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Almost-stationary motions and gauge conditions in general relativity

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An almost-stationary gauge condition is proposed with a view to numerical relativity applications. The time lines are defined as the integral curves of the timelike solutions of the harmonic almost-Killing equation. This vector equation is derived from a variational principle, by minimizing the deviations from isometry. The corresponding almost-stationary gauge condition allows us to put the field equations in hyperbolic form, both in the free-evolution ADM and in the Z4 formalisms.

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I. INTRODUCTION

It is well known that there is no preferred set of coordinate frames in Einstein's theory of gravitation, which is known as general relativity precisely because of this fact. In practical applications, however, one is forced to consider specific values of the gravitational field components (spacetime metric, curvature tensor) and this can be done only after choosing a specific coordinate system. One just expects that this gauge choice will not hide the physics of the problem behind a mask of nontrivial coordinate effects.

Harmonic coordinate systems have deserved much interest since the very beginning of Einstein's theory [1,2]. The coordinates themselves are defined by a set of four harmonic spacetime functions $\{\Phi^\mu\}$, that is

$$\Box \Phi^\mu = 0, \tag{1}$$

where the box stands for the wave operator acting on functions. In an harmonic coordinate system one takes $x^\mu \equiv \Phi^\mu$, so that Einstein's field equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2}$$

get a very convenient form, in which the principal part is simply a scalar wave equation for every metric component [1,2]. This fact has been used by Yvonne Choquet-Bruhat for proving the well-posedness of the Cauchy problem in general relativity [3].

In recent years, this simplification has been used for building up numerical relativity codes, with interesting related developments also on the theoretical side [4–7]. On the applications side, the ‘‘gauge sources’’ variant of the harmonic gauge [8] has been used by Pretorius in a numerical simulation of the evolution of a binary black-hole system: a full quasicircular orbit has been achieved [9,10]. In this simulation, however, a grid velocity has been introduced in order to get into the corotating frame. This means that computational nodes are rotating with respect to the coordinate system. In other words, harmonic coordinates by themselves are not following the overall rotation pattern of the black-hole binary system.

Binary systems provide a good example of almost-stationary configurations. Take for instance the well known pulsar 1913 + 16: it would be a perfect clock if we could just neglect the (very small) energy loss due to gravitational radiation, getting then a sort of steady system. One would like to choose a corotating frame in order to get a clean view of symmetry deviations. This case is representative of many other situations in which there is not an exact symmetry. In these cases, the idea of approximate symmetry, or that of almost-Killing vectors, would be of great help in numerical relativity applications.

A precise implementation of the concept of almost-symmetry has been provided by Matzner [11]. Starting from a variational principle, it defines a measure of the symmetry deviation of any given spacetime. This idea has been applied by Isaacson [12] to the study of high-frequency gravitational waves, by defining a steady coordinate system in which the radiation effects can be easily separated from the background metric. More recently [13], the same measure has been considered as an inhomogeneity index of the spacetime, which can be related with some entropy concept.

We are not interested here, however, in studying the spacetime properties or in comparing different spacetimes. We will focus instead in characterizing motions in arbitrary spacetimes. By a motion we mean a congruence of time lines, that can then be associated to the world lines of a system of observers. For instance, geodesic motions (associated with freely falling observers) or harmonic motions (associated with the observers at rest in harmonic coordinate systems). Our goal is then to characterize the motions that correspond to the physical idea of almost-symmetry and to study the adapted coordinate systems with a view to numerical relativity applications.

In this sense, we will see that our approach is more directly related with the almost-Killing equation. This is a generalization of the Yano-Bochner equation [14]

$$\nabla_\nu[\xi^{\mu;\nu} + \xi^{\nu;\mu}] = 0, \tag{3}$$

which has been considered by York [15] and others in order to identify physically meaningful tensor components in asymptotically flat spacetimes (transverse-traceless decomposition). The same idea has also been applied to the identification of asymptotic Killing vectors in Kerr spacetime [16].

These Killing-like equations are briefly reviewed in the next section, where the almost-Killing equation is derived from a variational approach. The main new results start however in the third section, where harmonic almost-Killing motions are shown to provide a convenient generalization of the standard harmonic motions. This generalization is implemented through a true vector condition, in contrast with the standard harmonic case, in which one deals instead with the set of four scalar conditions (1). This point is illustrated by considering spherically symmetric spacetimes, where spherical coordinates are incompatible with the standard harmonic condition but perfectly allowed by the proposed generalization.

In section four, we consider the related problem of using harmonic almost-stationary motions as gauge conditions, with a view to numerical relativity applications. The adapted coordinate system is used in order to show the hyperbolicity of the full system: Einstein's field equations plus gauge conditions. Of course, any hyperbolicity proof requires a specific formulation for the field equations. We have chosen here the Z4 formalism [17] just for simplicity, although the proposed almost-stationary gauge condition should also work out with other hyperbolic formalisms. The specific implementation we provide could be used as a guide for any other particular choices. In order to illustrate this point, we also show the hyperbolicity of the standard ADM free-evolution approach when supplemented with the proposed almost-stationary gauge condition.

II. ALMOST-KILLING VECTOR FIELDS

A. Killing-like equations

Killing vectors can be defined as the solutions of the Killing equation:

$$\mathcal{L}_{\xi}(g_{\mu\nu}) = \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0. \tag{4}$$

Their physical meaning can be better understood by considering an adapted coordinate system. In the timelike case, for instance, we can choose the time lines to be the integral curves of ξ and the time coordinate to be the special choice of the affine parameter on these curves such that

$$\xi = \partial_t. \tag{5}$$

Then, the Killing equation (4) reads simply

$$\partial_t g_{\mu\nu} = 0, \tag{6}$$

meaning that the metric is stationary, so that spacetime geometry is preserved along the integral curves of ξ.

A well known generalization of the Killing Eq. (4) is given by the affine Killing vectors (AKV), namely, the solutions of

$$\nabla_{\rho}[\mathcal{L}_{\xi}(g_{\mu\nu})] = 0. \tag{7}$$

The physical meaning is again more transparent if we express it the adapted coordinate system (5). Then, Eq. (7) amounts to

$$\partial_t \Gamma^{\mu}_{\rho\sigma} = 0. \tag{8}$$

This means that the affine structure of the spacetime, given by the connection coefficients $\Gamma^{\mu}_{\rho\sigma}$, is preserved along the integral curves of ξ.

An interesting subset of AKV is that of the homothetic Killing vectors, defined as the solutions of

$$\mathcal{L}_{\xi}(g_{\mu\nu}) = 2g_{\mu\nu} \tag{9}$$

(the factor 2 in the right-hand side can be changed to any nonzero value by a suitable rescaling of ξ). The physical relevance of the homothetic Killing vectors comes from the invariance of Eq. (9) under a rescaling of the metric. This translates the idea that there is no preferred length (time) scale in the spacetime, allowing scale-invariant (self-similar) processes to develop. These processes have been seen to arise in connection with critical phenomena in general relativity [18,19].

B. A variational-principle approach: almost-Killing vectors

We will consider here a further generalization of (4), the almost-Killing equation (AKE) given by [16]

$$\nabla_{\nu}\left[\xi^{(\mu;\nu)} - \frac{\lambda}{2}(\nabla\cdot\xi)g^{\mu\nu}\right] = 0, \tag{10}$$

where the round brackets denote symmetrization. The solution space includes Killing vectors and AKV for any value of the constant λ. The simplest parameter choice ($\lambda = 0$), corresponds to the Yano-Bochner equation [14]. The case $\lambda = \frac{1}{2}$ corresponds to the Conformal AKE, which includes conformal Killing vectors ($\lambda = \frac{2}{3}$ for a three-dimensional manifold, as in Ref. [15]).

The term "almost-Killing" is justified by the fact that the AKE equation (10) can be obtained from a standard variational principle

$$\delta S = 0, \qquad S \equiv \int L\sqrt{g}d^4x, \tag{11}$$

where the Lagrangian density L is given by

$$L = \xi_{(\rho;\sigma)}\xi^{(\rho;\sigma)} - \frac{\lambda}{2}(\nabla\cdot\xi)^2, \tag{12}$$

and the variations of the field ξ are considered in a fixed spacetime. The covariant conservation law (10) provides then a precise meaning to the heuristic concept of approximate Killing vectors. This was not obvious *a priori*, because the Lagrangian (12) is not a positive-definite quantity: the outcome of the minimization process was not granted to include the zeros of (12).

The mathematical structure of the AKE is more transparent if we rewrite it in the equivalent form

$$\Box\xi_\mu + R_{\mu\nu}\xi^\nu + (1-\lambda)\nabla_\mu(\nabla\cdot\xi) = 0, \qquad (13)$$

where we have just reversed the order of some covariant derivatives in (10). This "wave equation" form of the AKE can be alternatively obtained from the Lagrangian

$$L' = \xi_{\rho;\sigma}\xi^{\rho;\sigma} - R_{\rho\sigma}\xi^\rho\xi^\sigma + (1-\lambda)(\nabla\cdot\xi)^2, \qquad (14)$$

which is of course equivalent to the original one, modulo a four-divergence:

$$L' = 2L - \nabla_\sigma[(\xi\cdot\nabla)\xi^\sigma - \xi^\sigma(\nabla\cdot\xi)]. \qquad (15)$$

The principal symbol of the differential operator in either form of the AKE can be written in Fourier space as

$$k^2\delta^\mu_\nu + (1-\lambda)k^\mu k_\nu, \qquad (16)$$

so that:

(i) The characteristic hypersurfaces are the light cones ($k^2 = 0$).

(ii) The symbol (16) is singular for $\lambda = 2$. For vacuum spacetimes, this case corresponds to Maxwell's equations for the electromagnetic potential [16]. A supplementary condition (such as the "Lorentz condition" $\nabla\cdot\xi = 0$) would be then required in order to get a unique solution.

(iii) On noncharacteristic hypersurfaces, the symbol (16) can be algebraically inverted for $\lambda \neq 2$.

The last point is a strong indication of the existence of solutions for ξ in any given spacetime, for every set of noncharacteristic initial data. Of course, this is not a 100% rigorous proof because the straightforward passage to Fourier space ignores the coordinate dependence of the metric (the standard "frozen coefficients" approach). But this gives us a sound basis for assuming in what follows that solutions for ξ may be constructed in any given spacetime.

To be more specific, the initial value problem can be expressed as follows:

(i) We can freely choose the values of ξ on a given initial hypersurface (let us say $t = 0$). The space derivatives of ξ can then be computed from these values.

(ii) The time derivative of ξ can also be freely specified on the initial hypersurface. In this way, we have the full set of first covariant derivatives $\xi_{\mu;\nu}$ at $t = 0$.

(iii) In order to propagate these values along the time lines, we must compute the second time derivative of ξ from the second order equation (13). This can always be done for the space components ξ_i provided that $g^{00} \neq 0$, meaning that the initial hypersurface is not tangent to the local light cone. In the case of the ξ_0 component, we must require in addition that $\lambda \neq 2$, as it was to be expected from the results of the preceding paragraph.

In the timelike case, the integral curves of ξ can be interpreted as the world lines of a set of observers. As far as there is one solution for every set of (noncharacteristic) initial data, we can interpret the set of solutions as providing a set of motions that minimize the deviation from isometry along the congruence of time lines. This justifies the name of "almost-stationary motions" for the timelike solutions of the AKE.

III. HARMONIC ALMOST-STATIONARY MOTIONS

We will consider now the particular parameter choice $\lambda = 1$, in which the principal part of Eq. (13) is harmonic, that is

$$\Box\xi^\mu + R^\mu_\nu\xi^\nu = 0, \qquad (17)$$

so that the resulting timelike solutions will be called "harmonic almost-Killing motions." The conservation-law version (10) can then we written as

$$\nabla_\nu\left[\frac{1}{\sqrt{g}}\mathcal{L}_\xi(\sqrt{g}g^{\mu\nu})\right] = 0. \qquad (18)$$

Notice that the Lagrangian L' in this case, namely

$$L' = \xi_{\rho;\sigma}\xi^{\rho;\sigma} - R_{\rho\sigma}\xi^\rho\xi^\sigma, \qquad (19)$$

gets an interesting "kinetic minus gravitational" form.

Both forms (17) and (18) of the Harmonic AKE equation (HAKE) suggest a close relationship with harmonic coordinates. This relationship is again more transparent in the adapted coordinate system, where (18) leads to

$$g^{\rho\sigma}\partial_t\Gamma^\mu_{\rho\sigma} = 0 \qquad (20)$$

[compare with Eq. (8) for AKV], whereas the harmonic coordinates condition (1) reads just

$$\Gamma^\mu \equiv g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = 0 \qquad (21)$$

in adapted coordinates. One can then write (20) as

$$\partial_t\Gamma^\mu = \Gamma^\mu_{\rho\sigma}\partial_t(g^{\rho\sigma}), \qquad (22)$$

so that it is clear that the flow associated with harmonic coordinates will provide a first integral for the HAKE in the weak field limit (where only linear terms are retained). This fact can be relevant for the characterization of the gravitational waves degrees of freedom in asymptotically flat spacetimes [12].

Note also that the HAKE (18) is a true vector equation, whereas the standard harmonic condition (1) is rather a set of four scalar equations. This difference is important, because the congruence of time lines in a given motion is defined by its tangent vector field. One can, for instance, relabel the particular time lines by an arbitrary time-independent coordinate transformation, namely

$$x^i = f^i(y^j) \qquad i, j = 1, 2, 3, \qquad (23)$$

while keeping the same expression (5) for the timelike tangent vector. The transformation (23) allows one to select the type of space coordinate system (cylindrical, spherical, or whatsoever) which is more adapted to any specific problem. In numerical simulations, the transformation (23) corresponds to the freedom of choosing an arbitrary space coordinate system on the initial hypersurface.

This is not the case for the standard harmonic coordinates choice. In order to illustrate this, we will consider for instance a spherically symmetric line element, namely

$$ds^2 = -\alpha^2 dt^2 + X^2 dr^2 + Y^2[d\theta^2 + \sin^2(\theta) d\varphi^2], \quad (24)$$

where all the metric functions (α, X, Y) depend only on (t, r). In this case, the time and radial components of the HAKE (20) provide conditions for the corresponding (t, r) coordinates, whereas the angular components are identically satisfied in the adapted coordinate system form (20). In the standard harmonic case, however, one gets

$$\Gamma^\varphi = 0, \qquad \Gamma^\theta = -\frac{\cot(\theta)}{Y^2} \neq 0, \quad (25)$$

so that spherical coordinates happen to be incompatible with the harmonic condition (21).

This lack of versatility of the standard harmonic coordinates can be a serious drawback in numerical relativity applications, where one could be unable to fully adapt the coordinate frame to the features of the physical system under consideration. We hope that the proposed almost-stationary generalization (20) will contribute to avoid this complication.

IV. ALMOST-STATIONARY GAUGE CONDITIONS

Standard harmonic motions, as defined by (21), have been used recently in advanced numerical relativity applications [4–7,9,10]. Note however that in this context one is building up the coordinate system and the spacetime itself at the same time, whereas the spacetime was supposed to be given in the previous sections. To be more precise, Einstein's field equations (2) must be coupled with the gauge condition. In the harmonic case, this condition is given by Eq. (21).

The principal part of Einstein's equations can be written in the DeDonder-Fock form [1,2], namely

$$\Box g_{\mu\nu} - \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu = \cdots \quad (26)$$

so that, by choosing the spacetime coordinates to be the solutions of (1), one ensures the vanishing of Γ^μ and the field equations can be relaxed to a system whose principal part consists in a scalar wave equation for every metric component, namely

$$\Box g_{\mu\nu} = \dots . \quad (27)$$

Note that the metric in (21) and (27) is overdetermined, because one gets in all 14 equations for only 10 metric components. This can be better understood by introducing a "zero vector" Z^μ as an additional dynamical field (Z4 system [17]), so that the field equations read

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 8\pi\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right). \quad (28)$$

The principal part of the Z4 field equations in the DeDonder-Fock form reads now

$$\Box g_{\mu\nu} - \partial_\mu(\Gamma_\nu + 2Z_\nu) - \partial_\nu(\Gamma_\mu + 2Z_\mu) = \cdots , \quad (29)$$

so that the relaxed system (27) is recovered by setting the values of Z^μ to be

$$Z^\mu = -\frac{1}{2}\Gamma^\mu, \quad (30)$$

which can be considered just an extension of the harmonic coordinates condition (1), namely

$$\Box \Phi^\mu = 2Z^\mu. \quad (31)$$

As far as one recovers in this way exactly the same relaxed system (27), one gets exactly the same solutions for the metric. The extra quantities Z^μ just allow us to monitor to which extent the harmonic coordinates condition (30), when considered as a constraint on the computed metric, is actually verified. In this sense, it is useful to consider the four-divergence of the Z4 field equations (28), namely

$$\Box Z_\mu + R_{\mu\nu} Z^\nu = 0, \quad (32)$$

which can be interpreted as the constraint-propagation law. It follows that the constraint-violation vector Z^μ obeys precisely the HAKE (17).

Note that the Z4 system has been used here just as a convenient analysis tool in order to discuss the overlap between the 10 equations of the relaxed system and the 4 equations of the harmonic coordinates condition. The same overlap will occur when replacing the standard harmonic coordinates condition (21) by the HAKE (20). This suggests to consider, in the Z4 context, the analogous replacement of (30) by

$$g^{\rho\sigma}\partial_t(\Gamma^\mu_{\rho\sigma}) + 2\partial_t Z^\mu = 0. \quad (33)$$

A covariant expression for (33) is given by

$$\nabla_\nu\left[\frac{1}{\sqrt{g}}\mathcal{L}_\xi(\sqrt{g}g^{\mu\nu})\right] = 2\mathcal{L}_\xi(Z^\mu). \quad (34)$$

The principal part of (33) can be written now simply as

$$\partial_t(\Gamma^\mu + 2Z^\mu) = \cdots, \quad (35)$$

so that (the principal part of) the full system (29) and (35) gets a triangular form. The characteristic lines can easily be identified:

(i) the time lines, as it follows from the fact that the "gamma sector" equations (35) are yet in diagonal form

(ii) the light rays, as it follows from the fact that the only nondiagonal terms in the ''metric sector'' equations (29) are just coupling terms with the Gamma sector, which is itself in diagonal form.

We can conclude that the full differential system formed by the Z4 system (28) and the (extended) HAKE condition (33) is hyperbolic. The only trouble can arise at the specific points where the harmonic almost-stationary vector ξ is lightlike ($\xi^2 = 0$), so that the corresponding time line gets tangent to the light cone. The nondiagonal coupling terms in (29) would then prevent the full diagonalization of the characteristic matrix at this specific point. This would be for instance the case of the apparent horizon in stationary spacetimes, when the Killing vector is selected as a solution of the HAKE equation.

Let us note again that we are using here the Z4 formalism just as an analysis tool, which allows us to monitor the evolution of the constraint violations. The proposed coordinate conditions (20) are actually independent of the hyperbolic formalism one likes to choose for the field equations. In order to illustrate this point, let us take for instance the standard ADM free-evolution approach:

(i) The original field equations (26) are considered. However, only the space components

$$\Box g_{ij} - \partial_i \Gamma_j - \partial_j \Gamma_i = \cdots \qquad (36)$$

are kept, because the four remaining combinations provide just constraints which are not solved in the free-evolution approach.

(ii) The almost-stationary conditions (20), with principal part

$$\partial_t \Gamma^0 = \cdots \qquad \partial_t \Gamma_i = \cdots \qquad (37)$$

which provide the missing evolution equations for the remaining metric components g^{00}, g_{0i}, respectively.

We can see by inspection that the principal part of the full system (36) and (37) is also in triangular form. The characteristic lines are again either the time lines (''gamma sector'') and the light cones (''metric sector''). The resulting ADM system, when supplemented with the almost-stationary gauge condition, is then hyperbolic, provided that the time lines do not get tangent to the light cones.

We can conclude that the same qualitative behavior is obtained both in the Z4 and in the ADM frameworks when using the almost-stationary gauge condition. We do not expect then any essential difficulty in adapting this coordinate condition to other hyperbolic formalisms which are being used in numerical relativity applications.

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