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Stability of modified gravity models

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Conditions for the existence and stability of de Sitter space in modified gravity are derived by considering inhomogeneous perturbations in a gauge-invariant formalism. The stability condition coincides with the corresponding condition for stability with respect to homogeneous perturbations, while this is not the case in scalar-tensor gravity. The stability criterion is applied to various modified gravity models of the early and the present universe.

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I. INTRODUCTION

The 1998 discovery that the expansion of the Universe is accelerated, obtained with the study of type Ia supernovae [1], has prompted many theoretical models to explain this phenomenon. Most of these models can be classified in three classes: dark energy models, modified gravity models, and brane-world models. In the first class it is assumed that a form of *dark energy* or *quintessence* of unknown nature has come to dominate the dynamics of the Universe at recent times (redshifts $z \leq 1$). These models are usually explored in the context of Einstein's theory of general relativity [2], or possibly in its scalar-tensor generalizations (*extended quintessence*) [3]. Dark energy must necessarily have exotic properties in order to generate acceleration. In the spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) line element describing our Universe according to the recent cosmic microwave background experiments [4], and given by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{1.1}$$

in comoving coordinates (t, x, y, z), the acceleration equation [5]

$$\frac{\ddot{a}}{a} = \frac{\kappa}{6}(\rho + 3P) \tag{1.2}$$

holds, where $\kappa = 8\pi G$ and ρ and P are the total energy density and pressure of the cosmic fluid, respectively. An overdot denotes differentiation with respect to the comoving time t. If dark energy were the only form of energy of the Universe, acceleration $\ddot{a} > 0$ would require an exotic negative pressure $P < -\rho/3$; and if ordinary matter and dark matter contributing to the dynamics are taken into account, the pressure of dark energy must be even more negative to compensate. As a matter of fact, dark energy is even more exotic: the best fit to the supernovae data favors an extreme form of dark energy called *phantom energy* or *superquintessence* with $P < -\rho$—or with an effective equation of state parameter $w \equiv P/\rho < -1$. Furthermore, the effective equation of state should evolve with time [6]. If confirmed, this fact would definitely rule out the cosmological constant Λ as an explanation of the cosmic acceleration because $w_\Lambda = -1$ is strictly constant (the cosmological constant model is anyway disfavored because of the cosmological constant problem [7] and of the cosmic coincidence problem [8] that accompany it). Most models of dynamical dark energy are based on a scalar field ϕ rolling in a potential $V(\phi)$, a way to implement cosmic acceleration that is well known from inflationary theory in the early universe [9]. However, a canonical scalar field minimally coupled to the Ricci curvature R in Einstein gravity cannot explain an equation of state parameter $w < -1$, which is equivalent to *superacceleration* $\dot{H} > 0$, where $H \equiv \dot{a}/a$ is the Hubble parameter [10]. (This name distinguishes a regime in which the Hubble parameter increases from an "ordinary" acceleration regime in which $\ddot{a} = a(\dot{H} + H^2) > 0$ and $\dot{H} \leq 0$.) In fact, the energy density and pressure of such a scalar field are

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \qquad P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{1.3}$$

and the Einstein-Friedmann equation of general relativity

$$\dot{H} = -\frac{\kappa}{6}(P + \rho) = -\frac{\kappa}{2}\dot{\phi}^2 \tag{1.4}$$

yields $\dot{H} \leq 0$ for a universe dominated by such a scalar (the upper bound $H = \text{const}$ is attained by de Sitter space). In order to model an equation of state parameter $w < -1$ corresponding to $\dot{H} > 0$, which is the situation favored by the observational data, a phantom field with negative kinetic energy [11] or a scalar field coupled nonminimally to gravity [12] have been used. Both of these theories can be seen as special cases of scalar-tensor gravity, described by the action

$$S = \int d^4x\sqrt{-g}\left[\psi(\phi)R - \frac{\omega(\phi)}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right], \tag{1.5}$$

where $\psi(\phi)$ and $\omega(\phi)$ are arbitrary coupling functions. The exotic properties of dark energy or of its extreme form, phantom energy, have led some authors to a different approach and to the second class of models mentioned above. Instead of postulating an exotic form of dark energy of mysterious nature, these authors [13,14] consider the

possibility that gravity deviates from Einstein gravity at large scales and assume that the Einstein-Hilbert Lagrangian is modified by corrections that become important only at late times in the history of the Universe, i.e., when the curvature becomes small. This class of theories, called *modified gravity*, is described by the gravitational action

$$S_g = \int d^4x\sqrt{-g}f(R), \tag{1.6}$$

where $f(R)$ is a nonlinear function of R. The first model proposed had the form $f(R) = R - \mu^4/R$, in which the correction in R^{-1} becomes important only at low curvatures $R \to 0$. The general form (1.6) of the action also includes quantum gravity corrections to Einstein's theory originally introduced to improve renormalizability [15,16] and used in inflationary models of the early universe [17]. In addition to the desired phenomenological properties of modified gravity in cosmology, there is some motivation for these models from M theory [18].

In both classes of models, depending on the arbitrary functions and parameters adopted, there are solutions describing universes that accelerate forever, other solutions in which the Universe ends its existence in a finite time in the future in a big rip singularity [19] or encounters another type of ''sudden future singularity'' [20]. The fate of the Universe depends on whether attractor solutions that are forever accelerating or big rip attractors exist in the phase space, and on the size of their respective attraction basins. In many models of both dark energy and modified gravity there are de Sitter attractors accelerating forever. In this paper we focus on modified gravity and in Sec. III we use scalar-tensor gravity for a comparison of properties. We determine whether de Sitter attractors exist in the phase space of modified gravity by deriving conditions for the existence and stability of these solutions. Throughout most of this paper we consider general nonlinear actions of the form (1.6) with $\partial^2 f/\partial R^2 \neq 0$ and, in the final part of this paper, we apply our general results to specific scenarios and forms of $f(R)$ proposed in the literature.

While it is straightforward to study the stability of de Sitter space with respect to homogeneous perturbations, which only depend on time, it is physically more significant to assess stability with respect to more general inhomogeneous perturbations, which depend on both space and time. This goal is more ambitious because of the gauge-dependence problems associated with this type of cosmological perturbations [21] and one expects the stability condition with respect to inhomogeneous perturbations to be more restrictive than the corresponding condition for stability with respect to homogeneous perturbations. It comes therefore as a surprise that these conditions coincide, as shown in Sec. III and briefly reported in a previous communication [22]—this result cannot be guessed or justified *a priori*. A similar analysis shows instead that in scalar-tensor theories the stability condition with respect to homogeneous perturbations is indeed more restrictive than the corresponding one for homogeneous perturbations [22].

Certain modified gravity models are ruled out on the basis of instabilities that manifest on short timescales [23–27]. The stability condition derived here has the advantage of being applicable to any nonlinear Lagrangian of the form $\sqrt{-g}f(R)$ and is useful in the study of the phase space and dynamics of modified gravity scenarios. Another motivation for our study is that, in order to be viable, modified gravity models need to have the correct Newtonian and post-Newtonian limit, and currently there is disagreement on whether certain models pass or not this test [28–31]. Because many models do not admit a Minkowski solution around which to expand the weak-field metric, an expansion around the de Sitter background is used instead [25,26,29]. This is meaningful when a de Sitter solution exists and is stable.

The plan of this paper is the following: In Sec. II we derive a stability condition for de Sitter space with respect to inhomogeneous perturbations by using a covariant and gauge-invariant formalism suitable for generalized gravity (including scalar-tensor and modified gravity and possibly mixed models). In Sec. III we derive the much simpler stability condition with respect to homogeneous perturbations in modified gravity, and we interpret our results. In Sec. IV the stability condition derived for modified gravity is applied to various scenarios widely discussed in the literature, while Sec. V contains a discussion and the conclusions.

II. STABILITY OF DE SITTER SPACE WITH RESPECT TO INHOMOGENEOUS PERTURBATIONS

We can consider at once modified gravity and scalar-tensor theories, which we will use for a comparison with modified gravity, by studying the gravitational action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(\phi, R) - \frac{\omega(\phi)}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right]. \tag{2.1}$$

This action contains also possible combinations of modified and scalar-tensor gravity if both $\partial f/\partial\phi$ and $\partial^2 f/\partial R^2$ are nonvanishing: such mixed scenarios have received little attention in the literature so far [32]. Scalar-tensor gravity is the special case in which f is linear in R, i.e., $f(\phi, R) = \psi(\phi)R$, while modified gravity corresponds to setting $\phi = 1$ and $\partial^2 f/\partial R^2 \neq 0$. In the spatially flat FLRW metric (1.1) the field equations assume the form

$$H^2 = \frac{1}{3F}\left(\frac{\omega}{2}\dot{\phi}^2 + \frac{RF}{2} - \frac{f}{2} + V - 3H\dot{F}\right), \tag{2.2}$$

$$\dot{H} = -\frac{1}{2F}(\omega\dot{\phi}^2 + \ddot{F} - H\dot{F}), \tag{2.3}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega}\left(\frac{d\omega}{d\phi}\dot{\phi}^2 - \frac{\partial f}{\partial \phi} + 2\frac{dV}{d\phi}\right) = 0, \quad (2.4)$$

where $F \equiv \partial f/\partial R$. It is natural to use (H, ϕ) as dynamical variables and the equilibrium points of the dynamical system (2.2)–(2.4) are de Sitter spaces with constant scalar field (H_0, ϕ_0): they exist subject to the conditions

$$R_0 F_0 = 2(f_0 - V_0), \quad (2.5)$$

$$f_0' = 2V_0', \quad (2.6)$$

where $R_0 = 12H_0^2$, $f_0 \equiv f(\phi_0, R_0)$, $F_0 \equiv F(\phi_0, R_0)$, $V_0 = V(\phi_0)$, $V_0' = \frac{dV}{d\phi}|_{\phi_0}$, and a prime denotes differentiation with respect to ϕ. In modified gravity, there is only the condition (2.5) for the existence of de Sitter solutions because there is only one arbitrary function $f(R)$.

In order to describe inhomogeneous perturbations of the de Sitter fixed points (H_0, ϕ_0) we use the covariant and gauge-invariant formalism of Bardeen-Ellis-Bruni-Hwang-Vishniac [21,33] in the version given by Hwang and Hwang and Noh [34] for generalized gravity. The metric perturbations A, B, H_L, and H_T are defined by the relations

$$g_{00} = -a^2(1 + 2AY), \quad (2.7)$$

$$g_{0i} = -a^2 B Y_i, \quad (2.8)$$

$$g_{ij} = a^2[h_{ij}(1 + 2H_L) + 2H_T Y_{ij}], \quad (2.9)$$

where the scalar harmonics Y are the eigenfunctions of the eigenvalue problem $\bar{\nabla}_i \bar{\nabla}^i Y = -k^2 Y$, and where h_{ij} is the three-dimensional metric of the FLRW background, $\bar{\nabla}_i$ is the covariant derivative associated with h_{ij}, and k is the eigenvalue. The vector and tensor harmonics Y_i and Y_{ij} satisfy the equations

$$Y_i = -\frac{1}{k}\bar{\nabla}_i Y, \quad (2.10)$$

$$Y_{ij} = \frac{1}{k^2}\bar{\nabla}_i \bar{\nabla}_j Y + \frac{1}{3} Y h_{ij}. \quad (2.11)$$

We need the Bardeen gauge-invariant potentials [21]

$$\Phi_H = H_L + \frac{H_T}{3} + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right), \quad (2.12)$$

$$\Phi_A = A + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right) + \frac{a}{k}\left[\dot{B} - \frac{1}{k}(a\dot{H}_T)^{\cdot}\right], \quad (2.13)$$

and the Ellis-Bruni variable [33]

$$\Delta\phi = \delta\phi + \frac{a}{k}\dot{\phi}\left(B - \frac{a}{k}\dot{H}_T\right), \quad (2.14)$$

while equations similar to Eq. (2.14) define the gauge-invariant variables Δf, ΔF, and ΔR. The first order equations satisfied by the gauge-invariant perturbations are [34]

$$\Delta\ddot{\phi} + \left(3H + \frac{\dot{\phi}}{\omega}\frac{d\omega}{d\phi}\right)\Delta\dot{\phi} + \left[\frac{k^2}{a^2} + \frac{\dot{\phi}^2}{2}\frac{d}{d\phi}\left(\frac{1}{\omega}\frac{d\omega}{d\phi}\right) - \frac{d}{d\phi}\left(\frac{1}{2\omega}\frac{\partial f}{\partial\phi} - \frac{1}{\omega}\frac{dV}{d\phi}\right)\right]\Delta\phi = \dot{\phi}(\dot{\Phi}_A - 3\dot{\Phi}_H) + \frac{\Phi_A}{\omega}\left(\frac{\partial f}{\partial\phi} - 2\frac{dV}{d\phi}\right) + \frac{1}{2\omega}\frac{\partial^2 f}{\partial\phi\partial R}\Delta R, \quad (2.15)$$

$$\Delta\ddot{F} + 3H\Delta\dot{F} + \left(\frac{k^2}{a^2} - \frac{R}{3}\right)\Delta F + \frac{F}{3}\Delta R + \frac{2}{3}\omega\dot{\phi}\Delta\dot{\phi} + \frac{1}{3}\left(\dot{\phi}^2\frac{d\omega}{d\phi} + 2\frac{\partial f}{\partial\phi} - 4\frac{dV}{d\phi}\right)\Delta\phi = \dot{F}(\dot{\Phi}_A - 3\dot{\Phi}_H) + \frac{2}{3}(FR - 2f + 4V)\Phi_A, \quad (2.16)$$

$$\ddot{H}_T + \left(3H + \frac{\dot{F}}{F}\right)\dot{H}_T + \frac{k^2}{a^2}H_T = 0, \quad (2.17)$$

$$-\dot{\Phi}_H + \left(H + \frac{\dot{F}}{2F}\right)\Phi_A = \frac{1}{2}\left(\frac{\Delta\dot{F}}{F} - H\frac{\Delta F}{F} + \frac{\omega}{F}\dot{\phi}\Delta\phi\right), \quad (2.18)$$

$$\left(\frac{k}{a}\right)^2\Phi_H + \frac{1}{2}\left(\frac{\omega}{F}\dot{\phi}^2 + \frac{3}{2}\frac{\dot{F}^2}{F^2}\right)\Phi_A = \frac{1}{2}\left\{\frac{3}{2}\frac{\dot{F}\Delta\dot{F}}{F^2} + \left(3\dot{H} - \frac{k^2}{a^2} - \frac{3H}{2}\frac{\dot{F}}{F}\right)\frac{\Delta F}{F} + \frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \frac{1}{2F}\left[\dot{\phi}^2\frac{d\omega}{d\phi} - \frac{\partial f}{\partial\phi} + 2\frac{dV}{d\phi} + 6\omega\dot{\phi}\left(H + \frac{\dot{F}}{2F}\right)\right]\Delta\phi\right\}, \quad (2.19)$$

$$\Phi_A + \Phi_H = -\frac{\Delta F}{F}, \quad (2.20)$$

$$\ddot{\Phi}_H + H\dot{\Phi}_H + \left(H + \frac{\dot{F}}{2F}\right)(2\dot{\Phi}_H - \dot{\Phi}_A) + \frac{1}{2F}(f - 2V - RF)\Phi_A = -\frac{1}{2}\left[\frac{\Delta\ddot{F}}{F} + 2H\frac{\Delta\dot{F}}{F} + (P - \rho)\frac{\Delta F}{2F} + \frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \frac{1}{2F}\left(\dot{\phi}^2\frac{d\omega}{d\phi} + \frac{\partial f}{\partial\phi} - 2\frac{dV}{d\phi}\right)\Delta\phi\right], \quad (2.21)$$

and

$$\Delta R = 6\left[\ddot{\Phi}_H + 4H\dot{\Phi}_H + \frac{2}{3}\frac{k^2}{a^2}\Phi_H - H\dot{\Phi}_A - \left(2\dot{H} + 4H^2 - \frac{k^2}{3a^2}\right)\Phi_A\right]. \quad (2.22)$$

In the de Sitter background (H_0, ϕ_0), these equations assume the considerably simpler form

$$\Delta\ddot{\phi} + 3H_0\Delta\dot{\phi} + \left[\frac{k^2}{a^2} - \frac{1}{2\omega_0}(f_0'' - 2V_0'')\right]\Delta\phi = \frac{f_{\phi R}}{2\omega_0}\Delta R, \tag{2.23}$$

$$\Delta\ddot{F} + 3H_0\Delta\dot{F} + \left(\frac{k^2}{a^2} - 4H_0^2\right)\Delta F + \frac{F_0}{3}\Delta R = 0, \tag{2.24}$$

$$\ddot{H}_T + 3H_0\dot{H}_T + \frac{k^2}{a^2}H_T = 0, \tag{2.25}$$

$$-\dot{\Phi}_H + H_0\Phi_A = \frac{1}{2}\left(\frac{\Delta\dot{F}}{F_0} - H_0\frac{\Delta F}{F_0}\right), \tag{2.26}$$

$$\Phi_H = -\frac{1}{2}\frac{\Delta F}{F_0}, \tag{2.27}$$

$$\Phi_A + \Phi_H = -\frac{\Delta F}{F_0}, \tag{2.28}$$

$$\ddot{\Phi}_H + 3H_0\dot{\Phi}_H - H_0\dot{\Phi}_A - 3H_0^2\Phi_A = -\frac{1}{2}\frac{\Delta\ddot{F}}{F_0} - H_0\frac{\Delta\dot{F}}{F_0} + \frac{3H_0^2}{2}\frac{\Delta F}{F_0}, \tag{2.29}$$

to first order, whereas

$$\Delta R = 6\left[\ddot{\Phi}_H + 4H_0\dot{\Phi}_H + \frac{2}{3}\frac{k^2}{a^2}\Phi_H - H_0\dot{\Phi}_A + \left(\frac{k^2}{3a^2} - 4H_0^2\right)\Phi_A\right]. \tag{2.30}$$

To first order and in the absence of ordinary matter [35], vector perturbations do not have any effect. Expanding de Sitter spaces with $H_0 > 0$ are always stable, to first order, with respect to tensor perturbations [36], as can be seen from Eq. (2.17). On the other hand, contracting de Sitter spaces with $H_0 < 0$ are always unstable [36] and will not be considered further. There remain scalar perturbations, which we set out to examine.

In modified gravity theories with $f = f(R)$, $\phi \equiv 1$, and $f_{RR} \neq 0$, Eqs. (2.27) and (2.28) yield

$$\Phi_H = \Phi_A = -\frac{\Delta F}{2F_0}, \tag{2.31}$$

whereas Eqs. (2.30) and (2.31) yield

$$\Delta R = 6\left[\ddot{\Phi}_H + 3H_0\dot{\Phi}_H + \left(\frac{k^2}{a^2} - 4H_0^2\right)\Phi_H\right], \tag{2.32}$$

and $a = a_0 e^{H_o t}$ is the scale factor of the unperturbed de Sitter space, with a_0 a constant. In the de Sitter background the gauge-invariant variables reduce, to first order, to

$$\Delta\phi = \delta\phi, \qquad \Delta R = \delta R, \qquad \Delta F = \delta F, \qquad \Delta f = \delta f. \tag{2.33}$$

Since the function F depends only on R one has

$$\frac{\Delta F}{F_0} = \frac{f_{RR}}{F_0}\Delta R, \tag{2.34}$$

where

$$f_{RR} \equiv \frac{\partial^2 f}{\partial R^2}\bigg|_{R_0}, \tag{2.35}$$

and therefore

$$\Delta R = -\frac{2F_0}{f_{RR}}\Phi_H. \tag{2.36}$$

The perturbations $\Phi_H = \Phi_A$ then evolve according to Eq. (2.29), which becomes

$$\ddot{\Phi}_H + 3H_0\dot{\Phi}_H + \left(\frac{k^2}{a^2} - 4H_0^2 + \frac{F_0}{3f_{RR}}\right)\Phi_H = 0, \tag{2.37}$$

where $a = a_0 e^{H_0 t}$, with a_0 a constant. At late times the term k^2/a^2 can be neglected and stability is achieved if the coefficient of Φ_H in the last term of the left-hand side of Eq. (2.37) is positive or zero: upon use of the value of the Hubble parameter given by Eq. (2.5) for the unperturbed de Sitter space, this condition reduces to

$$\frac{F_0^2 - 2f_0 f_{RR}}{F_0 f_{RR}} \geq 0. \tag{2.38}$$

This inequality was presented in a previous communication without details of the derivation [22]. We now comment on the physical meaning of the approximation leading to (2.38). Information on the spatial dependence of the inhomogeneous scalar perturbations are contained in the eigenvector k of the spherical harmonics, and the fact that the only term containing k in Eq. (2.37) becomes negligible as time progresses in a de Sitter background implies that the spatial dependence effectively disappears from the analysis. One may be tempted to conclude that the stability condition (2.38) with respect to inhomogeneous perturbations could be obtained in a much quicker way by considering the simpler homogeneous perturbations: this would be incorrect as one would not be able to guess *a priori*, in a homogeneous perturbation analysis, the structure of Eq. (2.37) and the fact that the spatial dependence disappears. Furthermore, in the parallel case of scalar-tensor gravity, the stability condition with respect to homogeneous perturbations differs from the corresponding one for inhomogeneous perturbations, and one would expect the same to happen for modified gravity. This is the subject of the next section.

III. HOMOGENEOUS PERTURBATIONS IN MODIFIED GRAVITY AND IN SCALAR-TENSOR THEORIES

We now derive the stability condition of de Sitter space with respect to *homogeneous* perturbations in modified gravity, in order to compare it with the result (2.38) of the previous section. The field equations reduce to

$$H^2 = \frac{1}{3F}\left(\frac{RF - f}{2} - 3H\dot{F}\right), \tag{3.1}$$

$$\dot{H} = -\frac{1}{2F}(\ddot{F} - H\dot{F}). \tag{3.2}$$

By assuming that $H(t) = H_0 + \delta H(t)$ and using the first order expansions

$$\begin{aligned} &R = R_0 + \delta R, \qquad \delta R = 6(\delta\dot{H} + 4H_0\delta H), \\ &F = F_0 + f_{RR}\delta R, \qquad f = f_0 + F_0\delta R, \end{aligned} \tag{3.3}$$

and Eq. (2.5), one obtains the evolution equation for the homogeneous perturbation δH

$$\delta\ddot{H} + \left(4H_0 - \frac{f_0}{6H_0F_0}\right)\delta\dot{H} + \frac{1}{3}\left(\frac{F_0}{f_{RR}} - \frac{2f_0}{F_0}\right)\delta H = 0. \tag{3.4}$$

The ansatz $\delta H = \epsilon e^{st}$ yields an algebraic equation for s with roots

$$s_\pm = \frac{1}{2}\left[-\frac{f_0}{2H_0F_0} \pm \sqrt{\left(\frac{f_0}{2H_0F_0}\right)^2 - \frac{4}{3}\left(\frac{F_0}{f_{RR}} - \frac{2f_0}{F_0}\right)}\right]. \tag{3.5}$$

Assuming $f_0 > 0$ and $H_0 > 0$, if also $F_0 > 0$, then $-f_0/(2H_0F_0) < 0$ and there is stability if and only if

$$\frac{F_0}{f_{RR}} - \frac{2f_0}{F_0} \geq 0, \tag{3.6}$$

which is equivalent to the condition (2.38). If (3.6) is not satisfied, the root s_+ is real and positive, corresponding to an unstable mode growing exponentially in time.

The case $F_0 < 0$ does not correspond to a de Sitter solution when $f_0 > 0$ because Eq. (2.5), which reduces to $R_0F_0 = 2f_0$ cannot be satisfied in this case.

Why the stability condition with respect to homogeneous perturbations coincides with the corresponding stability condition with respect to inhomogeneous perturbations? A naïve answer would be that the spatial dependence of the inhomogeneous perturbations can be eliminated safely in the analysis of Eq. (2.37) or, in other words, inhomogeneities are redshifted away, and the results must coincide. This would be intuitive: even initial anisotropies are known to be smoothed out by de Sitter-like expansion [37] in general relativity and in some scalar-tensor cosmologies [38], but this is not the correct explanation. In fact, if it were true, it should hold also in the case of scalar-tensor gravity in which $f(\phi, R) = \psi(\phi)R$, but this is not the case as we are going to show. The stability condition with respect to inhomogeneous perturbations in scalar-tensor gravity has been derived in Ref. [36] by analyzing the equation for the gauge-independent Bardeen potentials and for the Ellis-Bruni variable $\Delta\phi$,

$$\Delta\ddot{\phi} + 3H_0\Delta\dot{\phi} + \left[\frac{k^2}{a^2} - \frac{\frac{f_0''}{2} - V_0'' + \frac{6f_{\phi R}^2H_0^2}{F_0}}{\omega_0(1 + \frac{3f_{\phi R}^2}{2\omega_0F_0})}\right]\Delta\phi = 0, \tag{3.7}$$

where

$$f_{\phi R} \equiv \left.\frac{\partial^2 f}{\partial\phi\partial R}\right|_{(\phi_0,R_0)}, \qquad f_0'' \equiv \left.\frac{\partial^2 f}{\partial\phi^2}\right|_{(\phi_0,R_0)}. \tag{3.8}$$

This equation is obtained from Eqs. (2.23)–(2.30) if $1 + 3f_{\phi R}^2/(2\omega_0F_0) \neq 0$. The stability condition that ensues is [36]

$$\frac{\frac{f_0''}{2} - V_0'' + \frac{6f_{\phi R}^2H_0^2}{F_0}}{\omega_0(1 + \frac{3f_{\phi R}^2}{2\omega_0F_0})} \leq 0. \tag{3.9}$$

For the sake of comparison, let us derive the corresponding stability condition with respect to homogeneous perturbations in scalar-tensor gravity. Assuming that $f(\phi, R) = \psi(\phi)R$, the homogeneous perturbations

$$H(t) = H_0 + \delta H(t), \qquad \phi(t) = \phi_0 + \delta\phi(t) \tag{3.10}$$

satisfy the first order evolutions equations

$$\delta\dot{H} = -\frac{1}{2\psi_0}(\psi_0'\delta\ddot{\phi} - H_0\psi_0'\delta\dot{\phi}), \tag{3.11}$$

$$\delta\ddot{\phi} + 3H_0\delta\dot{\phi} + \frac{1}{2\omega_0}(2V_0'' - f_0'')\delta\phi = 0. \tag{3.12}$$

By contrast, in general relativity with a minimally coupled scalar field the perturbations have no effect to first order. This is due to the noncanonical form of the effective energy-momentum tensor of the scalar field appearing in the left-hand side of the field equations of scalar-tensor gravity when these are written in the form $G_{ab} = 8\pi T_{ab}^{(\mathrm{eff})}[\phi]$ (see Refs. [39,40] for a discussion). The stability condition with respect to homogeneous perturbations can be read off of Eq. (3.12),

$$\frac{\frac{f_0''}{2} - V_0''}{\omega_0} \leq 0. \tag{3.13}$$

The stability condition (3.9) with respect to inhomogeneous perturbations is more restrictive than (3.13) and, in spite of having neglected a term $k^2/a^2 = k^2e^{-2H_0t}/a_0^2$ in Eq. (3.9), the final stability condition (3.9) retains a memory of the spatial dependence of the inhomogeneous perturbations, which is instead lost in the homogeneous perturbation analysis leading to (3.13). *A priori*, one should expect a similar situation for modified gravity, and the fact that the two stability conditions coincide for these theories appears to be coincidental.

For the particular class of scalar-tensor theories described by the action

$$S = \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi} g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] \tag{3.14}$$

and containing a single arbitrary coupling function $\omega(\phi)$, the stability conditions (3.9) and (3.13) coincide [41]. However, these two conditions fail to coincide in the general scalar-tensor theory (1.5) because, in the right-hand side of Eq. (2.23), curvature perturbations ΔR act as sources for the perturbations $\Delta\phi$ (which, in the de Sitter background, coincide with $\delta\phi$). On the contrary, such a term is absent in the homogeneous perturbation analysis of the Klein-Gordon equation (2.4), and an analogous term does not appear in Eq. (2.37) for the gauge-independent Bardeen potential Φ_H in modified gravity because there is no scalar field in this case and $f_{\phi R} = 0$—Eq. (2.23) then becomes homogeneous.

IV. APPLICATION TO SPECIFIC MODIFIED GRAVITY SCENARIOS

We now proceed to apply the stability condition (2.38) to certain specific modified gravity scenarios that have been proposed in the literature, for which a de Sitter space is relevant.

A. $f(R) = R - \frac{\mu^4}{R}$

This theory [13,14,18,23,24,42], with the mass scale $\mu_0 \simeq H_0 \simeq 10^{-33}$ eV, was the first candidate proposed to explain the cosmic acceleration, and it is known to be subject to an instability that develops on a time scale of order 10^{-26} s [24]. For this theory, the stability condition (2.38) reduces to

$$1 + \frac{6\mu^4}{R_0^2} - \frac{3\mu^8}{R_0^4} \leq 0, \tag{4.1}$$

and it is clear that by taking $\mu_0 \approx H_0$ with $R_0 = 12H_0^2$, the stability condition is impossible to satisfy. More precisely, the condition for the existence of de Sitter solutions is $R_0 = \sqrt{3}\mu^2$ (see also Ref. [14]) and the stability condition (2.38) reduces to $8/3 \leq 0$, which obviously cannot be satisfied. We stress that this instability of de Sitter space arises in the gravitational sector of the theory, while the instability discovered in Ref. [24] arises in the matter sector and would disappear in vacuum. The instability of de Sitter space can be stabilized by adding a term of the form ϵR^2 with $0 < \epsilon < \mu^4$ to the Lagrangian density, as shown in the following.

B. $f(R) = R - \frac{\mu^4}{R} + aR^2$

In this theory [43] the condition for the existence of de Sitter solutions (2.5) becomes [43,44]

$$R_0 = \sqrt{3}\mu^2, \tag{4.2}$$

independent of the parameter of the quadratic correction—therefore this condition holds true also for $a = 0$ as seen in the previous case. For general values of the parameter a, upon use of Eq. (4.2), the stability condition (2.38) reduces to

$$\frac{1}{3\sqrt{3}a\mu^2 - 1} \geq 0, \tag{4.3}$$

and therefore de Sitter space is stable if $a > (3\sqrt{3}\mu^2)^{-1}$ and unstable if $a < (3\sqrt{3}\mu^2)^{-1}$ (in particular for $a = 0$, which is the previous case). Therefore, adding a quadratic correction with $a < (3\sqrt{3}\mu^2)^{-1}$ (and, in particular, with a negative a, which reinforces the effect of the term $-\mu^4/R$) leads to instability. Now, if $\mu \sim H_0 \sim 10^{-33}$ eV, the parameter a must be larger than $\sim 10^{65}$ $(\text{eV})^{-2}$ for stability, which appears to be huge in natural units—stability is achieved at the price of fine-tuning the parameters.

C. $f(R) = R^n$

This modified gravity theory has been pursued in the literature [13,45], especially for $n = -1$ and for $n = 3/2$, in which case it is conformally equivalent to Liouville field theory [13]. The model has been used to explain the cosmic acceleration and it is also interesting because ordinary inflation with a minimally coupled scalar field and an exponential potential (power-law inflation) can be rewritten as a theory $f(R) = R^n$ [43]. For generic values of $n \neq 0, 1/2, 1$, the theory yields power-law inflation $a \propto t^\alpha$ with

$$\alpha = \frac{-2n^2 + 3n - 1}{n - 2}. \tag{4.4}$$

This theory does not admit de Sitter solutions if $n \neq 2$ [36,46] unless a cosmological constant, corresponding to a term with $n = 0$, is added to the action [46]. However, Minkowski space (the trivial de Sitter space) is a solution without cosmological constant for any positive value of n [47]. In fact, the condition (2.5) for the existence of de Sitter solutions yields $nR_0 = 2R_0^n$, which is only satisfied for $n = 2$ or $R_0 = 0$. The stability condition (2.38) yields

$$R_0\frac{2 - n}{n(n - 1)} \geq 0 \tag{4.5}$$

and is satisfied for $1 < n \leq 2$ and for $n < 0$; it is identically satisfied for $n = 2$, without imposing constraints on the Hubble parameter H_0 of de Sitter space. The Minkowski spaces $R_0 = 0$ are stable for any $n > 0$.

D. $f(R) = R + \epsilon R^2$

Quadratic corrections to the Einstein-Hilbert Lagrangian density are motivated by renormalizability [15,16] and higher order corrections are unavoidable near the Planck scale [48]. This theory was used in one of the earliest inflationary scenarios [17], not requiring an inflaton field. The constant has dimensions $\epsilon \sim M^{-2}$, where $M \sim 10^{12}$ GeV. The condition for the existence of de Sitter

solutions allows only the trivial Minkowski space $H_0 = 0$. However, there are nontrivial de Sitter solutions if a cosmological constant is added to $f(R)$ [36]. The stability condition (2.38) yields

$$\frac{1}{\epsilon(1 + 2\epsilon R_0)} \geq 0, \tag{4.6}$$

which, for Minkowski space, gives stability if $\epsilon > 0$ and instability for $\epsilon < 0$. The case $\epsilon = 0$ corresponding to Einstein's theory must be studied separately and it is concluded that Minkowski space is stable in this case [36].

E. $f(R) = R + \epsilon R^2 - 2\Lambda$

By adding a cosmological constant to the previous theory, de Sitter solutions become possible and are given by $R_0 = 4\Lambda$, or $H_0 = \sqrt{\Lambda/3}$ as in general relativity, because the condition on H_0 does not depend on the parameter ϵ and coincides with the corresponding condition for $\epsilon = 0$. The stability condition (2.38) reduces to $\epsilon > 0$: a positive quadratic correction acts in the same direction as the term R in the Lagrangian, whereas a negative quadratic correction with $|\epsilon|$ arbitrarily small makes de Sitter space unstable.

F. $f(R) = R + \epsilon R^n$

This theory [14,49], with $\epsilon \sim M^{2(1-n)}$, where M is a mass scale, comprises quantum gravity-motivated corrections to the Einstein-Hilbert Lagrangian for $n > 0$, as well as the theory $f(R) = R - \mu^4/R$ already discussed, or similar theories, if $n < 0$. The condition for the existence of de Sitter space is either $R_0 = 0$ (Minkowski space) or

$$R_0^{n-1} = (12H_0^2)^{n-1} = \frac{1}{\epsilon(n-2)} \tag{4.7}$$

for $n \neq 2$ (the case $n = 2$ has already been considered). In the case of a nontrivial de Sitter space $H_0 \neq 0$ the stability condition (2.38) yields, using Eq. (4.7),

$$\frac{-n^2 + 2n - 2}{\epsilon n(n-1)} \geq 0 \tag{4.8}$$

or $\epsilon n < 0$, and therefore de Sitter space is stable if $\epsilon n < 0$ and unstable otherwise. In particular, it is stable if $\epsilon > 0$ and $n < 0$ (or if $\epsilon < 0$ and $n > 0$), which comprises the case $f(R) = R - \mu^4/R$ (as $n = 2$ is excluded from this analysis, these results do not contradict the previous statements on the stability of Minkowski space when $n = 2$). Therefore, any theory of the form $f(R) = R - \mu^{2(1-n)}/R^m$ with $m > 0$ exhibits the same instability in the gravitational sector as the model $f(R) = R - \mu^4/R$ and, likely, the same instability reported in Ref. [24] for the matter sector.

G. $f(R) = a \ln(\frac{R}{b})$

This theory [30,50,51], which does not admit a Minkowski space or other solutions with vanishing Ricci curvature, admits a de Sitter solution only if $R_0 = 12H_0^2 = be^{b/2}$, where b is a positive parameter. The stability condition (2.38) reduces to

$$b\left[b + 2\ln\left(\frac{R_0}{b}\right)\right] \leq 0, \tag{4.9}$$

which, for $b > 0$, is equivalent to $e^b \leq 0$ and obviously is never satisfied: the de Sitter space $H_0 = \sqrt{be^{b/2}/12}$ is unstable.

V. DISCUSSION AND CONCLUSIONS

The general stability condition (2.38) of de Sitter space in modified gravity derived in Sec. II allows one to quickly assess the stability of de Sitter space in specific scenarios. Of course, when the number of terms in the Lagrangian density grows, so does the volume of parameter space to be searched and this analysis becomes cumbersome—it would be greatly helped by an estimate of the range of values of the parameters involved. In the future we plan to extend to power-law solutions the study carried out here for de Sitter spaces.

The comparison of the modified gravity results with the analogous results in scalar-tensor gravity shows that, although there exists a dynamical equivalence between modified gravity and scalar-tensor gravity [52], this should not be taken too literally. The stability conditions with respect to homogeneous and inhomogeneous perturbations coincide in modified gravity but not in scalar-tensor gravity, due to the different detailed structure of the equations satisfied by the perturbations.

The scope of the stability analysis can perhaps be extended to more general theories of the form $f(R, R^2, R_{ab}R^{ab}, R_{abcd}R^{abcd})$ containing string-motivated corrections (see, e.g., Refs. [53–55]); there are, however, doubts on certain choices of the string corrections to the Einstein-Hilbert action, due to ghosts or light, long-ranged, gravitational scalars that potentially violate solar system bounds [25,26].

Finally, it should be stressed that, while we have analyzed linear stability with respect to inhomogeneous perturbations described by gauge-invariant variables, other definitions of stability can be considered: for example, stability with respect to black hole nucleation [44] or quantum fluctuations [56]. Sometimes these different definitions yield results that are qualitatively similar to our stability condition (2.38) (e.g., [44]). Another possibility is to search for a positive-definite energy functional [57], although one should probably look for energies bounded from below rather than positive energies [58]; and so on. The fact that these stability criteria are inequivalent is not surprising since the physical processes considered are quite different and, even from the mathematical point of view,

several inequivalent definitions of stability exist for dynamical systems [59].

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Note added in proof.—After completion of this work we learned of Ref. [61] containing a study of theories with additional quadratic invariants of the Riemann tensor and showing that theories of the form $f(R, R_{ab}R^{ab} - 4R_{abcd}R^{abcd})$ are ghost-free. A stability condition similar to (2.38) also is obtained. The existence and stability of de Sitter solutions are studied also in Ref. [62].

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