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Renormalized stress tensor for trans-Planckian cosmology

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Finite expressions for the mean value of the stress tensor corresponding to a scalar field with a generalized dispersion relation in a Friedmann-Robertson-Walker universe are obtained using adiabatic renormalization. Formally divergent integrals are evaluated by means of dimensional regularization. The renormalization procedure is shown to be equivalent to a redefinition of the cosmological constant and the Newton constant in the semiclassical Einstein equations.

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INTRODUCTION

One of the most important goals of inflationary scenarios [1] is that they provide a causal explanation for the large scale structure of the Universe and for the anisotropy in the cosmic microwave background (CMB). The mechanism is based in the stretching that the exponential (or quasiexponential) expansion produces in the physical wavelengths. Therefore, a density fluctuation of cosmological scale today originated on scales much smaller than the Hubble radius during inflation.

If the inflationary period lasts sufficiently long to solve the causality and other related problems, the scales of interest today are not only within the horizon but are also *sub-Planckian* at the beginning of inflation [2]. This fact, known as the *trans-Planckian problem*, is a potentially interesting window to observe consequences of the Planck-scale physics. The inflationary models may turn the Universe into a Planck-scale ''microscope.'' For this reason, since the formulation of this problem, many authors [3] have studied the possibility of observing signatures of Planckian physics in the power spectrum of the CMB and in the evolution of the Universe. In the absence of a full quantum theory of gravity, the analysis is necessarily phenomenological. For instance, modified dispersion relations for the modes of quantum fields might arise in loop quantum gravity [4] or due to the interaction with gravitons [5]. It is then important to test the robustness of inflationary predictions under such modifications. Another possibility is to consider an effective field theory approach in which the trans-Planckian physics is encoded in the state of the quantum fields when they leave sub-Planckian scales [6,7]. One could also consider space-space or space-time noncommutativities [8].

In this paper we will consider the first approach, i.e. we will analyze quantum fields with nonstandard dispersion relations in Friedmann-Robertson-Walker (FRW) backgrounds. Within this framework, in simple models with a single quantum scalar field ϕ, the information on the power spectrum of the CMB is contained in the vacuum expectation value $\langle\phi^2\rangle$. Moreover, the backreaction of the scalar field is contained in the expectation value $\langle T_{\mu\nu}\rangle$. Note that both $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ are in general divergent quantities.

There is a debate in the literature about whether the backreaction of trans-Planckian modes affects significantly the background space-time metric or not [6,9–11]. If sub-Hubble but super-Planck modes are excited during inflation, its energy density may be of the same order of magnitude that the background energy density, and prevent inflation. This fact would put a bound on the occupation numbers of the excited modes, and therefore on the effect that trans-Planckian physics may have on the power spectrum of the CMB. This argument has been disputed in Ref. [12], where the authors point out some subtleties regarding the choice of the ultraviolet cutoff and the equation of state of the trans-Planckian modes.

A consistent solution of this controversy should be based on a careful evaluation of the expectation value of the pressure and the energy density, and in the analysis of the solutions of the semiclassical Einstein equations (SEE), in which $\langle T_{\mu\nu}\rangle$ acts as a source. Any physically meaningful prediction requires the obtention of finite quantities starting from the formal divergent expression for $\langle T_{\mu\nu}\rangle$. In previous works [12,13], a particular renormalization prescription has been used, which consists essentially of subtracting the ground state energy of each Fourier mode. This prescription may lead to inconsistencies for quantum fields in curved spaces [14]. The purpose of the present paper is to carefully study this problem and provide a correct definition of such finite quantities.

The renormalization procedure for quantum fields satisfying the standard dispersion relations in curved backgrounds is of course well established [14–16]. For example, in the point-splitting regularization technique [17], $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ can be computed in terms of the

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coincidence limit of the two-point function $G^{(1)}(x, x') = \langle\{\phi(x), \phi(x')\}\rangle$ and its derivatives. The renormalized values are obtained by using the subtracted function $G^{(1)}_{\rm sub}(x, x') = G^{(1)}(x, x') - G^{(1)}_{\rm Had}(x, x')$, where $G^{(1)}_{\rm Had}(x, x')$ is a two-point function with the Hadamard singularity structure [14], truncated at the fourth adiabatic order [18]. The limit $x' \to x$ is taken at the end of the calculation. Alternatively, using dimensional regularization one can work with $x' = x$ from the beginning. This renormalization procedure is covariant, and the divergences of $\langle T_{\mu\nu}\rangle$ can be absorbed into redefinitions of the coupling constants of the theory in the SEE. In order to absorb all divergences it is necessary to include terms quadratic in the curvature in the classical gravitational action.

The method of renormalization described above can be applied in principle in any space-time metric. However, in the particular case of FRW metrics, the adiabatic subtraction is simpler and more adequate for numerical calculations [15,19,20]. Instead of subtracting the two-point function, the idea is to subtract the adiabatic expansion of the modes of the quantum fields. Adiabatic subtraction must be complemented with a covariant regularization, as for instance dimensional regularization. It has been shown that this method is equivalent to the previous one [21].

For a quantum field with generalized dispersion relations, the covariance is lost unless one introduces an additional dynamical degree of freedom u^μ that defines a preferred rest frame [22]. One usually works within this preferred frame, in which the space-time metric has the FRW form and the additional degree of freedom does not contribute to the energy-momentum tensor. In this particular frame, because the stress tensor of the quantum field is the source in the SEE, we must demand that it fulfills the same conservation equation of the Einstein tensor, i.e. ${G^{\mu\nu}}_{;\nu} = 0$ implies $\langle T^{\mu\nu}\rangle_{;\nu} = 0$. We stress that the conservation equation for $\langle T_{\mu\nu}\rangle$ is not necessarily valid in other frames, since the complete energy-momentum tensor may contain an additional part coming from u^μ.

Therefore, the renormalization should be compatible with the structure of the SEE in the preferred frame. The divergent contributions to be subtracted must have the form of geometric conserved tensors, in order to be absorbed into redefinitions of the bare constants. To ensure this, we shall follow the adiabatic renormalization procedure described above, that is, we shall evaluate the divergent contributions of the adiabatic expansion of the stress tensor, and define the renormalized stress tensor as $\langle T^{\mu\nu}\rangle - \langle T^{\mu\nu}\rangle_{\rm Ad}$. We shall show that because fourth or higher adiabatic order contributions are already finite for the dispersion relations considered, no additional terms must be included in the SEE, and only a redefinition of the cosmological constant and the Newton constant is required.

The paper is organized as follows. In Section II we generalize the WKB expansion to fields with a nonstandard dispersion relation in an arbitrary number of dimensions. In Section III we discuss the adiabatic renormalization of the energy-momentum tensor for a generic dispersion relation, and then compute the explicit expressions for the dimensionally regularized counterterms in some particular cases. We include our conclusions in Section IV. In the appendix we describe the simpler problem of the renormalization of $\langle\phi^2\rangle$.

Throughout the paper we set $c = 1$ and adopt the sign convention denoted $(+++)$ by Misner, Thorne, and Wheeler [23].

II. THE WKB EXPANSION

We begin by computing the WKB expansion for the modes of a scalar field ϕ with a nonstandard dispersion relation. The action of the theory is given by [13]:

$$S = \int d^n x\sqrt{-g}(\mathcal{L}_\phi + \mathcal{L}_{\rm cor} + \mathcal{L}_u), \tag{1}$$

where n is the space-time dimension, $\mathcal{L}_\phi$ is the standard Lagrangian of a free scalar field

$$\mathcal{L}_\phi = -\tfrac{1}{2}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (m^2 + \xi R)\phi^2], \tag{2}$$

$\mathcal{L}_{\rm cor}$ is the corrective Lagrangian that gives rise to a generalized dispersion relation

$$\mathcal{L}_{\rm cor} = -\sum_{s,p\leq n} b_{sp}(\mathcal{D}^{2s}\phi)(\mathcal{D}^{2p}\phi), \tag{3}$$

with $\mathcal{D}^2\phi \equiv \perp_\mu^\lambda \nabla_\lambda \perp_\gamma^\mu \nabla^\gamma\phi$ ($\perp_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$, where ∇_μ is the corresponding covariant derivative), and $\mathcal{L}_u$ describes the dynamics of the additional degree of freedom u^μ whose explicit expression is not necessary for our present purposes.

We work with a general spatially flat FRW metric given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \equiv -(u_\mu dx^\mu)^2 + \perp_{\mu\nu} dx^\mu dx^\nu \\ &= C(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j], \end{aligned} \tag{4}$$

where $C^{1/2}(\eta)$ is the scale factor given as a function of the conformal time η, and $u_\mu \equiv C^{1/2}(\eta)\delta^\eta_\mu$.

The generalized dispersion relation takes the form

$$\omega_k^2 = k^2 + C(\eta)\left[m^2 + 2\sum_{s,p}(-1)^{s+p} b_{sp}\left(\frac{k}{C^{1/2}(\eta)}\right)^{2(s+p)}\right], \tag{5}$$

where b_{sp} are arbitrary coefficients, with $p \leq s$.

The Fourier modes χ_k corresponding to the scaled field $\chi = C^{(n-2)/4}\phi$ satisfy

$$\chi_k'' + [(\xi - \xi_n)RC + \omega_k^2]\chi_k = 0, \tag{6}$$

with the usual normalization condition

$$\chi_k \chi_k'^* - \chi_k' \chi_k^* = i. \tag{7}$$

Here primes stand for derivatives with respect to the conformal time η, R is the Ricci scalar, and in the conformal coupling case we have $\xi = \xi_n \equiv (n-2)/(4n-4)$, while $\xi = 0$ corresponds to minimal coupling. The normalization condition implies that the field modes χ_k can be expressed in the well-known form

$$\chi_k = \frac{1}{\sqrt{2W_k}} \exp\left(-i \int^\eta W_k(\tilde{\eta}) d\tilde{\eta}\right). \tag{8}$$

Substitution of Eq. (8) into Eq. (6) yields

$$W_k^2 = \Omega_k^2 - \frac{1}{2}\left(\frac{W_k''}{W_k} - \frac{3}{2}\frac{W_k'^2}{W_k^2}\right), \tag{9}$$

$$\Omega_k^2 = \omega_k^2 + (\xi - \xi_n)CR. \tag{10}$$

The nonlinear differential equation for W_k can be solved iteratively by assuming that it is a slowly varying function of η. This is the adiabatic or WKB approximation, and the number of time derivatives of a given term is called the adiabatic order. Thus, if we work up to the second adiabatic order (which is the highest order which will be required; see below), we can replace W_k by ω_k on the right-hand side of Eq. (9). Then, with the use of (5) and (9), we straightforwardly obtain the second order solution for a generic evolution of the scale factor:

$$\begin{aligned} W_k^2 = {} & \omega_k^2 + (\xi - \xi_n)(n-1)\left(\frac{C''}{C} + \frac{(n-6)}{4}\frac{C'^2}{C^2}\right) \\ & - \frac{1}{4}\frac{C''}{C}\left(1 - \frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right) - \frac{1}{4}\frac{C'^2}{C^2}\frac{k^4}{\omega_k^2}\frac{d^2\omega_k^2}{d(k^2)^2} \\ & + \frac{5}{16}\frac{C'^2}{C^2}\left(1 - \frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)^2, \end{aligned} \tag{11}$$

where we have used that ω_k^2/C is a function of k^2/C to write the temporal derivatives of ω_k in terms of derivatives with respect to k^2. We see that $W_k^2 = \omega_k^2 + \epsilon_k$, where ϵ_k is already of second adiabatic order (that is, it includes second derivatives or the square of first derivatives of the scale factor). In the calculations below we shall need the squared modes $|\chi_k|^2 = \chi_k\chi_k^* = (2W_k)^{-1}$ and, up to second adiabatic order, we can simply work with the approximation $(W_k)^{\pm 1} \approx (\omega_k)^{\pm 1}[1 \pm \epsilon_k/(2\omega_k^2)]$.

In what follows it will be relevant to know the dependence with k of the different adiabatic orders. From Eq. (11) it is clear that while the zeroth adiabatic order scales as ω_k^2, the second adiabatic order scales as ω_k^0. Using an inductive argument it can be shown that the $2j$-adiabatic order scales as ω_k^{2-2j}.

III. RENORMALIZATION OF THE STRESS TENSOR

Motivated by the work in Ref. [13], we start from the following expressions for the vacuum expectation values of the energy density ρ and pressure p, which we have generalized to arbitrary dimension n and coupling ξ to the scalar curvature:

$$\begin{aligned} \langle\rho\rangle = {} & \frac{1}{\sqrt{C}} \int \frac{d^{n-1}k\mu^{4-n}}{(2\pi\sqrt{C})^{(n-1)}} \left\{ \frac{C^{(n-2)/2}}{2} \left| \left(\frac{\chi_k}{C^{(n-2)/4}}\right)' \right|^2 + \frac{\omega_k^2}{2}|\chi_k|^2 + \xi G_{\eta\eta}|\chi_k|^2 \right. \\ & \left. + \xi\frac{(n-1)}{2}\left[\frac{C'}{C}(\chi_k'\chi_k^* + \chi_k\chi_k'^*) - \frac{C'^2}{C^2}\frac{(n-2)}{2}|\chi_k|^2\right] \right\}, \end{aligned} \tag{12}$$

$$\begin{aligned} \langle p\rangle = {} & \frac{1}{\sqrt{C}} \int \frac{d^{n-1}k\mu^{4-n}}{(2\pi\sqrt{C})^{(n-1)}} \left\{ \left(\frac{1}{2} - 2\xi\right) C^{(n-2)/2} \left| \left(\frac{\chi_k}{C^{(n-2)/4}}\right)' \right|^2 + \xi G_{11}|\chi_k|^2 + \left[\left(\frac{k^2}{n-1}\right)\frac{d\omega_k^2}{dk^2} - \frac{\omega_k^2}{2}\right]|\chi_k|^2 \right. \\ & \left. - \xi(\chi_k''\chi_k^* + \chi_k\chi_k''^*) + \xi\frac{C'}{2C}(\chi_k'\chi_k^* + \chi_k\chi_k'^*) - \xi\frac{(n-2)}{2}\left(\frac{C''}{C} - \frac{(8-n)}{4}\frac{C'^2}{C^2}\right)|\chi_k|^2 \right\}. \end{aligned} \tag{13}$$

Here μ is an arbitrary parameter with mass dimension introduced to ensure that χ has the correct dimensionality, and $G_{\eta\eta}$ and G_{11} ($= G_{22} = G_{33}$) are the nontrivial components of the Einstein tensor

$$G_{\eta\eta} = \frac{(n-1)}{4}\frac{(n-2)}{2}\frac{C'^2}{C^2}, \tag{14}$$

$$G_{11} = G_{22} = G_{33} = \frac{(n-2)}{2}\left[\frac{C'^2}{C^2}\left(\frac{(n-1)}{4} - \frac{(n-4)}{2}\right) - \frac{C''}{C}\right]. \tag{15}$$

After introducing the form of the Fourier modes given in Eq. (8) in the expressions for the vacuum expectation values $\langle\rho\rangle$ and $\langle p\rangle$ of Eqs. (12) and (13) we find

$$\langle\rho\rangle = \frac{\Omega_{n-1}}{2\sqrt{C}}\int\frac{dk\,k^{n-2}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\left\{\frac{[(W_k^2)']^2}{32W_k^5}+\frac{W_k}{2}+\frac{\omega^2}{2W_k}+\frac{(n-2)}{2}\left[\frac{C'^2(n-2)}{16W_kC^2}+\frac{C'(W_k^2)'}{8CW_k^3}\right]+\xi\frac{G_{\eta\eta}}{W_k} - \xi\frac{(n-1)}{2}\left[\frac{C'^2}{C^2}\frac{(n-2)}{2W_k}+\frac{C'}{C}\frac{(W_k^2)'}{2W_k^3}\right]\right\}, \tag{16}$$

$$\langle p\rangle = \frac{\Omega_{n-1}}{2\sqrt{C}}\int\frac{dk\,k^{n-2}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\left\{\frac{[(W_k^2)']^2}{32W_k^5}+\frac{W_k}{2}-\frac{\omega^2}{2W_k}+\frac{(n-2)}{2}\left[\frac{C'^2(n-2)}{16W_kC^2}+\frac{C'(W_k^2)'}{8CW_k^3}\right]+\frac{k^2}{(n-1)W_k}\frac{d\omega^2}{dk^2}+\xi\frac{G_{11}}{W_k} + \xi\left[\frac{(W_k^2)''}{2W_k^3}-\frac{3}{4}\frac{[(W_k^2)']^2}{W_k^5}-\frac{(n-1)}{4}\frac{C'(W_k^2)'}{CW_k^3}\right]+\frac{(n-2)}{2}\frac{\xi}{W_k}\left[\frac{C''}{C}-\frac{3}{2}\frac{C'^2}{C^2}\right]\right\}, \tag{17}$$

where we have defined the factor $\Omega_{n-1}\equiv 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$ coming from the angular integration.

The dependence with k of the $2j$-adiabatic order has been described at the end of the previous section. From that result it is possible to check that, for $\omega_k^2\sim k^r$ with $r\geq 8$, all contributions of second or higher adiabatic order are finite. The divergences come only from the zeroth adiabatic terms contained in $\langle\rho\rangle$ and $\langle p\rangle$. Instead, in the cases $\omega_k^2\sim k^6$ and $\omega_k^2\sim k^4$, though no fourth order divergencies appear, second adiabatic order terms include, in principle, divergent contributions. Therefore we only need to work up to second adiabatic order. Since $W_k^2=\omega_k^2+\epsilon_k$, where ϵ_k is of adiabatic order two, we can substitute $(W_k^2)'$ by $(\omega^2)'$ in equations above. Then, with the help of Eq. (11) and the explicit form of ω_k^2 (given in Eq. (5)), we obtain the following expressions for the zeroth and second adiabatic order contributions:

$$\langle\rho\rangle^{(0)} = \frac{1}{2\sqrt{C}}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk\,k^{n-2}\omega_k, \tag{18}$$

$$\langle p\rangle^{(0)} = \frac{1}{2\sqrt{C}}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk\frac{k^{n-2}}{(n-1)}\frac{k^2}{\omega_k}\frac{d\omega_k^2}{dk^2}, \tag{19}$$

$$\langle\rho\rangle^{(2)}_{\xi=0} = \frac{1}{2\sqrt{C}}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\left(\frac{C'}{C}\right)^2\int dk\,k^{n-2}\left\{\frac{1}{32\omega_k}\left(1-\frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)^2+\frac{(n-2)}{32\omega_k}\left(n-\frac{2k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)\right\}, \tag{20}$$

$$\langle p\rangle^{(2)}_{\xi=0} = \langle\rho\rangle^{(2)}_{\xi=0} - \frac{1}{2\sqrt{C}}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk\,k^{n-2}\frac{1}{2\omega_k}\left(1-\frac{k^2}{(n-1)\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)\left\{\frac{(n-2)}{4}\left(\frac{C''}{C}+\frac{(n-6)}{4}\frac{C'^2}{C^2}\right) + \frac{C''}{4C}\left(1-\frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)+\frac{C'^2}{4C^2}\left[\frac{k^4}{\omega_k^2}\frac{d^2\omega_k^2}{d(k^2)^2}-\frac{5}{4}\left(1-\frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)^2\right]\right\}, \tag{21}$$

$$\langle\rho\rangle^{(2)}_{\xi} = \frac{\xi}{2\sqrt{C}}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk\,k^{n-2}\left\{\frac{G_{\eta\eta}}{\omega_k}-\frac{C'^2(n-1)}{4C^2\omega_k}\left(n-1-\frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)\right\}, \tag{22}$$

$$\langle p\rangle^{(2)}_{\xi} = \frac{\xi}{2\sqrt{C}}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk\,k^{n-2}\left\{\frac{G_{11}}{\omega_k}+\frac{1}{2\omega_k}\left[\frac{2C''}{C}+\frac{(n-6)}{4}\frac{C'^2}{C^2}\right]\left(1-\frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)+3\frac{(n-2)}{2}+\frac{3}{2}\left(1-\frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)^2 + \frac{1}{2\omega_k}\frac{C'^2}{C^2}\left[\frac{k^4}{\omega_k^2}\frac{d^2\omega_k^2}{d(k^2)^2}-\frac{(n-1)}{2}\left(1-\frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)\right]\right\}, \tag{23}$$

which include divergences coming from different powers of the wave vector k. Here the superscripts stand for the adiabatic order, and we have separated the second adiabatic order which appears in the minimal coupling case $\langle.\rangle_{\xi=0}$ from the one proportional to the coupling constant ξ.

A. Zeroth adiabatic order

The divergences in the components of the stress tensor that come from the zeroth order in the adiabatic expansion can be removed by renormalization of the cosmological constant in the SEE. This can already be verified as follows: Up to zeroth order we have that $\langle p\rangle$ is given by Eq. (19), which can be recast as

$$\langle p\rangle^{(0)} = \frac{1}{2\sqrt{C}}\frac{\Omega_{n-1}}{(2\pi\sqrt{C})^{n-1}}\int dk\frac{k^{n-1}\mu^{4-n}}{(n-1)}\frac{d\omega_k}{dk}, \tag{24}$$

so that one can integrate by parts to obtain

$$\langle p\rangle^{(0)} = \frac{1}{2\sqrt{C}}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\left\{\int dk\frac{d}{dk}\left(\frac{\omega_k k^{n-1}}{n-1}\right) - \int dk k^{n-2}\omega_k\right\}. \tag{25}$$

Then, since in dimensional regularization the integral of a total derivative vanishes [24], we find that

$$\langle p\rangle^{(0)} = -\frac{1}{2\sqrt{C}}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk\, k^{n-2}\omega_k = -\langle\rho\rangle^{(0)}. \tag{26}$$

To exhibit more clearly the dependence of this adiabatic order on C, by rescaling the integration variable with a factor $C^{-1/2}$, we rewrite it as

$$\langle\rho\rangle^{(0)} = -\langle p\rangle^{(0)} = \frac{\Omega_{n-1}\mu^{4-n}}{2(2\pi)^{n-1}}\int dk\, k^{n-2}\tilde{\omega}_k, \tag{27}$$

where $\tilde{\omega}_k = \omega_k/\sqrt{C}$. Thus, as we are working with the metric in the conformal form, we see that $\langle T_{\mu\nu}\rangle^{(0)} = N_0 g_{\mu\nu}$ (with N_0 a divergent factor) so that we can define $\langle\tilde{T}_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle^{(0)}$ and the SEE

$$G_{\mu\nu} + \Lambda_B g_{\mu\nu} = 8\pi G(\langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle^{(0)} + \langle T_{\mu\nu}\rangle^{(0)}) \tag{28}$$

can be recast in the form

$$G_{\mu\nu} + \Lambda_R g_{\mu\nu} = 8\pi G\langle\tilde{T}_{\mu\nu}\rangle, \tag{29}$$

where $\Lambda_R = \Lambda_B - 8\pi G N_0$ is the renormalized cosmological constant.

Since in the case of a generalized dispersion relation for which $\omega_k^2 \sim k^r$ with $r \geq 8$, the energy-momentum tensor can be renormalized by subtracting the zeroth adiabatic order, we can make the identification $\langle T_{\mu\nu}\rangle_{\rm Ren} \equiv \langle\tilde{T}_{\mu\nu}\rangle$ (i.e., $\langle\rho\rangle_{\rm Ren} = \langle\rho\rangle - \langle\rho\rangle^{(0)}$ and $\langle p\rangle_{\rm Ren} = \langle p\rangle - \langle p\rangle^{(0)}$). In such a case, as these expressions are already finite, in order to evaluate them in terms of the modes of the scalar field it is not necessary to work in n dimensions: one can first take the limit $n \to 4$ and then perform the momentum integration, that is

$$\langle\rho\rangle_{\rm Ren} = \frac{1}{C^2}\int\frac{d^3k}{(2\pi)^3}\left\{\frac{C}{2}\left|\left(\frac{\chi_k}{\sqrt{C}}\right)'\right|^2 + \frac{\omega_k^2}{2}|\chi_k|^2 + \xi G_{\eta\eta}|\chi_k|^2 + \frac{3}{2}\xi\left[\frac{C'}{C}(\chi_k'\chi_k^* + \chi_k\chi_k'^*) - \frac{C'^2}{C^2}|\chi_k|^2\right] - \frac{1}{2}\omega_k\right\}, \tag{30}$$

$$\langle p\rangle_{\rm Ren} = \frac{1}{C^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left(\frac{1}{2} - 2\xi\right)C\left|\left(\frac{\chi_k}{\sqrt{C}}\right)'\right|^2 + \xi G_{11}|\chi_k|^2 + \left[\left(\frac{k^2}{3}\right)\frac{d\omega^2}{dk^2} - \frac{\omega_k^2}{2}\right]|\chi_k|^2 - \xi(\chi_k''\chi_k^* + \chi_k\chi_k''^*) + \xi\frac{C'}{2C}(\chi_k'\chi_k^* + \chi_k\chi_k'^*) - \xi\left(\frac{C''}{C} - \frac{C'^2}{C^2}\right)|\chi_k|^2 - \frac{k^2}{6\omega_k}\frac{d\omega_k^2}{dk^2}\right\}. \tag{31}$$

The adiabatic renormalization procedure works only for the vacuum states of the field that coincide with the adiabatic vacuum up to the order subtracted [25]. If we assume that the scalar field is in the vacuum state near the initial singularity $C \to 0$, this means that the exact solution χ_k of Eq. (6) should coincide with the WKB solution up to that order for $C \to 0$. This fact ensures that the above integrals are finite.

B. Second adiabatic order

So far we have shown that the zeroth adiabatic order of the vacuum expectation values of the energy density and pressure satisfy Eq. (27) and, hence, that they can be absorbed by a redefinition of the cosmological constant. In what follows, we shall see that the WKB expansion of $\langle\rho\rangle$ and $\langle p\rangle$, up to second adiabatic order, have the appropriate structure required to remove the divergences in the stress tensor that appear in the cases $\omega_k^2 \sim k^6$ and $\omega_k^2 \sim k^4$. Therefore, all divergences will be absorbed renormalizing the cosmological and Newton constants in the SEE. More specifically, with the use of dimensional regularization, we shall show that the second adiabatic orders of $\langle\rho\rangle$ and $\langle p\rangle$ are proportional to the components $G_{\eta\eta}$ and G_{11} of the Einstein tensor, respectively, yielding a renormalization of the Newton constant.

Using integration by parts and some algebra one can find expressions for $\langle\rho\rangle^{(2)}$ and $\langle p\rangle^{(2)}$ that involve only the two integrals

$$I_1 = \int dx\frac{x^{(n-3)/2}}{\tilde{\omega}_k}, \qquad I_2 = \int dx\frac{x^{(n+1)/2}}{\tilde{\omega}_k^3}\frac{d^2\tilde{\omega}_k^2}{dx^2}, \tag{32}$$

where $x \equiv k^2/C$ and, as above, $\tilde{\omega}_k = \omega_k/\sqrt{C}$.

As an example, let us consider the following integral

$$I \equiv \int dk\frac{k^{n-2}}{C^{(n-2)/2}\omega_k}\left(1 - \frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)^2, \tag{33}$$

which contributes to both $\langle\rho\rangle^{(2)}$ and $\langle p\rangle^{(2)}$. In order to rewrite it we can proceed as follows

$$I = \frac{1}{2}\int dx \frac{x^{(n-3)/2}}{\tilde{\omega}_k} + 2\int dx\, x^{(n-1)/2}\frac{d\tilde{\omega}_k^{-1}}{dx} - \frac{1}{3}\int dx x^{(n+1)/2}\frac{d\tilde{\omega}_k^{-3}}{dx}\frac{d\tilde{\omega}_k^2}{dx} = \frac{[(n-3)^2-1]}{6}\int dx \frac{x^{(n-3)/2}}{\tilde{\omega}_k} + \frac{1}{3}\int dx \frac{x^{(n+1)/2}}{\tilde{\omega}_k^3}\frac{d^2\tilde{\omega}_k^2}{dx^2} = \frac{[(n-3)^2-1]}{6}I_1 + \frac{1}{3}I_2, \tag{34}$$

where the first equality follows after the change of variables $x = k^2/C$ and some rearrangements of the integrand, while the second one is obtained, with the use of dimensional regularization, after two integrations by parts.

Applying a similar procedure to the other integrals we get, after a long calculation,

$$\langle\rho\rangle^{(2)}_{\xi=0} = \frac{G_{\eta\eta}}{C}\frac{\Omega_{n-1}\mu^{4-n}}{4(2\pi)^{n-1}}\left\{-\frac{[n+2+n(n-4)]}{6(n-1)(n-2)}I_1 + \frac{1}{6(n-1)(n-2)}I_2\right\}, \tag{35}$$

$$\langle p\rangle^{(2)}_{\xi=0} = \langle\rho\rangle^{(2)}_{\xi=0}\left[1 - 2\frac{(n-4)}{(n-1)} - \frac{4}{(n-1)}\frac{C''}{C}\frac{C^2}{C'^2}\right] = \frac{G_{11}}{G_{\eta\eta}}\langle\rho\rangle^{(2)}_{\xi=0}, \tag{36}$$

$$\langle\rho\rangle^{(2)}_{\xi} = \frac{\xi G_{\eta\eta}}{C}\frac{\Omega_{n-1}\mu^{4-n}}{4(2\pi)^{n-1}}I_1, \tag{37}$$

$$\langle p\rangle^{(2)}_{\xi} = \frac{\xi G_{11}}{C}\frac{\Omega_{n-1}\mu^{4-n}}{4(2\pi)^{n-1}}I_1. \tag{38}$$

Notice that in the case of the usual dispersion relation, the integral containing the second derivative of ω_k^2 vanishes and we recover the known second adiabatic order results [20]. It is worth mentioning here that, for $\omega_k^2 \sim k^6$, the leading terms in the second adiabatic order cancel out for large k, and thus for minimal coupling $\xi = 0$ the second adiabatic order is finite as $n \to 4$. This can be seen directly from Eq. (35) or from Eq. (20).

Equations (35)–(38) above are enough for our purposes, since it is clear from them that the second adiabatic order of $\langle\rho\rangle$ and $\langle p\rangle$ are proportional to $G_{\eta\eta}$ and G_{11}, respectively. If we write $\langle T_{\mu\nu}\rangle^{(n)}$ for the term of adiabatic order n of the stress tensor, we find that

$$\langle T_{\mu\nu}\rangle^{(0)} = N_0 g_{\mu\nu}, \tag{39}$$

where N_0 is in principle a divergent factor, so that the corresponding contributions can be removed by introducing a renormalized cosmological constant Λ_R. On the other hand, we have

$$\langle T_{\mu\nu}\rangle^{(2)} = N_2 G_{\mu\nu}, \tag{40}$$

where N_2 is another divergent factor, and hence these contributions can be absorbed in a renormalization of the Newton gravitational constant. Therefore we can define

$$\langle T_{\mu\nu}\rangle_{\rm Ren} = \langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle^{(0)} - \langle T_{\mu\nu}\rangle^{(2)} \tag{41}$$

and write the SEE as

$$G_{\mu\nu} + \Lambda_R g_{\mu\nu} = 8\pi G_R\langle T_{\mu\nu}\rangle_{\rm Ren}. \tag{42}$$

Differing from the case of standard dispersion relations $\omega_k^2 \sim k^2$, now all contributions of adiabatic orders higher than the second are finite, so that no additional terms must be included in the SEE in order to deal with physically meaningful quantities.

C. Evaluation of the regularized integrals

The explicit expression for the constants N_0 and N_2 can be obtained by a direct computation in Eqs. (18)–(23) or equivalently from Eqs. (27) and (35)–(38).

As a first example, let us consider the case of a massless field with a dispersion relation of the form $\omega_k^2 = k^2 + 2b_{11}k^4/C + 2|b_{12}|k^6/C^2$ (we will assume that $|b_{12}| > b_{11}^2/2$ to avoid zeros of the frequency). In this case the divergent contributions come from the zeroth adiabatic order and from the terms proportional to ξ of the second one, since (as we already mentioned) the second adiabatic order is finite for $\xi = 0$. Then, after computing the integrals we obtain [26]

$$\langle\rho\rangle^{(0)} = -\langle p\rangle^{(0)} = \frac{\mu^{4-n}(2|b_{12}|)^{(2-n)/4}}{4\Gamma[-\frac{1}{2}](4\pi)^{(n-1)/2}}\left\{\frac{1}{\sqrt{2|b_{12}|}}\Gamma\left[\frac{n}{4}\right]\Gamma\left[-\frac{1}{2}-\frac{n}{4}\right]{}_2F_1\left[-\frac{1}{2}-\frac{n}{4},\frac{n}{4};\frac{1}{2};\frac{b_{11}^2}{2|b_{12}|}\right] - \frac{b_{11}}{|b_{12}|}\Gamma\left[\frac{n}{4}+\frac{1}{2}\right]\Gamma\left[-\frac{n}{4}\right]{}_2F_1\left[\frac{n}{4}+\frac{1}{2},-\frac{n}{4};\frac{3}{2};\frac{b_{11}^2}{2|b_{12}|}\right]\right\}, \tag{43}$$

$$\langle\rho\rangle^{(2)}_{\xi=0} = \langle p\rangle^{(2)}_{\xi=0}\frac{G_{\eta\eta}}{G_{11}}$$
$$= \frac{G_{\eta\eta}}{C}\frac{\mu^{4-n}(2|b_{12}|)^{(2-n)/4}}{24(4\pi)^{(n-1)/2}}\bigg\{\frac{b_{11}}{\sqrt{2|b_{12}|}}\frac{\Gamma[\frac{1}{2}-\frac{n}{4}]\Gamma[\frac{n}{4}]}{4\Gamma[\frac{1}{2}]\Gamma[\frac{n}{2}+\frac{1}{2}]}\bigg({}_2F_1\bigg[\frac{3}{2}-\frac{n}{4},\frac{n}{4};\frac{1}{2};\frac{b_{11}^2}{2|b_{12}|}\bigg]$$
$$-\frac{(n-4)(n-2)}{2}F_1\bigg[\frac{3}{2}-\frac{n}{4},\frac{n}{4};\frac{3}{2};\frac{b_{11}^2}{2|b_{12}|}\bigg]\bigg)+\frac{\Gamma[2-\frac{n}{4}]\Gamma[\frac{n}{4}-\frac{1}{2}]}{\Gamma[\frac{1}{2}]\Gamma[\frac{n}{2}+\frac{1}{2}]}\bigg({}_2F_1\bigg[1-\frac{n}{4},\frac{n}{4}-\frac{1}{2};\frac{1}{2};\frac{b_{11}^2}{2|b_{12}|}\bigg]$$
$$+\frac{b_{11}^2}{4|b_{12}|}F_1\bigg[2-\frac{n}{4},\frac{n}{4}+\frac{1}{2};\frac{3}{2};\frac{b_{11}^2}{2|b_{12}|}\bigg]\bigg)\bigg\}, \tag{44}$$

$$\langle\rho\rangle^{(2)}_{\xi} = \langle p\rangle^{(2)}_{\xi}\frac{G_{\eta\eta}}{G_{11}}$$
$$= \xi\frac{G_{\eta\eta}}{C}\frac{\mu^{4-n}(2|b_{12}|)^{(4-n)/4}}{4\Gamma[\frac{1}{2}](4\pi)^{(n-1)/2}}\bigg\{\frac{1}{\sqrt{2|b_{12}|}}\Gamma\bigg[1-\frac{n}{4}\bigg]\Gamma\bigg[-\frac{1}{2}+\frac{n}{4}\bigg]{}_2F_1\bigg[1-\frac{n}{4},-\frac{1}{2}+\frac{n}{4};\frac{1}{2};\frac{b_{11}^2}{2|b_{12}|}\bigg]$$
$$-\frac{b_{11}}{|b_{12}|}\Gamma\bigg[\frac{3}{2}-\frac{n}{4}\bigg]\Gamma\bigg[\frac{n}{4}\bigg]{}_2F_1\bigg[\frac{3}{2}-\frac{n}{4},\frac{n}{4};\frac{3}{2};\frac{b_{11}^2}{2|b_{12}|}\bigg]\bigg\}, \tag{45}$$

where we have used some properties of the gamma Γ and hypergeometric ${}_2F_1$ functions [27].

The behavior of $\langle\rho\rangle^{(0)}$ and $\langle\rho\rangle^{(2)}$ in the limit $n\to 4$ is

$$\langle\rho\rangle^{(0)} = -\frac{b_{11}(b_{11}^2-2|b_{12}|)}{32\sqrt{2}\pi^2|b_{12}|^{5/2}}\bigg[\frac{1}{n-4}-\ln(|b_{12}|^{1/4}\mu)\bigg]+\mathcal{O}(n-4), \tag{46}$$

$$\langle\rho\rangle^{(2)} = -\frac{\xi G_{\eta\eta}}{C(2\pi)^2\sqrt{2|b_{12}|}}\bigg[\frac{1}{n-4}-\ln(|b_{12}|^{1/4}\mu)\bigg]+(\text{finite }\mu\text{-independent terms as }n\to 4), \tag{47}$$

where we have redefined μ to absorb a constant term.

Let us now consider a dispersion relation of the form $\omega_k^2 = k^2 + Cm^2 + 2b_{11}k^4/C$, with $b_{11} > 0$. The integrals in Eqs. (27) and (35)–(38) can be computed explicitly. Recalling some properties of the Gamma and hypergeometric functions [27], in the limit $n\to 4$ we obtain

$$\langle\rho\rangle^{(0)} = -\langle p\rangle^{(0)}$$
$$= \frac{m^{3/2}}{2^{1/4}64b_{11}^{5/4}\pi^{5/2}}\bigg\{\Gamma\bigg[-\frac{3}{4}\bigg]\Gamma\bigg[\frac{5}{4}\bigg]{}_2F_1\bigg[-\frac{3}{4},\frac{5}{4};\frac{3}{2};\frac{1}{8b_{11}m^2}\bigg]-\sqrt{2b_{11}m^2}\Gamma\bigg[-\frac{5}{4}\bigg]\Gamma\bigg[\frac{3}{4}\bigg]{}_2F_1\bigg[-\frac{5}{4},\frac{3}{4};\frac{1}{2};\frac{1}{8b_{11}m^2}\bigg]\bigg\}, \tag{48}$$

$$\langle\rho\rangle^{(2)}_{\xi=0} = \langle p\rangle^{(2)}_{\xi=0}\frac{G_{\eta\eta}}{G_{11}}$$
$$= \frac{\sqrt{m}}{2^{3/4}2304b_{11}^{3/4}\pi^{5/2}}\bigg\{4\sqrt{8b_{11}m^2}\Gamma\bigg[\frac{1}{4}\bigg]^2\bigg[-2\frac{(1-12b_{11}m^2)}{1-8b_{11}m^2}{}_2F_1\bigg[-\frac{3}{4},\frac{1}{4};\frac{1}{2};\frac{1}{8b_{11}m^2}\bigg]$$
$$+3{}_2F_1\bigg[\frac{1}{4},\frac{1}{4};\frac{1}{2};\frac{1}{8b_{11}m^2}\bigg]\bigg]+\Gamma\bigg[-\frac{1}{4}\bigg]^2\bigg[2{}_2F_1\bigg[-\frac{1}{4},\frac{3}{4};\frac{1}{2};\frac{1}{8b_{11}m^2}\bigg]+{}_2F_1\bigg[\frac{3}{4},\frac{3}{4};\frac{1}{2};\frac{1}{8b_{11}m^2}\bigg]\bigg]\bigg\}, \tag{49}$$

$$\langle\rho\rangle^{(2)}_{\xi} = \langle p\rangle^{(2)}_{\xi}\frac{G_{\eta\eta}}{G_{11}}$$
$$= \frac{1}{2^{1/4}32b_{11}^{5/4}\sqrt{m}\pi^{5/2}}\bigg\{\sqrt{2b_{11}m^2}\Gamma\bigg[-\frac{1}{4}\bigg]\Gamma\bigg[\frac{3}{4}\bigg]{}_2F_1\bigg[-\frac{1}{4},\frac{3}{4};\frac{1}{2};\frac{1}{8b_{11}m^2}\bigg]-\Gamma\bigg[\frac{1}{4}\bigg]\Gamma\bigg[\frac{5}{4}\bigg]{}_2F_1\bigg[\frac{1}{4},\frac{5}{4};\frac{3}{2};\frac{1}{8b_{11}m^2}\bigg]\bigg\}. \tag{50}$$

In the case of a massless field, the results simplify to

$$\langle \rho \rangle^{(0)} = -\langle p \rangle^{(0)} = \frac{1}{32\pi^2 b_{11}^2} \frac{\Gamma(-5/2)}{\Gamma(-1/2)} = \frac{1}{120 b_{11}^2 \pi^2}, \tag{51}$$

$$\langle \rho \rangle^{(2)} = -\frac{(1-18\xi)}{288\pi^2 b_{11}} \frac{\Gamma(-1/2)}{\Gamma(1/2)} \frac{G_{\eta\eta}}{C} = \frac{(1-18\xi)}{144\pi^2 b_{11}} \frac{G_{\eta\eta}}{C}, \tag{52}$$

$$\langle p \rangle^{(2)} = -\frac{(1-18\xi)}{288\pi^2 b_{11}} \frac{\Gamma(-1/2)}{\Gamma(1/2)} \frac{G_{11}}{C} = \frac{(1-18\xi)}{144\pi^2 b_{11}} \frac{G_{11}}{C}. \tag{53}$$

All the contributions which are in principle divergent give finite results when evaluated by means of dimensional regularization (all negative arguments appearing in the gamma functions are noninteger). This could have been anticipated, because the dependence $\omega_k^2 \sim k^4$ of the dispersion relation leads to integral expressions which are formally equivalent (for large k or in the massless limit) to which would be obtained in $2+1$ dimensions for a standard dispersion relation, and in this case dimensional regularization leads to finite results for integrals which are in principle divergent [28].

IV. CONCLUSIONS

We have given a prescription for obtaining finite, physically meaningful, expressions for the components of the stress tensor for a field of arbitrary coupling, with generalized dispersion relations, in a FRW background. We have followed the usual procedure of subtracting from the exact components of the energy-momentum tensor the, in principle, divergent contributions of the corresponding expressions obtained from the adiabatic expansion, which we have evaluated by means of dimensional regularization.

We have seen that, differing from the usual case corresponding to standard dispersion relations with $\omega_k^2 \sim k^2$, the fourth adiabatic order is convergent. Consequently, additional terms proportional to the geometric tensors $H_{\mu\nu}^{(1)}$ and $H_{\mu\nu}^{(2)}$ associated with corrections of second order in the curvature are not necessary in the SEE, and the renormalization does not require more than the redefinition of the cosmological constant and the Newton constant. At first sight it may look surprising that for generalized dispersion relations with $\omega_k^2 \sim k^r$, $r \geq 4$ the divergences are milder than for the standard case. The reason is that, while the divergence of the zeroth adiabatic order is stronger, the higher orders are suppressed by powers of ω_k^{-2}. Therefore, for the cases $\omega_k^2 \sim k^6$ and $\omega_k^2 \sim k^4$, the fourth adiabatic orders are already finite.

In the case of dispersion relations of the form $\omega_k^2 \sim k^r$ with $r \geq 8$ or $\omega_k^2 \sim k^6$ with $\xi = 0$, the second adiabatic order is finite and the divergences are contained in the zeroth adiabatic order. Therefore, in such cases, the adiabatic renormalization is equivalent to the subtraction of the zero point energy of each field Fourier mode, as done in Ref. [12].

There are several issues that would deserve further investigation. From a formal point of view, it would be interesting to extend the renormalization of the stress tensor to interacting theories with nonstandard dispersion relations. From a "phenomenological" point of view, the renormalized SEE obtained in this paper should be the starting point to evaluate whether the backreaction of trans-Planckian modes prevents inflation or not.

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APPENDIX: RENORMALIZATION OF $\langle \phi^2 \rangle$

We shall consider the simple problem of the mean squared field in order to illustrate how one can effectively make the subtraction leading to a finite quantity starting from a divergent integral. We will consider the dispersion relation (5) in the particular case $\omega_k^2 \sim k^4$ for a de Sitter evolution $C(\eta) = \alpha^2/\eta^2$. The equation for the associated field modes χ_k reads

$$\frac{\partial^2 \chi_k}{\partial \eta^2} + \left(k^2 + \frac{\tilde{\mu}^2 \alpha^2}{\eta^2} + \frac{2 b_{11} k^4 \eta^2}{\alpha^2}\right)\chi_k = 0, \tag{A1}$$

where $\tilde{\mu}^2 = m^2 + n(n-1)(\xi - \xi_n)/\alpha^2$. This equation can be solved exactly in the case $\tilde{\mu}^2 = 0$ or for arbitrary values of $\tilde{\mu}^2$ in the limit $\eta \to -\infty$. Indeed, in both cases, with the substitution $s = (2b_{11})^{1/4}\alpha^{-1/2}k\eta$ and introducing the constant $\lambda = \alpha(2b_{11})^{-1/2}$, the equation to be solved becomes

$$\frac{\partial^2 \chi_k}{\partial s^2} + (\lambda + s^2)\chi_k = 0. \tag{A2}$$

The solution is of the parabolic form

$$\chi_k(s) = D_{-[(1+i\lambda)/2]}[\pm(1+i)s] \tag{A3}$$

(see Ref. [26] for the definition of the parabolic function D).

For our purposes it is enough with the expansion for large $|s|$, which corresponds to $\eta \to -\infty$; this expansion has the form

$$D_p(z) \approx e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{8z^4} - \cdots\right), \tag{A4}$$

where $p = -(1+i\lambda)/2$ and $z = \pm(1+i)s$. After imposing the normalization condition (7), by power counting (and recalling that $s \sim \eta k$) it is easy to see that the only divergence comes from the leading order term of the expansion. This term is

$$\chi_k^{(0)} = \frac{1}{\sqrt{k}}\left(\frac{\lambda}{2}\right)^{1/4}(\sqrt{2}s)^{-[(1+i\lambda)/2]}\exp\left[-\frac{i}{2}\left(\frac{\pi}{4}+s^2\right)\right], \tag{A5}$$

so that

$$|\chi_k^{(0)}|^2 = \frac{\alpha}{2k^2|\eta|\sqrt{2b_{11}}} \tag{A6}$$

and when substituted in the integral for the vacuum expectation value

$$\langle\phi^2\rangle = \frac{\sqrt{C}}{2}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk k^{n-2}|\chi_k|^2 \tag{A7}$$

gives a linear divergence. The contribution of the other terms of the expansion Eq. (A4) is finite.

For the WKB solutions we have $|\chi_k|^2 = \chi_k\chi_k^* = 1/(2W_k)$. Because for large values of k we have $\omega_k^2 \sim k^4$, then the only divergence appearing in the inner product comes from the lowest order of the expansion. This contribution is given by

$$\langle\phi^2\rangle^{(0)} = \frac{\sqrt{C}}{2}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int\frac{dk\,k^{n-2}}{2\omega_k} \tag{A8}$$

which clearly diverges linearly. Once this quantity is calculated, then the renormalized mean squared field can be defined by subtracting the adiabatic expansion from the exact result:

$$\langle\phi^2\rangle_{\rm Ren} = \langle\phi^2\rangle - \langle\phi^2\rangle^{(0)}. \tag{A9}$$

For the sake of illustration, let us evaluate explicitly Eq. (A8) in n dimensions

$$\langle\phi^2\rangle^{(0)} = \frac{\sqrt{C}}{4}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk\,k^{n-3}\left(\frac{C}{2b_{11}}\right)^{1/2} \times\frac{1}{(k^2+\frac{C}{2b_{11}})^{1/2}}. \tag{A10}$$

Note that (up to the factor $(C/2b_{11})^{1/2}$) Eq. (A10) is formally analogous, in the limit $n \to 4$, to what one would obtain in the case of a field of nonvanishing "mass" given by $m^2 = 1/2b_{11}$ in $2+1$ dimensions. Thus we expect a finite result after applying the usual formulae for dimensional regularization [28]. Indeed, we obtain

$$\langle\phi^2\rangle^{(0)} = \frac{1}{16b_{11}}\frac{\Gamma(-1/2)}{\Gamma(1/2)} = -\frac{1}{8b_{11}}, \tag{A11}$$

which is finite.

To obtain the finite result for $\langle\phi^2\rangle_{\rm Ren}$ one should be able to compute the integral of the modes in Eq. (A7), which should also be finite in dimensional regularization. This is not appealing from a practical point of view, since in general there will be no analytical expression for this integral. However, as the difference

$$\langle\phi^2\rangle_{\rm Ren} = \frac{\sqrt{C}}{2}\frac{\Omega_{n-1}\mu^{4-n}}{(2\pi\sqrt{C})^{n-1}}\int dk\,k^{n-2}\left[|\chi_k|^2 - \frac{1}{2\omega_k}\right] \tag{A12}$$

is convergent , one can take the limit $n \to 4$ inside the integral and evaluate numerically both χ_k and the momentum integral already at $n = 4$.

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