tau}. \tag{47}$$

Performing the integral over σ gives

$$\int_0^{2\pi} d\sigma G_L(v|\sigma) = 1, \tag{48}$$

which establishes the result

$$\begin{aligned} \int_0^{2\pi} d\sigma K_L(\rho, \phi, \theta|\sigma, \tau) &= \oint_C \frac{dz}{i2\pi z} F_L(w|\tau) \\ &= F_L(w|\tau)|_{z=0} \\ &= K_L^{\mathrm{LLM}}(\rho, \phi, \theta|\tau). \end{aligned} \tag{49}$$

V. CONCLUSION

We have in the present work considered the simple a complex two matrix model with a purpose of developing further its correspondence with AdS eigenstates. We develop a (hybrid) formalism to construct a two dimensional sequence of invariant matrix model eigenstates. Here one of the (matrix) degrees of freedom is treated in a density representation (in a manner analogous to the one matrix collective field theory), while the other is represented in the coherent state picture. This leads to a sequence of (integral) equations which we then solve for the case of the oscillator potential. The two dimensional set of eigenstates extends the one dimensional space representing the eigenstates of free fermions. As such this extension allows a nontrivial probe of one further extra dimension . This as we argue can be mapped into either the radial coordinate of AdS or the radial coordinate of the sphere.

The mapping between states of the matrix model and the wave functions of SUGRA is one to one. As such it differs from the holographic map where one of the dimensions is projected out. In the present case the map can be described by a (two dimensional) kernel in paralel with similar maps in the case of 2d noncritical string theory. We also note that leg factors of this kind were found in the pp-wave map of [51].

When applied to a one dimensional subspace of $1/2$ BPS wave functions our kernel reduces to the (linearized) map of LLM. In the construction of the extended map one seemingly requires a cutoff providing an interesting implementation of the 'exclusion principle'. The understanding of this cutoff is clearly of further interest.

It should be commented that much like in the $1/2$ BPS case of free fermions the model considered is that of simple decoupled harmonic oscillators. Yang-Mills type interactions present in the full theory might be of relevance but are not included in our study. For the case of $1/2$ BPS correlators there are theorems regarding the absence of coupling constant corrections. It can be hoped that this will persist for the present set of states. Certainly, the effect of coupling constant correction deserves to be investigated (e.g., [52]). It is also of interest to extend the present map to a still larger set of eigenstates.

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APPENDIX A: EXPANDING THE LLM SOLUTION IN FLUCTUATIONS

In this section we would like to expand the circular droplet solution in "off-shell" fluctuations of the matrix model and see the equations of motion these fluctuations satisfy from the bosonic equations of motion of gravity. This analysis was also performed independently in a recent paper [26].

The general 1/2 BPS LLM solution for the metric is determined by the function

$$u(x_1, x_2, y) = \frac{y^2}{\pi}\int d\tilde{x}^2 u(\tilde{x}_1, \tilde{x}_2, 0)\frac{1}{[(\vec{x}-\tilde{\vec{x}})^2 + y^2]^2},$$

with $u(\tilde{x}_1, \tilde{x}_2, 0)$ being the phase space distribution of the fermions in the matrix model picture. Parametrizing the boundary of the fermi surface using the polar coordinates representation

$$\tilde{x}_1^2(\phi, t) + \tilde{x}_2^2(\phi, t) = R_{\rm AdS}^2 + \sum_{n>0} p_n(t)\sin(n\phi) + nq_n(t) \times \cos(n\phi),$$

the phase space density becomes

$$u(r, \phi, 0, t) = -\theta\left(\sqrt{R_{\rm AdS}^4 + \sum_{n>0} p_n(t)\sin(n\phi) + nq_n(t)\cos(n\phi)} - r\right) + \frac{1}{2},$$

and after approximating at first order in perturbations p_n, q_n the distribution becomes.

$$u(r, \phi, 0, t) \approx \frac{1}{2} - \theta(R_{\rm AdS}^2 - r) - \delta(R_{\rm AdS}^2 - r)\left[\sum_{n>0}\frac{p_n(t)}{2R_{\rm AdS}^2} \times \sin(n\phi) + n\frac{q_n(t)}{2R_{\rm AdS}^2}\cos(n\phi)\right].$$

The field that is produced is given then by

$$u(r, \phi, y, t) = u_{\rm AdS}(r, \phi, y, t) + \tilde{u}(r, \phi, y, t)$$

$$u(r, \phi, y, t) = u_{\rm AdS}(r, \phi, y, t) - \frac{y^2}{2\pi}\int_0^{2\pi} d\tilde{\phi} \times \frac{\sum_{n>0} p_n(t)\sin(n\tilde{\phi}) + nq_n(t)\cos(n\tilde{\phi})}{[R_{\rm AdS}^4 + r^2 + y^2 - 2rR_{AdS}^2\cos(\tilde{\phi}-\phi)]^2}.$$

The above integral can be computed from the more general

$$\int_0^{2\pi} d\phi \frac{e^{in\phi}}{(a - 2\cos(\phi))^2} = \frac{1}{i}\oint_C dz \frac{z^{n+1}}{(z^2 - az + 1)^2} = \frac{1}{i}\oint_C dz\frac{z^{n+1}}{(z-z_+)^2(z-z_-)^2} = 2\pi\frac{d}{dz}\frac{z^{n+1}}{(z-z_+)^2}\Big|_{z=z_-} = 2\pi\frac{z_-^n}{(z_- - z_+)^2}\left[n + \frac{z_+ + z_-}{z_+ - z_-}\right],$$

$$z_\pm = \frac{a \pm \sqrt{a^2-4}}{2}.$$

Where the contour C is the unit circle on the complex plane of integration and we only picked the contribution from z_- which is the root that is inside the circle for $a > 2$.

After setting

$$y = R_{AdS}^2 \sinh\rho\sin\theta \qquad r = R_{\rm AdS}^2\cosh\rho\cos\theta,$$

the result is given by

$$u(\rho, \phi, \theta, t) = u_{\rm AdS}(\rho, \phi, \theta, t) - \frac{1}{R_{AdS}^4}\frac{\sinh^2\rho\sin^2\theta}{(\cosh^2\rho - \cos^2\theta)^2} \times \sum_{n>0}\left(\frac{\cos\theta}{\cosh\rho}\right)^n\left[n + \frac{\cosh^2\rho + \cos^2\theta}{\cosh^2\rho - \cos^2\theta}\right] \times [p_n(t)\sin(n\phi) + nq_n(t)\cos(n\phi)],$$

where

$$u_{\rm AdS}(\rho, \phi, \theta, t) = \frac{1}{2}\frac{\sinh^2\rho - \sin^2\theta}{\sinh^2\rho + \sin^2\theta}.$$

The perturbation of the metric on S^5 is given by

$$\frac{1}{R_{\rm AdS}^2}d\tilde{s}_{S^5}^2 = -\frac{2}{\sinh\rho\sin\theta}(\cosh^2\rho\sin^2\theta + \sinh^2\rho\cos^2\theta) \times\frac{u_{\rm AdS}}{\sqrt{1-4u_{\rm AdS}^2}}\tilde{u}d\theta^2\frac{4\cosh\rho\sinh\rho\sin^2\theta V_\phi^{\rm AdS}}{\sqrt{1-4u_{\rm AdS}^2}} \times\tilde{V}_r d\theta d\phi - \left[\frac{4\sinh\rho\sin\theta V_\phi^{\rm AdS}}{\sqrt{1-4u_{\rm AdS}^2}}\tilde{V}_\phi \times + \frac{8\sinh\rho\sin\theta u_{\rm AdS}V_\phi^{\rm AdS^2}}{\sqrt{1-4u_{\rm AdS^3}^2}}\tilde{u} \times + \frac{2\cosh^2\rho\cos^2\theta u_{\rm AdS}}{\sinh\rho\sin\theta\sqrt{1-4u_{\rm AdS}^2}}\tilde{u}\right]d\phi^2 \times -2\sinh\rho\sin\theta\sqrt{\frac{1+2u_{\rm AdS}}{1-2u_{\rm AdS}}}\frac{1}{(1+2u_{\rm AdS})^2}\tilde{u}d\tilde{\Omega}_3^2.$$

At this point we would like to show that the degrees of freedom q_n, p_n turn on the chiral primary fields σ^I of IIB SUGRA on $AdS_5 \times S^5$. After performing the field dependent coordinate transformation

$$\theta \to \theta - \frac{1}{2R_{\rm AdS}^4}\frac{\sin\theta\cos\theta}{\cosh^2\rho - \cos^2\theta} \times \sum_{n>0}\left(\frac{\cos\theta}{\cosh\rho}\right)^n [p_n(t)\sin(n\phi) + nq_n(t)\cos(n\phi)]$$

$$\rho \to \rho + \frac{1}{R_{\rm AdS}^4}\frac{\cos^2\theta\tanh\rho}{\cosh^2\rho - \cos^2\theta} \times \sum_{n>0}\left(\frac{\cos\theta}{\cosh\rho}\right)^n [p_n(t)\sin(n\phi) + nq_n(t)\cos(n\phi)]$$

$$t \to t + \frac{1}{R_{\rm AdS}^4}\sum_{n>0}\left(\frac{\cos\theta}{\cosh\rho}\right)^n [p_n(t)\cos(n\phi) - q_n(t)\sin(n\phi)],$$

the $\theta\theta$ and $\tilde{S}_3$ components of the first order perturbed metric are scaled by

$$\frac{2}{R_{\rm AdS}^4}(n+1)\sum_{n>0}\left(\frac{\cos\theta}{\cosh\rho}\right)^n [p_n(t)\sin(n\phi) + nq_n(t)\cos(n\phi)].$$

After this observation we may identify the chiral primary fields

$$\sigma^{\pm n} = \frac{1}{8R_{\rm AdS}^4}\frac{n+1}{n}[nq_n \mp ip_n]\left(\frac{\cos\theta}{\cosh\rho}\right)^n.$$

The correctly normalized action for the chiral primaries as given by Seiberg *et al.* reads

$$S = \sum_n \frac{8R_{\rm AdS}^8 n(n-1)}{(n+1)^2}\int_{{\rm AdS}^5} dx^5\sqrt{g_{{\rm AdS}^5}} \times [\sigma^{-n}\Box\sigma^{+n} - n(n-4)\sigma^{-n}\sigma^{+n}],$$

which after performing the spatial integral on AdS_5 gives

$$S = \sum_{n>0}\frac{1}{2}\int dt\left[\frac{1}{n^2}\dot{p}_n^2 + \dot{q}_n^2 - n^2 q_n^2 - p_n^2\right],$$

and for each n we have a four dimensional phase space. Supersymmetry requires that $(\partial_t - \partial_\phi)\sigma = 0$ which for our $0+1$ dimensional variables means that the "angular momentum" is equal to the energy. Choosing an opposite chirality for the fermions we would have had the condition $(\partial_t + \partial_\phi)\sigma = 0$ which would flip the sign in the relation between energy and "angular momentum".

APPENDIX B: HERMITICITY AND THE ZERO IMPURITY SECTOR

The zero impurity sector is the usual single matrix problem for M_{ij}. We are interested in fluctuations about this single matrix background. As is now well known [39–41], this background is only exhibited as the stationary point of an explicitly Hermitian effective potential. We recall the construction of this effective Hamiltonian [39].

In order to take into account the nontrivial Jacobian J involved in the change from the original variables to loop variables, one needs to implement the similarity transformation (i is a generic loop variable)

$$\partial_i \to J^{1/2}\partial_i J^{-1/2} = \partial_i - \frac{1}{2}\partial_i \ln J.$$

The Jacobian satisfies [39]

$$\Omega_{ij}\partial_j \ln J = \omega_i - \partial_j\Omega_{ji}.$$

The terms of the kinetic energy operator that are sufficient to generate the background and fluctuations are then [53–55]

$$-\tfrac{1}{2}\partial_i\Omega_{ij}\partial_j + \tfrac{1}{8}\omega_i\Omega_{ij}^{-1}\omega_j. \qquad \text{(B1)}$$

In the zero impurity sector,

$$\omega(k,0) = -k\int_0^k dk'\psi(k',0)\psi(k-k',0)$$
$$\Omega(k,0;k',0) = -kk'\psi(k+k',0) \qquad \text{(B2)}$$

The x representation of $\psi(k,0)$ is the usual density of eigenvalues:

$$\psi(x,0) = \Sigma_i\delta(x-\lambda_i),$$

and

$$\Omega(x,0;y,0) = \partial_x\partial_y(\psi(x,0)\delta(x-y))$$
$$\omega(x,0) = -2\partial_x\left(\psi(x,0)\int dz\frac{\psi(z,0)}{x-z}\right).$$

From (B1) we then obtain the form of the effective Hamiltonian which is sufficient for the discussion of background generation and fluctuations:

$$H = \int dx\int dy\left(-\frac{1}{2}\frac{\partial}{\partial\psi(x,0)}\Omega(x,0;y,0)\frac{\partial}{\partial\psi(y,0)} + \frac{1}{8}\omega(x,0)\Omega^{-1}(x,0;y,0)\omega(y,0)\right) + \int dx\psi(x,0)\left(\frac{x^2}{2}-\mu\right)\Bigg),$$

where the Lagrange multiplier μ enforces the constraint

$$\int dx\psi(x,0) = N. \qquad \text{(B3)}$$

Since

$$\partial_x\partial_y\Omega^{-1}(x,0;y,0) = \frac{\delta(x-y)}{\psi(x,0)},$$

and

$$\int dx\psi(x,0)\left(\int dy\frac{\psi(y,0)}{x-y}\right)^2 = \frac{\pi^2}{3}\int dx\psi^3(x,0), \qquad \text{(B4)}$$

the effective Hamiltonian becomes:

$$-\frac{1}{2}\int dx\partial_x\frac{\partial}{\partial\psi(x,0)}\psi(x,0)\partial_x\frac{\partial}{\partial\psi(x,0)} + \int dx\left(\frac{\pi^2}{6}\psi^3(x,0)+\psi(x,0)\left(\frac{x^2}{2}-\mu\right)\right). \tag{B5}$$

To exhibit explicitly the N dependence, we rescale

$$x\to\sqrt{N}x \qquad \psi(x,0)\to\sqrt{N}\psi(x,0)$$
$$-i\frac{\partial}{\partial\psi(x,0)}\equiv\Pi(x)\to\frac{1}{N}\Pi(x) \qquad \mu\to N\mu, \tag{B6}$$

and obtain

$$H^0_{\rm eff}=\frac{1}{2N^2}\int dx\partial_x\Pi(x)\psi(x,0)\partial_x\Pi(x) + N^2\left(\int dx\frac{\pi^2}{6}\psi^3(x,0)+\psi(x,0)\left(\frac{x^2}{2}-\mu\right)\right), \tag{B7}$$

which is Eq. (7) in the main text.

APPENDIX C: MARCHESINI-ONOFRI KERNEL

We consider the problem of finding the spectrum of the operator

$$\int_{-\sqrt{2}}^{\sqrt{2}}dy\frac{\phi_0(y)}{(x-y)^2}(f(x)-f(y)) = \left(-\frac{d}{dx}\int_{-\sqrt{2}}^{\sqrt{2}}dy\frac{\phi_0(y)}{(x-y)}\right)f(x)+\frac{d}{dx}\times\int_{-\sqrt{2}}^{\sqrt{2}}dy\frac{\phi_0(y)f(y)}{(x-y)}. \tag{C1}$$

We start with the second term and consider the following integral, in "time of flight" coordinates:

$$\int_{-\pi}^{\pi}\frac{dq}{\pi}\pi\phi_0(q)\frac{e^{inq}}{x(q_0)-x(q)}, \qquad n>0.$$

Note that the range of the integral extends over a full period $2L=2\pi$ of the classical motion. Therefore, the integral above can be calculated by the residue theorem, by choosing a vertical path from $-\pi+i\infty$ to $-\pi$, then along the real axis from $-\pi$ to π, and then along a vertical path from π to $\pi+i\infty$, "closing" at $+i\infty$. The contribution fom the vertical paths cancel, due to the periodicity of the classical motion. The origin of the "time of flight" can always be chosen so that the only poles on the real axis occur at $q=q_0$ and $q=-q_0$, corresponding to an even (in q) "displacement" $x(q)$ and odd "velocity" $\pi\phi_0(q)$. We alwyas choose a principal value prescription for poles on the real axis (half of the residue). If there are no other poles, as is the case in general for stabilized potentials, we obtain the result:

$$\int_{-\pi}^{\pi}\frac{dq}{\pi}\pi\phi_0(q)\frac{e^{inq}}{x(q_0)-x(q)}=2i\int_0^{\pi}\frac{dq}{\pi}\pi\phi_0(q)\times\frac{\sin(nq)}{x(q_0)-x(q)} = -2i\cos(nq_0).$$

In other words

$$\int_0^{\pi}\frac{dq}{\pi}\pi\phi_0(q)\frac{\sin(nq)}{x(q_0)-x(q)}=-\cos(nq_0). \tag{C2}$$

Therefore

$$\partial_q\int\frac{dq'}{\pi}\frac{\sin(nq')}{x(q)-x(q')}\equiv -i|\partial_q|(\sin(nq))=n(\sin(nq)).$$

In x space,

$$\int_{-\sqrt{2}}^{\sqrt{2}}dy\frac{\sin(nq(y))}{(x-y)}=-\cos(nq(x)).$$

It follows that the eigenvalue equation

$$\frac{d}{dx}\int_{-\sqrt{2}}^{\sqrt{2}}\frac{dy}{\pi}\frac{\pi\phi_0(y)f_n(y)}{(x-y)}=\epsilon_n f_n,$$

has solutions

$$f_n(x)=\frac{\sin(nq(x))}{\pi\phi_0}=\frac{\sin(nq(x))}{\sqrt{2}\sin(q(x))},$$
$$x(q)=-\sqrt{2}\cos(q), \qquad \epsilon_n=n.$$

This follows from the observation that in terms of time of flight coordinates, the above spectrum equation takes the form

$$\partial_q\int\frac{dq'}{\pi}\frac{\pi\phi_0(q')f_n(q')}{x(q)-x(q')}\equiv -i|\partial_q|(\pi\phi_0(q)f_n(q)).$$

Concerning the first term in (C1), we have already seen for the main text that it can be obtained straightforwardly from the result (Eq. (10))

$$\int dz\frac{\phi_0(z)}{(x-z)}=x. \tag{C3}$$

This equation is solved by the well known methods of Ref. [46]. We point out that first term of (C1) can also be obtained in general by considering the integral

$$\int_{-\pi}^{\pi}\frac{dq}{\pi}\frac{(\pi\phi_0(q))^2}{(x(q_0)-x(q))^2},$$

along the contour described above. There is now a contribution from "infinity", and one obtains the result that the above integral equals -1

APPENDIX D: THREE IMPURITIES

For three impurities, we have

$$\Psi(k,3) = \int_0^k dk_2 \int_0^{k_2} dk_1 \mathrm{Tr}(Be^{ik_1M}Be^{i(k_2-k_1)M}Be^{i(k-k_2)M}) = -3\Sigma_{i,j,k}(V^+BV)_{ij}(V^+BV)_{jk}(V^+BV)_{ki} \times \frac{e^{ik\lambda_j}}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}, \quad \text{(D1)}$$

and

$$\psi(x,3) = -3\Sigma_{i,j,k}(V^+BV)_{ij}(V^+BV)_{jk}(V^+BV)_{ki} \times \frac{\delta(x-\lambda_j)}{(x-\lambda_i)(x-\lambda_k)}. \quad \text{(D2)}$$

After some algebra, one obtains

$$\Omega(k_0,0{:}k,3) = -k_0\Sigma_{i,j,k}(V^+BV)_{ij}(V^+BV)_{jk}(V^+BV)_{ki}\bigg[\frac{-3ke^{i(k+k_0)\lambda_i}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} - 3ie^{i(k+k_0)\lambda_i}\bigg(\frac{1}{(\lambda_i-\lambda_k)^2(\lambda_i-\lambda_j)} + \frac{1}{(\lambda_i-\lambda_j)^2(\lambda_i-\lambda_k)}\bigg) + \frac{3ie^{ik_0\lambda_j}e^{ik\lambda_i}}{(\lambda_j-\lambda_i)^2(\lambda_i-\lambda_k)} + \frac{3ie^{ik_0\lambda_k}e^{ik\lambda_i}}{(\lambda_k-\lambda_i)^2(\lambda_i-\lambda_j)}\bigg],$$

and

$$\Omega(x,0{:}y,3) = \Sigma_{i,j,k}(V^+BV)_{ij}(V^+BV)_{jk}(V^+BV)_{ki}\bigg[-3\partial_x\partial_y\bigg(\delta(x-y)\frac{\delta(y-\lambda_i)}{(y-\lambda_j)(y-\lambda_k)}\bigg) - 3\partial_x\bigg(\delta(x-\lambda_i)\delta(y-\lambda_i) \times \bigg(\frac{1}{(y-\lambda_k)^2}\frac{1}{(y-\lambda_j)}\frac{1}{(y-\lambda_j)^2}\frac{1}{(y-\lambda_k)}\bigg)\bigg) + 3\partial_x\bigg(\frac{\delta(x-\lambda_j)\delta(y-\lambda_i)}{(\lambda_j-y)^2(y-\lambda_k)}\bigg) + 3\partial_x\bigg(\frac{\delta(x-\lambda_k)\delta(y-\lambda_i)}{(\lambda_k-y)^2(y-\lambda_j)}\bigg)\bigg].$$

The $\Omega(x,0{:}y,3)$ term in (13) takes the form

$$\frac{1}{2}\int dx \int dy \Omega(x,0{:}y,3)\frac{\partial \ln J}{\partial\psi(x,0)}\frac{\partial}{\partial\psi(y,3)} = -\frac{3}{2}\Sigma_{i,j,k}(V^+BV)_{ij}(V^+BV)_{jk}(V^+BV)_{ki}\int dx\int dy\bigg[\delta(x-y) \times \frac{\delta(y-\lambda_i)}{(y-\lambda_j)(y-\lambda_k)}\partial_x\frac{\partial \ln J}{\partial\psi(x,0)}\partial_y\frac{\partial}{\partial\psi(y,3)} - \delta(x-\lambda_i)\delta(y-\lambda_i) \times \bigg(\frac{1}{(y-\lambda_k)^2}\frac{1}{(y-\lambda_j)}\frac{1}{(y-\lambda_j)^2}\frac{1}{(y-\lambda_k)}\bigg)\partial_x\frac{\partial \ln J}{\partial\psi(x,0)}\frac{\partial}{\partial\psi(y,3)} \times \bigg(\frac{\delta(x-\lambda_j)\delta(y-\lambda_i)}{(\lambda_j-y)^2(y-\lambda_k)} + \frac{\delta(x-\lambda_k)\delta(y-\lambda_i)}{(\lambda_k-y)^2(y-\lambda_j)}\bigg)\partial_x\frac{\partial \ln J}{\partial\psi(x,0)}\frac{\partial}{\partial\psi(y,3)}\bigg].$$

For the harmonic oscillator potential, we use the results (11) and (10), so that we can write

$$\partial_x\frac{\partial \ln J}{\partial\psi(x,0)} = 2\int dz\frac{\phi_0(z)}{(x-z)} = 2z.$$

Then

$$\frac{1}{2}\int dx\int dy\Omega(x,0{:}y,3)\frac{\partial \ln J}{\partial\psi(x,0)}\frac{\partial}{\partial\psi(y,3)} = -3\int dx\Sigma_{i,j,k}(V^+BV)_{ij}(V^+BV)_{jk}(V^+BV)_{ki}\frac{\delta(x-\lambda_i)}{(x-\lambda_j)(x-\lambda_k)} \times \int dz\frac{\phi_0(z)}{(x-z)}\partial_x\frac{\partial}{\partial\psi(x,3)} + 3\int dy\Sigma_{i,j,k}(V^+BV)_{ij}(V^+BV)_{jk} \times (V^+BV)_{ki}\delta(y-\lambda_i)\frac{\partial}{\partial\psi(y,3)}\bigg(\frac{\lambda_i}{(y-\lambda_k)^2}\frac{1}{(y-\lambda_j)} + \frac{\lambda_i}{(y-\lambda_j)^2}\frac{1}{(y-\lambda_k)} - \frac{\lambda_j}{(\lambda_j-y)^2(y-\lambda_k)} - \frac{\lambda_k}{(\lambda_k-y)^2(y-\lambda_j)}\bigg)$$
$$= \int dx\int dz\frac{\phi_0(z)}{(x-z)}\psi(x,3)\partial_x\frac{\partial}{\partial\psi(x,3)} - 2\int dx\psi(x,3)\partial_x\frac{\partial}{\partial\psi(x,3)}.$$

Again, the first term above cancels the similar term in (13), and we obtain for the quadratic Hamiltonian in the 3 impurity sector:

$$H_2^{s=3} = \int dx \int dz \frac{\phi_0(z)\psi(x,3) - \psi(z,3)\phi_0(x)}{(x-z)^2} \frac{\partial}{\partial\psi(x,3)} - 2\int dx\psi(x,3)\frac{\partial}{\partial\psi(x,3)}. \quad \text{(D3)}$$

This is again a shifted Marchesini-Onofri operator. The spectrum and eigenfunctions of this operator are

$$w_n = n-3; \qquad \phi_n^{s=3} = \frac{\sin(nq)}{\sqrt{2}\sin(q)}; \qquad n = 1, 2, \ldots.$$

Adding the contribution from the $\mathrm{Tr}(B\partial/\partial B)$ term of the Hamiltonian we obtain

$$w_n = n; \qquad \phi_n^{s=3} = \frac{\sin(nq)}{\sqrt{2}\sin(q)}; \qquad n = 1, 2, \ldots. \quad \text{(D4)}$$

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