

Resonant amplification of magnetic seed fields by gravitational waves in the early universe

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Inflation is known to produce both gravitational waves and seed magnetic fields on scales well beyond the size of the horizon. The general relativistic study of the interaction between these two sources after the end of inflation, showed a significant amplification of the initial magnetic seed which brought the latter within the currently accepted dynamo limits. In the present article we revisit this gravitomagnetic interaction and argue that the observed strong growth of the field is the result of resonance. More specifically, we show that the maximum magnetic boost always occurs when the wavelength of the inducing gravitational radiation and the scale of the original seed field coincide. We also look closer at the physics of the proposed Maxwell-Weyl coupling, consider the implications of finite electrical conductivity for the efficiency of the amplification mechanism and clarify further the mathematics of the analysis.

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I. INTRODUCTION

Observations have repeatedly verified the widespread presence of magnetic fields in the universe [1]. Large-scale fields have been found in galaxies, galaxy clusters, superclusters and also in high-redshift radio galaxies. The typical magnetic strengths vary between a few and several μG , while the associated coherence lengths are comparable to those of the virialized hosts. Despite their ubiquitous presence, however, the origin of these fields is still a matter of open debate [2]. Over the years many scenarios of cosmic magnetogenesis have appeared in the literature. These range from eddies, density fluctuations and reionization effects in the post-recombination plasma to cosmological phase-transitions, inflationary and superstring/M-theory inspired scenarios [3]. Historically, the study of magnetic generation has been motivated by the need to explain the origin of the large-scale galactic fields. The structure of these fields, particularly those seen in spiral galaxies, supports the galactic dynamo idea [4]. Although the efficiency of the mechanism has been criticized, it is generally believed that galactic dynamos can substantially amplify preexisting weak magnetic seeds. The origin of the seed fields, however, is still elusive. When the dynamo amplification is efficient, the initial field can be as low as $\sim 10^{-23}$ G at the time of completed galaxy formation [5]. This limit is relaxed to $\sim 10^{-30}$ G in universes dominated by dark energy [6]. In the absence of dynamo, however, magnetic seeds of the order of 10^{-12} G, or even 10^{-8} G, are required. The scale of the initial field is also an issue, since galactic dynamos require a minimum coherence length of ~ 100 pc to guarantee the stability of the process [7]. In summary, the lowest current theoretical requirement for the dynamo to work is a magnetic seed close to 10^{-30} G on a collapsed scale of ~ 100 pc. This corresponds to a

field of approximately 10^{-34} G with a comoving length of roughly 10 kpc.

The possible cosmological origin of the initial magnetic seeds is an appealing suggestion because it can explain both the fields seen in nearby galaxies and those detected in galaxy clusters and high-redshift condensations. Causality, however, means that the coherence length of any field generated between inflation and (roughly) recombination is unacceptably small. A mechanism known as “inverse cascading” can solve this problem [8], but it requires large amounts of helicity to operate successfully. Inflation has long been suggested as a solution to the causality problem because it naturally achieves correlations on superhorizon scales. The problem with inflation is that the residual magnetic field is far too weak to sustain the galactic dynamo. The reason is the “adiabatic,” a^{-2} depletion rate of the field (a is the cosmological scale factor) during the de Sitter phase. One can get around this obstacle by slowing down the decay of the primordial seed. The effect is known as “superadiabatic amplification” and it is usually achieved by breaking away from classical electromagnetic theory [9]. There are more than one ways of doing that, which explains the variety of the proposed mechanisms in the literature [10,11].

It should be noted that when the Friedman-Robertson-Walker (FRW) model has open spatial curvature, the coupling between the field and the background geometry can slow down the magnetic decay without violating classical electromagnetism [12]. This occurs during the poorly conductive phase of de Sitter inflation and affects lengths close to the curvature scale and beyond. As a result, magnetic fields spanning those lengths decay as a^{-1} (or slower) instead of a^{-2} . Even if the universe is only marginally open today, this mechanism can produce large-scale fields with astrophysically interesting strengths. For example, assuming $1 - \Omega \sim 10^{-2}$ at present, GUT-scale inflation and a reheating temperature of $\sim 10^9$ GeV, one obtains a

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residual field of the order of 10^{-35} G on a scale $\sim 10^4$ Mpc today [12]. Moreover, the aforementioned final strength can increase by lowering the reheating temperature. Therefore, breaking away from Maxwell's theory is not a necessary requirement for the superadiabatic amplification of cosmological magnetic fields in perturbed FRW universes.

A common feature among almost all inflationary models is the production of gravitational radiation over a wide range of wavelengths. The interaction of these waves with large-scale magnetic fields soon after the end of inflation was originally considered in [13]. That study showed that the gravitationally induced shear can amplify the initial magnetic seed and the boost was found to be proportional to the square of the field's initial scale. The latter immediately suggested that large-scale primordial magnetic fields could be substantially amplified by Weyl curvature distortions alone. Seed fields spanning a current scale of ~ 10 kpc, such as those produced in [11] for example, were boosted by up to 14 orders of magnitude. In the present paper we revisit the aforementioned gravitomagnetic interaction and discuss the mathematics and the physics of the mechanism in further detail. We argue that the achieved strong magnetic growth results from the resonant coupling of the two interacting sources. More specifically, we show that the maximum amplification always occurs when the original seed field interacts with gravitational waves of the same scale. The maximum boost is determined at the onset of the gravitomagnetic interaction, which for our purposes coincides with the end of inflation. Once the parameters of the adopted inflationary model are fixed, the resonant growth factor is proportional to the initial magnetic scale, relative to the horizon size at the time. Also the whole process is shown to operate in cosmological environments of low electrical conductivity. All these make the proposed amplification mechanism a highly efficient "geometric dynamo" during the early stages of reheating. In this respect, the Maxwell-Weyl resonance discussed here resembles the magnetic amplification via parametric resonance proposed in [14,15]. Given that the universe has been a good conductor for most of its lifetime, we examine the role of finite conductivity and establish the threshold at which the electrical resistivity of the medium becomes unimportant. In agreement with the numerical results of [15], our analytical approach suggests that the gravitomagnetic resonance is suppressed in highly conductive cosmological environments.

In Secs. II and III we provide a description of the model, of the Maxwell-Weyl interaction and of the resulting magnetic amplification. We follow the presentation of [13], where the reader is referred to for details, and provide additional mathematical clarifications and physical insight. The gravitomagnetic resonance is shown in Sec. IV and an expression for the resonant growth factor is given. Section V applies the proposed amplification mechanism

to several inflation-produced magnetic seeds, while the role of finite electrical conductivity is discussed in Sec. VI.

II. GRAVITOMAGNETIC INTERACTION

A. Background equations

Consider a spatially flat FRW universe containing a barotropic perfect fluid of density ρ and isotropic pressure $p = p(\rho)$. Following [13], allow for the presence of a weak magnetic field B_a with $B^2 = B_a B^a \ll \rho$. At this limit, the field has negligible contribution to the background dynamics, which is described by

$$\kappa\rho - \frac{1}{3}\Theta^2 = 0, \quad (1a)$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}\kappa\rho(1 + 3w) = 0, \quad (1b)$$

$$\dot{\rho} + (1 + w)\Theta\rho = 0, \quad (2a)$$

and

$$\dot{B}_a + \frac{2}{3}\Theta B_a = 0. \quad (2b)$$

In the above $\kappa = 8\pi G$, $\Theta = 3\dot{a}/a = 3H$ is the expansion scalar (H is the Hubble parameter) and $w = p/\rho$ is the barotropic index of the fluid [13]. Also, throughout this paper we use natural units with $c = 1 = \hbar$ and $G^{-1/2} = m_{\text{Pl}} \approx 10^{19}$ GeV.

We perturb the aforementioned background by allowing for the propagation of weak gravitational waves, which are covariantly monitored via the electric (E_{ab}) and the magnetic (H_{ab}) Weyl components [16]. In the magnetic presence, one isolates the linear pure-tensor perturbations by imposing the conditions $D_a B^2 = 0 = \varepsilon_{abc} B^b \text{curl} B^c$ [17].¹ These, together with the standard constraints of the perfect-fluid models (e.g. see [20]), guarantee that all traceless second-rank tensors are also transverse.

B. Linear equations

Adopting the aforementioned weakly magnetized FRW background, we find that the linear magnetic evolution in the presence of gravity-wave perturbations is governed by the system.²

$$\begin{aligned} \ddot{B}_a + \frac{5}{3}\Theta\dot{B}_a + \frac{1}{3}(1 - w)\Theta^2 B_a - D^2 B_a \\ = 2(\dot{\sigma}_{ab} + \frac{2}{3}\Theta\sigma_{ab})\tilde{B}^b + \text{curl} J_a, \end{aligned} \quad (3)$$

$$\ddot{\sigma}_{ab} + \frac{5}{3}\Theta\dot{\sigma}_{ab} + \frac{1}{6}(1 - 3w)\Theta^2\sigma_{ab} - D^2\sigma_{ab} = 0, \quad (4)$$

¹The gradient $D_a = h_a^b \nabla_b$, with $h_{ab} = g_{ab} + u_a u_b$, is the covariant derivative operator orthogonal to the observer 4-velocity u_a . Also, $\text{curl} B_a = \varepsilon_{abc} D^b B^c$, where ε_{abc} is the spatial permutation tensor (i.e. $\varepsilon_{abc} u^a = 0$). For more details and an extensive covariant discussion of cosmological magnetic and electromagnetic fields the reader is referred to [18,19].

²The most straightforward derivation of Eq. (3) is by linearizing the nonlinear magnetic wave equation given in [19] [see Eq. (40) there]. On the other hand, one can obtain Eq. (4) directly from expression (3.11) in [20].

where J_a is the 3-current and σ_{ab} is the gravitationally induced shear [13]. Note that \tilde{B}_a is the original magnetic field, and B_a is the one induced by the coupling between \tilde{B}_a and gravity-wave distortions. The induced field vanishes in the background, which frees our study from potential gauge-related ambiguities. Hence, to first order, only the background magnetic field contributes to the right-hand side of (3).

At this stage we will ignore the current term in Eq. (3). This confines our analysis to a medium of zero electrical conductivity or to an electrically neutral one.³ In Sec. VI, however, we will show that our results also hold in cosmological environments of finite but relatively low electrical conductivity. We have also ignored the magnetic backreaction in (4) because it does not affect the dominant linear mode of the gravitationally induced shear [17]. Finally, we note that gravity-wave perturbations are the driving force behind the magnetic adulation described by Eq. (3). In particular, one can explicitly show that the Weyl field alone triggers fluctuations in an otherwise homogeneous magnetic field distribution (see [18]).

Expression (2b) means that $\tilde{B}_a = \tilde{B}_a^0(a_0/a)^2$, with $\tilde{B}_a^0 = 0$. By splitting the zero-order field as $\tilde{B}_a = \tilde{B}_{(n)}\tilde{Q}_a^{(n)}$, we assign the finite physical scale $\lambda_{\tilde{B}} = a/n$ to \tilde{B}_a [13,21]. This, however, does not mean that the background field is treated as a propagating wave of any sort. Then, for $\sigma_{ab} = \sigma_{(k)}Q_{ab}^{(k)}$ and $B_a = B_{(\ell)}Q_a^{(\ell)}$, where $Q_{ab}^{(k)}$ and $Q_a^{(\ell)} = Q_{ab}^{(k)}\tilde{Q}_{(n)}^b$ are tensor and vector harmonics, respectively, we have [13]

$$\begin{aligned} \ddot{B}_{(\ell)} + \frac{5}{3}\Theta\dot{B}_{(\ell)} + \left[\frac{1}{3}(1-w)\Theta^2 + \frac{\ell^2}{a^2}\right]B_{(\ell)} \\ = 2\left(\dot{\sigma}_{(k)} + \frac{2}{3}\Theta\sigma_{(k)}\right)\tilde{B}_0^{(n)}\left(\frac{a_0}{a}\right)^2. \end{aligned} \quad (5)$$

$$\ddot{\sigma}_{(k)} + \frac{5}{3}\Theta\dot{\sigma}_{(k)} + \left[\frac{1}{6}(1-3w)\Theta^2 + \frac{k^2}{a^2}\right]\sigma_{(k)} = 0. \quad (6)$$

Here, the zero suffix indicates the onset of the gravitomagnetic interaction. Also, $\ell = (k^2 + n^2 + 2kn \cos\varphi)^{1/2}$ is the comoving wave number of the induced magnetic field and $\varphi \in [0, \pi/2)$ is the angle between the two interacting sources. Finally, setting $\mathcal{B}_{(\ell)} = \kappa^{1/2}B_{(\ell)}/\Theta$, $\Sigma_{(k)} = \sigma_{(k)}/\Theta$, using conformal time (η , with $\dot{\eta} = a^{-1}$) and primes to indicate differentiation with respect to η , the above recast as [13]

³Mathematically speaking, the gravitomagnetic interaction is independent of the electrical resistivity if $\text{curl}J_a = 0$ [13]. Physically, however, the assumption of a curl-free current field appears rather arbitrary and we will not pursue it here.

$$\begin{aligned} \mathcal{B}_{(\ell)}'' + (1-3w)\left(\frac{a'}{a}\right)\mathcal{B}_{(\ell)}' - \left[\frac{3}{2}(1-3w)w\left(\frac{a'}{a}\right)^2 - \ell^2\right]\mathcal{B}_{(\ell)} \\ = 2\kappa^{1/2}a\left[\Sigma_{(k)}' + \frac{1}{2}(1-3w)\left(\frac{a'}{a}\right)\Sigma_{(k)}\right]\tilde{B}_0^{(n)}\left(\frac{a_0}{a}\right)^2, \end{aligned} \quad (7)$$

and

$$\begin{aligned} \Sigma_{(k)}'' + (1-3w)\left(\frac{a'}{a}\right)\Sigma_{(k)}' \\ - \left[\frac{3}{2}[1+(2-3w)w]\left(\frac{a'}{a}\right)^2 - k^2\right]\Sigma_{(k)} = 0, \end{aligned} \quad (8)$$

respectively (see also the appendix).

C. Scale of the induced magnetic field

The comoving wave number of the induced magnetic field depends on the coherence length of the background field, on the wavelength of the inducing gravitational radiation and on the interaction angle between these two sources. To be precise,

$$\ell = n\left[1 + \left(\frac{k}{n}\right)^2 + 2\left(\frac{k}{n}\right)\cos\varphi\right]^{1/2}, \quad (9)$$

since n takes finite values only. Assuming that k and therefore ℓ are also finite, the wavelengths $\lambda_{\tilde{B}} = a/n$, $\lambda_{GW} = a/k$ and $\lambda_B = a/\ell$ are all well defined and finite. Then, Eq. (9) provides the following expression:

$$\lambda_B = \lambda_{\tilde{B}}\left[1 + \left(\frac{\lambda_{\tilde{B}}}{\lambda_{GW}}\right)^2 + 2\left(\frac{\lambda_{\tilde{B}}}{\lambda_{GW}}\right)\cos\varphi\right]^{-1/2}, \quad (10)$$

for the coherence scale of the induced field. Given that $0 \leq \varphi < \pi/2$, this means $\lambda_B \leq \lambda_{\tilde{B}}$ always. In particular, we find $\lambda_B \sim \lambda_{\tilde{B}}$ when $\lambda_{\tilde{B}} \sim \lambda_{GW}$ and $\lambda_{\tilde{B}} \ll \lambda_{GW}$, whereas $\lambda_{GW} \ll \lambda_{\tilde{B}}$ implies $\lambda_B \ll \lambda_{\tilde{B}}$.

III. GRAVITOMAGNETIC DYNAMO

A. Magnetic amplification

After inflation the only period of low conductivity is that of early reheating, when the effective equation of state corresponds to a pressureless fluid. For $p = 0$ we have $w = 0$, $a = H_0^2 a_0^3 \eta^2/4$ and $a'/a = 2/\eta$. Then, expressions (7) and (8) simplify to

$$\mathcal{B}_{(\ell)}'' + \frac{2}{\eta}\mathcal{B}_{(\ell)}' + \ell^2\mathcal{B}_{(\ell)} = \frac{8\alpha_1}{\eta^2}\left(\Sigma_{(k)}' + \frac{1}{\eta}\Sigma_{(k)}\right) \quad (11)$$

and

$$\Sigma_{(k)}'' + \frac{2}{\eta}\Sigma_{(k)}' - \left(\frac{6}{\eta^2} - k^2\right)\Sigma_{(k)} = 0, \quad (12)$$

respectively (with $\alpha_1 = \kappa^{1/2}\tilde{B}_0^{(n)}/a_0H_0^2$). Superhorizon-sized gravity waves, with $\lambda_{GW} \gg \lambda_H = 1/H$, have $k\eta \ll 1$ and the dominant mode in the solution of Eq. (12) is $\Sigma_{(k)} = \Sigma_0^{(k)}(\eta/\eta_0)^2$. Substituted into (11) the latter gives

$$\mathcal{B}''_{(\ell)} + \frac{2}{\eta} \mathcal{B}'_{(\ell)} + \ell^2 \mathcal{B}_{(\ell)} = \frac{6\beta_1}{\eta}, \quad (13)$$

where $\eta_0^2 = 4/a_0^2 H_0^2$ and $\beta_1 = \kappa^{1/2} a_0 \tilde{B}_0^{(n)} \Sigma_0^{(k)}$. This describes a forced oscillation with a damping effect due to the expansion. The force depends on the strength of the background magnetic field and on the gravitationally induced shear at the beginning of the gravitomagnetic interaction. When $\ell \neq 0$ we obtain

$$\mathcal{B}^{(\ell)} = \mathcal{B}^{(\ell)}(\eta) = \mathcal{B}_0^{(\ell)} [\cos(\ell\eta) + \sin(\ell\eta)] \left(\frac{\eta_0}{\eta} \right) + \frac{6\beta_1}{\ell^2 \eta}, \quad (14)$$

which on super-Hubble scales [i.e. for $\ell\eta \ll 1$ and $\cos(\ell\eta) + \sin(\ell\eta) \simeq 1 + \ell\eta \simeq 1$] reduces to

$$\mathcal{B}^{(\ell)} = \mathcal{B}_0^{(\ell)} \left(\frac{\eta_0}{\eta} \right) + \frac{6\beta_1}{\ell^2 \eta}. \quad (15)$$

Finally, recalling that $\mathcal{B}^{(\ell)} = \kappa^{1/2} B^{(\ell)}/\Theta$ and using the relations $\eta^2 = 4a/H_0^2 a_0^3$ and $\Theta = 24/H_0^2 a_0^3 \eta^3$ of the $w = 0$ era, Eq. (15) gives⁴

$$B^{(\ell)} = 9 \Sigma_0^{(k)} \left(\frac{\lambda_B}{\lambda_H} \right)_0^2 \tilde{B}_0^{(n)} \left(\frac{a_0}{a} \right)^2 = 9 \Sigma_0^{(k)} \left(\frac{\lambda_B}{\lambda_H} \right)_0^2 \tilde{B}^{(n)}, \quad (16)$$

where λ_B is the scale of the induced field. Also, since B_a vanishes in the background (see Sec. II B) we have set $B_0^{(\ell)} = 0$. Accordingly, the gravitomagnetic interaction can lead to a substantial amplification of the B-field when $9 \Sigma_0^{(k)} (\lambda_B/\lambda_H)_0^2 \gg 1$. For inflation-produced, super-horizon-sized magnetic fields this is a realistic possibility. In other words, the Maxwell-Weyl coupling discussed here can act as an effective large-scale dynamo during the early stages of reheating.

It should be noted that the above results also apply to the post-recombination universe, provided that the plasma effects are negligible (e.g. when $\text{curl} \mathbf{J}_a = 0$). In that case the radiation era solution of (7) and (8) is almost identical to Eq. (16) [see Eq. (25) in [13]].

B. Gravitationally induced shear

A common feature in almost all inflationary models is the production of gravitational radiation with wavelengths extending over a wide range of scales. In fact, a relic gravity-wave spectrum is perhaps the only direct signature of inflation that may still be observable today. The energy density of a linearized gravity-wave mode produced during

⁴In [13] all the solutions were obtained under the assumption that $\lambda_{GW} \sim \lambda_{\tilde{B}}$. This ensured that $\lambda_B \sim \lambda_{GW} \sim \lambda_{\tilde{B}}$ [with all three wavelengths finite—see Eq. (9)] and allowed us to replace λ_B with $\lambda_{\tilde{B}}$ in the final expressions [e.g. see Eq. (21) in [13]]. As we will show in Sec. VI, this special case corresponds to the maximum (resonant) magnetic amplification. Also, the initial conditions of [13] assumed $B_0^{(\ell)} = \tilde{B}_0^{(n)}$ instead of setting $B_0^{(\ell)}$ to zero.

a period of de Sitter expansion is (e.g. see [22])

$$\kappa \rho_{GW} = \frac{1}{2} \int [(\Delta h_+)^2 + (\Delta h_\times)^2] k^* dk^* = \frac{2k^{*2}}{\pi} \left(\frac{H}{m_{\text{Pl}}} \right)^2. \quad (17)$$

where k^* is the physical wave number of the mode. Also, $\Delta h_{+,\times} = (2/\pi^{1/2})(H/m_{\text{Pl}})$ is the mean fluctuation of the metric perturbation $h_{+,\times}$ and m_{Pl} is the Planck mass [22]. Clearly, $\rho_{GW} \rightarrow 0$ as $k^* \rightarrow 0$.

To proceed further we note that the energy density of gravitational wave perturbations is related to the magnitude of the transverse and trace-free part of the shear tensor by $\kappa \rho_{GW} = \sigma^2$ [17]. Then, expression (17) takes the form

$$\Sigma = \left(\frac{2}{9\pi} \right)^{1/2} \left(\frac{\lambda_H}{\lambda_{GW}} \right) \left(\frac{H}{m_{\text{Pl}}} \right), \quad (18)$$

where $\Sigma = \sigma/\Theta$ and $\lambda_{GW} = 1/k^*$. The above measures the shear anisotropy due to gravitational radiation of inflationary origin. As expected, the anisotropy depends on the parameters of the underlying model of inflation (i.e. on the value of H/m_{Pl}) and it is inversely proportional to the scale of the wave mode.

IV. GRAVITOMAGNETIC RESONANCE

Hyperhorizon-sized magnetic fields emerge naturally by the end of inflation, when subhorizon quantum fluctuations in the Maxwell field are stretched outside the Hubble radius and then freeze-in as classical electromagnetic waves. At that time the universe is also permeated by large-scale gravitational waves; the inevitable prediction of almost all inflationary scenarios. Following Eq. (17), the effect of the linear interaction between these two sources depends on the gravitationally induced shear anisotropy. For inflation-produced gravitons the latter is inversely proportional to their wavelength [see (18)]. Thus, combining relations (16) and (18) we obtain

$$B^{(\ell)} \sim \left(\frac{\lambda_B}{\lambda_H} \right)_0 \left(\frac{\lambda_B}{\lambda_{GW}} \right)_0 \left(\frac{H}{m_{\text{Pl}}} \right)_0 \tilde{B}^{(n)}. \quad (19)$$

Note that the zero suffix marks the beginning of the gravitomagnetic interaction, which here is the end of inflation. According to expression (19), we have a substantial amplification of the geometrically induced B-field if

$$\mathcal{A} \equiv \left(\frac{\lambda_B}{\lambda_H} \right)_0 \left(\frac{\lambda_B}{\lambda_{GW}} \right)_0 \left(\frac{H}{m_{\text{Pl}}} \right)_0 \gg 1, \quad (20)$$

where \mathcal{A} may be seen as the amplification factor. Given that the ratio $(H/m_{\text{Pl}})_0$ is fixed by the adopted model of inflation, the effect of the Maxwell-Weyl coupling depends on the initial relation between λ_B , λ_{GW} and λ_H . In particular, since we are confined to superhorizon scales, the magnitude of the amplification factor depends primarily on λ_B and λ_{GW} . These are related to each other and also to the scale of the background field by Eq. (10), which trans-

forms expression (20) into

$$\mathcal{A} = \mathcal{A}(\chi) = 10^\alpha \left(\frac{H}{m_{\text{Pl}}}_0 \right) [\chi(1 + \chi^2 + 2\chi \cos \phi)^{-1}], \quad (21)$$

with $\chi = (\lambda_{\bar{B}}/\lambda_{GW})_0$ by definition. The latter varies between $0 < \chi < \infty$ and determines the scale ratio of the two interacting sources. The parameter α determines the coherence length of \bar{B}_0 , relative to the horizon length at the time, according to $(\lambda_{\bar{B}}/\lambda_H)_0 = 10^\alpha$. Typically $\alpha \gg 1$, since $(\lambda_{\bar{B}})_0 \gg (\lambda_H)_0$ by the end of inflation. Once α and $(H/m_{\text{Pl}})_0$ are fixed, the point of maximum amplification is decided by the function within the brackets. It is then straightforward to show that $\mathcal{A}(\chi)$ takes its maximum value at $\chi = 1$, or equivalently for $(\lambda_{\bar{B}})_0 = (\lambda_{GW})_0$. In other words, the maximum magnetic boost is achieved when the two interacting sources are in resonant coupling. For $\chi = 1$ the amplification factor becomes $\mathcal{A} = \mathcal{A}_{\text{max}} \approx 10^\alpha (H/m_{\text{Pl}})_0$. When $\chi \ll 1$ or $\chi \gg 1$, on the other hand, expression (21) ensures that $\mathcal{A} \ll \mathcal{A}_{\text{max}}$. Thus, the maximum magnitude of the gravitationally induced magnetic field is

$$B^{(\ell)} = (B^{(\ell)})_{\text{max}} \sim 10^\alpha \left(\frac{H}{m_{\text{Pl}}}_0 \right) \bar{B}^{(n)}, \quad (22)$$

where $\alpha \gg 1$. All these mean that the interaction between inflation-produced magnetic seeds and gravitational waves in the poorly conductive environment of early reheating, can lead to the resonant amplification of the former. Following (21), the maximum magnetic growth occurs at $\chi = 1$ irrespective of the value of ϕ . The latter determines only the shape of the amplification curve. In other words, the gravitomagnetic resonance is independent of the interaction angle between the two sources.

Expression (22) provides the spectrum of the gravitationally amplified magnetic field at the end of the resonant-growth phase. The latter occurs during the early stages of reheating when the electrical conductivity of the cosmic fluid is low. Once the conductivity has grown beyond a certain threshold, however, the plasma effects become important (see Sec. VI). When this happens the electric current term in Eq. (3) needs to be accounted for and our analysis no longer holds. For our purposes, the gravitomagnetic resonance and the geometric amplification of the induced B-field cease at that point.

V. AMPLIFICATION OF INFLATIONARY MAGNETIC SEEDS

Our results so far have shown that the maximum growth of the gravitationally induced magnetic field depends on the scale and the magnitude of the initial seed, as measured at the end of inflation, and on the adopted inflation model. Given that inflation has stretched these fields well outside the horizon, their amplification can be very substantial. In what follows we will consider three alternative scenarios of magnetogenesis and calculate the strengths of the resonantly amplified seeds in each case. Our aim is to obtain

a first estimate of the resonant magnetic growth in each case and to illustrate the potential of the Maxwell-Weyl coupling as a very efficient early-universe dynamo.

Large-scale magnetic fields of inflationary origin are generally extremely weak. Typically, the current energy density of a primordial field that survived a phase of de Sitter expansion (in spatially flat FRW universes) is $\rho_B = 10^{-104} \lambda_{\text{Mpc}}^{-4} \rho_\gamma$, where $\rho_B = B^2/8\pi$, ρ_γ is the radiation energy density and λ_{Mpc} is the field's physical scale today [9]. For a magnetic field with a coherence length of ~ 10 kpc, which is the minimum required for the galactic dynamo to work, the corresponding strength is roughly 10^{-53} G. Such seeds are too weak to support the dynamo and are therefore treated as astrophysically irrelevant. However, the interaction of the aforementioned field with gravity waves during the early stages of reheating can lead to the resonant amplification of the former according to Eq. (22). Since physical lengths are inversely proportional to the radiation temperature, a scale of $\lambda_{\bar{B}} \sim 10$ kpc today translates into $\lambda_{\bar{B}}/\lambda_H \sim 10^{21}$ at the end of inflation. The latter is obtained by assuming GUT-scale inflation with a typical value of $H_0 \sim 10^{13}$ GeV, which corresponds to a temperature of approximately 10^{15} GeV at the time.⁵ These mean $\alpha \approx 21$ and $(H/m_{\text{Pl}})_0 \sim 10^{-6}$ for the resonant amplification parameters of Eq. (21). As a result, the associated maximum-growth factor is of the order of 10^{15} and the initial magnetic seed is amplified to $\sim 10^{-38}$ G. Despite this, the residual field is still some 4 orders of magnitude below the minimum required strength of $\sim 10^{-34}$ G (see [6]). Put another way, for workable results we need a stronger initial seed.

When dealing with spatially flat FRW backgrounds, inflationary magnetic seeds stronger than 10^{-53} G are usually obtained outside the framework of classical electromagnetic theory. Such an inflation-based scenario of large-scale magnetogenesis was recently suggested in [11]. The proposed mechanism operates within the standard model, despite breaking the conformal invariance of the Maxwell field, and produces magnetic seeds of $\sim 10^{-34}$ G on scales of approximately 10 kpc. However, 10^{-34} G is the minimum strength required for the galactic dynamo to work, and this only in universes dominated by a dark-energy component. Nevertheless, the interaction of the above field with gravitational wave perturbations soon after inflation should boost its amplitude in accordance with Eq. (22). Given the scale of the original seed and using the parameters of [11] (i.e. $H_0 \sim 10^{13}$ GeV and $T_0 \sim 10^{15}$ GeV), the resonant amplification factor is 10^{15} (see also above). The latter brings the residual magnetic field up to $\sim 10^{-19}$ G, which lies very comfortably within the ga-

⁵At the end of inflation the scale factor corresponds to a temperature (T) given by the formula $H = (8\pi^{3/2} g_*^{1/2} / \sqrt{90}) \times (T^2/m_{\text{Pl}})$, where $g_* = g_*(T) \approx 10^2$ is the number of the relativistic degrees of freedom (e.g. see [11]).

lactic dynamo requirements for dark-energy dominated cosmologies [6]. Moreover, comoving seeds of 10^{-19} G can sustain the dynamo in conventional universes as well, especially when the enhancement of the field during the protogalactic collapse is accounted for.⁶

In spatially open universes, standard inflation can produce magnetic seeds stronger than the typical 10^{-53} G fields without the need to modify Maxwell's theory. In that case the general relativistic coupling between electromagnetism and the geometry of the 3-space changes the adiabatic depletion rate of the magnetic component naturally [12,19]. To be precise, on lengths near the curvature scale, a field that goes through a period of de Sitter inflation in a perturbed FRW cosmology with negative spatial curvature decays as a^{-1} instead of a^{-2} . Then, assuming a marginally open universe (i.e. setting $1 - \Omega \sim 10^{-2}$ today), GUT-scale inflation and a reheating temperature of $\sim 10^9$ GeV, one obtains a residual field of approximately 10^{-35} G on a current scale close to 10^4 Mpc (see [12] for details). For a field on this scale we have $(\lambda_{\tilde{B}}/\lambda_H)_0 \sim 10^{27}$, which implies a resonant amplification factor of the order of 10^{21} and a residual strength of $\sim 10^{-14}$ G today. The latter is easily within the galactic dynamo limits.⁷

The above quoted strengths correspond to resonant amplification. In other words, we have implicitly assumed that a background magnetic field of a given length interacts with gravitational waves of comparable scale. When the two sources have very different coherence lengths, however, the associated amplification factors are considerably smaller and the resulting fields much weaker than those given above [see Eq. (22)]. In general, of course, the background magnetic seed will interact with gravity-wave modes of various wavelengths [recall that $0 < \chi < \infty$ in (22)]. On these grounds, we expect the magnitude of the gravitationally induced field to show a smooth scale-distribution with peak at the point of gravitomagnetic resonance (i.e. at $\chi = 1$).

VI. CONDUCTIVITY EFFECTS

A. Low conductivity

To this point the gravitomagnetic interaction has been free of current effects, which limits our results to cosmological environments of very low electrical conductivity.

⁶In a spherically symmetric protogalactic collapse, a magnetic field that is frozen-in with the highly conductive plasma grows as $B \propto \rho^{2/3}$. This rate, which implies an amplification of the B-field by three or 4 orders of magnitude, can increase in the more-realistic case of an anisotropically collapsing protogalaxy due to shearing effects alone [23]

⁷Despite their very substantial growth the gravitationally amplified magnetic fields always remain very weak compared to the matter component (i.e. $B^2 \ll \rho$ at all times). This ensures that our initial weak-field approximation (see Sec. II) is never in any doubt and preserves the symmetries of the FRW background to very high accuracy.

The early reheating phase of the universe offers such a poorly conductive stage. As reheating progresses, however, the copious production of particles continuously increases the conductivity of the universe and plasma effects become important.

Consider a medium of finite electrical conductivity σ_c . Phenomenologically, the conductivity effects are accounted for by means of the electric currents. Using the covariant form of Ohm's law, in particular, one expresses the 3-current as (e.g. see [19,24])

$$J_a = \sigma_c E_a, \quad (23)$$

where E_a is the electric field seen by the observer. Assuming that the spatial variation of σ_c is small, which is a good approximation on large scales, the above means that $\text{curl} J_a = \sigma_c \text{curl} E_a$ to linear order and introduces the conductivity into the right-hand side of Eq. (3). Moreover, in a medium of finite conductivity the magnetic induction equation reads

$$\dot{B}_a + \frac{2}{3}\Theta B_a = \sigma_{ab} \tilde{B}^b - \text{curl} E_a, \quad (24)$$

to first order (e.g. see [18,19]). Note that the time derivative of the above leads to the linearized wave Eq. (3) (see [19] for details). Employing the auxiliary relations (23) and (24), Eq. (3) reads

$$\begin{aligned} \ddot{B}_a + \frac{5}{3} \left(1 + \frac{3\sigma_c}{5\Theta} \right) \Theta \dot{B}_a + \frac{1}{3} \left(1 - w + 2\frac{\sigma_c}{\Theta} \right) \Theta^2 B_a - D^2 B_a \\ = 2 \left[\dot{\sigma}_{ab} + \frac{2}{3} \left(1 + \frac{3\sigma_c}{4\Theta} \right) \Theta \sigma_{ab} \right] \tilde{B}^b, \end{aligned} \quad (25)$$

with the dimensionless ratio σ_c/Θ measuring the electrical conductivity of the expanding background. Hence, the linear evolution of the gravitationally induced B field depends on the value of σ_c/Θ in a rather involved way. Nevertheless, the gravitomagnetic interaction proceeds as if the conductivity were zero as long as $\sigma_c/\Theta \ll 1$. This qualitative result was also obtained in [9].

B. High conductivity

As particle production progresses and the universe heats up, the conductivity of the cosmic medium increases beyond the $\sigma_c/\Theta \sim 1$ threshold and we can no longer ignore the current term in the right-hand side of Eq. (3). Moreover, once the universe enters its standard big-bang evolution, the electrical resistivity is believed to remain very low [25]. When $\sigma_c/\Theta \gg 1$ the evolution of the gravitationally induced magnetic field depends largely on the electrical properties of the fluid [see Eq. (25)]. The precise role of finite conductivity during reheating lies beyond the scope of this article, as its study involves highly sophisticated quantum field theory techniques, it is model dependent and requires numerical methods to evaluate [26]. In what follows we will provide an analytical approach that helps to outline the implications of a highly conductive cosmologi-

cal environment for the proposed gravitomagnetic amplification. For $w = 0$ and $\sigma_c/\Theta \gg 1$, which correspond to the late stages of reheating, Eq. (25) gives

$$\begin{aligned} \mathcal{B}''_{(\ell)} + \frac{\sigma_c}{\Theta} \frac{6}{\eta} \mathcal{B}'_{(\ell)} + \left(\frac{\sigma_c}{\Theta} \frac{6}{\eta^2} + \ell^2 \right) \mathcal{B}_{(\ell)} \\ = \frac{8\alpha_1}{\eta^2} \left(\Sigma'_{(k)} + \frac{\sigma_c}{\Theta} \frac{3}{\eta} \Sigma_{(k)} \right), \end{aligned} \quad (26)$$

where $\alpha_1 = \kappa^{1/2} \tilde{B}_0^{(n)}/a_0 H_0^2$ and $\Sigma_{(k)}$ is monitored by (12) [see Secs. IIB and IIIA and also the appendix]. At the $\sigma_c/\Theta \gg 1$ limit, the ℓ dependence of the third term in the left-hand side of the above is only important on sufficiently small wavelengths (i.e. for $\ell\eta \gg 1$). Here, however, we are looking at superhorizon scales where $n\eta, k\eta$ and $\ell\eta \ll 1 \ll \sigma_c/\Theta$. On these wavelengths $\Sigma \propto \eta^2$ (see Sec. IIIA) and expression (26) reduces to

$$\mathcal{B}'' + \frac{\sigma_c}{\Theta} \frac{6}{\eta} \mathcal{B}' + \frac{\sigma_c}{\Theta} \frac{6}{\eta^2} \mathcal{B} = \frac{\sigma_c}{\Theta} \frac{6\beta_1}{\eta}, \quad (27)$$

with $\beta_1 = \kappa^{1/2} a_0 \Sigma_0 \tilde{B}_0$. Note that we are considering the resonant scenario, with $k = n = \ell$, which allows us to drop the wave number indices in (27). Contrary to the case of poor electrical conductivity [see Eq. (13)], we have arrived at a scale independent expression. This is due to the highly conductive environment, which washes out the ℓ dependence of (26) on sufficiently long wavelengths (i.e. when $\ell^2 \eta^2 \ll \sigma_c/\Theta$). To solve Eq. (27) analytically, consider a brief period of expansion and assume that in the interval the ratio σ_c/Θ varies very slowly with time (i.e. set $\sigma_c/\Theta \simeq \text{constant} \gg 1$). Then, since $\mathcal{B} = \kappa^{1/2} B/\Theta$, the solution of (27) approaches the form

$$B = B_0 \left(\frac{a_0}{a} \right)^2 + B_0 \left(\frac{a_0}{a} \right)^{3\sigma_c/\Theta} + 3\Sigma_0 \tilde{B}_0 \left(\frac{a_0}{a} \right), \quad (28)$$

where B_0 can be seen as the gravitationally induced magnetic field at the onset of the highly conductive regime. The first mode of the above corresponds to the adiabatic depletion of the field, while the second carries the plasma effects and decays very quickly when $\sigma_c/\Theta \gg 1$. Hence, for low electrical resistivity and in the absence of the gravitomagnetic interaction (i.e. for $\Sigma_0 = 0$) we recover the familiar a^{-2} -law. The third mode in (28) describes the effect of the Maxwell-Weyl coupling on the B field. Compared to the low conductivity case [e.g. see results (15) or (19)], there is no scale dependence and the gravitomagnetic resonance has been completely suppressed. Therefore, as the reheating of the universe progresses and the electrical resistivity of the cosmic medium drops, the large-scale effects of the Maxwell-Weyl resonance should fade away. This suggests that the proposed gravitomagnetic dynamo is only effective at the early stages of reheating. Similar results, showing an analogous damping of electromagnetic modes in highly conductive environments, have been obtained in the past. More specifically, numerical calculations of the magnetic

amplification due to the parametric resonance of the field with an oscillating complex scalar field during preheating, showed a very substantial decrease in the magnetic growth with increasing electrical conductivity [15].

Although the plasma effects may have overwhelmed the gravitomagnetic resonance, the third mode of Eq. (28) also shows that the Maxwell-Weyl coupling slows down the decay rate of the field. Interestingly, the same effect was also obtained through the relativistic coupling of the B -field with the spatial curvature of an open FRW cosmology [12]. This time, however, the geometrically induced superadiabatic magnetic amplification is not efficient. Indeed, ignoring the fast decaying second mode in (28), the latter reads

$$B = \left[B_0 + 3\Sigma_0 \tilde{B}_0 \left(\frac{a}{a_0} \right) \right] \left(\frac{a_0}{a} \right)^2. \quad (29)$$

Amplification is therefore achieved only when $3\Sigma_0 \tilde{B}_0 (a/a_0) > B_0$. Typically, $\Sigma_0 \ll 10^{-6}$ [see Sec. III B] and $\tilde{B}_0 \leq B_0$, which implies that the above given condition is satisfied at late times only (i.e. for $a/a_0 \gg 1$). As the time interval of the interaction increases, however, the assumption that $\sigma_c/\Theta \simeq \text{constant}$ becomes more difficult to maintain and this affects the accuracy of our results. Having said that, the same superadiabatic-type magnetic amplification is also observed at the infinite conductivity limit (see footnote 3 in [13]). All these raise the intriguing possibility of a change in the adiabatic a^{-2} law due to gravity-wave effects alone and irrespective of the conductivity of the cosmological environment.

VII. DISCUSSION

Inflation can naturally achieve superhorizon correlations from small-scale microphysics. This property has been exploited by several authors in order to produce primordial magnetic fields with the desired large coherence lengths. The drawback of inflation is the dramatic dilution of the magnetic energy density during the accelerated expansion phase. For a field that survived inflation and spans a scale of ~ 10 kpc today, the typical strength is roughly 10^{-53} G. On that scale the minimum required strength for the galactic dynamo to work is 10^{-34} G, assuming that the universe is dark-energy dominated. Otherwise the magnetic seed should be at least as strong as $\sim 10^{-23}$ G. The most common solution to the strength problem is by slowing down the adiabatic, a^{-2} decay rate of the B-field. When dealing with spatially flat FRW backgrounds, this usually means breaking the conformal invariance of the Maxwell field and in the majority of cases it is achieved outside the standard model.

Inflation also produces a background of large-scale gravitational radiation. The interaction of these waves with inflationary produced magnetic seeds soon after the end of inflation was first considered in [13]. The initial

results argued for a very significant growth, by many orders of magnitude, of the primordial field. Here, we have revisited this gravitomagnetic interaction in an attempt to understand and explain the physics of the amplification mechanism further. Our analysis has revealed that the very strong magnetic growth found in [13], reflects the resonant coupling of the two interacting sources in cosmological environments of poor electrical conductivity. We have shown, in particular, that the maximum amplification of the B field occurs always when the coherence scale of the latter coincides with the wavelength of the inducing gravitational radiation.

The proposed Maxwell-Weyl interaction and the resulting amplification mechanism are rather simple in concept. At the end of inflation the universe is permeated by large-scale gravity waves and by a very weak, large-scale primordial magnetic field. The general relativistic interaction of these two sources during early reheating leads to a gravitationally induced magnetic component. When the associated scales are comparable this field is resonantly amplified. In general, of course, the original magnetic seed will interact with gravitational radiation of various wavelengths. This means that the strength of the induced field will have a smooth scale-dependent spectrum with peak at the point of resonance. The maximum strength of the geometrically amplified magnetic component is determined at the onset of the gravitomagnetic interaction. Once the parameters of the adopted inflationary model are fixed, the resonant growth factor is proportional to the scale of the initial field. This makes the proposed amplification mechanism particularly effective when operating on superhorizon-sized magnetic seeds. In particular, for a field with current physical scale close to 10 kpc, which is the minimum required for the dynamo to work, the resonant growth is of the order of 10^{15} . Although very substantial, such a boost cannot bring the typical inflation-produced magnetic field of $\sim 10^{-53}$ G (see [9]) within the galactic dynamo requirements. Nevertheless, when applied to seeds of $\sim 10^{-34}$ G and $\sim 10^{-35}$ G, like those produced in [11,12] for example, the gravitomagnetic resonance leads to residual fields of $\sim 10^{-19}$ G and $\sim 10^{-14}$ G respectively. The latter can support the galactic dynamo even in conventional universes with zero dark-energy contribution.

The resonant amplification of the initial seed field by many orders of magnitude, makes the proposed gravitomagnetic coupling a very promising early-universe dynamo. Given that, it is worth looking into the specifics of the mechanism in more detail. A key issue is the role of electrical conductivity near and beyond the $\sigma_c/\Theta \sim 1$ threshold. Here we found that at the $\sigma_c/\Theta \approx \text{constant} \gg 1$ limit the Maxwell-Weyl resonance is suppressed, which is in qualitative agreement with analogous earlier studies (e.g. see [14,15]). One could improve on this result by adopting a specific model for the conductivity of the reheating era. The nature of adopted inflationary scenario

and of the associated reheating process is also an issue. Nonoscillatory models, for example, may provide a longer period of low electrical conductivity and an enhanced gravity-wave spectrum. Another key question is the magnetic backreaction on the gravity waves themselves and its potential impact on the amplification mechanism itself. This issue acquires particular interest in view of the work of [27]. There large-scale stochastic magnetic fields were found capable of efficiently converting their energy into gravitational radiation, as they reenter the cosmological horizon. It is conceivable that the simultaneous study of the two processes will point towards a preferred saturation level for the combined gravitomagnetic interaction.

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APPENDIX: EXPANSION-NORMALIZED GRAVITOMAGNETIC EQUATIONS

The linear evolution of the magnetic mode $B_{(\ell)}$ induced by gravity-wave perturbations on a weakly magnetized, spatially flat FRW universe is governed by the system [see Eqs. (5) and (6) in Sec. IIB]

$$\begin{aligned} \ddot{B}_{(\ell)} + \frac{5}{3}\Theta\dot{B}_{(\ell)} + \left[\frac{1}{3}(1-w)\Theta^2 + \frac{\ell^2}{a^2}\right]B_{(\ell)} \\ = 2\left(\dot{\sigma}_{(k)} + \frac{2}{3}\Theta\sigma_{(k)}\right)\tilde{B}_0^{(n)}\left(\frac{a_0}{a}\right)^2, \end{aligned} \quad (\text{A1})$$

$$\ddot{\sigma}_{(k)} + \frac{5}{3}\Theta\dot{\sigma}_{(k)} + \left[\frac{1}{6}(1-3w)\Theta^2 + \frac{k^2}{a^2}\right]\sigma_{(k)} = 0. \quad (\text{A2})$$

Here $\tilde{B}_0^{(n)}$ is the background field, $\sigma_{(k)}$ is the shear anisotropy due to the gravitational waves and the zero suffix indicates the beginning of the gravitomagnetic interaction. To zero perturbative order the background expansion is monitored by the Friedmann and the Raychaudhuri equations, given by expressions (1a) and (1b), respectively. When combined these reduce Raychaudhuri's formula to

$$\dot{\Theta} = -\frac{1}{2}(1+w)\Theta^2. \quad (\text{A3})$$

Consider the expansion-normalized, dimensionless variables $\mathcal{B}_{(\ell)} = \kappa^{1/2}B_{(\ell)}/\Theta$ and $\Sigma_{(k)} = \sigma_{(k)}/\Theta$ defined in Sec. IIB. Using (A3) we obtain the auxiliary relations

$$\kappa^{1/2}\dot{B}_{(\ell)} = \Theta\dot{\mathcal{B}}_{(\ell)} - \frac{1}{2}(1+w)\Theta^2\mathcal{B}_{(\ell)}, \quad (\text{A4})$$

$$\kappa^{1/2}\ddot{B}_{(\ell)} = \Theta\ddot{\mathcal{B}}_{(\ell)} - (1+w)\Theta^2\dot{\mathcal{B}}_{(\ell)} + \frac{1}{2}(1+w)^2\Theta^3\mathcal{B}_{(\ell)}, \quad (\text{A5})$$

between the proper-time derivatives of $\mathcal{B}_{(\ell)}$ and $B_{(\ell)}$.

Similarly, the first and second derivatives of $\sigma_{(k)}$ give

$$\dot{\sigma}_{(k)} = \Theta \dot{\Sigma}_{(k)} - \frac{1}{2}(1+w)\Theta^2 \Sigma_{(k)}, \quad (\text{A6})$$

$$\ddot{\sigma}_{(k)} = \Theta \ddot{\Sigma}_{(k)} - (1+w)\Theta^2 \dot{\Sigma}_{(k)} + \frac{1}{2}(1+w)^2 \Theta^3 \Sigma_{(k)}. \quad (\text{A7})$$

Results (A4)–(A7) transform expressions (A1) and (A2) into

$$\begin{aligned} \ddot{\mathcal{B}}_{(\ell)} + \frac{1}{3}(2-3w)\Theta \dot{\mathcal{B}}_{(\ell)} - \left[\frac{1}{6}(1-3w)w\Theta^2 - \frac{\ell^2}{a^2} \right] \mathcal{B}_{(\ell)} \\ = 2\kappa^{1/2} \left[\dot{\Sigma}_{(k)} + \frac{1}{6}(1-3w)\Theta \Sigma_{(k)} \right] \tilde{B}_0^{(n)} \left(\frac{a_0}{a} \right)^2, \end{aligned} \quad (\text{A8})$$

and

$$\begin{aligned} \ddot{\Sigma}_{(k)} + \frac{1}{3}(2-3w)\Theta \dot{\Sigma}_{(k)} \\ - \left\{ \frac{1}{6} [1 + (2-3w)w] \Theta^2 - \frac{k^2}{a^2} \right\} \Sigma_{(k)} = 0, \end{aligned} \quad (\text{A9})$$

respectively.

The final step is to introduce the conformal-time variable η , with $\dot{\eta} = 1/a$ by definition. Then $\Theta = 3a'/a^2$, where

the prime indicates conformal-time derivatives. Accordingly,

$$\dot{\mathcal{B}}_{(\ell)} = \frac{1}{a} \mathcal{B}'_{(\ell)} \quad \text{and} \quad \ddot{\mathcal{B}}_{(\ell)} = \frac{1}{a^2} \mathcal{B}''_{(\ell)} - \frac{a'}{a^3} \mathcal{B}'_{(\ell)}, \quad (\text{A10})$$

with exactly analogous expressions for $\dot{\Sigma}_{(k)}$ and $\ddot{\Sigma}_{(k)}$ respectively. These relations recast Eqs. (A8) and (A9) in terms of conformal-time derivatives as

$$\begin{aligned} \mathcal{B}''_{(\ell)} + (1-3w) \left(\frac{a'}{a} \right) \mathcal{B}'_{(\ell)} - \left[\frac{3}{2}(1-3w)w \left(\frac{a'}{a} \right)^2 - \ell^2 \right] \mathcal{B}_{(\ell)} \\ = 2\kappa^{1/2} a \left[\Sigma'_{(k)} + \frac{1}{2}(1-3w) \left(\frac{a'}{a} \right) \Sigma_{(k)} \right] \tilde{B}_0^{(n)} \left(\frac{a_0}{a} \right)^2 \end{aligned} \quad (\text{A11})$$

and

$$\begin{aligned} \Sigma''_{(k)} + (1-3w) \left(\frac{a'}{a} \right) \Sigma'_{(k)} \\ - \left\{ \frac{3}{2} [1 + (2-3w)w] \left(\frac{a'}{a} \right)^2 - k^2 \right\} \Sigma_{(k)} = 0, \end{aligned} \quad (\text{A12})$$

respectively [compare to formulas (7) and (8) of Sec. IIB].

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