Two sum rules for the thermal n-point functions

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In real-time, thermal field theory there are 2^n different n-point functions, but there are two linear combinations that sum to zero. Simple proofs of the two sum rules are given. Each sum rule has two forms: one for the one-particle-reducible functions and one for the amputated functions.

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I. INTRODUCTION

At nonzero temperature there are four two-point functions in the real-time formulation [1–3]. All four can be expressed in terms of the retarded propagator $D_R(p)$ and the advanced propagator $D_A(p)$:

$$G_{11}(p) = (1+n)D_R(p) - nD_A(p)$$
$$G_{12}(p) = e^{\sigma p^0} n[D_R(p) - D_A(p)]$$
$$G_{21}(p) = e^{(\beta-\sigma)p^0} n[D_R(p) - D_A(p)]$$
$$G_{22}(p) = nD_R(p) - (1+n)D_A(p),$$

where $n = 1/[e^{\beta p_0} - 1]$ is the Bose-Einstein function and σ is a real parameter in the range $0 \le \sigma \le \beta$. Since the two-point functions are expressed in terms of $D_R(p)$ and $D_A(p)$, there must be two linear relations among the $G_{ab}(p)$. These relations can be expressed as two sum rules:

$$0 = G_{11}(p) - e^{-\sigma p^0} G_{12}(p) - e^{\sigma p^0} G_{21}(p) + G_{22}(p); \tag{1.1}$$

$$0 = G_{11}(p) - e^{(\beta-\sigma)p^0} G_{12}(p) - e^{(\sigma-\beta)p^0} G_{12}(p) + G_{22}(p). \tag{1.2}$$

The first sum rule can be extended to nonequilibrium systems; the second cannot. It is surprising that these rather trivial relations generalize to thermal n-point functions.

Field theory computations at nonzero temperature require thermal averages in which some fields are anti-time-ordered and others are time-ordered. Instead of one n-point function there are 2^n. There are two different linear combinations of these 2^n functions that vanish. These two sum rules have not been discussed much in the literature. The first sum rule was proved by Chou, Su, Hao, and Yu [4] for nonequilibrium systems using the Keldysh basis [5] for the closed-time-path functional integral. Evans [6] used their result and a time-reversed version of the closed time path to obtain the second sum rule. A clear statement of the both sum rules was given by van Eijck, Kobes, and van Weert [7], who noted that they could be proved using the circled/uncircled graph formalism of Kobes and Semenoff [8]. See also [9–12]. Most of the references deal with either the unamputated functions, e.g. [4,6], or with the amputated functions, e.g. [7].

The purpose of this note is to provide simple and explicit derivations of the two sum rules. The first sum rule is given below in Eq. (4.8) for unamputated functions and in Eq. (4.11) for amputated functions. The second sum rule is given in Eq. (5.2) for unamputated functions and in Eq. (5.4) for amputated functions.

II. OPERATOR IDENTITIES

The sum rules will be proven here using an operator method used many years ago by Nishijima [13] in two papers discussing composite particles and dispersion relations. Let $\phi(x)$ denote a self-adjoint field operator in an interacting theory. Define the functional

$$Z[J] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n J(x_1) \ldots J(x_n) \times T[\phi(x_1) \ldots \phi(x_n)], \tag{2.1}$$

where T denotes time ordering. The adjoint functional involves anti-time-ordering $\tilde{T}$:

$$Z^\dagger[J] = 1 + \sum_{n=1}^{\infty} \frac{(i)^n}{n!} \int d^4x_1 \ldots d^4x_n J(x_1) \ldots J(x_n) \times \tilde{T}[\phi(x_1) \ldots \phi(x_n)]. \tag{2.2}$$

These satisfy $Z^\dagger Z = 1$. Differentiating this gives

$$0 = (-i)^n \frac{\delta^n [Z^\dagger Z]}{\delta J(x_1)\delta J(x_2) \ldots \delta J(x_n)} = \sum_{\ell=0}^{n} \sum_{\text{comb}} (-1)^\ell \tilde{T}[\phi(x_1) \ldots \phi(x_\ell)] T[\phi(x_{\ell+1}) \ldots \phi(x_n)]. \tag{2.3}$$

The sum over combinations means that, for each value of ℓ, include the $n!/\ell!(n-\ell)!$ possible choices of the x's that appear in the anti-time-ordered bracket. When summed over ℓ the identity contains 2^n products since each $\phi(x_j)$ may either be in the anti-time-ordered bracket or in the time-ordered bracket. For $n = 2$ the identity is

$$0 = T[\phi(x_1)\phi(x_2)] - \phi(x_1)\phi(x_2) - \phi(x_2)\phi(x_1) + \tilde{T}[\phi(x_1)\phi(x_2)]. \quad (2.4)$$

It is easy to check that this sum is identically zero. If $t_1 > t_2$ then the first term cancels the second and the third term cancels the fourth. If $t_1 < t_2$ then the first term cancels the third and the second term cancels the fourth.

A second, independent identity results from differentiating the relation $Z^\dagger Z = 1$:

$$0 = (i)^n \frac{\delta^n[Z^\dagger Z]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)} = \sum_{\ell=0}^{n}\sum_{\text{comb}}(-1)^\ell T[\phi(x_{\ell+1})\ldots\phi(x_n)]\tilde{T}[\phi(x_1)\ldots\phi(x_\ell)]. \quad (2.5)$$

This second identity is not the adjoint of Eq. (2.3). For $n = 2$ it gives the same as Eq. (2.4) but for $n \geq 3$ the relations are different.

One can also prove the vanishing of the sums in Eqs. (2.3) and (2.5) by mathematical induction.

III. THERMAL n-POINT FUNCTIONS

Thermal n-point functions are defined [1–3] in the complex time plane on a contour that consists of two parts: C_1 runs along the real-time axis from $-\infty$ to ∞; C_2 runs antiparallel to the real-time axis from $\infty - i\sigma$ to $-\infty - i\sigma$. The parameter σ lies in the range $0 \leq \sigma \leq \beta$. A scalar field $\phi(x)$ may be defined on contours C_1 and C_2 by introducing

$$\Phi_\alpha(x) = \begin{cases} \phi(x) & \text{if } \alpha = 1 \\ e^{\sigma H}\phi(x)e^{-\sigma H} & \text{if } \alpha = 2. \end{cases}$$

The thermal Green functions are

$$G_{\alpha_1\alpha_2\ldots\alpha_n}(x_1, x_2, \ldots x_n) = (-i)^{n-1}\sum_a P_a\langle a|T_c[\Phi_{\alpha_1}(x_1)\ldots\Phi_{\alpha_n}(x_n)]|a\rangle, \quad (3.1)$$

where T_c denotes contour ordering. The states $|a\rangle$ have energy E_a and thermal probability

$$P_a = e^{-\beta E_a}/\text{Tr}(e^{-\beta H}).$$

When all the fields are on contour C_1, the result is the thermal average of the time-ordered product of n fields:

$$G_{1\ldots1}(x_1, \ldots x_n) = (-i)^{n-1}\sum_a P_a\langle a|T[\phi(x_1)\ldots\phi(x_n)]|a\rangle.$$

When all the fields are on C_2, the result is thermal average of the anti-time-ordered product:

$$G_{2\ldots2}(x_1, \ldots x_n) = (-i)^{n-1}\sum_a P_a\langle a|\tilde{T}[\phi(x_1)\ldots\phi(x_n)]|a\rangle.$$

In the general case, ℓ fields are on contour C_2 and the remaining $n-\ell$ are on contour C_1. For example, if $x_1, x_2, \ldots x_\ell$ are the coordinates of the fields on C_2 and $x_{\ell+1}, \ldots x_n$ are the coordinates of the fields on C_1, the contour-ordered Green function is

$$G_{\underbrace{2\ldots2}_{\ell}\underbrace{1\ldots1}_{n-\ell}}(x_1, \ldots x_n) = (-i)^{n-1}\sum_{a,b} C_{ab}\langle a|\tilde{T}[\phi(x_1)\ldots\phi(x_\ell)]|b\rangle\langle b|T[\phi(x_{\ell+1})\ldots\phi(x_n)]|a\rangle, \quad (3.2)$$

where

$$C_{ab} = e^{\sigma(E_a - E_b)}P_a. \quad (3.3)$$

The subscripts 2 or 1 designate whether the corresponding field point $\phi(x_j)$ is in the anti-time-ordered or the time-ordered bracket. Equation (3.2) is invariant under permutations among the first ℓ coordinates and invariant under permutations among the last $n-\ell$ coordinates.

The first operator identity, Eq. (2.3), is closely related to the contour-ordered functions since the anti-time-ordered products occur to the left of the time-ordered products. The second operator identity, Eq. (2.5), would be related to functions defined on a contour in which the direction of the contour is reversed [6].

IV. FIRST SUM RULE

A. Closed time path: $\sigma = 0$

In the case $\sigma \to 0$, the contour is called the closed time path and $C_{ab} \to P_a$. Multiplying the operator identity Eq. (2.3) by $e^{-\beta H}/\text{Tr}(e^{-\beta H})$ and taking the trace gives the sum rule

$$0 = \sum_{\ell=0}^{n}\sum_{\text{comb}}(-1)^\ell G^{\text{CTP}}_{\underbrace{2\ldots2}_{\ell}\underbrace{1\ldots1}_{n-\ell}}(x_1, \ldots x_n). \quad (4.1)$$

The sums over ℓ and over combinations is equivalent to summing over the subscripts 1 and 2:

$$0 = \sum_{\alpha_1=1}^{2}\cdots\sum_{\alpha_n=1}^{2}(-1)^{a_1+\cdots a_n}G^{\text{CTP}}_{a_1\ldots a_n}(x_1, \ldots x_n), \quad (4.2)$$

since $a_1 + \cdots a_n = n + \ell$. [Equation (4.2) holds also for nonequilibrium systems, since a general density operator could replace $e^{-\beta H}/\mathrm{Tr}(e^{-\beta H})$.]

With the Fourier transform defined so that all energy momenta are incoming,

$$G^{\mathrm{CTP}}_{a_1\ldots a_n}(p_1,\ldots p_n) = \int \prod_{j=1}^{n}[d^4x_j e^{ip_j\cdot x_j}]G^{\mathrm{CTP}}_{a_1\ldots a_n}(x_1,\ldots x_n),$$

time-translation invariance of the equilibrium system leads to energy conservation:

$$p_1^0 + \cdots p_n^0 = 0. \tag{4.3}$$

The sum rule in momentum space is

$$0 = \sum_{a_1=1}^{2}\cdots\sum_{a_n=1}^{2}(-1)^{a_1+\cdots a_n}G^{\mathrm{CTP}}_{a_1\ldots a_n}(p_1,\ldots p_n). \tag{4.4}$$

This is equivalent to the result of Chou *et al.* [4], which is stated in the Keldysh basis.

B. General path: $\sigma \neq 0$

When $\sigma \neq 0$ the Fourier transform of Eq. (3.2) constrains $E_a - E_b$ to satisfy

$$E_a - E_b = -\sum_{j=1}^{\ell} p_j^0 = \sum_{j=\ell+1}^{n} p_j^0. \tag{4.5}$$

Thus the dependence on σ is trivial in momentum space:

$$G_{\underbrace{2\ldots 2}_{\ell}\underbrace{1\ldots 1}_{N-\ell}}(p_1,\ldots p_n) = \exp\Big[-\sigma\sum_{j=1}^{\ell}p_j^0\Big]G^{\mathrm{CTP}}_{\underbrace{2\ldots 2}_{\ell}\underbrace{1\ldots 1}_{n-\ell}}(p_1,\ldots p_n).$$

The same relation with an arbitrary ordering of the 2 and 1 indices becomes

$$G_{a_1\ldots a_n}(p_1,\ldots p_n) = e^{-\sigma(p_1^0 a_1+\cdots p_n^0 a_n)}G^{\mathrm{CTP}}_{a_1\ldots a_n}(p_1,\ldots p_n), \tag{4.6}$$

after using energy conservation in the form

$$\sum_{a_j=2\text{ only}} p_j^0 = \sum_{j=1}^{n} p_j^0 a_j. \tag{4.7}$$

The sum rule in Eq. (4.4) can be expressed as

$$0 = \sum_{a_1=1}^{2}\cdots\sum_{a_n=1}^{2}[-e^{\sigma p_1^0}]^{a_1}\ldots[-e^{\sigma p_n^0}]^{a_n}G_{a_1\ldots a_n}(p_1,\ldots p_n). \tag{4.8}$$

This is the version as given in [6,11].

C. Amputated functions

The one-particle-reducible functions G are related to the amputated functions Γ by

$$G_{a_1\ldots a_n}(p_1,\ldots p_n) = \sum_{b_1=1}^{2}\cdots\sum_{b_n=1}^{2}G_{a_1b_1}(p_1)\ldots G_{a_nb_n}(p_n) \times \Gamma_{b_1\ldots b_n}(p_1,\ldots p_n).$$

Substitution into Eq. (4.8) gives

$$0 = \sum_{a_1,b_1}\cdots\sum_{a_n,b_n}[-e^{\sigma p_1^0}]^{a_1}G_{a_1b_1}(p_1)\ldots[-e^{\sigma p_n^0}]^{a_n}G_{a_nb_n}(p_n) \times \Gamma_{b_1\ldots b_n}(p_1,\ldots p_n). \tag{4.9}$$

The propagators $G_{ab}(p)$ satisfy

$$\sum_{a=1}^{2}[-e^{\sigma p^0}]^a G_{ab}(p) = -D_A(p)[e^{\sigma p^0}]^b, \tag{4.10}$$

where $D_A(p)$ is the advanced propagator. Using this and canceling an overall factor

$$(-1)^n D_A(p_1)\ldots D_A(p_n),$$

allows the sum rule to be expressed in terms of the amputated functions Γ:

$$0 = \sum_{b_1=1}^{2}\cdots\sum_{b_n=1}^{2}[e^{\sigma p_1^0}]^{b_1}\ldots[e^{\sigma p_n^0}]^{b_n}\Gamma_{b_1\ldots b_n}(p_1,\ldots p_n). \tag{4.11}$$

Note that the $\pm$ signs that occur in Eq. (4.8) are not present in (4.11). This is the version given in [7].

V. SECOND SUM RULE

A. Closed time path: $\sigma = 0$

The second operator identity, Eq. (2.5), gives an independent sum rule for thermal n-point functions. The thermal trace of Eq. (2.5) is

$$0 = \sum_{\ell=0}^{n}\sum_{\text{comb}}(-1)^{\ell}\sum_{a,b}P_b\langle b|T[\phi(x_{\ell+1})\ldots\phi(x_n)]|a\rangle \times \langle a|\tilde{T}[\phi(x_1)\ldots\phi(x_\ell)]|b\rangle. \tag{5.1}$$

For a fixed ℓ and a fixed combination of x's, the necessary Fourier transform is

$$\int \prod_{j=1}^{n} [d^4 x_j e^{ip_j\cdot x_j}] \sum_{a,b} P_b \langle b|T[\phi(x_{\ell+1})\dots\phi(x_n)]|a\rangle\langle a|\tilde{T}[\phi(x_1)\dots\phi(x_\ell)]|b\rangle = \exp\left[-\beta\sum_{j=1}^{\ell} p_j^0\right] G^{\rm CTP}_{\underbrace{2\dots 2}_{\ell}\underbrace{1\dots 1}_{n-\ell}}(p_1,\dots p_n),$$

where P_b has been rewritten using Eq. (4.5) as

$$P_b = \exp\left[-\beta\sum_{j=1}^{\ell} p_j^0\right] P_a.$$

Therefore Eq. (5.1) becomes a sum rule for closed-time-path functions:

$$0 = \sum_{a_1=1}^{2}\cdots\sum_{a_n=1}^{2}[-e^{-\beta p_1^0}]^{a_1}\dots[-e^{-\beta p_n^0}]^{a_n} G^{\rm CTP}_{a_1\dots a_n}(p_1,\dots p_n).$$

B. General path: $\sigma \neq 0$

When $\sigma \neq 0$ the sum rule is

$$0 = \sum_{a_1=1}^{2}\cdots\sum_{a_n=1}^{2}[-e^{(\sigma-\beta)p_1^0}]^{a_1}\dots[-e^{(\sigma-\beta)p_n^0}]^{a_n} \times G_{a_1\dots a_n}(p_1,\dots p_n), \qquad (5.2)$$

because of Eq. (4.6). This form is given in [6].

C. Amputated functions

The one-particle-reducible functions G in Eq. (5.2) can be expressed in terms of the one-particle-irreducible functions Γ as before. It is necessary to use a different relation among propagators:

$$\sum_{a=1}^{2}[-e^{(\sigma-\beta)p^0}]^a G_{ab}(p) = -D_R(p)[e^{(\sigma-\beta)p^0}]^b. \qquad (5.3)$$

Equation (5.2) then acquires an overall factor

$$(-1)^n D_R(p_1)\dots D_R(p_n).$$

After cancellation, the second sum rule for amputated functions Γ becomes

$$0 = \sum_{b_1=1}^{2}\cdots\sum_{b_n=1}^{2}[e^{(\sigma-\beta)p_1^0}]^{b_1}\dots[e^{(\sigma-\beta)p_n^0}]^{b_n} \times \Gamma_{b_1\dots b_n}(p_1,\dots p_n). \qquad (5.4)$$

This is the version give in [7].

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