

Graviton mass or cosmological constant?

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To describe a massive graviton in 4D Minkowski space-time one introduces a quadratic term in the Lagrangian. This term, however, can lead to a readjustment or instability of the background instead of describing a massive graviton on flat space. We show that for all local 4D Lorentz-invariant mass terms Minkowski space is unstable. The instability can develop in a time scale that is many orders of magnitude shorter than the inverse graviton mass. We start with the Pauli-Fierz (PF) term that is the only local mass term with no ghosts in the linearized approximation. We show that nonlinear completions of the PF Lagrangian give rise to instability of Minkowski space. We continue with the mass terms that are not of a PF type. Although these models are known to have ghosts in the linearized approximations, nonlinear interactions can lead to background change in which the ghosts are eliminated. In the latter case, however, the graviton perturbations on the new background are not massive. We argue that a consistent theory of a massive graviton on flat space can be formulated in theories with extra dimensions. They require an infinite number of fields or nonlocal description from a 4D point of view.

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I. INTRODUCTION

One expects that a massive state mediates the Yukawa interaction at distances larger than its Compton wavelength.¹ To describe a massive particle one adds a quadratic in fields term to the Lagrangian. This guarantees that at least in classical theories with no gravity large distance interactions are of a Yukawa type. Likewise, to describe a massive graviton in 4D Minkowski space-time one would introduce a quadratic mass term. However, this term could lead to a change of the gravitational background, instead of describing a massive graviton on flat space. Below we will discuss local Lorentz-invariant quadratic “mass terms” for gravity in 4D. We will argue that in all the cases of physical relevance the mass term leads to the instability of Minkowski space. The instability can set in within a time scale that is arbitrarily small compared to the inverse graviton mass. The resulting theory either has no stable vacuum at all, or the original Minkowski space is readjusted to a curved background. Along the way, we draw attention to an interesting phenomenon: a theory that has a ghost in the linearized approximation can become ghost free due to nonlinear interactions that lead to the readjustment of the gravitational background.

We will argue that a natural way to have a massive graviton on flat background is to invoke certain theories with infinite-volume extra dimensions. The latter have an infinite number of states in the spectrum at an arbitrarily low-energy scale. From the point of view of 4D they are nonlocal field theories.

The paper is organized as follows: in Sec. II we start with the Pauli-Fierz (PF) massive gravity. We show

that the PF term, and any of its nonlinear polynomial completion, gives rise to instabilities of flat space. We find new cosmological solutions in empty space that describe the instability of the Minkowski background and we discuss the time scale in which the instability can set in.

In Sec. III we discuss non-PF quadratic terms. These terms are traditionally discarded since they give rise to ghosts already in the linearized approximation. We show that a reparametrization invariant nonlinear completion of at least one of these models gives rise to a background change. There are no ghosts on a new background, however, a graviton does not mediate the Yukawa potential at large distances.

In light of our findings, in Sec. IV we comment on the strong coupling problem in massive gravity. The issue should be addressed on a stable (or long-lived metastable) background if such a background exists. A conclusion on whether the strong coupling problem is present or not depends in general on the properties of the background itself. For 4D PF gravity, however, we could not find any convincing arguments in favor that the theory possesses a stable (or very long-lived) ground state in which the problem could be studied.

Finally in Sec. V we discuss how a model of massive gravity on a flat background can be obtained in theories with infinite-volume extra dimensions. We emphasize certain distinctive features that enable these models to accommodate a flat space massive graviton. A brief summary of main results is given in Sec. VI.

II. PAULI-FIERZ GRAVITY

In the linearized approximation the PF term is introduced as follows [1]:

¹An exception is a massive photon in the Maxwell-Chern-Simons theory in $(2 + 1)$ dimensions where the potential is powerlike.

$$\begin{aligned}
S_{\text{PF}}^L = & \frac{M_{\text{Pl}}^2}{8} \int d^4x (\partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} - 2\partial_\alpha h_\mu^\alpha \partial_\beta h^{\mu\beta} \\
& + 2\partial_\alpha h_\mu^\alpha \partial^\mu h_\beta^\beta - \partial_\alpha h_\mu^\mu \partial^\alpha h_\nu^\nu) \\
& - \frac{M_{\text{Pl}}^2 m_g^2}{8} \int d^4x (h_{\mu\nu}^2 - (h_\mu^\mu)^2), \quad (1)
\end{aligned}$$

where m_g stands for the graviton mass and $h_{\mu\nu}$ denotes graviton perturbation on a flat background. The first term in the parentheses on the right-hand side (r.h.s.) of (1) is the linearized Einstein-Hilbert term.

The action (1) describes a consistent theoretical model of a free massive spin-2 state with five physical degrees of freedom. This can easily be seen by making the reparametrization invariance of this action manifest using Stückelberg's method. This action could be useful for, e.g., a spin-2 glueball in QCD with $m_g \sim 2$ GeV and $M_{\text{Pl}} \rightarrow M_{\text{QCD}} \sim 1$ GeV; however, the action (1) cannot describe observable gravity. This is primarily because of the van Dam-Veltman-Zakharov (vDVZ) discontinuity [2,3] (see also [4]), and because (1) does not contain nonlinear gravitational interactions that are being measured in gravity observables. The nonlinearities could cure the vDVZ discontinuity problem as well [5]. Therefore, a nonlinear completion of the action (1) is needed. However, this in general leads to problems [6]. The simplest strategy is to complete the kinetic term in (1) to a nonlinear Einstein-Hilbert term:

$$S_m = -\frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{g} R(g) - \frac{M_{\text{Pl}}^2 m_g^2}{8} \int d^4x (h_{\mu\nu}^2 - (h_\mu^\mu)^2), \quad (2)$$

where we define $h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu})$. Note that with this definition of h the mass term in the action (2) is regarded as an exact term and not as a leading term in a small h expansion. Furthermore, higher powers in h could be arbitrarily added to the mass term since there is no principle, such as reparametrization invariance, that could fix the arbitrariness in choosing those terms. For definiteness, we can assume that the indices in the mass term are raised and lowered by $\eta_{\mu\nu}$; using $g_{\mu\nu}$ instead, would result in differences that appear only in the cubic and higher orders in h which are ambiguous anyway.

One may attempt to find a more satisfactory than (2) completion of the PF term by expressing $h_{\mu\nu}$ in terms of the invariant curvatures in a certain nonlocal way. However, because of the specifics of the PF term this is not conceivable. Indeed, consider the equation of motion that follows from the variation of (1). Let us take a derivative of both sides of the equation. Since the Einstein tensor is identically conserved, this gives a new constraint arising from the mass term. This is an analog of the Proca condition for massive gauge fields. In the case of the PF term the Proca condition reads

$$\partial^\mu h_{\mu\nu} = \partial_\nu h_\alpha^\alpha. \quad (3)$$

An important fact is that for any field that satisfies (3) the Ricci scalar is zero in the linearized approximation. Hence, the field h cannot be expressed via the Ricci scalar. Let us now look at the Ricci tensor. For the fields that satisfy (3) we find

$$R_{\mu\nu} = \hat{P}_{\mu\alpha} h_\nu^\alpha, \quad \hat{P}_{\mu\alpha} \equiv \partial^2 \eta_{\mu\alpha} - \partial_\mu \partial_\alpha. \quad (4)$$

Hence, the Ricci tensor and h are related by a projector operator \hat{P} which is not invertible for general configurations. Therefore, $h_{\mu\nu}$ cannot be expressed via the Ricci tensor either.

We continue with the action (2). Once this completion is adopted problems emerge. On a flat background the nonlinear theory (2) describes a massive spin-2 state with 5 degrees of freedom plus a ghostlike spin-0 state that appears only on a nonlinear level [6]. The Hamiltonian for h is not positive semidefinite [6]. This indicates that Minkowski space should be unstable.

At first glance one might think that the typical time scale for the instability should be of the order of the inverse graviton mass, since this is the only new dimensionful parameter in the Lagrangian. If this were true, then the theory with the graviton mass as small as $m_g \sim H_0 \sim 10^{-42}$ GeV would have been almost stable for all practical purposes. However, as we will show shortly, this is not so. Below we derive exact *empty-space solutions* of PF gravity that take the background away from Minkowski space and show that the time scale for setting in this instability can be arbitrarily short.

To exhibit the instability of Minkowski space it is enough to focus on the following restricted class of metrics:

$$ds^2 = N^2 dt^2 - a^2 d\mathbf{x}^2, \quad (5)$$

where $N = N(t)$ and $a = a(t)$ are some functions of the time coordinate. The Lagrangian for these configurations takes the form (below we set $M_{\text{Pl}} = 1$):

$$L = -N^{-1} a \dot{a}^2 - m_g^2 F, \quad (6)$$

where $F = F(a, N)$ denotes a general mass term. From the above Lagrangian we calculate the Hamiltonian and find the conserved energy:

$$E = -N^{-1} a \dot{a}^2 + m_g^2 F. \quad (7)$$

The function N should also satisfy a constraint

$$N^{-2} a \dot{a}^2 = m_g^2 \partial_N F. \quad (8)$$

It is convenient to make a change of variables $N dt \rightarrow dt$ (note that this is not a coordinate transformation under

which the massive theory is invariant, this is just a formal change of variables used for technical simplifications). In terms of the new time variable the constant energy reads

$$E = -Na\dot{a}^2 + m_g^2 F, \quad (9)$$

and the constraint takes the form

$$a\dot{a}^2 = m_g^2 \partial_N F. \quad (10)$$

Let us now turn to the PF mass term. For the metric (5) the PF term takes the form $F = (N^2 + a^2 - 2)(1 - a^2)/4$. The corresponding conserved energy is

$$m_g^2 E = -\frac{a^2 \dot{a}^4}{1 - a^2} - \frac{m_g^4}{4} (2 - a^2)(1 - a^2). \quad (11)$$

The latter expression can be rewritten as follows:

$$a^2 \dot{a}^4 + m_g^2 (1 - a^2)(E + m_g^2 (2 - a^2)(1 - a^2)/4) = 0. \quad (12)$$

Minkowski space, that is $a = 1$, $N = 1$, is certainly a solution of the above equation. However, perturbations take the solution far away from Minkowski background, as we will see below.

To see the instability of Minkowski space manifestly we need to study perturbations for both a and N near the point $a = 1$, $N = 1$. For this, let us take $E = -m_g^2 \epsilon/2$, with small positive $0 < \epsilon \ll 1$. For small $\delta \equiv 1 - a$ we find

$$\delta^4 = m_g^4 \delta (\epsilon - \delta). \quad (13)$$

The above equation describes oscillations of the delta between 0 and ϵ . On the other hand, one can show that for $\delta \neq 0$

$$N^2 = (\epsilon - \delta)/\delta. \quad (14)$$

When $\delta = \epsilon/2$, we have $N = 1$. That is the solution passes close to the Minkowski region. However, at the turning points, $N \rightarrow 0$ or $N \rightarrow \infty$, the system moves away from Minkowski space.² Consider the regime when $\delta \ll \epsilon$. The time dependence of the small δ takes the form $\delta \sim (m_g t)^{4/3} \epsilon^{1/3}$, and N scales as follows: $N^2 \sim \epsilon^{2/3}/(m_g t)^{4/3} \gg 1$. Thus, a small departure from $a = 1$

²We would like to point out again that the nonlinear theory lacks reparametrization invariance, and, hence, different choices of coordinates could lead to different physical spaces. Our interval is defined by (5), and what we call Minkowski space corresponds to the point $a = 1$, $N = 1$. Note that geodesic equations for matter fields are not modified as compared to the standard GR and, therefore, external sources moving along the geodesics would not distinguish between the coordinate systems. However, from the point of view of pure PF gravity, different coordinates can be physically different.

leads to a large deviation from the $N = 1$ point (i.e., from Minkowski space). Note that a typical time scale for the system to complete one cycle between the turning points is $T \sim \sqrt{\epsilon}/m_g$. The latter can be arbitrarily small. Therefore, the instability of Minkowski space could develop almost instantaneously. The appearance of a new short time scale is due to the integration constant (i.e., the energy E) which does not enter as a parameter in the Lagrangian.

There are two comments to be made here: (1) The scale of the instability—the frequency of the oscillations of the metric $\omega_g \sim m_g/\sqrt{\epsilon}$ —could be smaller than the UV cutoff of the theory $\Lambda_3 \sim (m_g^2 M_{\text{pl}})^{1/3}$ [7]. By tuning the value of ϵ the energy of the space E can be made arbitrarily close to zero, while simultaneously the frequency of the oscillations ω_g can be kept smaller than the cutoff Λ_3 by tuning the value of m_g . All the above suggests that the oscillatory instabilities are present in a low-energy effective theory [7], and as such, are unlikely to be removable by a new UV physics that can enter at Λ_3 . (2) The above instabilities are not *a priori* related to other known instabilities, e.g., a decay of a space by the bubble nucleation etc.; what we are finding are the oscillations of the metric that for arbitrarily small values of the energy create large metric perturbations.

Actually, one can show that perturbations around Minkowski space-time with negative energies exist for arbitrary (nonlinearly completed) polynomial mass terms. For nonlinear completions of the special form, when the mass term in the action is a function $f(h_{\mu\nu}^2 - (h_\mu^\mu)^2)$ this was shown in [6]. This can be generalized for an arbitrary mass term, as we have shown in the Appendix.

For the PF mass term there also exists a curious “cosmological” solution. Consider a universe with $E = -m_g^2/2$. For this case Eq. (12) simplifies and we get

$$\dot{a}^4 = m_g^4 (1 - a^2)(3 - a^2)/4. \quad (15)$$

This describes an expanding and then recollapsing universe. The early time expansion law $a = m_g t/\sqrt{2}$ corresponds to the equation of state $p = -\rho/3$. Note also that the Minkowski space, $a = 1$, $N = 1$, is formally a solution of the system (10) and (15); however, for small $\delta = 1 - a \ll 1$ the perturbation of N is huge, $N \sim 1/\sqrt{\delta}$, and the corresponding energy is negative. Therefore, small perturbations in a move the system from Minkowski space away to a collapsing universe.

III. NON-PF TERMS

We showed above that Minkowski space is unstable for the PF theory. Therefore, there is no reason to prefer the PF term over any other non-PF quadratic terms, for which it is known that ghosts appear already in the linearized approximation [8]. On the other hand, choosing non-PF terms one might hope to find a nonlinear completion for which the

ghost will be eliminated by nonlinear interactions. We will discuss this possibility below.

Let us first start with a general non-PF quadratic term³

$$\frac{M_{\text{Pl}}^2 m_g^2}{8} \int d^4x (h_{\mu\nu}^2 - a(h_\mu^\mu)^2), \quad (16)$$

where $a \neq 1$. In this case the Proca condition takes the form

$$\partial^\mu h_{\mu\nu} = a \partial_\nu h_\alpha^\alpha. \quad (17)$$

As a result, the 4D curvature in the linearized theory is not identically zero, $R \sim (a-1)\partial^2 h_\alpha^\alpha$ (unlike the case of the PF term). However, for $a \neq 1$ the term (16) gives rise to a ghost. The easiest way to see this is to focus on the scalar ϕ where $h_{\mu\nu} = \partial_\mu \partial_\nu \phi$. For this scalar the integrand in (16) reads

$$(1-a)(\partial^2 \phi)^2. \quad (18)$$

The energy density that follows from (18)

$$\mathcal{E} \propto (1-a)[(\partial_0^2 \phi)^2 - (\partial_i^2 \phi)^2], \quad (19)$$

is not positive definite irrespective of the sign of $(1-a)$. In terms of a propagator for ϕ , one finds a pole with a negative residue—a ghost. Is it possible to overcome this inconsistency of the theory? This question can be given a positive answer at least for a certain choice $a = 1/2$. This is due to a mechanism that we will describe briefly below (a similar mechanism was used in a higher-dimensional context to stabilize ghosts in Ref. [9]). To focus on the main idea in as simple terms as possible consider a scalar field theory in the absence of gravity:

$$\mathcal{L} = \mathcal{G}(\Phi, \chi) \partial^\alpha \Phi \partial_\alpha \Phi - V(\Phi, \chi), \quad (20)$$

where \mathcal{G} encodes nonlinear interactions of Φ , its derivatives and/or other fields collectively denoted by χ :

$$\mathcal{G}(\Phi, \chi) \equiv -\frac{1}{2} + \mathcal{O}(\Phi; \partial\Phi; \chi; \partial\chi). \quad (21)$$

The sign of the first term on the r.h.s. of (21) is such that small perturbations of Φ around $\Phi = 0$ are unstable, i.e., these perturbations have a negative signature kinetic term and are ghostlike. However, due to nonlinear interactions one can change the signature of the kinetic term (21). This can be done in a few ways:

(i) Consider an example

$$\mathcal{G}(\Phi) \equiv -\frac{1}{2} + \frac{\Phi^2}{v^2}. \quad (22)$$

Furthermore, let the potential V in (21) take the form:

$$V(\Phi) = \lambda(\Phi^2 - v^2)^2. \quad (23)$$

³This form of the mass term does not include the case when the $h_{\mu\nu}^2$ term is absent. However, this should be similar to the other generic $a \neq 1$ cases.

Then the vacuum solution is $\Phi = v$ and a small perturbation σ around the vacuum, $\Phi = v + \sigma$, acquires a kinetic term with a positive signature⁴ (as long as $|\sigma| \ll v$):

$$\left(\frac{1}{2} + \frac{2\sigma}{v} + \frac{\sigma^2}{v^2}\right)(\partial\sigma)^2 + \dots \quad (24)$$

(ii) The second example is similar to the first one, but it is due to higher derivatives. Consider

$$\mathcal{G}(\chi) \equiv -\frac{1}{2} + \frac{\partial^2 \chi}{v^3}. \quad (25)$$

Suppose that for certain dynamical reasons the χ field develops the following condensate:

$$\langle \partial^2 \chi \rangle = v^3. \quad (26)$$

This condensate leads to the “signature change” for the kinetic term of the Φ field and small perturbations of the Φ field about the correct vacuum state will have a positive sign of energy.

(iii) Finally, nonlinear interactions of a single tensor field could be a reason for the elimination of the ghost. Below we will discuss such a mechanism for a graviton. For this we will restrict ourselves to the case $a = 1/2$ in (16). This choice is somewhat special for reasons that will become clear shortly. We will also comment on the other $a \neq 1$ cases below.

Thus, we consider the linearized action:

$$\begin{aligned} S_{g\text{PF}}^L &= \frac{M_{\text{Pl}}^2}{8} \int d^4x (\partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} - 2\partial_\alpha h_\mu^\alpha \partial_\beta h^{\mu\beta} \\ &\quad + 2\partial_\alpha h_\mu^\alpha \partial^\mu h_\beta^\beta - \partial_\alpha h_\mu^\mu \partial^\alpha h_\nu^\nu) \\ &\quad - \frac{M_{\text{Pl}}^2 m_g^2}{8} \int d^4x \left(h_{\mu\nu}^2 - \frac{1}{2} (h_\mu^\mu)^2 \right). \end{aligned} \quad (27)$$

In the quadratic approximation the above action can be rewritten as

$$\begin{aligned} S_{g\text{PF}}^L &= \frac{M_{\text{Pl}}^2}{8} \int d^4x (\partial_\alpha \tilde{h}_{\mu\nu} \partial^\alpha \tilde{h}^{\mu\nu} - 2\partial_\alpha \tilde{h}_\mu^\alpha \partial_\beta \tilde{h}^{\mu\beta} \\ &\quad + 2\partial_\alpha \tilde{h}_\mu^\alpha \partial^\mu \tilde{h}_\beta^\beta - \partial_\alpha \tilde{h}_\mu^\mu \partial^\alpha \tilde{h}_\nu^\nu) \\ &\quad - \frac{M_{\text{Pl}}^2 m_g^2}{2} \int d^4x \left(-1 - \frac{\tilde{h}}{2} + \frac{1}{4} (\tilde{h}_{\mu\nu}^2 - \frac{1}{2} (\tilde{h}_\mu^\mu)^2) \right), \end{aligned} \quad (28)$$

where $\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu}$, and we used the relation:

$$\frac{1}{4} \left(h_{\mu\nu}^2 - \frac{1}{2} (h_\mu^\mu)^2 \right) = -1 - \frac{\tilde{h}}{2} + \frac{1}{4} \left(\tilde{h}_{\mu\nu}^2 - \frac{1}{2} (\tilde{h}_\mu^\mu)^2 \right). \quad (29)$$

⁴Note that the phases of the above model with $\Phi = 0$ and $\Phi = v$ can in general be disconnected from each other (superselection sectors), however, this is not a matter of our discussions.

We can imagine that the action (28) is our starting point in which matter couples to \tilde{h} in a conventional way. The actions (27) and (28) describe a free massive spin-2 state plus a massive spin-0 ghost. This could be seen by calculating a one-particle exchange amplitude between two conserved sources $T_{\mu\nu}$ and $T'^{\mu\nu}$. The momentum space amplitude of the linearized theory contains the following terms:

$$\frac{T_{\mu\nu}T'^{\mu\nu} - \frac{1}{3}TT'}{m_g^2 - p^2 - i\epsilon} - \frac{1}{6} \frac{TT'}{m_g^2 - p^2 - i\epsilon}, \quad (30)$$

with p^2 being the transfer-momentum square. The first term corresponds to an exchange of a massive spin-2 state while the second term gives rise to a repulsive interaction due to a massive spin-0 ghost. Therefore, the model (27), [or (28)] as it stands, cannot be a consistent theory of gravity.

Being motivated by the scalar field example discussed above we will add new terms to (27) and (28) to eliminate the ghost. This procedure, as we will see, also eliminates the longitudinal polarizations of a massive graviton and leads to a theory of a massless graviton on a curved background. Thus, we expect that

$$(h_{\mu\nu}^2 - \frac{1}{2}(h_{\mu}^{\mu})^2) + V(h; \eta_{\mu\nu}) \quad (31)$$

can describe a theory with no ghosts for certain choices of V . The key observation is that

$$\sqrt{|\det h_{\mu\nu}|} = (h_{\mu\nu}^2 - \frac{1}{2}(h_{\mu}^{\mu})^2) + V(h_{\mu\nu} - \eta_{\mu\nu}), \quad (32)$$

where V is a known polynomial of its argument. The above relation can be established by using an identity

$$\sqrt{|\det h_{\mu\nu}|} \equiv \sqrt{|\det(\eta_{\mu\nu} + h_{\mu\nu} - \eta_{\mu\nu})|},$$

and formally expanding it in powers of $h_{\mu\nu} - \eta_{\mu\nu}$:

$$\begin{aligned} \sqrt{|\det(\eta_{\mu\nu} + h_{\mu\nu} - \eta_{\mu\nu})|} &= \exp\left(\frac{1}{2} \text{Tr} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \right. \\ &\quad \left. \times (h_{\mu\nu} - \eta_{\mu\nu})^n\right). \end{aligned}$$

It is certainly true that V in (32) contains an infinite number of constant, linear, quadratic and higher powers of h ; nevertheless, the above procedure is a nonlinear completion for the action (28) that is written in terms of the variable $\tilde{h}_{\mu\nu}$, since the function V contains only cubic and higher powers in $\tilde{h}_{\mu\nu}$. The polynomial terms in $V(\tilde{h})$ trigger the background change for the ghost field by a mechanism similar to the one described above. The fact that this is the case is easy to understand without a term-by-term calculation of V . This is because the resulting nonlinear theory can be written in a simple way

$$S = -\frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{\tilde{h}} (R(\tilde{h}) + 2m_g^2). \quad (33)$$

The above functional is nothing but the action of a theory with nonzero cosmological constant equal to m_g^2 . It admits solutions with curved background but does not admit flat solutions. The spectrum of the theory on the curved background (either de Sitter (dS) or anti-de Sitter) has no ghosts. The graviton in (33), unlike a 4D massive spin-2 state, propagates 2 physical degrees of freedom.

The results obtained above could be also understood in the following way: Let us start with the Einstein-Hilbert action with a nonzero cosmological constant (33). Let us expand this action *formally* around a flat background, $\eta_{\mu\nu} + \tilde{h}_{\mu\nu}$. Note that we are expanding around a background that is *not* a solution of the equations of motion. Because of this we should anticipate certain inconsistencies to emerge. As we will see, the way the inconsistencies appear is very instructive, so we continue with our expansion. We truncate this expansion at the quadratic order in \tilde{h} (the covariant derivative in this expansion is just a simple derivative). The resulting theory is (28) that has the quadratic mass term in \tilde{h} , the linear term, and the constant term. We regard the resulting model as a certain free theory of \tilde{h} . Minkowski space, i.e., $\tilde{h} = 0$, is certainly not a solution of the above linearized theory (this can also be understood as an impossibility to obtain a cosmological term starting from flat background and considering consistent self-coupling requirements for the linearized action [10]). However, it is remarkable that in the linearized theory (28), the constant, linear, and quadratic nonderivative terms can be rearranged as a non-PF term with $a = 1/2$ by a formal change of variables [see action (27)]. The latter is a non-PF mass term for h on a flat background. It has a ghost in spite of the fact that the original nonlinear theory (33) was ghost free.

Two important comments are in order.

(1) The above derivation applies to the $a = 1/2$ case only. The question is whether the same conclusions remain valid for any other $a \neq 1$ case. It is certainly true that the linearized theory is unstable (has ghosts) for any $a \neq 1$. Therefore, to make sense of such models the signature change has to take place. If so, the background will also be changed. Then, we come to similar conclusions—either these models are inconsistent, or they describe curved space, but none of these models can describe a massive graviton on a flat space.

(2) So far we have been dealing with the classical effects only. However, quantum corrections can be important in discussing the issues of massive gravity. Let us start with the PF mass term again. This term is set in a classical theory by adjusting the coefficients of the $h_{\mu\nu}^2$ and $(h_{\mu\nu}^{\alpha})^2$ terms to be equal. However, there is no reason for quantum *gravitational* loops to preserve this condition after the appropriate wave-function renormalization is performed.

These coefficients are different in quantum theory and one in general is back to the $a \neq 1$ case.

A question arises why the infinite number of terms in V that we add in (31) and (32) are stable with respect to quantum corrections. The answer is that the reparametrization invariance of the complete theory (33) protects these terms from being renormalized. Therefore the procedure described for the $a = 1/2$ case is stable under loop corrections. These corrections just renormalize the wave function h , Newton's constant $G_N \equiv 1/8\pi M_{\text{pl}}^2$, "graviton mass" m_g^2 (i.e., the cosmological constant), and give rise to higher-derivative terms.

How do these arguments change for the other $a \neq 1$ models? It is clear that unless the other $a \neq 1$ models also have a reparametrization invariant completion, similar to that of the $a = 1/2$ case, any finely adjusted nonlinear addition to these models will in general be destroyed by gravitational loop effects.

IV. ON THE STRONG COUPLING PROBLEM IN MASSIVE GRAVITY

It has been known for some time that perturbative expansion in G_N breaks down in nonlinear diagrams at a scale that is parametrically lower than the UV cutoff of the theory [5] (see also [11]). This can be understood as a consequence of strongly interacting longitudinal modes of a massive graviton [7]. At the classical level, the calculations can still be performed by means of resummation of the tree-level perturbation theory in G_N , or by using a perturbative expansion in a different parameter [5,11]. However, the question whether the same can or cannot be done in full quantum PF theory remains open. If the resummation is not possible in the quantum PF theory, then there will appear higher-derivative operators in the theory that are suppressed by a phenomenologically unacceptable low scale [7].

The above results are obtained by considering perturbative expansion on a flat background. However, as we discussed above, the Minkowski background is unstable in PF gravity. Moreover, the instability of Minkowski space can set in within a time scale that can be arbitrarily short. Therefore, to understand whether the problem is truly present in the PF theory, the issue should be studied on a stable (or a long-lived metastable) ground state, if such a state exists. At the moment the existence of such a ground state is not obvious. If such a state does not exist, then the nonlinear version of the PF gravity should be discarded as an inconsistent model. The present work has nothing new to add in this regard; all we have shown is that the Minkowski space is certainly not a candidate for such a background. On the other hand, if some stable curved ground state exists, then the vDVZ discontinuity and the strong coupling problems could in principle be cured by the background curvature effects [7,12,13].

It is also instructive to mention in this regard different solutions of PF gravity that exist in the literature. One starts with an empty space and puts a static and spherically symmetric source in it. In the linear theory (1) this source produces a static potential on a flat space that has the Yukawa behavior at infinity. However, all this changes in the nonlinear theory (2) where we look for a spherically symmetric and static solution of nonlinear equations. Moreover, we require that the solution gives rise to a $1/r$ potential for distances $r \ll m_g^{-1}$, and the $\exp(-m_g r)/r$ potential at larger scales. It has been known for some time [14,15] that the solutions of massive gravity in the above two asymptotic regimes are hard to match together. Moreover, recent numerical studies [16] show explicitly that the matching is possible only at the expense of introducing a naked singularity at a finite proper distance from a completely regular source. This is certainly unacceptable. However, there exist solutions [15] for which the potential is similar to that of a de Sitter-Schwarzschild metric in the static coordinate system (in that system $g_{00} = 1 - r_g/r - \Lambda r^2$, and $g_{rr} = 1/g_{00}$).⁵ These solutions also exist in the absence of the source, i.e., when $r_g = 0$. The de Sitter curvature Λ is determined by the graviton mass and a certain integration constant $\Lambda = (m_g u)^2$, where $u > 3/4$. The presence of this arbitrary integration constant is reminiscent of an arbitrary constant E in the solutions found in Sec. II. Furthermore, unlike the solutions found in the previous section, these solutions have a smooth limit as $m_g \rightarrow 0$ [15]. The solution can be interpreted as follows. The *gravitational mass* term itself acts as a source for gravity and produces effects that are somewhat similar to those of a cosmological constant. An open question remains whether the dS-Schwarzschild solution itself is stable with respect to small perturbations. If it is stable and its curvature is bigger than m_g^2 , then there is neither the vDVZ nor the strong coupling problems in this case [7,12,13]. However, irrespective of whether the curved background is stable or not, our main conclusion holds unchanged—the PF mass term at best leads to a change of the background, but it in no way describes a flat space massive graviton.

V. HOW DO EXTRA DIMENSIONS HELP?

In conventional compactifications of theories with extra dimensions one obtains a massless graviton that is interacting with an infinite number of massive spin-2 states. In the linearized approximation the mass terms for each of

⁵More precisely, the solution in Ref. [15] was found in a different coordinate system in which the off-diagonal terms in the metric are not zero. The above solution is reducible to the static-patch dS-Schwarzschild solution by a formal change of coordinates. However, since the reparametrization invariance is absent, these two coordinate systems are not physically equivalent. Nevertheless, we will refer to these metrics as dS-Schwarzschild solutions keeping in mind the above disclaimer.

these massive spin-2 Kaluza-Klein (KK) modes have the PF form. As we argued in Sec. II, flat space is unstable for any nonlinear completion of the PF mass term. On the other hand, the original higher-dimensional theory is a reparametrization invariant model and can be shown to have no instabilities of the type obtained in Sec. II. The resolution of this seeming contradiction is in the fact that one gets an infinite number of massive spin-2 KK states upon compactification and truncation of this tower to any finite order leads to inconsistencies [17,18]. The manifest reparametrization invariance of a higher-dimensional theory is a convenient bookkeeping tool to utilize to see these properties. The reparametrization invariance at each KK level is maintained on the same KK level only in the linearized approximation. Nonlinear effects mix different KK levels under the coordinate transformations [17,18]. Hence, the consistency of the theory is achieved by means of an infinite number of four-dimensional reparametrization invariances. Any truncation of the theory to a finite number of massive spin-2 fields leads to an explicit breakdown of all the massive gauge invariances, including the ones that correspond to the massive fields that are retained in the low-energy description. As a result, in the truncated theory the problems of PF gravity will arise. Therefore, a consistent theory should maintain all the infinite number of fields.

In conventional compactifications one obtains a massless graviton. In this case, large distance gravity is indistinguishable from 4D general relativity. Our goal, however, is to present a model of a massive graviton (with no massless mode).

A *generally covariant* model that shares many properties of massive gravity, but retains all the attractive features of a higher-dimensional reparametrization invariant theory is the DGP model [19]. In five-dimensional context it described a metastable graviton with no mass. In higher-dimensional generalizations of the DGP model [20,21] the graviton has an effective mass that is much larger than its width [22,23]. Thus, the model introduces a reparametrization invariant mass term for a graviton. Such models have string theory realization [24] (see Refs. [25–32] for interesting cosmological and astrophysical studies).

Gravitational dynamics encoded in the model can be inferred both from the four-dimensional as well as $(4 + N)$ -dimensional standpoints. From the 4D perspective, gravity on the brane is mediated by an infinite number of the Kaluza-Klein modes that have no mass gap. Under conventional circumstances (i.e., with no brane kinetic term) this would lead to higher-dimensional interactions. However, the large Einstein-Hilbert term on the brane suppresses the wave functions of heavier KK modes, so that in effect they do not participate in the gravitational interactions on the brane at observable distances [33]. Only light KK modes, with masses $m_{\text{KK}} \lesssim m_g \sim 10^{-42}$ GeV, remain essential, and they collectively act as an effective

4D graviton with a typical mass of the order of m_g . At present, the $N \geq 2$ DGP models [20,21] seem to be the only consistent model of a massive graviton on flat space (the $N = 1$ model is not massive). This model has no ghosts in the linearized theory [21,34] and possesses a reparametrization invariant nonlinear action. Because of this, unlike nonlinear PF gravity, the ghosts do not appear in the nonlinear theory and instabilities of the PF gravity are not present.

The above models evade the problems of the 4D PF theory because at any low-energy scale they contain an infinite number of KK gravitons with no mass gap. In other words, these models can be thought of as nonlocal models from the 4D point of view [25].⁶ Indeed a massive graviton in 4D can be described by a nonlocal equation $(1 + m_g^2/\nabla^2)G_{\mu\nu} = T_{\mu\nu} + \dots$, where dots stand for some other terms that are needed to restore the Bianchi identities.

We also note that in $(2 + 1)$ dimensions a unitary and causal theory of a massive graviton is topologically massive gravity [36].

VI. CONCLUSIONS

Summarizing, the PF term is the only quadratic term that has no ghosts in the linearized theory. Any polynomial nonlinear completion of this term, however, gives rise to instabilities. We found empty-space solutions that manifestly show the instability of Minkowski space in PF gravity. Therefore, there is no reason to restrict ourselves to the PF term and one might start with a non-PF quadratic terms. The latter have ghosts already in the linear theory. Nevertheless, the ghosts can be eliminated by higher-derivative terms via the background rearrangement. The resulting nonlinear theory has no classical instabilities, however, it does not describe graviton mediating Yukawa interactions on flat 4D space. It is very likely that out of all candidates for “massive” local 4D theories, only one has a nonlinear completion that is radiatively stable (the $a = 1/2$ case). However, in this case the “mass” term is nothing but the cosmological term. A natural way to account for a massive graviton on flat space is to invoke theories with extra dimensions. The latter evade the problems of the 4D massive gravity because they are nonlocal theories from the 4D point of view.

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⁶As was proposed in [35], nonlocalities postulated in pure 4D theory (for whatever reasons they might appear), can solve an “old” cosmological constant problem [35], and give rise to new mechanisms for the present-day acceleration of the universe [35].

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APPENDIX

We will consider nonlinear oscillations of a massive graviton condensate, meaning that all components of the metric are functions of t only. It is convenient to use the Arnowitt-Deser-Misner parametrization of the metric:

$$ds^2 = N^2 dt^2 - \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (\text{A1})$$

Then the Lagrangian reads

$$L = \frac{1}{2}\sqrt{\gamma}N^{-1}(\dot{\gamma}_{ij}\dot{\gamma}^{ij} - (\dot{\gamma}_j^j)^2) - m_g^2 F, \quad (\text{A2})$$

where $F(N, N_i, \gamma_{ij})$ represents a general nonlinearly completed Pauli-Fierz term. The N_j constraints simply remove the N_j dependence of F :

$$F \rightarrow F(N, \gamma_{ij}). \quad (\text{A3})$$

It remains to remove the N constraint. Diagonalizing the 3-metric, $\gamma_{ij} = \text{diag}(e^{2a}, e^{2b}, e^{2c})$, and writing $N^2 = e^{2d}$, we obtain a simple Lagrangian

$$L = -e^{a+b+c}e^{-d}(\dot{a}\dot{b} + \dot{a}\dot{c} + \dot{b}\dot{c}) - m_g^2 F(a, b, c, d). \quad (\text{A4})$$

The quadratic part of F is the Pauli-Fierz quadratic form

$$F = -(ab + ac + bc) - d(a + b + c) + G, \quad (\text{A5})$$

and G contains cubic and higher order terms. This is easy to check. In the linearized approximation, the Lagrangian becomes

$$L = -(\dot{a}\dot{b} + \dot{a}\dot{c} + \dot{b}\dot{c}) + m_g^2((ab + ac + bc) + d(a + b + c)). \quad (\text{A6})$$

The d constraint gives $a + b + c = 0$, leading to

$$L = \frac{1}{2}(\dot{a}^2 + \dot{b}^2 + \dot{c}^2) - \frac{1}{2}m_g^2(a^2 + b^2 + c^2), \quad (\text{A7})$$

which describes constrained positive-energy harmonic oscillators of mass m_g .

In the generic nonlinear case, the d constraint gives

$$e^{a+b+c}e^{-d}T - m_g^2\partial_d F = 0, \quad (\text{A8})$$

where

$$T \equiv \dot{a}\dot{b} + \dot{a}\dot{c} + \dot{b}\dot{c}. \quad (\text{A9})$$

The energy is

$$E = -e^{a+b+c}e^{-d}T + m_g^2 F = m_g^2(F - \partial_d F), \quad (\text{A10})$$

where $d = d(a, b, c, T)$ from (A8).

To show that the energy can be negative for arbitrary F , we assume that a, b, c, d , and T are infinitesimals of the same order. Then we can linearize both the constraint Eq. (A8) and the energy expression (A10). The constraint equation is

$$T = -m_g^2(a + b + c). \quad (\text{A11})$$

Our assumption that a, b, c, d , and T are infinitesimals of the same order will be correct only if we choose a, b, c , and T that satisfy (A11). Then the energy is

$$E = -T, \quad (\text{A12})$$

which is not positive semidefinite.

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