Primordial magnetic seed field amplification by gravitational waves

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Using second-order gauge-invariant perturbation theory, a self-consistent framework describing the nonlinear coupling between gravitational waves and a large-scale homogeneous magnetic field is presented. It is shown how this coupling may be used to amplify seed magnetic fields to strengths needed to support the galactic dynamo. In situations where the gravitational wave background is described by an ''almost'' Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology we find that the magnitude of the original magnetic field is amplified by an amount proportional to the magnitude of the gravitational wave induced shear anisotropy and the square of the field's initial comoving scale. We apply this mechanism to the case where the seed field and gravitational wave background are produced during inflation and find that the magnitude of the gravitational boost depends significantly on the manner in which the estimate of the shear anisotropy at the end of inflation is calculated. Assuming a seed field of 10^{-34} G spanning a comoving scale of about 10 kpc today, the shear anisotropy at the end of inflation must be at least as large as 10^{-40} in order to obtain a generated magnetic field of the same order of magnitude as the original seed. Moreover, contrasting the weak-field approximation to our gauge-invariant approach, we find that while both methods agree in the limit of high conductivity, their corresponding solutions are otherwise only compatible in the limit of infinitely long-wavelength gravitational waves.

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I. INTRODUCTION

The origin of cosmological magnetic fields that are prevalent throughout galaxies clusters, disk and spiral galaxies and high-redshift condensations has generated much debate in recent years, with the majority of this work being focused on providing mechanisms that generate these galactic fields on large scales (see [1,2] and references therein). The candidate mechanisms are diverse, often depending on the required seed field strengths. It has been suggested that the fields observed today could be a result of the amplification of a relatively large seed field through protogalactic collapse at the onset of structure formation [3]. As the gas collapses to current measured densities, the flux lines of the frozen-in cosmological magnetic field get compressed, inducing adiabatic amplification. Another popular mechanism, which requires a relatively weaker pre-existing seed field, is amplification via the galactic dynamo by means of parametric resonance [4]. The combined effect of differential rotation across the disk and the cyclonic turbulent motions of the ionized gas is believed to lead to the exponential amplification of a smaller primordial field until the back-reaction of the plasma opposes further growth. Although the dynamo mechanism is strongly supported by the close correlation between the observed structure of the galactic fields and the spiral pattern of galaxies, there is some argument over its efficiency and hence the amount of amplification that can occur through this process. The major problem with all of these mechanisms is that they assume the presence of a pre-existing seed field whose origin is still to be established. A further idea relies on turbulence (disrupted flow) and shocks, which occur during the stages of structure formation, inducing weaker magnetic fields via battery-type mechanisms, which operate as a result of large-scale misalignments of gradients in electron number density and pressure (or temperature) [5].

There have been numerous attempts to generate early, pre-recombination, magnetic fields with strengths suitable to support and maintain the dynamo by exploiting the different out-of-equilibrium epochs that are believed to have taken place between the end of the inflationary era and decoupling [6]. These fields are facilitated by currents that arise from local charge separation generated by vortical velocity fields prevalent in the early plasma (cf. also [7]).

One problem with the above mechanisms is that they are casual in nature so the scales over which the fields are coherent cannot exceed the particle horizon during that epoch. Given that such phase transitions took place at very early times, where the comoving horizon size was small, tight constraints must be placed on the coherence length of these magnetic fields. However, pre big bang models based on string theory [8], in which vacuum fluctuations of the magnetic field are amplified by the dilaton field, predict superhorizon fields.

Inflation has long been suggested as a solution to the causality problem, since it naturally achieves correlations on superhorizon scales, however adjustments to the standard inflationary models need to be made since magnetic

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fields surviving this epoch are small on account of the inability of vector fields to couple gravitationally to the conformally flat metric resulting from the exponentially fast expansion. A way around this obstacle is by breaking the conformal invariance of electromagnetism since this alters the way the underlying gauge fields couple to gravity. There are many ways of doing this which explains the variety of the proposed mechanisms in the literature [9]. Such inflationary scenarios have not been without critique, though [10].

It has also been proposed that inflation is followed by a period of preheating in which the parametric resonance of the causal oscillations of the inflaton field and the accompanying perturbations can lead to amplification on superhorizon scales [11]. Other authors have advocated the breakdown of Lorentz invariance either in the context of string theory and noncommutative varying speed of light theories, or due to the dynamics of large extra dimensions [12]. The success of these proposals, however, is usually achieved at the expense of simplicity.

In order for these proposed mechanisms to be viable, they must, in addition, produce seed fields that satisfy the criteria for the subsequent amplification processes to work. To be a candidate seed field for the galactic dynamo, the induced field must exceed a minimum coherence scale in order to prevent the destabilization of the dynamo action. The time scale over which the amplification takes place also dictates a minimum field strength, for example, in the case of a dark-energy dominated Universe we obtain $B \sim 10^{-34}$ G on a coherence scale of 10 kpc. Davis *et al.* [13] proposed an inflationary mechanism that exploits the natural coupling between the Z-boson and the gravitational background. Unfortunately, the fields produced only just fall within dynamo limits in the case of a dark energy dominated Universe. Recently, the production of a magnetic seed field due to the rotational velocity of ions and electrons, caused by the nonlinear evolution of primordial density perturbations in the cosmic plasma during pre-recombination radiation and matter eras, was investigated in [14] and a rms amplitude $B \approx 10^{-23}(\lambda/\mathrm{Mpc})^{-2}$ G at recombination on comoving scales $\lambda \gtrsim 1$ Mpc was reported.

In this paper, we offer an alternative mechanism that looks at the interaction of a *pre-existing* field, such as the one proposed by Davis *et al.*, with a gravitational wave (GW) spectrum which accompanies most inflationary scenarios. This builds on earlier work by Tsagas *et al.* [15] in which this idea was first introduced within the weak-field approximation. Our aim is to investigate whether this interaction can produce a sufficiently large amplification of a seed field present at the end of inflation to meet the above mentioned requirements for the dynamo to work.

The issue of how to deal with the coupling between gravitational waves and the seed magnetic field is rather subtle. A commonly used approximation in the literature is to assume that the magnetic field is weak and that its contribution to the energy-momentum tensor is such that it does not disturb the isotropy of the Friedmann-Lemaître-Robertson-Walker (FLRW) background [16]. This is done by assuming that the energy density of the magnetic field $\tilde{B}_a$ is much less than the matter energy density: $\tilde{B}^2 \ll \mu$ and that its anisotropic pressure is negligible: $\pi_{ab} \equiv -\tilde{B}_{\langle a}\tilde{B}_{b\rangle} \approx 0$.[1] The problem with this approximation is that it is not gauge-invariant in a strict mathematical sense, so one can therefore not guarantee that, when calculating the magnetic field which arises through its coupling with linear perturbations of FLRW (such as gravitational waves), it leads to physically meaningful results. In order to solve this problem we develop a self-consistent framework based on second-order perturbation theory, employing the methods initiated by recent work of Clarkson [17] and Clarkson *et al.* [18]. Here the seed magnetic field is treated as a *on average* homogeneous linear perturbation of the background FLRW model and couplings to gravitational degrees of freedom that arise when perturbing the background are taken to be second order in the perturbation theory. Adopting this approach allows us to write Maxwell's equations in a way that makes them manifestly gauge-invariant to second order with interaction terms that clearly describe the modes induced by the gravity wave-magnetic field interaction. The restriction to a homogeneous seed field leads to simplification on the technical level but still encapsulates the main features of the gravito-magnetic interaction. The implementation of an inhomogeneous seed is reserved to a future article.

The results show that, in the presence of gravitational radiation, the magnitude of the magnetic field is amplified proportionally to the shear distortion caused by the propagating waves. Once the amplification is saturated, the magnetic field dissipates adiabatically as usual. The gravitational boost is also proportional to the square of the field's original scale, which suggests that the proposed mechanism could lead to significant amplification in the case of large-scale magnetic fields. Indeed, when applied to fields of roughly 10^{-34} G spanning a comoving scale of about 10 kpc today (see, for example, the fields produced in [13]) , the mechanism leads to an amplification of up to 13 orders of magnitude (depending on the calculation of the shear distortion), bringing these magnetic fields well within the galactic dynamo requirements, without the need for extra amplification during reheating. We thus qualitatively and quantitatively rediscover in a gauge-invariant fashion the main results reported in [15].

In order to contrast the two different approaches in detail, we compare our solutions with the corresponding solutions obtained using the weak-field approximation

[1] Here the angle bracket represents the projected symmetric trace-free (PSTF) part of any tensor: $A_{\langle ab\rangle} \equiv h^c_{(a}h^d_{b)}A_{cd} - \frac{1}{3}h_{ab}A^c_c$.

[16] and find that while both methods agree in the limit of high conductivity, their corresponding solutions are otherwise only compatible in the limit of infinitely long-wavelength gravitational waves when merely the dominant contribution is considered.

The units employed in this paper are $c = h = 1$ and $\kappa = 8\pi G = 1$, the exception being section V, where natural units are used.

II. PERTURBATION SCHEME

If we wish to study the interaction between gravitational waves and a magnetic field in a cosmological setting, we immediately face a second-order problem in perturbation theory because both the magnetic field as well as GW are absent in the exact FLRW background, and may thus be individually regarded as first-order perturbations. Using the $1 + 3$ covariant approach [19], we therefore develop a two parameter expansion in two smallness parameters: ϵ_B represents the magnitude of a homogeneous magnetic field and ϵ_g represents the magnitude of the GW. The magnitude of the interaction GW $\times$ magnetic field is of order $\mathcal{O}(\epsilon_B\epsilon_g)$ as is the magnitude of the in such a manner generated electromagnetic fields. However, at second-order level, only terms of order $\mathcal{O}(\epsilon_B\epsilon_g)$ are kept while terms of order $\mathcal{O}(\epsilon_g^2)$ and $\mathcal{O}(\epsilon_B^2)$ are discarded. In fact, when dealing with the gravito-magnetic interaction, these discarded terms would always appear multiplied by a first-order quantity and are thus irrelevant for our considerations.

Whence, the perturbation spacetimes are divided up and denoted in the following way:

(i) $\mathcal{B} =$ Exact FLRW as background spacetime, $\mathcal{O}(\epsilon^0)$;

(ii) $\mathcal{F}_1 =$ Exact FLRW perturbed by a homogeneous magnetic field whose energy density and curvature are neglected, $\mathcal{O}(\epsilon_B)$;

(iii) $\mathcal{F}_2 =$ Exact FLRW with gravitational perturbations $\mathcal{O}(\epsilon_g)$;

(iv) $\mathcal{S} = \mathcal{F}_1 + \mathcal{F}_2$ allows for inclusion of interactions terms of order $\mathcal{O}(\epsilon_B\epsilon_g)$.

We will generally refer to terms of order $\mathcal{O}(\epsilon_B)$ and $\mathcal{O}(\epsilon_g)$ appearing in $\mathcal{F}$ as ‘first-order’ and to variables of mixed order $\mathcal{O}(\epsilon_B\epsilon_g)$ appearing in $\mathcal{S}$ as ‘second-order’.

It should be noticed that the absence of an electric field in $\mathcal{F}_1$ does not necessarily imply that there is no electric field at all but rather that the electric field is *perturbatively* smaller than the magnetic field. This is in accordance with the standard assumption that the very early Universe was a good conductor (see, for example, [20] for an example of how this works). The inclusion of an electric field in $\mathcal{F}_1$ is possible, in principle, but would require to alter the perturbation scheme because then interactions between gravitational waves and the electric field needed to be taken into account as well. However, a more realistic way of describing the interaction between gravitational waves and electromagnetic fields should employ a multifluid description [7], which allows for modelling the currents, but that is beyond the scope of the present paper.

Having outlaid the different stages we turn to review the concomitant equations. We keep them as general as possible, which will allow us to illuminate the effects of spatial geometry, cosmological constant Λ and equation of state for the matter on the interaction. We limit ourselves to the irrotational case, that is, we require the vorticity ω_{ab} to vanish throughout.

A. FLRW background

The FLRW models are characterized by a perfect fluid matter tensor and the condition of everywhere-isotropy. Thus, relative to the congruence of fundamental observers with 4-velocity u^a ($u^a u_a = -1$), the kinematical variables have to be locally isotropic, which implies the vanishing of the 4-acceleration $\dot{u}_a \equiv u^b \nabla_b u_a$, shear $\sigma_{ab} \equiv \mathrm{D}_{<a} u_{b>}$ and vorticity $\omega_{ab} \equiv \mathrm{D}_{[a} u_{b]}$ $(0 = \dot{u}_a = \sigma_{ab} = \omega_{ab})$. Furthermore, the models have to be not only conformally flat, that is, the electric and magnetic components of the Weyl tensor vanish ($0 = E_{ab} = H_{ab}$), but also spatially homogeneous implying the vanishing of the spatial gradients of the energy density μ, the pressure p and the expansion $\Theta \equiv \mathrm{D}_a u^a$ $(0 = \mathrm{D}_a \mu = \mathrm{D}_a \Theta = \mathrm{D}_a p)$. As usual, the spatial derivative $\mathrm{D}_a \equiv h_a{}^b \nabla_b$ is obtained by projection of the spacetime covariant derivative ∇_a onto the 3-space (with metric $h_{ab} \equiv g_{ab} + u_a u_b$) orthogonal to the observer's worldline. As a consequence, the key background equations are the energy conservation equation

$$\dot{\mu} + \Theta(\mu + p) = 0, \tag{1}$$

the Raychaudhuri equation

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) + \Lambda, \tag{2}$$

and the Friedmann equation

$$\mu + \Lambda = \frac{1}{3}\Theta^2 + \frac{3K}{a^2}, \tag{3}$$

where the constant K indicates the geometry of the spatial sections.

B. First-order perturbations

1. The homogeneous magnetic field $\tilde{B}_a$

We assume the magnetic field $\tilde{B}_a$ to be spatially homogeneous at first order ($\mathrm{D}_a \tilde{B}_b = 0$) and thus consider the gradient of $\tilde{B}_a$ as well as the magnetic anisotropy $\Pi_{ab} = -\tilde{B}_{\langle a}\tilde{B}_{b\rangle}$ as being of second order. We presuppose that such a field was produced by some primordial process, which left a relic field on average homogeneous over a typical coherence length. Since there are no electric fields or charges in the $\mathcal{F}_1$ perturbation spacetime, the magnetic induction equation takes the form

$$\beta_a \equiv \dot{\tilde{B}}_{\langle a\rangle} + \tfrac{2}{3}\Theta\tilde{B}_a = 0. \tag{4}$$

As a result, the magnetic field scales as

$$\tilde{B}_a = \tilde{B}_a^0\left(\frac{a_0}{a}\right)^2, \tag{5}$$

where a denotes the scale factor, e.g., $\Theta = 3\dot{a}/a = 3H$, where H denotes the inverse Hubble length.

2. Gravitational waves

Gravitational waves are covariantly described via transverse parts of the electric (E_{ab}) and magnetic (H_{ab}) Weyl components, which are PSTF tensors [21]. The pure tensor modes are transverse, obtained by switching off scalar and vector modes ($0 = \mathrm{D}_a\mu = \mathrm{D}_a\Theta = \mathrm{D}_a p = \omega_a = \dot{u}_a$), which results in the constraints[2]

$$0 = \mathrm{D}^a\sigma_{ab} = \mathrm{D}^a E_{ab} = \mathrm{D}^a H_{ab} = H_{ab} - \mathrm{curl}\sigma_{ab}. \tag{6}$$

The propagation equations for these tensor modes are simply

$$\dot{\sigma}_{\langle ab\rangle} + \tfrac{2}{3}\Theta\sigma_{ab} = -E_{ab}, \tag{7}$$

$$\dot{E}_{\langle ab\rangle} + \Theta E_{ab} = \mathrm{curl}(\mathrm{curl}\sigma_{ab}) - \tfrac{1}{2}(\mu + p)\sigma_{ab}, \tag{8}$$

together with the background equations for Θ and μ. Since every FOGI tensor satisfies the linearized identity

$$\mathrm{curl}(\mathrm{curl}T_{ab}) = -\mathrm{D}^2 T_{ab} + \tfrac{3}{2}\mathrm{D}_{\langle a}\mathrm{D}^c T_{b\rangle c} + (\mu + \Lambda - \tfrac{1}{3}\Theta^2)T_{ab}, \tag{9}$$

we see that the gravitational waves are completely determined by a closed wave equation for the shear, namely

$$\ddot{\sigma}_{ab} - \mathrm{D}^2\sigma_{ab} + \tfrac{5}{3}\Theta\dot{\sigma}_{ab} + (\tfrac{1}{9}\Theta^2 + \tfrac{1}{6}\mu - \tfrac{3}{2}p + \tfrac{5}{3}\Lambda)\sigma_{ab} = 0. \tag{10}$$

C. The interaction

Maxwell's equations govern the interaction between GW and magnetic fields. If we require charge neutrality and neglect currents as well as the back-reaction of induced second-order magnetic fields with the shear, we obtain

$$\dot{E}_{\langle a\rangle} + \tfrac{2}{3}\Theta E_a = \mathrm{curl}B_a, \tag{11}$$

$$\dot{B}_{\langle a\rangle} + \tfrac{2}{3}\Theta B_a = \sigma_{ab}\tilde{B}^b - \mathrm{curl}E_a, \tag{12}$$

$$D^a E_a = 0, \tag{13}$$

$$D^a B_a = 0. \tag{14}$$

Observe that the EM fields have to be divergence-free at all orders due to neglecting vorticity effects. Moreover, the system is not gauge-invariant because it contains a mixture of second-order (E_a, $\mathrm{curl}E_a$, $\mathrm{curl}B_a$) and first-order terms (σ_{ab}), while B_a now comprises the full magnetic field (the first-order contribution plus the induced field). The situation we are interested in is the interaction between the shear σ_{ab} and the first-order magnetic field, neglecting the back-reaction with the induced magnetic field. How does one then disentangle the different magnetic field perturbations in a consistent way?

In special relativity, the standard procedure would be to use a power series expansion of the magnetic field,

$$B^a = \epsilon_B B_1^a + \epsilon_g\epsilon_B B_2^a + \mathcal{O}(\epsilon_g^2, \epsilon_B^2), \tag{15}$$

where the first-order field B_1^a satisfies the magnetic induction Eq. (4). Although insertion of this expansion into the above system yields only second-order terms, the procedure does not work in general relativity since the commutation relations for the various differential operators (cf. the appendix) can not be consistently satisfied. To illustrate this important point clearly, we consider the commutation relation between the (proper) time derivative and the spatial gradient applied to the magnetic field. It is evident that the case where the commutator relation is introduced after the expansion of B^a,

$$(D^b B^a)^{\cdot}_{\perp} = \epsilon_g\epsilon_B(D^b B^a)^{\cdot}_{\perp} = \epsilon_g\epsilon_B[D^b\dot{B}_2^a - \tfrac{1}{3}\Theta D^b B_2^a], \tag{16}$$

does not agree with the case where the linearized identity for $(D^a B^b)^{\cdot}$ is substituted before using the power series expansion (15):

$$\begin{aligned}(D^b B^a)^{\cdot}_{\perp} &= D^b\dot{B}^a - \tfrac{1}{3}\Theta D^b B^a + H^{bd}\epsilon_{dac}B^c + \sigma^d{}_c D^c B_a \\ &= \epsilon_g\epsilon_B[D^b\dot{B}_2^a - \tfrac{1}{3}\Theta D^b B_2^a] + \epsilon_B H^b{}_d\epsilon^{dac}B_c^1.\end{aligned} \tag{17}$$

Here, $\perp$ denotes projection onto the fundamental observer's rest space. This inconsistency can only be resolved if all interaction terms are zero. It is via the commutation relations that Weyl curvature is brought in through the back door which couples to the magnetic field and thus affects the interaction. It is this feature that renders the power series procedure faulty.

The difficulty arises because the magnetic field B^a is not gauge-invariant in $\mathcal{S}$ as it does not vanish in $\mathcal{F}_1$. We therefore need to define a new second-order gauge-invariant (SOGI) variable which satisfactorily describes the effects that we wish to investigate. However, a look at Maxwell's equations above reveals that $\beta_a \equiv \dot{B}_{\langle a\rangle} + \frac{2}{3}\Theta B_a$ is the sought SOGI variable which has to be used at second order instead of the magnetic field B_a. We chose to describe the interaction in terms of the variable $I_a \equiv \sigma_{ab}\tilde{B}^b$. Hence, Maxwell's equations can be written in truly gauge-invariant terms at second-order, namely

[2] We use $\mathrm{curl}V_a \equiv \epsilon_{abc}D^b V^c$ to denote the curl of a vector and $\mathrm{curl}W_{ab} \equiv \epsilon_{cd\langle a}\mathrm{D}^c W_{b\rangle}{}^d$ to denote the covariant curl of a second-rank PSTF tensor, where ϵ_{abc} is the volume element of the 3-space. Finally, the covariant spatial Laplacian is $\mathrm{D}^2 \equiv \mathrm{D}^a\mathrm{D}_a$.

$$\dot{E}_{\langle a\rangle} + \tfrac{2}{3}\Theta E_a = \mathrm{curl} B_a, \tag{18}$$

$$\beta_a + \mathrm{curl} E_a = I_a. \tag{19}$$

Observe that the standard constraints $0 = \mathrm{D}^a B_a = \mathrm{D}^a E_a$, which hold at all orders, imply

$$\mathrm{D}^a \beta_a = \mathrm{D}^a I_a = \sigma_{ab} \mathrm{D}^a \tilde{B}^b = 0, \tag{20}$$

where the last equality is only true as long as spatial gradients of $\tilde{B}^a$ are regarded as second-order. Clearly, if the idealized assumption of infinite conductivity is made so that all electric fields vanish, Maxwell's equations reduce to $\beta_a = I_a$. In this specific case, once the solution for I_a is known, the (not gauge-invariant) generated magnetic field measured by the fundamental observer can be obtained via a standard integration of β_a. However, it is important to stress that β_a is the fundamental variable, whose deviation from zero quantifies the evolution of the magnetic field at second-order in a truly gauge-invariant manner.

III. WAVE EQUATIONS FOR THE MAIN VARIABLES

Having written the key Maxwell's equations as a system of differential equations of purely SOGI variables, we now turn to the derivation of wave equations for the electric and magnetic fields. In doing this we make no assumptions about the spatial geometry or the equation of state and also keep the cosmological constant; this has the advantage of allowing us to draw some conclusions about how these parameters influence the interaction between GW and magnetic fields. In particular, it will turn out that neglecting the current in Maxwell's equations and at the same time requiring a homogeneous magnetic field at first-order level leads to consistent equations in spatially flat models only.

A. Wave equation for the interaction variable

Let us first derive the wave equation for the interaction variable $I_a = \sigma_{ab}\tilde{B}^b$. Even though the shear σ_{ab} belongs to $\mathcal{F}_2$ and the magnetic field $\tilde{B}_a$ to $\mathcal{F}_1$, the commutator relations do not lead to ambiguities for I_a since they manifest themselves only at third-order in this case. In order to derive an evolution equation for I_a, we need the auxiliary quantity $J_a \equiv E_{ab}\tilde{B}^b$. Then, using Eqs. (4) and (7)–(9), we arrive at the system

$$\dot{I}_{\langle a\rangle} + \tfrac{4}{3}\Theta I_a = -J_a, \tag{21}$$

$$\dot{J}_{\langle a\rangle} + \tfrac{5}{3}\Theta J_a = -\mathrm{D}^2 I_a + [\tfrac{1}{2}(\mu - p) + \Lambda - \tfrac{1}{3}\Theta^2] I_a, \tag{22}$$

where we employed that spatial gradients of the magnetic field are second-order and thus $\mathrm{D}^2 I_a = \mathrm{D}^2(\sigma_{ab}\tilde{B}^b) = (\mathrm{D}^2\sigma_{ab})\tilde{B}^b$. Eliminating the auxiliary variable J_a, the general closed wave equation for I_a is found to be

$$\ddot{I}_{\langle a\rangle} - \mathrm{D}^2 I_a + 3\Theta \dot{I}_{\langle a\rangle} + [\tfrac{13}{9}\Theta^2 - \tfrac{1}{6}\mu - \tfrac{5}{2}p + \tfrac{7}{3}\Lambda] I_a = 0. \tag{23}$$

In the case of infinite conductivity, the solution to Eq. (23) instantly yields the solution of β_a, from which the induced magnetic field measured by the fundamental observer might be obtained by integration.

B. Wave equation for the electric field

To derive the wave equation for the induced electric field, we first differentiate Eq. (18) and equate the result with the second-order identity

$$(\mathrm{curl} B_a)^{\cdot}_{\perp} = -\Theta \mathrm{curl} B_a + \mathrm{curl}\beta_a - H_{ab}\tilde{B}^b \tag{24}$$

to obtain

$$\ddot{E}_{\langle a\rangle} + \tfrac{5}{3}\Theta \dot{E}_{\langle a\rangle} + [\tfrac{4}{9}\Theta^2 - \tfrac{1}{3}(\mu + 3p) + \tfrac{2}{3}\Lambda] E_a = \mathrm{curl}\beta_a - H_{ab}\tilde{B}^b. \tag{25}$$

Secondly, using Eq. (19) to substitute for $\mathrm{curl}\beta_a$ above and the expansion

$$\mathrm{curl}(\mathrm{curl} E_a) = -\mathrm{D}^2 E_a - [\tfrac{2}{9}\Theta^2 - \tfrac{2}{3}(\mu + \Lambda)] E_a, \tag{26}$$

we find a forced wave equation for the induced electric field, namely

$$\ddot{E}_{\langle a\rangle} - \mathrm{D}^2 E_a + \tfrac{5}{3}\Theta \dot{E}_{\langle a\rangle} + [\tfrac{2}{9}\Theta^2 + \tfrac{1}{3}(\mu - 3p) + \tfrac{4}{3}\Lambda] E_a = K_a, \tag{27}$$

where the forcing term $K_a \equiv \mathrm{curl} I_a - H_{ab}\tilde{B}^b = \epsilon_{cd[a}\mathrm{D}\sigma^c_{b]}B^b$ has no divergence. It is possible to show that the forcing term K_a, as well as $\mathrm{curl} I_a$ and $H_{ab}\tilde{B}^b$, respectively, can be found from the wave equation

$$\ddot{K}_{\langle a\rangle} - \mathrm{D}^2 K_a + \tfrac{11}{3}\Theta \dot{K}_{\langle a\rangle} + [\tfrac{22}{9}\Theta^2 - \tfrac{1}{3}(\mu + 9p) + \tfrac{8}{3}\Lambda] K_a = 0. \tag{28}$$

For example, the wave equation for $\mathrm{curl} I_a$ follows by taking the curl of Eq. (23) and using the expansion (26), while the case $H_{ab}\tilde{B}^b$ is similar to the derivation of the wave equation for the interaction term I_a.

It will be useful for later purposes to consider the electric field's rotation. By taking the curl of Eq. (27), we immediately arrive at

$$(\mathrm{curl} E_a)^{\cdot\cdot}_{\perp} - \mathrm{D}^2(\mathrm{curl} E_a) + \tfrac{7}{3}\Theta(\mathrm{curl} E_a)^{\cdot}_{\perp} + [\tfrac{7}{9}\Theta^2 + \tfrac{1}{6}(\mu - 9p) + \tfrac{5}{3}\Lambda]\mathrm{curl} E_a = \mathrm{curl} K_a. \tag{29}$$

Because $\mathrm{curl}(H_{ab}\tilde{B}^b) = -\mathrm{D}^2 I_a + [-\tfrac{5}{18}\Theta^2 + \tfrac{5}{6}\times(\mu + \Lambda)] I_a$ holds, we note the interesting result

$$\mathrm{curl} K_a = [\tfrac{1}{18}\Theta^2 - \tfrac{1}{6}(\mu + \Lambda)] I_a. \tag{30}$$

That is, for a cosmological model with flat spatial sections we have $\mathrm{curl} K_a = 0$ and, therefore, the electric field's rotation is not induced by the interaction between magnetic fields and GWs at second-order—the generated electric field is curl-free. As a consequence, the interaction between magnetic field and GW produces the same magnetic field in a case of a spatially flat Universe as in the limit of high conductivity.

However, upon closer inspection of the forcing term K_a in Eq. (27) one discovers that this term is actually identically zero because of the identity [22]

$$0 = \epsilon^{abc} V_b (\mathrm{D}_d A_c{}^d) - 2 V_b \epsilon^{cd[a} (\mathrm{D}_c A^{b]}{}_d), \tag{31}$$

which holds for any vector V_a and tensor $A_{ab} = A_{\langle ab \rangle}$ perpendicular to the congruence u_a. Thus, Eq. (30) implies that our chosen perturbative scheme is only consistent if the cosmological model is spatially flat (cf. also footnote IV B below). In essence, we see that the requirement of having a spatially homogeneous and thus a curl-free magnetic field at first-order can only be achieved when the Universe is spatially flat. Furthermore, the interaction between GW and a magnetic field generates in this particular case no electric fields (at least to second order in the perturbation scheme).

C. The generated magnetic field

We have already pointed out that for spatially flat models the generated magnetic field follows directly from the interaction variable since in this case we have $\beta_a = I_a$. For closed or open models, however, a wave equation for β_a is needed to determine the induced magnetic field. The sought after equation may be obtained by applying the constraint Eq. (19) to Eq. (29) and substituting for $\mathsf{curl} K_a$ via Eq. (30). This leads to

$$\begin{aligned} &\ddot{\beta}_{\langle a \rangle} - \mathrm{D}^2 \beta_a + \tfrac{7}{3} \Theta \dot{\beta}_{\langle a \rangle} + [\tfrac{7}{9} \Theta^2 + \tfrac{1}{6}(\mu - 9p) + \tfrac{5}{3}\Lambda] \beta_a \\ &\quad = \ddot{I}_{\langle a \rangle} - \mathrm{D}^2 I_a + \tfrac{7}{3} \Theta \dot{I}_{\langle a \rangle} + [\tfrac{13}{18} \Theta^2 + \tfrac{1}{3}\mu - \tfrac{3}{2} p + \tfrac{11}{6}\Lambda] I_a. \end{aligned} \tag{32}$$

Observe that for models with flat spatial sections the left-hand side and right-hand side of the above equation become identical—in agreement with the comment following Eq. (29). A slight simplification is achieved by employing Eq. (23) yielding finally a forced wave equation for β_a:

$$\begin{aligned} &\ddot{\beta}_{\langle a \rangle} - \mathrm{D}^2 \beta_a + \tfrac{7}{3} \Theta \dot{\beta}_{\langle a \rangle} + [\tfrac{7}{9} \Theta^2 + \tfrac{1}{6}(\mu - 9p) + \tfrac{5}{3}\Lambda] \beta_a \\ &\quad = -\tfrac{2}{3} \Theta \dot{I}_{\langle a \rangle} - [\tfrac{13}{18} \Theta^2 - \tfrac{1}{2}(\mu + 2p - \Lambda)] I_a. \end{aligned} \tag{33}$$

It is evident that the variable I_a and hence the gravitational waves source fluctuations in the magnetic field variable β_a. Another way to derive Eq. (33) consists of differentiating Maxwell's equation (19) twice, using Eq. (18) to get rid off the $\mathsf{curl} E_a$-term and applying the corresponding commutation relations. This clearly demonstrates the consistency of our approximation scheme.

IV. SOLUTIONS FOR FLAT UNIVERSES

After having derived the fundamental equations governing the interaction between GWs and magnetic fields as well as the generated electromagnetic fields, we turn to the task of solving them. For the sake of simplicity, we investigate the solutions only for spatially flat models with zero cosmological constant Λ. We assume the matter to obey a barotropic equation of state, $p = w\mu$, with constant barotropic index w.

A. A useful time variable

The background equations (1)–(3) subject to the assumptions stated above imply the following evolution equation for the scale factor:

$$\frac{\ddot{a}}{a} + \frac{1}{2}(1 + 3w)\left(\frac{\dot{a}}{a}\right)^2 = 0. \tag{34}$$

By integrating once and choosing initial conditions such that $\Theta_0 \equiv \Theta(t_0) = 3H_0$ for some arbitrary initial time t_0 with $H = \dot{a}/a$ defining the Hubble radius, we obtain the following solution for the expansion

$$\frac{1}{3}\Theta = \frac{\dot{a}}{a} = \frac{2}{3(1 + w)(t - t_0) + 2/H_0}. \tag{35}$$

Integrating once more, we find for the scale factor the solution

$$a(t) = a_0 [\tfrac{3}{2} H_0 (1 + w)(t - t_0) + 1]^{2/3(1+w)}. \tag{36}$$

The introduction of a dimensionless time variable, τ, defined for $w \neq -1$ as

$$\tau \equiv \tfrac{3}{2} H_0 (1 + w)(t - t_0) + 1, \tag{37}$$

will turn out to be extremely useful as it simplifies the integration of almost all the equations considered later irrespective of the barotropic index while taking the initial conditions explicitly into account as well. For example, the scale factor evolves simply as $a = a_0 \tau^{2/(3(1+w))}$ and the Hubble radius is given by $H = H_0/\tau$. Moreover, $\tau = 1$ corresponds to the initial time t_0. Note however that the τ variable cannot be used in the de Sitter limit $w \rightarrow -1$, where the scale factor becomes $a(t) = a_0 \exp(H_0(t - t_0))$.

B. Generated magnetic field

Since we are only considering Universes with *flat* spatial geometry, the induced magnetic field can be found by integrating over β_a. To this end, it suffices to solve for the interaction variable I_a. A standard harmonic decomposition [23] is used to take care of the Laplacian operator. We expand the shear $\sigma_{ab} = \sum_k \sigma^{(k)} Q^{(k)}_{ab}$ in pure tensor harmonics, where as usual $\dot{Q}^{(k)}_{\langle ab \rangle} = 0$ and $\mathrm{D}^2 Q^{(k)}_{ab} = -(k^2/a^2) Q^{(k)}_{ab}$ hold. Moreover, each gravitational wave mode is associated with the physical wave length $\lambda_{\mathrm{GW}} = 2\pi a/k$. Since the magnetic field in $\mathcal{F}_1$ obeys $\mathsf{curl} \tilde{B}_a = 0$, it follows that $D^2\tilde{B}_a = -\mathsf{curl}(\mathsf{curl}\tilde{B}_a) = 0$ and therefore that the expansion of the magnetic field $\tilde{B}_a = \sum_n \tilde{B}^{(n)} Q_a^{(n)}$ in pure vector (solenoidal) harmonics reduces to $\tilde{B}_a = \tilde{B}^{(0)} Q_a^{(0)}$, where $\tilde{B}^{(0)} = \tilde{B}^0 (a_0/a)^2$. This just means that the magnetic field $\tilde{B}^a$ is spatially constant, e.g., in agreement with the assumption of homogeneity.[3] Of course, the solenoidal harmonics also obey the relations $\dot{Q}^{(n)}_{\langle a\rangle} = 0$ and $D^2 Q_a^{(n)} = -(n^2/a^2) Q_a^{(k)}$. Perturbations in S are conveniently decomposed with the vector harmonics[4] $V_a^{(\ell)} \equiv Q^{(k)}_{ab} Q^b_{(n)}$, which are readily verified to fulfill the standard requirements $\dot{V}^{(\ell)}_{<a>} = 0$ and $D^2 V_a^{(\ell)} = -(\ell^2/a^2) V_a^{(\ell)}$, where the wavenumber ℓ satisfies $\ell^2 = (k_a + n_a) \times (k^a + n^a)$. Because the magnetic field in $\mathcal{F}_1$ has got only the zero mode in our investigation, the wavenumber ℓ coincides with the wavenumber k of the shear.

Using the unified time variable τ and the harmonics explained above, we transform the wave Eq. (23) for the interaction variable I_a into an ordinary differential equation:

$$\frac{9}{4}(1+w)^2 I''_{(\ell)} + \frac{27(1+w)}{2\tau} I'_{(\ell)} + \left[\left(\frac{\ell}{a_0 H_0}\right)^2 \tau^{-[4/3(1+w)]} + \frac{25-15w}{2\tau^2}\right] I_{(\ell)} = 0, \quad (38)$$

where a prime means differentiation with respect to τ. Initial conditions are chosen as follows:

$$I_{(\ell)}(t_0) = \sigma_{(k)}(t_0)\tilde{B}_0, \quad (39)$$

$$I'_{(\ell)}(\tau = 1) = \tilde{B}_0\left[\sigma'_{(k)}(1) - \frac{4}{3(1+w)}\sigma_{(k)}(1)\right]; \quad (40)$$

here, $\tilde{B}_0$ is the initial amplitude of the first-order magnetic field and

$$\dot{\sigma}_{(k)}(t_0) = 3/2 H_0 (1+w) \sigma'_{(k)}(1) \quad (41)$$

was used. For every mode k we have initially $\sigma(t_0) = \sigma_0$ and $\sigma'(1) = \sigma'_0$.

1. Infinite-wavelength limit

In the infinite-wavelength limit ($\ell \to 0$), the solution of Eq. (38) is easily found to be

$$I^{(0)}(\tau) = C_1 \tau^{-[10/3(1+w)]} + C_2 \tau^{(-5+3w)/3(1+w)}, \quad (42)$$

where C_1 and C_2 are constants of integration. If the initial conditions (39) and (40) are chosen, the corresponding integration constants are

$$C_1 = \frac{(-5+3w) I_{(\ell)}(1) - 3(1+w) I'_{(\ell)}(1)}{5+3w}, \quad (43)$$

$$C_2 = \frac{10 I_{(\ell)}(1) + 3(1+w) I'_{(\ell)}(1)}{5+3w}. \quad (44)$$

We remark that this solution is in agreement with the result obtained by multiplying the first-order magnetic field (5) with the infinite-wavelength solution of the shear Eq. (10). Whence, the total magnetic field in the presence of infinite-wavelength GWs is

$$B^{(0)}(\tau) = \tilde{B}_0 \tau^{-[4/3(1+w)]}\left[1 - \frac{C_1}{\tilde{B}_0 H_0}\frac{2}{3(1-w)} \times (\tau^{(-1+w)/3(1+w)} - 1) + \frac{C_2}{\tilde{B}_0 H_0}\frac{1}{1+3w} \times (\tau^{(2+6w)/3(1+w)} - 1)\right], \quad (45)$$

where $\tilde{B}_0$ is the magnitude of the first-order magnetic field interacting with the GW at initial time t_0 and it is required for physical reasons that the induced magnetic field vanishes initially. We stress that the interaction always leads to an amplification of the magnetic field for any physically acceptable choice of equation of state because of the growing contribution in the second line of Eq. (45).

Let us look at some important special cases. For the sake of simplicity, we take $I'_{(\ell)}(1) = 0$ for granted. In the matter-dominated era, where the matter is accurately described as dust, $w = 0$ and $a = a_0 \tau^{2/3}$, this yields for the magnetic field mode

$$B^{(0)}_{\rm Dust}(a) = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left[1 + \frac{2}{3}\frac{\sigma_0}{H_0}\left\{\left(\frac{a_0}{a}\right)^{3/2} - 1\right\} + \frac{2\sigma_0}{H_0}\left\{\frac{a}{a_0} - 1\right\}\right], \quad (46)$$

whereas for a radiation-dominated era, where $w = 1/3$ and $a = a_0 \tau^{1/2}$, the magnetic field mode is

$$B^{(0)}_{\rm Rad}(a) = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left[1 + \frac{2}{3}\frac{\sigma_0}{H_0}\left\{\frac{a_0}{a} - 1\right\} + \frac{5}{6}\frac{\sigma_0}{H_0}\left\{\left(\frac{a}{a_0}\right)^2 - 1\right\}\right]. \quad (47)$$

It follows that in the infinite-wavelength limit the amplification depends mainly on the scale factor and the magnitude of the initial GW distortion relative to the Hubble parameter $(\sigma/H)_0$.

[3] In light of the commutator relation (26), which holds for $\tilde{B}^a$ in $\mathcal{F}_1$, $D_a\tilde{B}_b = 0$ (which also leads to $\mathsf{curl}\tilde{B}_a = 0$ in our approximation scheme) is only consistent for a spatially flat Universe—in an open or closed Universe, a current is needed to uphold the magnetic field's homogeneity.

[4] It should be kept in mind that all above introduced harmonics are exclusively defined on the background FLRW spacetime.

2. General case with $\ell \neq 0$

The general solution to the interaction Eq. (38) is:

$$I_{(\ell)}(\tau) = \tau^{(-5+w)/2(1+w)}\bigg[D_1 J_1\bigg(\frac{3w+5}{2(1+3w)}, \frac{\ell}{a_0 H_0} \times \frac{2}{1+3w}\tau^{(1+3w)/3(1+w)}\bigg) + D_2 J_2\bigg(\frac{3w+5}{2(1+3w)}, \frac{\ell}{a_0 H_0} \times \frac{2}{1+3w}\tau^{(1+3w)/3(1+w)}\bigg)\bigg], \qquad (48)$$

where D_1, D_2 are integration constants and J_1, J_2 denote Bessel functions of the first and second kind, respectively. Observe that in the limit of infinite wavelengths, $\ell \to 0$, the solution (42) is recovered. The generated magnetic field relative to the observer moving with 4-velocity u^a can be calculated from the solution (48) analytically for every barotropic parameter w. We will state here only the total magnetic field solution in the case of dust and radiation, respectively. For dust, where $w = 0$ and $a = a_0\tau^{2/3}$, the full magnetic field is

$$B^{(\ell)}_{\rm Dust}(a) = \tilde{B}_0\bigg(\frac{a_0}{a}\bigg)^2\bigg[1 + \frac{3}{4\pi^2}\bigg(\frac{\lambda_{\rm GW}}{\lambda_{\rm H}}\bigg)^2_0\bigg(\frac{\sigma_0}{H_0} + \frac{\sigma_0'}{2H_0}\bigg) + \mathcal{O}(a^{-1})\bigg], \qquad (49)$$

while for radiation, where $w = 1/3$ and $a = a_0\tau^{1/2}$, the total magnetic field modes obey

$$B^{(\ell)}_{\rm Rad}(a) = \tilde{B}_0\bigg(\frac{a_0}{a}\bigg)^2\bigg[1 + \frac{3}{4\pi^2}\bigg(\frac{\lambda_{\rm GW}}{\lambda_{\rm H}}\bigg)^2_0\bigg(\frac{\sigma_0}{H_0} + \frac{2\sigma_0'}{3H_0}\bigg) + \mathcal{O}(a^{-1})\bigg]. \qquad (50)$$

Here, we introduced the gravitational wavelength $\lambda_{\rm GW} = 2\pi a/k$ and the Hubble length $\lambda_{\rm H} = 1/H$. The undisplayed remainders $\mathcal{O}(a^{-1})$ in the expressions above contain oscillating functions which decay at least as fast as the inverse scale factor a^{-1}. Note that when the infinite-wavelength limit of the full solutions above is taken, the findings (46) and (47) are rediscovered. The results (49) and (50) clearly show how the generated magnetic field depends on the initial conditions and that the late time behavior is almost identical for both dust and radiation. It should be noted that the interaction can only be effective if the wavelength of the GW matches the size of the magnetic field region, $\lambda_{\rm GW} \sim \lambda_{\tilde{\rm B}}$: in the case of $\lambda_{\rm GW} \gg \lambda_{\tilde{\rm B}}$ the magnetic field cannot be physically affected by the GW, while for $\lambda_{\rm GW} \ll \lambda_{\tilde{\rm B}}$ the effect becomes negligible due to its quadratic dependence on $\lambda_{\rm GW}$. If we divide the findings (49) and (50) through the energy density of the background radiation, the dominant contribution can be summarized as follows,

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \bigg[1 + \frac{1}{10}\bigg(\frac{\lambda_{\tilde{\rm B}}}{\lambda_{\rm H}}\bigg)^2_0\bigg(\frac{\sigma}{H}\bigg)_0\bigg]\bigg(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\bigg)_0, \qquad (51)$$

where the wavenumber indices have been suppressed and $\sigma_0' = 0$ was assumed. At late times, a significant amplification of the original magnetic field can be achieved for superhorizon gravitational waves. Note that a result almost identical to (51) was obtained in [15], wherein the factor $1/10$ is replaced by 10 instead. However, our result holds for *any* finite gravitational wavelength, $\lambda_{\rm GW} \sim \lambda_{\tilde{\rm B}}$, while the result in [15] assumes $\lambda_{\rm H} \ll \lambda_{\rm GW}$. Moreover, [15] used somewhat contrived initial conditions leading to an abrupt amplification of the field whereas we chose initial conditions such that there is no generated magnetic field present when the interaction kicks in at the end of inflation.

V. APPLICATION

In order to estimate the amplification of the seed field due to the interaction with GWs we reproduce the analysis presented in [15] using the same parameter values. We find it convenient to adopt natural units in this section.

Given that the evolution of the (spatially flat) Universe is dominated by a dark-energy component such as a cosmological constant or quintessence, the minimum seed required for the dynamo mechanism to work is of the order of 10^{-30} G at the time of completed galaxy formation and coherent on a scale at least as large as the largest turbulent eddy, roughly $\sim$100 pc [24]. Such a collapsed magnetic field corresponds to a field $\tilde{B}$ of $\sim 10^{-34}$ G with coherence length $\lambda_{\tilde{\rm B}} \sim 10$ kpc on a comoving scale if the field remains frozen into the cosmic plasma from the epoch of radiation decoupling to galaxy formation. Its field strength compared to the energy density of the background radiation, μ_γ, gives rise to the ratio $\tilde{B}/\mu_\gamma^{1/2} \sim 10^{-29}$, which stays constant as long as the magnetic flux is conserved and the magnetic field is frozen into the cosmic medium.

During inflation, the Hubble parameter H remains constant and is taken to be $H \sim 10^{13}$ GeV [13]. The scale of the magnetic field therefore implies $\lambda_{\tilde{\rm B}}/\lambda_{\rm H} \sim 10^{20}$ at the end of inflation. A general prediction of all inflationary scenarios is the production of large-scale gravitational waves whose energy density per wavelength is roughly [15]

$$\mu_{\rm GW} \simeq m_{\rm Pl}^2\bigg(\frac{1}{\lambda_{\rm GW}}\bigg)^2\bigg(\frac{H}{m_{\rm Pl}}\bigg)^2. \qquad (52)$$

Here, $\lambda_{\rm GW}$ denotes the wavelength of the GW and $m_{\rm Pl}$ the Planck mass (see, for example, [25]). The total energy density of the gravity waves expressed in terms of the shear is (see footnote 4 in [26] for a neat discussion)

$$\mu_{\rm GW} = \frac{m_{\rm Pl}^2}{16\pi}\sigma_{ab}\sigma^{ab}, \qquad (53)$$

which implies an induced shear anisotropy [15]

$$\left(\frac{\sigma}{H}\right)_0 \simeq \left(\frac{\lambda_{\rm H}}{\lambda_{\rm GW}}\right)_0 \left(\frac{H}{m_{\rm Pl}}\right), \tag{54}$$

where the zero suffix indicates the end of the inflationary epoch. Typical inflationary models predict $H/m_{\rm Pl} \sim 10^{-6}$, which lies comfortably within the bound $H/m_{\rm Pl} \lesssim 10^{-5}$ stemming from the quadrupole anisotropy of the CMB.

The interaction of such a primordial magnetic field with GWs produced by inflation leads to a substantial amplification of the former. Resorting to our result (51) and applying (54), we find for the magnetic field [15]

$$\frac{B}{\mu_\gamma^{1/2}} \simeq \left[1 + \frac{1}{10}\left(\frac{\lambda_{\tilde{\rm B}}}{\lambda_{\rm H}}\right)_0 \left(\frac{H}{m_{\rm Pl}}\right)\right] \left(\frac{\tilde{B}}{\mu_\gamma^{1/2}}\right)_0. \tag{55}$$

Substituting $(\lambda_{\tilde{\rm B}}/\lambda_{\rm H})_0 \sim 10^{20}$ and $H/m_{\rm Pl} \sim 10^{-6}$ into the above expression, we obtain that GWs amplify the original magnetic field as much as 13 orders of magnitude. This mechanism thus brings an inflationary seed such as in [13] up to $\sim 10^{-21}$ G, which is comfortably within the requirements of the galactic dynamo mechanism [24]. In Universes with zero cosmological constant, the minimum seed for the dynamo has to be raised from $\sim 10^{-30}$ G to $\sim 10^{-23}$ G [27]. However, the use of (52) and (53) to find the shear anisotropy is problematic in the sense that (52) holds strictly speaking only on scales up to the horizon; the energy stored in superhorizon modes cannot be measured by local observers. Once the initially superhorizon GWs re-enter the observer's horizon, they contribute to the measured energy density. Therefore, the correct procedure is to use the value of the shear anisotropy at horizon crossing, $(\sigma/H)_{\rm HC}$, and scale that value back to the end of inflation using its evolution equation. During the radiation-dominated era, the gravitational wave length varies with the scale factor (i.e. $\lambda_{\rm GW} \sim \tau^{1/2}$), while the horizon scales as the inverse of the square of the scale factor (i.e. $\lambda_{\rm H} = H^{-1} \sim \tau$). This gives

$$\left(\frac{\lambda_{\rm GW}}{\lambda_{\rm H}}\right) = \left(\frac{\lambda_{\rm GW}}{\lambda_{\rm H}}\right)_0 \tau^{-1/2}, \tag{56}$$

where $\tau = 1$ corresponds to the end of inflation after reheating. At the point in time where the gravitational wave crosses back into the horizon, its physical wavelength equals the Hubble scale $\lambda_{\rm H}$, that is $(\lambda_{\rm GW}/\lambda_{\rm H})_{\rm HC} = 1$. Substituting for the values from above, we obtain the crossing time $\tau_{\rm HC} = 10^{40}$, which hitherto leads to the Hubble parameter $H_{\rm HC} \sim 10^{-27}$ GeV at horizon crossing. During the radiation era, the temperature is proportional to the square of the Hubble parameter [25], $T \sim \sqrt{m_{\rm PL} H/10}$, yielding a temperature of $T_{\rm HC} \sim 10^{-5}$ GeV at horizon crossing. Since 1 GeV $\sim 10^{13}$ K, the actual temperature is $\sim 10^8$ K, confirming that the superhorizon mode under consideration indeed crosses back into the horizon during the radiation-dominated epoch. Whence, combining (52) and (53) gives a shear anisotropy

$$\Sigma_{\rm HC} \equiv \left(\frac{\sigma}{H}\right)_{\rm HC} \sim \left(\frac{H}{m_{\rm Pl}}\right)_0 \sim 10^{-6} \tag{57}$$

at the time of horizon crossing.

Assuming a flat model with no cosmological constant, the evolution of the shear anisotropy, $\Sigma = \sigma/H$, can be obtained by solving for the shear modes $\sigma_{(k)}$ from Eq. (10) and noting that $H \sim \tau^{-1}$. The result is

$$\begin{aligned}\Sigma_{(k)}(\tau) = \tau^{(-1+9w)/6(1+w)}\Bigg[&AJ_1\bigg(\frac{3w+5}{2(1+3w)}, \frac{k}{a_0 H_0} \\ &\times \frac{2}{1+3w}\tau^{(1+3w)/3(1+w)}\bigg) \\ &+ BJ_2\bigg(\frac{3w+5}{2(1+3w)}, \frac{k}{a_0 H_0} \\ &\times \frac{2}{1+3w}\tau^{(1+3w)/3(1+w)}\bigg)\Bigg],\end{aligned} \tag{58}$$

where A, B are constants of integrations and J_1, J_2 denote Bessel functions of the first and second kind, respectively. Since we are only interested in the growing mode, we may set $B = 0$ eliminating the decaying mode contribution. Specializing to the case of radiation, $w = 1/3$, remembering that horizon crossing for the mode under consideration (for which $k/(a_0 H_0) = 2\pi(\lambda_{\rm H}/\lambda_{\rm GW})_0 \sim 2\pi \times 10^{-20}$ holds) happens at $\tau_{\rm HC} = 10^{40}$ and using the estimate (57) for the shear anisotropy at horizon crossing, one determines the remaining constant to be $|A| \sim 10^{-16}\pi$. Hence, at the end of inflation ($\tau = 1$), one finally obtains for the sought shear anisotropy

$$\Sigma_0 = \left(\frac{\sigma}{H}\right)_0 \sim 10^{-45}, \tag{59}$$

where the approximation $J_1(\nu, x) \sim x^\nu$ for small arguments $x \ll 1$ of the Bessel function of the first kind has been employed. This is remarkably close to the result one would obtain by simply using the growing mode solution of (58) in the limit $k/(a_0 H_0) \ll 1$, that is $\Sigma_{(k)} = \Sigma_0 \tau$ in the case of radiation, which gives $\Sigma_0 \sim 10^{-46}$. If the above value for the shear anisotropy is used in (51), the gravitomagnetic amplification is completely negligible:

$$\frac{1}{10}\left(\frac{\lambda_{\tilde{\rm B}}}{\lambda_{\rm H}}\right)_0^2 \left(\frac{\sigma}{H}\right)_0 \sim 10^{-6}. \tag{60}$$

We stress that the efficiency of the mechanism depends crucially on the ratio between the coherence length $\lambda_{\tilde{\rm B}}$ of the initial magnetic field and the initial size of the horizon $\lambda_{\rm H}$. This ratio, however, disappears when the infinite-wavelength limit is taken (see Sec. IV B 1). Even though the solutions (46) and (47) show a growth proportional (quadratic) to the scale factor, the factor of proportionality $(\sigma/H)_0$ ($\sim 10^{-26}$ or $\sim 10^{-45}$ in our first and second examples, respectively) is far too small in order to achieve an effective amplification. It follows that the interaction between GWs and on average homogeneous magnetic fields is completely negligible in the limit of infinitely long-wavelength gravity waves.

VI. COMPARISON AND DISCUSSION

The interaction between GWs and magnetic fields in the cosmological setting has recently been investigated in [15], where the weak-field approximation [16] was used. Here one allows for a weak magnetic test field $\tilde{B}_a$ in the background, whose energy density, anisotropic stress and spatial dependence have negligible impact on the background dynamics: $\tilde{B}^2 \ll \mu$ and $\pi_{ab} = -\tilde{B}_{\langle a}\tilde{B}_{b\rangle} \simeq 0 \simeq \mathrm{D}_a\tilde{B}_b$ to zero order. In order to isolate linear tensor perturbations, it is necessary to impose $\mathrm{D}_a\tilde{B}^2 = 0 = \epsilon_{abc}\tilde{B}^b \mathrm{curl}\tilde{B}^c$ in addition to the standard constraints $\omega_a = 0 = \mathrm{D}_a\mu = \mathrm{D}_a p$ associated with pure perfect fluid cosmologies. In the weak-field approximation, the main equations governing the induced magnetic field arising from the interaction between a weak background magnetic field $\tilde{B}^a$ and GWs were derived in [15] for the case of a spatially flat Universe with vanishing cosmological constant Λ and a barotropic equation of state $p = w\mu$:

$$\ddot{B}_{(\ell)} + \frac{5}{3}\Theta\dot{B}_{(\ell)} + \left[\frac{1}{3}(1-w)\Theta^2 + \frac{\ell^2}{a^2}\right]B_{(\ell)} = 2\left(\dot{\sigma}_{(k)} + \frac{2}{3}\Theta\sigma_{(k)}\right)\tilde{B}_0^{(n)}\left(\frac{a_0}{a}\right)^2, \tag{61}$$

where the GWs are determined by the shear wave equation

$$\ddot{\sigma}_{(k)} + \frac{5}{3}\Theta\dot{\sigma}_{(k)} + \left[\frac{1}{6}(1-3w)\Theta^2 + \frac{k^2}{a^2}\right]\sigma_{(k)} = 0. \tag{62}$$

Here, the shear is harmonically decomposed as $\sigma_{ab} = \sigma_{(k)}Q_{ab}^{(k)}$, while for the induced magnetic field $B_a^{(\ell)} = B_{(\ell)}V_a^{(\ell)}$ with $V_a^{(\ell)} = Q_{ab}^{(k)}Q_{(n)}^b$ was adopted. The background magnetic field evolves as $\tilde{B}_a = \tilde{B}_a^0(a_0/a)^2$ and $\tilde{B}_a^0 = \tilde{B}_{(n)}^0 Q_a^{(n)}$ is assumed.

We want to compare our results with the corresponding ones in the weak-field approximation. For simplicity, we restrict ourselves here to the case of dust. As pointed out above, the only allowed magnetic wavenumber for the interacting magnetic field is $n = 0$, when $\mathrm{D}_a\tilde{B}_b = 0$, which leads to $\ell = k$. The published solution for the generated magnetic field in the weak-field approximation, e.g., equation (21) in [15], however, is not applicable in the limit $n \to 0$. This can be traced back to the choice for the initial conditions for the generated magnetic field made by the authors of [15] when solving Eqs. (61) and (62), see equation (19) in [15].

In what follows below, we solve Eqs. (61) and (62) again, including the full solution for the shear instead of merely keeping the dominant part as done in [15]. We specify the initial conditions by choosing for every mode k of the shear $\sigma_{(k)}(a_0) = \sigma_0$, $\dot{\sigma}_{(k)}(a_0) = 0$ and for every mode $\ell = k$ of the generated magnetic field $B_{(\ell)}(a_0) = 0 = \dot{B}_{(\ell)}(a_0)$. Note that this choice of initial conditions differs from that in [15] but agrees with our choice made in Sec. IV. The solution, including the background field, for an arbitrary wavenumber k of the shear has the structure

$$B_{\mathrm{Dust}}^{(\ell)}(a) = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left[1 + \frac{\sigma_0}{H_0}f(\sqrt{a};k) + \mathcal{O}(a^{-1/2})\right], \tag{63}$$

where the function $f(\sqrt{a};k)$ is built of several oscillatory terms with amplitude $(\lambda_{\mathrm{GW}}/\lambda_{\mathrm{H}})_0^2$ at most and the undisplayed part falls of at least as fast as $a^{-1/2}$. If this is compared with our result (49), one observes that it differs by having another time behavior. More strikingly, however, is that now the term $f(\sqrt{a};k)$ not only amplifies the seed field but also grows like $\sqrt{a}$ in the long wavelength limit $(k/a_0H_0 \ll 1)$. This is in clear contrast to the gauge-invariant result (49), where the seed undergoes amplification but then still decays adiabatically like a^{-2}. On the other hand, in the infinite-wavelength limit ($k \to 0$), the exact full solution is now

$$B_{\mathrm{Dust}}^{(0)}(a) = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left[1 + \frac{\sigma_0}{H_0}\left\{\frac{20}{3} - 14\left(\frac{a}{a_0}\right)^{1/2} + \frac{36}{5}\left(\frac{a}{a_0}\right) + \frac{2}{15}\left(\frac{a_0}{a}\right)^{3/2}\right\}\right]. \tag{64}$$

Again, we obtain a solution whose time behavior differs from that found in (46). However, the weak-field solutions agree with our presented solutions in the infinite-wavelength limit when only the dominant part of the solutions is considered, at least in the examples considered above. The reason why the solutions obtained within the weak-field approximation are in general not equivalent to the solutions found using the gauge-invariant approach developed in this paper results from the non-gauge-invariance of the weak-field approximation, where the magnetic field $\tilde{B}_a$ interacting with the GW is treated as a weak background field. However, gauge-invariance requires $\tilde{B}_a$ to vanish exactly in the FLRW background. We remind the reader once more that our procedure solves firstly for the gauge-invariant variable $\beta_a = \dot{B}_{\langle a\rangle} + \frac{2}{3}\Theta B_a$, from which the magnetic field B_a measured in the frame of reference of u^a can then subsequently be found.

A further important remark concerns the issue of conductivity. We have seen earlier that, within our assumptions and for spatially flat Universes, the gravito-magnetic interaction leads to an induced magnetic field which is independent of the conductivity of the cosmic medium. This is due to the fact that the interaction does not generate rotational electric field modes which might affect the magnetic field. In the weak-field approximation, however, the situation is completely different. If one assumes that the conductivity of the cosmic medium that high so that electric fields are quickly dissipated away, yielding a curl-free induced magnetic field, then Eq. (61) no longer applies and one simply has to use

$$\dot{B}_{(\ell)} + \frac{2}{3}\Theta B_{(\ell)} = \sigma_{(k)}\tilde{B}_0^{(n)}\left(\frac{a_0}{a}\right)^2 \tag{65}$$

instead, while the equation for the shear (62) is unaltered. This means that the weak-field approximation produces the same result as our gauge-invariant perturbation approach in the high conductivity limit, and for that case only. It is therefore evident that in the weak-field approximation the conductivity of the cosmic medium has a crucial bearing on the generated magnetic field, in stark contrast to the result of our gauge-invariant approach (see also [28]).

VII. CONCLUSION

In this paper we have investigated the properties of magnetic fields in the presence of cosmological gravitational waves, using a two parameter perturbation scheme. Using proper second-order gauge-invariant variables (SOGI), we were able to obtain results in terms of clearly defined quantities, with no ambiguity concerning the physical validity of the variables. The full set of equations determining the evolution of the gravitational waves and the generated electromagnetic fields was presented, and the integration shows an amplification of the induced magnetic field due to the interaction of a "background" magnetic field with gravitational waves. The magnitude of the original magnetic field is amplified by an amount proportional to the magnitude of the gravitational wave induced shear anisotropy and the square of the field's initial comoving scale Once the amplification saturates, the magnetic field dissipates adiabatically as usual. The results were discussed in different fluid regimes, in particular, dust and radiation, and it was established that the dominant contribution to the magnetic field is the same in both fluid regimes. We find that the magnitude of the gravitational boost depends significantly on the manner in which the estimate of the shear anisotropy at the end of inflation is calculated. For a seed field of 10^{-34} G spanning a comoving scale of about 10 kpc today, the shear anisotropy at the end of inflation (during which we assume $H \sim 10^{13}$ GeV) should be larger than 10^{-40} for any noticeable amplification of the seed field to arise at all.

Moreover, we further recalculated the induced magnetic field employing the weak-field approximation, thereby extending previous results in [15], and compared the solutions with ours derived in a gauge-invariant manner using SOGI variables. It was found that there is a significant difference in the growth behavior of the magnetic field when SOGI variables are used as compared to the case of a weak-field approximation scheme. While the two methods agree in the limit of high conductivity, they seem to be compatible otherwise only in the limit of infinitely long-wavelength gravitational waves when the dominant part of the solution is considered.

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APPENDIX: COMMUTATION RELATIONS

Here we present various commutator relations which have been used in the text. The relations are given up to second order in our perturbation scheme. The vanishing of vorticity, $\omega_{ab} = 0$, is assumed throughout in conjunction with the constraints $\mathrm{D}_a\mu = \mathrm{D}_a p = 0$ which isolate the pure tensor modes. All appearing tensors are PSTF, $S_{ab} = S_{\langle ab\rangle}$, and all vectors V_a, W_a are purely spatial.

Commutators for first-order vectors V_a:

$$(\mathrm{D}_a V_b)^{\cdot}_{\perp} = \mathrm{D}_a \dot{V}_b - \tfrac{1}{3}\Theta \mathrm{D}_a V_b - \sigma_a{}^c \mathrm{D}_c V_b + H_a{}^d \epsilon_{dbc} V^c \tag{A1}$$

$$(\mathrm{curl} V_a)^{\cdot}_{\perp} = \mathrm{curl}\dot{V}_a - \tfrac{1}{3}\Theta\,\mathrm{curl} V_a - \epsilon_{abc}\sigma^{bd}\mathrm{D}_d V^c - H_{ab}V^b \tag{A2}$$

$$\mathrm{D}_{[a}\mathrm{D}_{b]}V_c = [\tfrac{1}{9}\Theta^2 - \tfrac{1}{3}(\mu + \Lambda)]V_{[a}h_{b]c} + (\tfrac{1}{3}\Theta\sigma_{c[a} - E_{c[a})V_{b]} + h_{c[a}(E_{b]d} - \tfrac{1}{3}\Theta\sigma_{b]d})V^d \tag{A3}$$

Commutators for first-order tensors S_{ab}:

$$(\mathrm{D}_a S_{bc})^{\cdot}_{\perp} = \mathrm{D}_a \dot{S}_{bc} - \tfrac{1}{3}\Theta \mathrm{D}_a S_{bc} - \sigma_a{}^d \mathrm{D}_d S_{bc} + 2H_a{}^d \epsilon_{de(b}S_{c)}{}^e \tag{A4}$$

$$(\mathrm{D}^b S_{ab})^{\cdot}_{\perp} = \mathrm{D}^b \dot{S}_{ab} - \tfrac{1}{3}\Theta \mathrm{D}^b S_{ab} - \sigma^{bc}\mathrm{D}_c S_{ab} + \epsilon_{abc}H^b_d S^{cd} \tag{A5}$$

$$(\mathrm{curl} S_{ab})^{\cdot}_{\perp} = \mathrm{curl}\dot{S}_{ab} - \tfrac{1}{3}\Theta\,\mathrm{curl} S_{ab} - \sigma_e{}^c \epsilon_{cd(a}D^e S_{b)}{}^d + 3H_{c\langle a}S_{b\rangle}{}^c \tag{A6}$$

$$\mathrm{curl\,curl} S_{ab} = -D^2 S_{ab} + (\mu + \Lambda - \tfrac{1}{3}\Theta^2)S_{ab} + \tfrac{3}{2}D_{\langle a}D^c S_{b\rangle c} + 3S_{c\langle a}(E_{b\rangle}{}^c - \tfrac{1}{3}\Theta\sigma_{b\rangle}{}^c) \tag{A7}$$

Commutators for second-order vectors W_a:

$$(\mathrm{D}_a W_b)^{\cdot}_{\perp} = \mathrm{D}_a \dot{W}_b - \tfrac{1}{3}\Theta \mathrm{D}_a W_b \tag{A8}$$

$$\mathrm{D}_{[a}\mathrm{D}_{b]}W_c = [\tfrac{1}{9}\Theta^2 - \tfrac{1}{3}(\mu + \Lambda)]W_{[a}h_{b]c} \tag{A9}$$

$$\mathrm{curl\,curl} W_a = -\mathrm{D}^2 W_a + \mathrm{D}_a(\mathrm{div} W) + \tfrac{2}{3}(\mu + \Lambda - \tfrac{1}{3}\Theta^2)W_a \tag{A10}$$

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