

String pair production in a time-dependent gravitational field

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We study the pair creation of point particles and strings in a time-dependent, weak gravitational field. We find that, for massive string states, there are surprising and significant differences between the string and point-particle results. Central to our approach is the fact that a weakly curved spacetime can be represented by a coherent state of gravitons, and therefore we employ standard techniques in string perturbation theory. String and point-particle pairs are created through tree-level interactions between the background gravitons. In particular, we focus on the production of excited string states and perform explicit calculations of the production of a set of string states of arbitrary excitation level. The differences between the string and point-particle results may contain important lessons for the pair production of strings in the strong gravitational fields of interest in cosmology and black hole physics.

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I. INTRODUCTION

The aim of this paper is a calculation of the pair production of strings in a time-dependent background. This phenomenon is of paramount importance in cosmology, particularly during (p)reheating after inflation, and near cosmological singularities. Usually one studies pair production using an effective field theory approach. Here we study a spacetime with weak gravitational fields and compute the pair production of excited string states using world sheet methods. This “stringy” calculation gives significantly different results from those found in the effective field theory approximation. Our results suggest specific signatures of the string pair creation phenomenon that may survive into the regime of strong gravitational fields.

Recently [1–3] the pair creation of strings in a cosmological setting was estimated using effective field theory methods. In this work the excited states of the string are treated as massive particles for which the usual field theory calculation applies. While the production rate of a single massive state is small, the exponential Hagedorn growth of the number of string states can give rise to a significant cumulative effect. Here we test this effective field theory approximation in a weak-field limit, where string pair creation may be computed exactly. Although our real interest is strong gravitational fields, such as those that arise in cosmology, the formalism does not yet exist to calculate pair production of strings here (see [4] for an interesting approach to this and [5] for a calculation in a more specific background). Nevertheless interesting surprises can be seen already for weak fields. Among them, we find that the enhancement of the production of excited string states due to the growing density of states is greatly suppressed due to the very mild hard scattering behavior of

the string, which in turn arises from the fact that the string is an extended object.

To carry out our calculation of string pair production, we use the familiar idea that a curved geometry can be represented as a coherent state of gravitons [6,7]. For weak gravitational fields this gives a prescription relating S -matrix amplitudes on a weakly curved spacetime to a sum of those on Minkowski spacetime with multiple graviton vertex operators inserted. To lowest order in the closed string coupling, the pair production of strings is thus the four-point process $gg \rightarrow AB$, with two gravitons g interacting to produce two strings in states A and B . This approach has the advantage of leading to a controlled calculation using standard string perturbation theory, but there are many other conceptual and technical obstacles to an understanding of strings in general time-dependent backgrounds. There are subtleties involved in interpreting the S -matrix for strings on curved backgrounds (see, for example, [8,9]). More generally, in any quantum theory including gravity, the inevitable formation of singularities as predicted by the singularity theorems [10,11] implies the absence of asymptotically trivial “in” and “out” regions required to define an S -matrix. We construct a specific geometry describing the collision of two small amplitude plane waves which we embed in an exact nonlinear solution of Einstein’s equations. An analysis of the full solution shows the formation of a spacelike singularity in the future of the collision of the plane waves, and also how by making the amplitude of both waves sufficiently small we may push the singularity arbitrarily far into the future of the initial collision region. Consequently, perturbation theory is valid for an arbitrarily long period, and we believe the S -matrix formalism can be used to understand the physics in this region. This is as it should be, for the possible formation of a big crunch in our future should not prevent us from understanding physics today.

In this work we primarily focus on the pair production of long-wavelength string states. Since the theory of pair

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creation in field theory is well developed [12,13], and since string theory should have an effective field theory description valid for $\lambda \gg \sqrt{\alpha'}$, one might ask why we expect there to be differences between string and point-particle pair creation in this regime. When viewed as the four-point process $gg \rightarrow AB$, then certainly for massless states A and B , with wavelengths $\lambda \gg \sqrt{\alpha'}$, all momentum scales in the scattering process are well below the string scale. On the other hand, to produce excited string states, with masses $m \sim 1/\sqrt{\alpha'}$, then the typical graviton momenta and momentum exchange will in fact be greater than $1/\sqrt{\alpha'}$. This is beyond the validity of effective field theory, and indeed previous investigations have revealed uniquely stringy behavior in this regime [14–16]. Thus, it is reasonable to expect the pair production of massive *strings* might be fundamentally different from the pair production of massive *particles*.

Our calculation reveals two specific differences between the string and point-particle results. As we have mentioned above, and will discuss in more detail in Sec. IV, the pair production of strings is suppressed due to the mild hard scattering behavior of string amplitudes. Another intriguing difference relates to the production of identical string states. At tree level in field theory only identical particle pairs are produced. A surprising feature of the string case is that production of pairs of strings at the same excitation level are suppressed, going to zero as $\lambda \rightarrow \infty$. Instead, most created pairs are of different mass levels. Both of these features can be explained in the context of certain simple features of the string spectrum and scattering amplitudes, as we discuss in more detail below.

This paper is organized as follows: In Sec. II we discuss the perturbative approach to calculating pair production, using field theory as a guide to the string calculation. In Sec. III we discuss a specific example of a background, describing the collision of two plane waves, for which the perturbative calculation may be applied.

In Sec. IV we perform the explicit string calculation and discuss its properties. Finally, we present our conclusions in Sec. V. We also include two appendices with additional details on some elements of our work. The four-point amplitude describing pair production is discussed in Appendix A, and a more detailed account of perturbatively computing Bogoliubov coefficients is given in Appendix B.

II. PAIR PRODUCTION IN WEAK GRAVITATIONAL FIELDS

In this section we discuss how pair production may be studied perturbatively. Certainly there are examples where the production rate is nonperturbative; these include Schwinger's classic calculation of e^+e^- production in an electric field [17] and more recent calculations with open strings [18–20]. An example with a different behavior is provided by the pair production of scalars of mass m in a

flat Friedmann-Robertson-Walker spacetime that undergoes a brief period of expansion [21]. In this example we have the metric

$$ds^2 = a(\eta)^2(-d\eta^2 + dx_3^2), \quad a^2(\eta) = 1 + b \tanh(\rho\eta), \quad (1)$$

and the number of particles produced in each mode $k = (\omega, \vec{k})$ is found to be

$$N_{\vec{k}} = \frac{\sinh^2[\pi(\omega_{+\infty} - \omega_{-\infty})/2\rho]}{\sinh(\pi\omega_{+\infty}/\rho) \sinh(\pi\omega_{-\infty}/\rho)}, \quad (2)$$

where

$$\omega(\eta) = [|\vec{k}|^2 + a(\eta)^2 m^2]^{1/2}, \quad \omega_{\pm\infty} = \omega(\pm\infty). \quad (3)$$

For $b \ll 1$ the number of pairs produced goes like $\sim b^2 m^4$ for small b . Thus there are backgrounds in which it is plausible that pair creation could be studied within a perturbative framework.

A. Field theory

In the case of field theory, studying pair production can be realized quite straightforwardly through the addition of new vertices in the Feynman diagrams of the theory. This point of view is developed in more detail in Appendix B, where it is shown that this technique yields precisely the same results as the more standard Bogoliubov coefficient calculation. The field theory case is useful for our present purposes as it provides a prototype for the string calculation. One expands the metric about Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + \dots, \quad (4)$$

where $h_{\mu\nu}^{(1)}$ is the first-order metric perturbation, and $h_{\mu\nu}^{(2)} + \dots$, the corrections required at successively higher order so that the full metric $g_{\mu\nu}$ satisfies the Einstein equations. If we consider a massive minimally coupled scalar field with action

$$S_\phi = -\frac{1}{2} \int (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \sqrt{-g} d^D x, \quad (5)$$

then the expansion (4) of the metric gives a corresponding perturbative expansion of the action,

$$S_\phi = S_\phi^{(0)} + S_\phi^{(1)} + S_\phi^{(2)} + \dots \quad (6)$$

as well as a series of terms $S_\phi^{(n)}$ at n th order in the perturbation.

This allows a perturbative evaluation of the n -point functions in the background, denoted by $\langle \dots \rangle_b$, in terms of the same quantities $\langle \dots \rangle_0$ in Minkowski space with additional insertions. For any combination of fields \mathcal{O} , the expansion of the action (6) inserted into the path integral yields

$$\begin{aligned} \langle \mathcal{O} \rangle_b &= \langle \mathcal{O} \rangle_0 + \int \langle T \mathcal{O}_k \Sigma_{-k}^{(1)} \rangle_0 \frac{d^D k}{(2\pi)^D} + \int \langle T \mathcal{O}_{k'} \Sigma_{-k'}^{(2)} \rangle_0 \\ &\times \frac{d^D k'}{(2\pi)^D} + \dots, \end{aligned} \quad (7)$$

where $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are the new vertices that appear to first and second order in perturbations. These are determined by the expansion (6) of the action

$$\Sigma^{(1)} = iS_\phi^{(1)}, \quad (8a)$$

$$\Sigma^{(2)} = iS_\phi^{(2)} - \frac{1}{2}[S_\phi^{(1)}]^2. \quad (8b)$$

The statements that we have made for the n -point functions have analogues for the S -matrix. However, defining an S -matrix involves a choice of asymptotic “in” and “out” states. Therefore this prescription really only makes sense when the spacetime in question is asymptotically Minkowski. In other cases it may be necessary to modify the definition of asymptotic states.

At this point, it is possible to see how even a weak gravitational background can lead to pair production. Pair production is the process with no “in” particles and two “out” particles. While forbidden by momentum conservation in Minkowski space, in the perturbed spacetime each term in the S -matrix carries three momenta; the two momenta of the “out” particles and the momentum carried by the new vertices $\Sigma^{(n)}$. Thus two positive energy particles can appear in the “out” state if one of the terms $\Sigma_k^{(n)}$ supplies the “missing” momentum. So, the amplitude to pair produce two particles with momenta k_1, k_2 will be nonzero provided that we have $\Sigma_{-k_1-k_2}^{(n)} \neq 0$. More physically the missing momentum is coming from the background gravitons.

B. String theory

Following the same procedure as the field theory calculation, one may insert the perturbative expansion of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} + \dots$ into the conformal gauge string world sheet action

$$S = -\frac{1}{2\pi\alpha'} \int g_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu d^2z. \quad (9)$$

Now, the successive metric perturbations $h_{\mu\nu}^{(n)}$ are chosen to satisfy the string beta function equations.

As is well known [6,7], the expansion of the metric leads to the insertion of graviton vertex operators in world sheet correlation functions. For example, if we wish to study the S -matrix element involving two string states A and B , then we must compute the world sheet correlation function

$$\begin{aligned} \langle \mathcal{V}_A(k_A) \mathcal{V}_B(k_B) \rangle_b &= \langle \mathcal{V}_A(k_A) \mathcal{V}_B(k_B) \rangle_0 \\ &+ \frac{1}{4\pi g_c} \int h_{\mu\nu}^{(1)}(k') \langle \mathcal{V}_A(k_A) \mathcal{V}_B(k_B) \\ &\times \mathcal{V}_g^{\mu\nu}(k') \rangle_0 \frac{d^D k'}{(2\pi)^D} + \dots, \end{aligned} \quad (10)$$

where $\mathcal{V}_{A,B}(k_{A,B})$ are the vertex operators for the created strings in states A and B , $\mathcal{V}_g^{\mu\nu}(k')$ is a graviton vertex operator integrated over the world sheet representing the metric perturbation, and we have absorbed a factor of g_c in each of the vertex operators. The metric perturbation $h_{\mu\nu}^{(1)}$ appears as the polarization tensor of the background graviton.

The first-order contribution to pair creation (10) vanishes when A and B are massive, due to momentum conservation when the graviton is on-shell (as it must be for a consistent string amplitude). Nevertheless, (10) is related to the phenomenon of “particle transmutation,” by which a string can change its mass and spin as it passes through a gravitational background [22–25]. We will not investigate this phenomenon in the current work, but for weakly curved spacetimes it could in principle be studied within the S -matrix framework we apply herein.

Ideally we would be able to continue this expansion to higher order in the metric perturbation, by analogy to the field theory case. However, at second-order issues arise in the string theory that make the situation somewhat more complex. For one, consistency of the string amplitude (10) requires that each of the vertex operators be a conformal tensor of weight $(h, \bar{h}) = (1, 1)$, or in other words satisfy the physical state conditions. As is well known, this is equivalent to imposing the linearized Einstein equations on $h_{\mu\nu}^{(1)}$. At second order in the expansion of world sheet amplitudes, a graviton vertex operator with polarization $h_{\mu\nu}^{(2)}$ appears. By construction, the second-order perturbation $h_{\mu\nu}^{(2)}$ satisfies the second-order renormalization-group equations. These equations are not identical to the physical state conditions on the vertex operator. Therefore, in general $h_{\mu\nu}^{(2)}$ will *not* in fact satisfy the physical state conditions. Thus, the string amplitude with the corresponding vertex operator will not be consistent.

Another issue that arises is the presence of a nonzero “tadpole” for some fields. Taking our prescription literally, to compute the S -matrix elements on our background to second order involves inserting two graviton vertex operators into the relevant correlation functions. For massive fields A , the tadpole is zero at first order due to momentum conservation. At second order in the metric perturbation, the tadpole for some state A in the background is therefore given by

$$\langle \mathcal{V}_A(k_A) \rangle_b = \langle \mathcal{V}_A(k_A) \mathcal{V}_g(k_1) \mathcal{V}_g(k_2) \rangle + \dots, \quad (11)$$

with $\mathcal{V}_g(k) = \frac{1}{4\pi g_c} h_{\mu\nu}^{(1)} \mathcal{V}_g^{\mu\nu}(k)$. This expression does not

vanish for the examples we consider. A nonvanishing tadpole is an indication that the classical equations of motion are not exactly obeyed by the background solution. This is to be expected, as here we will use a background solution that solves the leading-order equations of motion for massless string excitations, but we have ignored the massive excitations and higher corrections. The presence of the tadpole is telling us that the equations of motion for these massive fields forbid setting them to zero throughout spacetime. Furthermore, in a curved background the vertex operators and states should change, which we have not taken into account. We expect that the failure of the tadpole to vanish is due to some combination of all of these effects.

Given these new complexities, it appears that problems are beginning to develop with the technique of inserting new vertex operators at second order. Nevertheless, by considering each of these new problems, we will argue in Sec. II C that useful results can be obtained from the second-order expansion. Therefore, for the purpose of studying pair production we will use the prescription

$$\begin{aligned} \langle AB|S|vac\rangle &= \frac{1}{(4\pi g_c)^2} \int h_{\mu\nu}^{(1)}(k') h_{\sigma\tau}^{(1)}(k'') \\ &\times \langle \mathcal{V}_A(k_A) \mathcal{V}_B(k_B) \mathcal{V}_g^{\mu\nu}(k') \\ &\times \mathcal{V}_g^{\sigma\tau}(k'') \rangle_0 \frac{d^D k'}{(2\pi)^D} \frac{d^D k''}{(2\pi)^D}, \end{aligned} \quad (12)$$

where $\mathcal{V}_{A,B}(k_{A,B})$ are the vertex operators for the pair created string states A and B . In defining this S -matrix element, it is understood as usual that the positions of three of the vertex operators are fixed and the remaining one is integrated over the world sheet. This prescription does not involve a term at first order in the metric perturbation, since the pair production of massive states is forbidden to this order by momentum conservation. Furthermore corrections to the vertex operators and background solution, which should be included in a more complete analysis, only affect this S -matrix element at higher orders in $h_{\mu\nu}^{(1)}$.

C. Coherent states

The existence of nonzero ‘‘tadpoles,’’ or 3-point functions $\langle \mathcal{V}_g \mathcal{V}_g \mathcal{V}_N \rangle$, where \mathcal{V}_N denote the vertex operators for level N string states, implies that turning on a nontrivial gravitational background will, at second order, produce a nontrivial background of excited states. The question we must address is whether we can still trust the pair production results that we will extract at this order. By studying an analogous situation in field theory, we will argue that the corrections to the pair production amplitudes appear at higher order, and thus our second-order results may be trusted.

The production of excited string states can be modeled in field theory as the pair production of a massive scalar field χ via interactions with a massless scalar field H , representing the graviton. This can be realized with the

action

$$\begin{aligned} S &= \int d^d x \left(-\frac{1}{2}(\partial H)^2 - \frac{1}{2}(\partial \chi)^2 \right. \\ &\quad \left. - \frac{1}{2}m^2 \chi^2 + \lambda H^2 \chi - \frac{1}{2}\mu H^2 \chi^2 \right). \end{aligned} \quad (13)$$

The presence of the λ coupling implies that we cannot turn on a background field for H without also exciting a background value for χ , as a consequence of the equations of motion

$$-\square H = 2\lambda H \chi + \dots \quad (14)$$

$$-\square \chi + m^2 \chi = \lambda H^2 + \dots \quad (15)$$

The μ coupling gives rise to pair creation of χ quanta in the presence of a nontrivial ‘‘graviton’’ background H . Before calculating this effect we must first correctly deal with the background. Assuming $\chi = 0 + O(H^2)$, these equations may be solved perturbatively to second order in H as [here $\int d\tilde{k} = \int d^{d-1}k/(2\omega)$, $\omega = |k^0|$ which is valid for massless and massive particles]

$$H(x) = H_0(x) = \int d\tilde{k} (\alpha^*(k) e^{ik \cdot x} + \alpha(k) e^{-ik \cdot x}) + O(H_0^3), \quad (16)$$

$$\chi(x) = \lambda \int d^d y G_{ret}(x, y) H_0^2(y) + O(H_0^3), \quad (17)$$

where $G_{ret}(x, y)$ is the retarded propagator for a scalar of mass m and we have chosen the following boundary conditions

$$\lim_{t \rightarrow -\infty} \chi(x) = 0. \quad (18)$$

It is well known that in field theory a nontrivial background may be described by a coherent state. The initial state contains only gravitons and may be described by

$$|vac, t \rightarrow -\infty\rangle = e^{-(1/2) \int d\tilde{k} |\alpha|^2} e^{\int d\tilde{k} \alpha a^\dagger} |0\rangle, \quad (19)$$

whereas the final state is a coherent state of both ‘‘gravitons’’ and χ quanta

$$\begin{aligned} |vac, t \rightarrow +\infty\rangle &= e^{-(1/2) \int d\tilde{k} |\beta|^2} e^{-\int d\tilde{k} (1/2) |\alpha|^2} \\ &\times e^{\int d\tilde{k} \alpha a^\dagger} e^{\int d\tilde{k} \beta d^\dagger} |0\rangle. \end{aligned} \quad (20)$$

Here a^\dagger/a , d^\dagger/d are the conventionally defined creation/annihilation operators for H and χ respectively, and $\beta(k)$ is the generated background for χ :

$$\lim_{t \rightarrow +\infty} \chi(x) = \int d\tilde{k} (\beta^*(k) e^{ik \cdot x} + \beta(k) e^{-ik \cdot x}). \quad (21)$$

We now compute the amplitude to create a pair of χ quanta. Because of the nontrivial background for χ as $t \rightarrow +\infty$, we must redefine the creation/annihilation operators

as $\tilde{d} = d - \beta$. This is the analogue of canceling the tadpole in the string theory. The β are also determined by the condition that the amplitude to create a single particle must vanish (tadpole cancellation):

$$\langle vac, t \rightarrow +\infty | \sqrt{2\omega} \tilde{d} S | vac, t \rightarrow -\infty \rangle = 0. \quad (22)$$

Now the amplitude to create a single pair of χ quanta with momenta (k_3, k_4) is

$$A[vac \rightarrow 2] = \langle vac, t \rightarrow +\infty | \sqrt{2\omega_3} \tilde{d}(k_3) \times \sqrt{2\omega_4} \tilde{d}(k_4) S | vac, t \rightarrow -\infty \rangle, \quad (23)$$

where S is the S -matrix. Utilizing the fact that the amplitude to create a single particle must vanish and expanding $A[vac \rightarrow 2]$ to second order, we find after making use of conservation of energy and momentum

$$A[vac \rightarrow 2] = \int d\tilde{k}_1 \int d\tilde{k}_2 \alpha(k_1) \alpha(k_2) \langle 0 | \sqrt{2\omega_3} d(k_3) \times \sqrt{2\omega_4} d(k_4) S a^\dagger(k_1) a^\dagger(k_2) | 0 \rangle + O(H_0^3). \quad (24)$$

This is a very simple result: to second order in perturbations, the amplitude to create a pair of particles is determined by the amplitude for two gravitons to convert to two χ quanta. Furthermore, at second order the result is independent of the modifications to the creation operators required to cancel the tadpole. This suggests that our string results will be reliable, even though we will not carry out the modifications to the vertex operators required to cancel the tadpole in the string case.

III. BACKGROUND

In this section we shall construct a specific example of a perturbative geometry for which our approach may be applied. In doing so we shall also uncover some of the inevitable limitations of the perturbative approach. The issue we must address is whether it makes sense to define a perturbative S -matrix on a time-dependent spacetime, due to the inevitable formation of spacelike singularities or black holes, as predicted by the singularity theorems [10,11].

A. Colliding plane waves

One of the simplest and best studied time-dependent geometries is the pp -wave. This is a solution of Einstein's vacuum equations with metric given by

$$ds^2 = 2dudv + H(u, X)du^2 + dX^i dX_i, \quad (25)$$

where $H(u, X)$ is a harmonic function $\nabla^2 H(u, X) = 0$. These solutions may easily be extended to include non-trivial dilaton and Neveu-Schwarz 2-form sources. However, for simplicity of presentation, we shall concentrate on pure vacuum solutions. We will focus on the class of "exact" plane waves for which $H(u, X) = A_{ij}(u)X^i X^j$

with $\sum_i A_{ii}(u) = 0$. All pp -wave spacetimes admit a null Killing vector $\frac{\partial}{\partial v}$ and this allows a global definition of null time. As a consequence, the natural "in" and "out" field theory vacua are chosen by decomposing modes into positive and negative frequency with respect to this time, and since this is a global definition we find that for a free field $|0\rangle_{\text{in}} = |0\rangle_{\text{out}}$ implying the absence of particle creation [26]. (This is true provided we make the conventional choice of vacua. If the plane wave geometry is not asymptotically Minkowski in past and future infinity then the reasons for this choice are less clear.)

In order to obtain a background with pair creation, we will consider a spacetime describing the collision of two plane waves. Treating the amplitude of each wave $A_{ij}^{(\alpha)}$ as small, then at the linearized level this will be described by the metric,

$$ds^2 = 2dudv + A_{ij}^+(u)X^i X^j du^2 + A_{ij}^-(v)X^i X^j dv^2 + dX^i dX_i + O(A^2). \quad (26)$$

At the linearized level, the two waves pass through each other undisturbed. The $O(A^2)$ terms will include interactions between the waves. If the amplitude of each wave is localized around a given null time, for example, if we take

$$A_{ij}^+(u) \approx A \exp(-u^2/L^2), \quad (27)$$

then at leading order the interaction region will be localized near $(|u| < L, |v| < L)$. However, at higher orders we will see that there is a long tail in the future light cone of the interaction region in which perturbations grow, eventually leading to a singularity.

Fortunately all this can be seen explicitly in a concrete example where A_{ij}^+ and A_{ij}^- are both diagonal. In this case it is straightforward to lift the linearized metric (26) to an exact nonlinear solution of Einstein's equations. To see this, first perform a linearized gauge transformation to take the metric to the form

$$ds^2 = 2dudv + (\delta_{ij} + h_{ij}(u, v))dX^i dX^j + O(A^2), \quad (28)$$

where

$$h_{ij}(u, v) = 2 \iint A_{ij}^+(u) du du + 2 \iint A_{ij}^-(v) dv dv. \quad (29)$$

This metric is a special case of a more general class of solutions given by

$$ds^2 = 2e^{\Gamma(u,v)} dudv + \alpha(u, v) \sum_i e^{2\beta_i(u,v)} dx_i^2, \quad (30)$$

where $\sum_i \beta_i = 0$. This is the form of an exact set of solutions to Einstein's equations known as Einstein-Rosen waves. They satisfy $\partial_u \partial_v \alpha = 0$ and $\partial_u (\alpha \partial_v \beta_i) + \partial_v (\alpha \partial_u \beta_i) = 0$. This form is invariant under transformations $u \rightarrow f(\bar{u})$, $v \rightarrow g(\bar{v})$ and $\exp(\Gamma) \rightarrow \exp(\bar{\Gamma})/(f_{,\bar{u}} g_{,\bar{v}})$. We may use this freedom to set $\alpha = (\bar{u} + \bar{v})\sqrt{2} = t$ and $z = (\bar{u} - \bar{v})\sqrt{2}$, then the β_i satisfy

$$\frac{1}{t} \partial_t (t \partial_t \beta_i) - \partial_z^2 \beta_i = 0. \quad (31)$$

It is straightforward to see that as $t \rightarrow 0$ the solutions to this equation behave as $\beta_i \rightarrow c_i(z) + d_i(z) \ln t$. The resulting solution describes a Kasner metric where the Kasner exponents are functions of z , consequently the generic collision of two plane waves will result in an anisotropic singularity.

The existence of a singularity should cause us no surprise; it arises as a simple consequence of the singularity theorems. Although the singularity theorems do not strictly apply here as we have no matter, compactifying the geometry on one direction gives us a theory with gravity and a scalar field which does satisfy the singularity criteria. In the present context it implies that perturbation theory will inevitably break down in a finite time period. In the traditional definition of an S -matrix theory, the ‘‘out’’ state is defined at future infinity, thus it would seem at first sight impossible to define an S -matrix on a spacetime with a singularity at future infinity.

B. Perturbative solution

Our approach to dealing with this fundamental issue is a pragmatic one, for if there exists a long period of time for which perturbation theory is valid, then the S -matrix should describe physics in that period. Let us now construct the perturbative solution to Einstein’s equations. This will enable us to check that nonlinear corrections remain small at low orders in the perturbation expansion, and to explore the breakdown of perturbation theory. To be consistent with the metric (28), we take $\beta^i \sim O(A)$, $\alpha \sim 1 + O(A^2)$ and $\Gamma \sim O(A^2)$. The perturbed metric up to and including third order in perturbations is

$$\alpha = 1 + A^2(\alpha_+(u) + \alpha_-(v)) + O(A^4), \quad (32)$$

$$\beta^i = A(\beta_+^i(u) + \beta_-^i(v)) - \frac{1}{2}A^3(\alpha_+(u)\beta_-^i(v) + \alpha_-(v)\beta_+^i(u)) + O(A^4), \quad (33)$$

$$\Gamma = -\frac{1}{4}A^2 \sum_i \beta_+^i(u)\beta_-^i(v) + O(A^4), \quad (34)$$

where

$$\alpha_+''(u) = -\frac{1}{4} \sum_i \beta_{+,u}^{i2}, \quad \alpha_-''(v) = -\frac{1}{4} \sum_i \beta_{-,v}^{i2}. \quad (35)$$

We are assuming that $\beta_+(u)$ and $\beta_-(v)$ are localized functions, for instance with a Gaussian profile $\beta_+ = \exp(-u^2/L^2)$ and $\beta_- = \exp(-v^2/L^2)$. From this it is clear that up to third order in A , β^i and Γ are both bounded functions localized near the interaction region $u = v = 0$. However, as a consequence of (35), α is not bounded. We choose the solution

$$\alpha_+(u) = -\frac{1}{4} \int_{-\infty}^u du_1 \int_{-\infty}^{u_1} \sum_i \beta_{+,u_2}^{i2}(u_2), \quad (36)$$

and similarly for $\alpha_-(v)$. From this it is clear that as $u \rightarrow +\infty$, then $\alpha_+(u) \rightarrow C_+ + D_+u$, where

$$D_+ = -\frac{1}{4} \int_{-\infty}^{\infty} du \sum_i \beta_{+,u}^{i2} \neq 0. \quad (37)$$

Trying to remove this divergence by adding the homogeneous solution $-D_+u$ causes $\alpha_-(u)$ to diverge as $u \rightarrow -\infty$. This is the first indication that perturbation theory breaks down at late times. On dimensional grounds, $D_+ \sim O(1)/L$ and so this coordinate system breaks down at $u \approx L/A^2$, $v \approx L/A^2$. Evidently by making A sufficiently small, we can push this region arbitrarily far into the causal future of the interaction region.

Remarkably, to third order in A , the growth of α is a coordinate artifact and does not reflect a genuine breakdown of perturbation theory. One may show this by removing the linear growth in u by means of the following coordinate transformation:

$$x^i \rightarrow x^i \left(1 + \frac{A^2}{2} (D_+u + D_-v) \right) + O(A^3), \quad (38)$$

$$u \rightarrow u + \frac{A^2}{4} D_+ \bar{x}^2 + O(A^3), \quad (39)$$

$$v \rightarrow v + \frac{A^2}{4} D_- \bar{x}^2 + O(A^3). \quad (40)$$

Similarly all components of the Riemann tensor are finite and supported only near $u = v = 0$. In fact we must go to fourth order in perturbations to see the first signal that perturbation theory is breaking down. An explicit calculation shows that certain components of the Riemann tensor diverge linearly for large u and v in any orthonormal frame. However, unsurprisingly this only occurs when $u \approx L/A^2$, $v \approx L/A^2$. So again we may push the inevitable breakdown of perturbation theory off to arbitrarily far in the future of the region of interest.

In what follows we shall only compute the string amplitudes to second order in the background perturbations. As we have seen, at second order the collision of two plane waves results in a localized interaction region and no pathologies occur. Thus there will be no problem interpreting the implications of our amplitudes. If we continue the amplitude calculations to higher order in perturbations, we may expect to see mildly pathological behavior associated with the eventual breakdown in perturbation theory. However, it should always be possible to separate this from the information which describes the interaction region. These above observations show the fundamental limitations of the application of the S -matrix formalism to a time-dependent geometry, these issues would not arise in a more Schrodinger-like prescription where we consider

the state at a given time, rather than the transition amplitude to a state at future infinity. This suggests that string field theory or a similar formalism may be a more appropriate way to consider time-dependent spacetimes.

IV. PAIR PRODUCTION

The first nonvanishing contribution to string pair production arises from the four-point amplitude with two incoming gravitons. In this section we discuss this amplitude for an explicit set of string states, and obtain results in accord with previous investigations on the high-energy behavior of string scattering amplitudes [14–16]. This suggests that the behavior we observe here may hold for more general excited string states as well. A detailed calculation of the relevant amplitude may be found in Appendix A.

We calculate the pair creation of “representative” string states given in oscillator notation by

$$|\epsilon; k\rangle = (N!)^{-1} \epsilon_{\mu_1 \bar{\mu}_1 \dots \mu_N \bar{\mu}_N} \left(\prod_{j=1}^N \alpha^{\mu_j} \bar{\alpha}^{\bar{\mu}_j} \right) |0; k\rangle, \quad (41)$$

where we assume that $N > 1$. This set of states includes a unique scalar state at each excitation level N , which provides the most direct comparison with field theory results.

Clearly the polarization ϵ must be symmetric under the interchange of two holomorphic indices. The physical state conditions are satisfied provided that

$$m^2 = \frac{4}{\alpha'} (N - 1), \quad (42a)$$

$$k^{\mu_j} \epsilon_{\mu_1 \bar{\mu}_1 \dots \mu_j \dots \mu_N \bar{\mu}_N} = 0, \quad \text{for all } j, \quad (42b)$$

$$\eta^{\mu_j \mu_k} \epsilon_{\dots \mu_j \dots \mu_k \dots} = 0, \quad \text{for all } j, k, \quad (42c)$$

along with similar conditions for the barred indices. These states will have unit norm provided that

$$\langle \epsilon; k | \epsilon; k \rangle = 1 \quad \text{if} \quad \epsilon^{\mu_1 \bar{\mu}_1 \dots \mu_j \dots \mu_N \bar{\mu}_N} \epsilon_{\mu_1 \bar{\mu}_1 \dots \mu_j \dots \mu_N \bar{\mu}_N} = 1. \quad (43)$$

The vertex operators corresponding to these states are given by

$$\mathcal{V}_N(k) = g_c \epsilon_{\mu_1 \bar{\mu}_1 \dots \mu_N \bar{\mu}_N} \left(\frac{2}{\alpha'} \right)^N : \left[\prod_{j=1}^N \partial X^{\mu_j} \bar{\partial} X^{\bar{\mu}_j} \right] e^{ik \cdot X}(z, \bar{z}) : \quad (44)$$

where we have included a factor of the closed string coupling g_c as is conventional.

As we are interested in comparing string results to those in field theory, we focus on the production of “long-wavelength” string states, or equivalently those whose spatial momenta are small in comparison with $1/\sqrt{\alpha'}$. Since we are also considering the production of very massive states, the relevant process is therefore one in which the pair of strings are created nearly on threshold.

In order to fix notation, we take k_{\pm} to be the momenta of the created strings, and $-k_{L,R}$ the momenta of the incoming gravitons, so that the equation for conservation of momentum is

$$k_+ + k_- + k_L + k_R = 0. \quad (45)$$

We further take the created strings to be scalar representative states of excitation levels N_{\pm} . For simplicity of exposition we take the incoming gravitons to possess momentum in the t, x plane only. Working in the center of mass frame, and focusing on the case where the outgoing strings are created nearly on threshold, it is convenient to parameterize the momenta as

$$k_+ = (m_+, 0, 0) + (\delta\omega^+, \delta k_x, \delta \vec{k}_T), \quad (46a)$$

$$k_- = (m_-, 0, 0) + (\delta\omega^-, -\delta k_x, -\delta \vec{k}_T), \quad (46b)$$

$$k_L = (-\omega, -\omega, 0) + (-\delta\omega^L, -\delta k'_x, 0), \quad (46c)$$

$$k_R = (-\omega, +\omega, 0) + (-\delta\omega^R, +\delta k'_x, 0). \quad (46d)$$

Momentum conservation and the mass-shell conditions imply that we may choose δk_x and $\delta \vec{k}_T$ to be the independent variables in the problem.

For our discussion, we will find it useful to define

$$a = N_+ - \frac{\alpha' t}{4}, \quad (47a)$$

$$b = N_- - \frac{\alpha' u}{4}, \quad (47b)$$

where t and u are the conventional Mandelstam variables. For the problem at hand, when the strings are produced with zero spatial momenta, these variables are given by

$$s = m_+^2 + m_-^2 + 2m_+ m_-, \quad (48a)$$

$$t = -m_+ m_-, \quad (48b)$$

$$u = -m_+ m_-. \quad (48c)$$

It will be important for our argument below that in the case where the pair is produced with zero spatial momentum, these variables are all $4/\alpha'$ multiplied by integers.

With this parametrization of the momenta, standard techniques in string perturbation theory lead to the leading-order term in the string S -matrix, given by

$$\begin{aligned} & \epsilon_{\mu_1 \bar{\mu}_1 \dots \mu_{N_+} \bar{\mu}_{N_+}}^+ \epsilon_{\nu_1 \bar{\nu}_1 \dots \nu_{N_-} \bar{\nu}_{N_-}}^- \epsilon_{\sigma \bar{\sigma}}^L \epsilon_{\tau \bar{\tau}}^R (N_+!)^{-1} (N_-!)^{-1} \\ & \times \frac{16\pi i}{\alpha'} g_c^2 \left(\frac{\alpha'}{2} \right)^{N_+ + N_-} \\ & \times \prod_{j=1}^{N_+} k_R^{\mu_j} k_R^{\bar{\mu}_j} \prod_{k=1}^{N_-} k_R^{\nu_k} k_R^{\bar{\nu}_k} \eta^{\sigma\tau} \eta^{\bar{\sigma}\bar{\tau}} \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \right]^2 \\ & \times \frac{\sin(\pi a) \sin(\pi b)}{\sin[\pi(a+b)]}. \end{aligned} \quad (49)$$

This amplitude is then multiplied by the metric perturbations and integrated according to Eq. (12). There are two

key features of this amplitude that differ substantially from the field theory case.

The first remarkable feature of this amplitude is that the production of identical string pairs vanishes at zero spatial momentum. This follows directly from the behavior of the sine functions appearing in the amplitude (49). When $N_+ = N_- = N$, and when the spatial momentum of the created pair vanishes, then using (47) and (48) and the string mass-shell condition we find

$$a = b = 2N - 1, \quad (50)$$

and therefore the trigonometric factors vanish. For field theory, examination of the S -matrix element arising from (7) reveals that the pair production amplitude is independent of $k_{x,T}$ in the long-wavelength limit. For strings, by contrast, it appears that the pair production vanishes as a power law in the long-wavelength limit. Similar behavior occurs whenever the condition $a, b \in \mathbf{Z}$ is satisfied, which occurs for pairs of excitation levels such as

$$(N_+, N_-) = (2, 5), (2, 10), (2, 17), (3, 9), \dots \quad (51)$$

in addition to the $N_+ = N_-$ cases corresponding to the creation of identical pairs. When N_{\pm} are such that $a \notin \mathbf{Z}$ and $b \notin \mathbf{Z}$, then the production of these string pairs are independent of spatial momentum in the long-wavelength limit, similar to the field theory case.

This behavior is a consequence of some fundamental features of string theory. Recall that the open string Veneziano amplitude is essentially uniquely determined by the presence of poles corresponding to excited string states, as well as world sheet duality. The closed string amplitudes are in a sense products of open string amplitudes, with additional sine factors to ensure that all of poles that occur in the product are simple ones. Again, the closed string amplitudes are essentially uniquely determined by basic features of the theory. Thus different sine factors are zero when the s , t or u Mandelstam variables correspond to a physical string state and compensate for some of the poles arising from the gamma functions in the amplitude. In our case, when the string pairs are produced with zero spatial momentum, the s Mandelstam variable corresponds to an on-shell string state, but the t and u variables are the negative of on-shell values. Thus the sine functions have zeros, but now the gamma functions have no poles, and so the amplitude vanishes. Of course, this argument requires that we establish exactly some additional integers appearing in the arguments to the gamma functions, and this is done in Appendix A. The vanishing of this amplitude holds not only for the leading term in the S -matrix, but for terms of all orders in k appearing in this amplitude.

The second key difference between the string and field theory pair production amplitudes is an exponential suppression of the string pair production rate. To explore this feature, we consider the case where excitation level of both strings is approximately N , in which case $a \sim b \sim 2N$

from (50). Applying Stirling's approximation to the gamma function factors in (49) we find

$$\frac{\Gamma(2N)^4}{\Gamma(4N)^2} = \frac{2\pi}{N} 2^{-8N} \left(1 + \frac{1}{8N} + \dots\right) \quad N \gg 1. \quad (52)$$

Therefore, the string amplitude falls off exponentially with m^2 , in contrast to the polynomial dependence of the field theory result. This behavior is characteristic of hard scattering string processes. As we have remarked earlier, typically we expect pair production rates in field theory to fall off exponentially with mass. However, recall that when studied with the S -matrix approach employed herein, there are actually two effects at play. The first is the structure of the field theory amplitude connecting the incoming gravitons to the outgoing particles. The second is the number of gravitons, or Fourier coefficients of the metric perturbations. In field theory, the former behaves polynomially with mass, while the second drops off exponentially. For strings on the same background (that is, with the same Fourier coefficients for metric perturbations) both factors fall off exponentially. Thus fewer pairs are produced.

Again, this may be understood using simple string physics. In studies of string amplitudes, one of the kinematic regions of interest [14–16] is the so-called ‘‘hard scattering’’ limit

$$s \rightarrow \infty, \quad t/s \text{ fixed}. \quad (53)$$

This is precisely the limit of interest when we look at particles of the same mass and look at the limit in which $m \rightarrow \infty$. In the hard scattering limit, string amplitudes behave as

$$\text{amplitude} \sim e^{-sf(\theta)}, \quad (54)$$

where $f(\theta) > 0$. This is precisely the behavior observed in our results. It indicates that, just as the scattering of strings is softer than that of point particles at high energy, so too the pair production of strings is reduced.

V. CONCLUSIONS

In this work we have studied the pair production of excited strings in a time-dependent background. The specific background we consider consists of two colliding plane gravitational waves. Through studying the singularity structure of this background, as well as the nature of corrections needed to cancel tadpole diagrams, we have concluded that reliable results may be obtained through standard S -matrix methods at second order in the gravitational wave amplitude. Our calculations revealed two essential differences between the field and string theory results. The pair production of identical string states is suppressed for certain pairs of excitation numbers of the outgoing strings; most significantly, the production of identical string states vanishes in the infinite wavelength limit. In addition, the overall production of strings is sup-

pressed relative to point-particle results, which may be viewed as a consequence of the mild hard scattering behavior of string amplitudes.

Our results are especially relevant to questions regarding the pair production of strings in cosmological spacetimes. In particular, these results may have implications for one motivation for the current work, in which the pair production of strings is studied using an effective field theory approximation [1–3]. The suppression of production of excited string states suggests that, despite the exponentially growing Hagedorn density of states, effective field theory provides an overestimate of the production of individual string states.

Of course, we have established these results using techniques that are reliable only when spacetime is nearly Minkowski. Nevertheless, our findings suggest phenomena that may also be present in the strong field regime. It would be quite interesting to know if these effects persist in more highly curved, time-dependent backgrounds. If so, it would be an example of a uniquely stringy signature of relevance to cosmology, providing further clues to the role of quantum gravity in the early universe.

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APPENDIX A: $gg \rightarrow NM$

Standard techniques in string perturbation theory give the form of the four-point string S -matrix amplitude as

$$\mathcal{M} = \frac{8\pi i}{\alpha' g_c^2} \langle : c(z_+) \tilde{c}(z_+) \mathcal{V}_{k_+}^+(z_+) :: c(z_-) \tilde{c}(z_-) \times \mathcal{V}_{k_-}^-(z_-) :: c(z_L) \tilde{c}(z_L) \mathcal{V}_{k_L}^L(z_L) :: \mathcal{V}_{k_R}^R(z_R) : \rangle, \quad (\text{A1})$$

where the fact that this expression is to be integrated over the unfixed vertex operator position is understood. We have also suppressed the momentum conservation factor, so that the full S -matrix is

$$S_4(k_+, k_-, k_L, k_R) = (2\pi)^{26} \delta^{26}(k_+ + k_- + k_L + k_R) \mathcal{M}. \quad (\text{A2})$$

In this work, we take $\mathcal{V}_{k_+}^+(z_+)$ and $\mathcal{V}_{k_-}^-(z_-)$ to be representative states at level N_+ and N_- , respectively, and $\mathcal{V}_{k_L}^L(z_L)$ and $\mathcal{V}_{k_R}^R(z_R)$ to be the massless graviton states. We fix the three vertex operator positions $z_+ = 0$, $z_- = 1$ and $z_L = \infty$. Next, using momentum conservation and the transverse property of the polarization tensors of each state, it is possible to put the matrix element in the form

$$\begin{aligned} & \left\langle : \prod_{j=1}^{N_+} \left(\partial X^{\mu_j} + \frac{i\alpha'}{2} \left[\kappa^{\mu_j} + \frac{k_R^{\mu_j}}{z} \right] \right) : \right. \\ & \times : \prod_{k=1}^{N_-} \left(\partial X^{\nu_k} - \frac{i\alpha'}{2} \left[\kappa^{\nu_k} + \frac{k_R^{\nu_k}}{1-z} \right] \right) : \\ & \times : \left(\partial X^\sigma + \frac{i\alpha'}{2} [z\kappa^\sigma - k_-^\sigma] \right) : \\ & \times : \left(\partial X^\tau - \frac{i\alpha'}{2} \left[\frac{\kappa^\tau}{z} - \frac{k_-^\tau}{z(1-z)} \right] \right) : \left. \right\rangle \\ & \times z^{\alpha'k_+ \cdot k_R/2} (1-z)^{\alpha'k_- \cdot k_R/2}, \quad (\text{A3}) \end{aligned}$$

multiplied by its complex conjugate from the antiholomorphic sector, multiplied by an overall factor

$$\frac{8\pi i g_c^2}{\alpha'} \left(\frac{2}{\alpha'} \right)^{N+M+2} \quad (\text{A4})$$

and of course the appropriate polarization tensor for each state. In this expression we have defined $\kappa^\sigma = k_+^\sigma + k_-^\sigma$. This expression for the amplitude results from evaluating the ghost path integral and contractions between the ∂X and e^{ikX} terms, as well as between two e^{ikX} terms. The remaining contractions are those between two ∂X terms.

Given this pattern of contractions, each possible contraction of operators results in an integral of the form

$$\begin{aligned} & \int_{\mathcal{C}} z^{a-1+m_1} \bar{z}^{a-1+n_1} (1-z)^{b-1+m_2} (1-\bar{z})^{b-1+n_2} d^2z \\ & = 2\pi \frac{\Gamma(a+m_1)\Gamma(b+m_2)\Gamma(1-a-b-n_1-n_2)}{\Gamma(a+b+m_1+m_2)\Gamma(1-a-n_1)\Gamma(1-b-n_2)}. \quad (\text{A5}) \end{aligned}$$

For $m_1, m_2, n_1, n_2 \in \mathbf{Z}$. We can simplify our discussion by focusing on the leading terms in this expression. Our concern in the present work is on the production of long-wavelength, highly excited string states. Thus, we parametrize the momenta in the problem as

$$k_+ = (m_+, 0, 0) + (\delta\omega^+, \delta k_x^+, \delta \vec{k}_T^+), \quad (\text{A6a})$$

$$k_- = (m_-, 0, 0) + (\delta\omega^-, \delta k_x^-, \delta \vec{k}_T^-), \quad (\text{A6b})$$

$$k_L = (-\omega, -\omega, 0) + (-\delta\omega^L, -\delta k_x^L, -\delta \vec{k}_T^L), \quad (\text{A6c})$$

$$k_R = (-\omega, +\omega, 0) + (-\delta\omega^R, -\delta k_x^R, -\delta \vec{k}_T^R), \quad (\text{A6d})$$

where we assume that $\delta k, \delta\omega \ll m_\pm$. We focus on the case where

$$0 = \delta \vec{k}_T^L = \delta \vec{k}_T^R, \quad (\text{A7a})$$

$$\delta k_x^- = \delta k_x^+ = -\delta k_x^-. \quad (\text{A7b})$$

Now applying the mass-shell conditions and momentum conservation, we find $\omega = \frac{m_+ + m_-}{2}$ and furthermore, $\delta \vec{k}_T^+ = \delta \vec{k}_T^- = -\delta \vec{k}_T^L$, and $\delta k_x^+ = \delta k_x^- = -\delta k_x^R$. We can take the two remaining free variables to be δk_x and $\delta \vec{k}_T$. The mass-shell conditions further imply

$$\delta\omega^\pm = \frac{(\delta k_x)^2 + (\delta \vec{k}_T)^2}{2m_\pm} + \mathcal{O}(k_{x,T}^4), \quad (\text{A8})$$

and $\delta\omega^L = \delta k'_x$, $\delta\omega^R = \delta k'_x$. So now we can reexpress all variables in terms of δk_x and $\delta \vec{k}_T$, by noting $\delta k'_x = \frac{\delta\omega^+ + \delta\omega^-}{2}$. Now returning to the expression (A5), we can see that

$$a - 1 = (\alpha'/2)k_+ \cdot k_R = N_+ - 1 + \Delta + \delta_{+R}, \quad (\text{A9a})$$

$$b - 1 = (\alpha'/2)k_- \cdot k_R = N_- - 1 + \Delta + \delta_{-R}, \quad (\text{A9b})$$

with $\Delta = [(N_+ - 1)(N_- - 1)]^{1/2}$ and

$$\delta_{-R} = \frac{\alpha'}{2} \left(\frac{m_-}{2} \delta\omega^+ + \left[\frac{m_+ + 2m_-}{2} \right] \delta\omega^- - \omega \delta k_x \right), \quad (\text{A10a})$$

$$\delta_{+R} = \frac{\alpha'}{2} \left(\frac{m_+}{2} \delta\omega^- + \left[\frac{m_- + 2m_+}{2} \right] \delta\omega^+ + \omega \delta k_x \right). \quad (\text{A10b})$$

At this point we have all of the information we need to compute each term in the amplitude. In order to find the dominant term, it suffices to examine each of the possible contractions between momentum vectors and polarization tensors. These are given by $(k_R \cdot e^+, k_R \cdot e^-) = \mathcal{O}(\sqrt{\alpha'} m)$ and $(\kappa \cdot e^+, \kappa \cdot e^-, \kappa \cdot e^L, k_- \cdot e^L, \kappa \cdot e^R, k_- \cdot e^R) = \mathcal{O}(\sqrt{\alpha'} \delta k)$. Thus it is seen that the term with the greatest number of k_R contracted with polarization tensors will be dominant. It will be convenient to rewrite the integral (A5) as

$$\begin{aligned} & \pi \frac{\Gamma(a + m_1)\Gamma(b + m_2)\Gamma(1 - a - b - n_1 - n_2)}{\Gamma(a + b + m_1 + m_2)\Gamma(1 - a - n_1)\Gamma(1 - b - n_2)} \\ &= \frac{\sin[\pi(a + n_1)]\sin[\pi(b + n_2)]}{\sin[\pi(a + b + n_1 + n_2)]} \\ & \times \frac{\Gamma(a + m_1)\Gamma(b + m_2)}{\Gamma(a + b + m_1 + m_2)} \frac{\Gamma(a + n_1)\Gamma(b + n_2)}{\Gamma(a + b + n_1 + n_2)}. \quad (\text{A11}) \end{aligned}$$

From our discussion above, the leading term will have the largest possible number of k_R contractions, implying $m_1 = n_1 = -N_+$ and $m_2 = n_2 = -N_-$. (This depends somewhat on the pattern of contractions involving the graviton polarization tensors, but this turns out to be unimportant in the final result.) Thus we find the value of the integral to be

$$\begin{aligned} & \frac{\sin[\pi(\Delta + \delta_{+R})]\sin[\pi(\Delta + \delta_{-R})]}{\sin[\pi(2\Delta + \delta_{+R} + \delta_{-R})]} \\ & \times \left[\frac{\Gamma(\Delta + \delta_{+R})\Gamma(\Delta + \delta_{-R})}{\Gamma(2\Delta + \delta_{+R} + \delta_{-R})} \right]^2. \quad (\text{A12}) \end{aligned}$$

From this it is possible to work backwards and obtain the full S -matrix expression, although we will not require the full amplitude.

A key feature of this amplitude is its behavior when $\Delta \in \mathbf{Z}$. The leading term derived above clearly vanishes as

$\delta_{\pm R} \rightarrow 0$ in this case, due to the structure of the zeros of the sine functions. While we have only displayed the leading-order term above, this behavior in fact holds for all of the terms contributing to this amplitude. This can be seen from the structure of the integral over the complex plane used to derive (A12). Regardless of the pattern of contractions corresponding to a given term, the arguments to all of the gamma functions appearing in (A12) are positive. Since the gamma function has no zeros, and only has poles for negative values of the argument, the amplitude must vanish when $\Delta \in \mathbf{Z}$ and $\delta_{\pm R} \rightarrow 0$.

APPENDIX B: PAIR PRODUCTION VIA THE S -MATRIX

In this appendix we carry out explicit calculations of the pair production rate using both S -matrix and Bogoliubov coefficient techniques. This provides a check that, while the two methods differ, the end result is precisely the same (for a related analysis, see also [27]). We study a minimally coupled scalar field in $D \neq 2$ spacetime dimensions, with action

$$S_\phi = -\frac{1}{2} \int (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \sqrt{-g} d^D x. \quad (\text{B1})$$

Our gravitational background is defined by

$$g_{\mu\nu} = a(t)^2 \eta_{\mu\nu}, \quad (\text{B2})$$

where the ‘‘scale factor’’ $a(t)$ is an arbitrary function of time. In order to better make contact with the examples considered in this work, we will assume that $a(t)$ is unity, except for a localized ‘‘bump’’ of small amplitude near $t = 0$. Thus, there will be well-defined ‘‘in’’ and ‘‘out’’ regions where the geometry is Minkowski space.

For future convenience we define a field variable ψ , given by

$$\phi(t, x_j) = a(t)^{1-D/2} \psi(t, x_j). \quad (\text{B3})$$

We Fourier expand ψ along the spatial coordinates, so $\psi(t, x_j) = \psi_k(t) e^{ik_j x^j}$, and define $g(t) = a^{1/n}$, $n = \frac{1}{D/2-1}$, and

$$\Delta(t) = (g(t)^{2n} - 1)m^2 - \frac{\ddot{g}}{g}. \quad (\text{B4})$$

With these replacements, the action for ψ becomes

$$S_\psi = -\frac{1}{2} \int (-\dot{\psi}_k^2 + \omega_k^2 \psi_k^2 + \Delta(t) \psi_k^2) \frac{d^{D-1}k}{(2\pi)^{D-1}} dt, \quad (\text{B5})$$

where $\omega_k^2 = m^2 + k^2$ as usual. The equation of motion for ψ_k is

$$\ddot{\psi} + [\omega_k^2 + \Delta(t)]\psi = 0. \quad (\text{B6})$$

1. Bogoliubov coefficient calculation

Now, we will calculate precisely what the β Bogoliubov coefficient is in the theory described in the previous section. Because the perturbation to the action is localized in time, the canonical Minkowski vacuum is a natural choice for the “in” and “out” regions, and so

$$\begin{aligned} u_j(t, x) &= \bar{u}_j(t, x) = e^{-i\omega t + ik_j x^j}, \\ u_j^*(t, x) &= \bar{u}_j^*(t, x) = e^{+i\omega t - ik_j x^j}. \end{aligned} \quad (\text{B7})$$

We now consider the following (approximate) solution to the equation of motion for ψ :

$$\psi_k(t) = \frac{1}{\sqrt{2\omega(k)}} (e^{-i\omega_k t} + \beta_{k,-k}(t) e^{i\omega_k t}) \quad (\text{B8})$$

along with the boundary condition that $\beta_{k,-k}(t) \rightarrow 0$ as $t \rightarrow -\infty$. This boundary condition ensures that $\psi_k(t)$ is a pure positive-frequency mode in the “in” region. Using the definitions above, it is clear that $\beta_{k,-k}(t)$ at $t = +\infty$ is the Bogoliubov $\beta_{k,-k}$ coefficient.

Substituting our solution (B8) into the wave equation, and making the assumption that $\beta \ll 1$, we find

$$\ddot{\beta}_{k,-k} + 2i\omega_k \dot{\beta}_{k,-k} = -\Delta(t) e^{-2i\omega_k t}. \quad (\text{B9})$$

This equation has the solution

$$\begin{aligned} \beta_{k,-k}(t) &= - \int_{t'=-\infty}^t e^{-2i\omega_k t'} \int_{t''=-\infty}^{t'} \Delta(t'') dt' dt'' \\ &\rightarrow \frac{i}{2\omega_k} \Delta(2\omega_k), \end{aligned} \quad (\text{B10})$$

where in the last step we have taken $t \rightarrow +\infty$. Note that our derivation is self-consistent even in $\Delta(t)$ is large; we have only assumed that $\beta_k(t) \ll 1$ and thus we only need require that $\Delta(\omega)/\omega$ is small.

To compare with the S -matrix calculation, we must switch to the Lorentz-invariant particle states, defined by

$$|k\rangle = \sqrt{2\omega_k} a_k^+ |0\rangle. \quad (\text{B11})$$

When this is done, one arrives at the invariant matrix element describing the particle production process,

$$\mathcal{M}(|0\rangle \rightarrow |k, -k\rangle) = -\Delta(2\omega_k). \quad (\text{B12})$$

We will see in the next section how precisely the same result is obtained using conventional field theory techniques.

2. S-matrix calculation

The S -matrix element may be calculated using the rules of standard flat-space quantum field theory. The calculation is much simpler than using the Bogoliubov coefficient method. In quantum field theory, if we have a perturbation to the action $\delta S[\psi]$, then

$$S = \int_{vol} \langle 0 | i \delta S[\psi] | k_1 k_2 \rangle, \quad (\text{B13})$$

where we “contract” any ψ appearing in the perturbation according to the rule,

$$\psi(x) |k_1\rangle = e^{-ik_1 \cdot x} |0\rangle. \quad (\text{B14})$$

In our situation, we have $\delta S = -\Delta(t)\psi^2/2$. Again, after including the factor of 2 from summing over the two possible contractions with the final state momenta, we find that,

$$S = -i \int e^{-i(k_1+k_2) \cdot x} \Delta(t) d^D x, \quad (\text{B15})$$

which is precisely the Fourier transform of $\Delta(t)$. Imposing momentum conservation, we find the invariant matrix element,

$$\mathcal{M}(|0\rangle \rightarrow |k, -k\rangle) = -\Delta(2\omega_k), \quad (\text{B16})$$

which is precisely the one found by other means in the previous section.

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