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Finite volume corrections to the two-particle decay of states with nonzero momentum

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We study the effects of finite volume on the two-particle decay rate of an unstable state with nonzero momentum. First, Lüscher's field-theoretic relation between the infinite-volume scattering phase shifts and the quantized energy levels of a finite-volume, two-particle system is generalized to the case of nonzero total momentum, confirming earlier results of Rummukainen and Gottlieb. We then use this result and the method of Lellouch and Lüscher to determine the corrections needed for a finite-volume calculation of a two-particle decay amplitude when the decaying particle has nonvanishing center-of-mass momentum.

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I. INTRODUCTION

A central problem in making standard model predictions for K meson decays, including the important CP violating amplitudes, is calculation of the decay $K \rightarrow \pi\pi$ into the $I = 0$ final state of two pions. In a standard calculation in which the K meson is at rest, the contribution of the two-pion final state is expected to be very difficult to extract from a lattice QCD calculation. The physical state has the same quantum numbers as the state with the two pions at rest (the difficulty emphasized by Maiani and Testa [1]). Of even greater concern is the fact that the QCD vacuum also has these same quantum numbers. A very appealing approach to deal with these problems is to study the decay of a K meson with nonzero momentum. For a calculation in finite volume, it is possible to adjust the momentum of the K meson and the box size so that both the transition amplitude to the vacuum must vanish (since the vacuum has zero momentum) and the final two-pion state must have physical relative momentum and total energy equal to that of the decaying K meson.

For such a calculation to be useful, we must be able to relate the decay amplitudes computed in finite volume using this technique to the infinite-volume matrix elements determined by experiment. To a large extent, the effects of finite volume on the particles involved are relatively mild. Typically the volumes used in such a calculation can be chosen sufficiently large that they do not significantly distort the K and π particles whose physical size will be much smaller than that of the box employed. The most obvious finite-volume effect will be the quantization of the energy levels of the two-pion final state. This is actually an advantage, permitting the box size to be adjusted to make one of the discrete, two-pion energies match precisely that of the K meson. For the case at hand, this can even be the lowest energy $\pi - \pi$ state.

However, the violation of rotational symmetry by the finite-volume boundary conditions does induce an important distortion in the computed decay rate. Because of the resulting nonconservation of angular momentum, the two-particle state into which the decay occurs is actually a mixture of states with many angular momenta. For typical lattice volumes, the actual decay of the K meson into these higher angular momentum states will be very small—angular momentum is effectively conserved at the short distances over which the decay occurs. However, the presence of these extra angular momentum states affects the normalization of the physical $\Delta J = 0$ amplitude which appears in the matrix element.

This finite-volume normalization problem has been solved by Lellouch and Lüscher [2] for the case of the decay of a K meson at rest. This problem has also been solved by a different approach in Ref. [3]. In this paper we will generalize the approach of Lellouch and Lüscher to obtain a result for states with nonzero total momentum. Central to their argument is an earlier treatment of Lüscher [4,5] which determines the allowed, finite-volume, two-particle energy eigenvalues in terms of the infinite-volume, two-particle scattering phase shifts for energies below all inelastic thresholds. This discussion must also be generalized to the case of nonzero center-of-mass (cm) momentum.

This topic has been studied earlier by Rummukainen and Gottlieb [6]. Their treatment involves an application of relativistic two-particle quantum mechanics. We believe that it is also important to study this problem starting from the equations of quantum field theory.

Our strategy follows closely that of Lüscher [4,5] and Lellouch and Lüscher [2]. We first discuss the energy quantization of finite-volume, interacting, two-particle states with nonzero center-of-mass momentum. Following Lüscher, this is first done in standard, two-particle, nonrelativistic quantum mechanics in Sec. II. In Sec. III we begin with the Bethe-Salpeter equation of relativistic field theory and, again following Lüscher, show how this equation when restricted to a particular 7-dimensional subspace of the 8-dimensional, 2-particle momentum space reduces to the standard Lippmann-Schwinger equation describing the earlier nonrelativistic system. Finally, in Sec. IV we use this result to generalize the argument of Lellouch and Lüscher to determine the finite-volume corrections to the decay amplitude computed for states with nonzero total momentum.

The issues addressed in this paper have also been considered by Kim, Sachrajda, and Sharpe. Using a related but different approach, they have also confirmed the validity of the results of Ref. [6] and derived the generalization of the result of Lellouch and Lüscher for the case of nonzero total momentum. Their paper [7] is being released simultaneously with the present article.

II. FINITE-VOLUME, NONRELATIVISTIC, TWO-PARTICLE STATES

We begin by considering a simple, nonrelativistic system of two distinguishable particles confined in a cubic box of side L and obeying periodic boundary conditions. The system is described by a wave function $\psi(\vec{r}_1, \vec{r}_2)$ which is periodic in $\vec{r}_1$ and $\vec{r}_2$ separately. An eigenstate of energy ψ_E obeys the Schrödinger equation:

$$\left\{-\frac{\nabla_1^2}{2m}-\frac{\nabla_2^2}{2m}+V(|\vec{r}_1-\vec{r}_2|)\right\}\psi_E(\vec{r}_1,\vec{r}_2)=E\psi_E(\vec{r}_1,\vec{r}_2), \tag{1}$$

where m is the identical mass of the two particles and $V(|\vec{r}_1 - \vec{r}_2|)$ their rotationally invariant interaction potential.

Such an equation is conventionally simplified by changing to center-of-mass and relative coordinates:

$$\vec{R}=\frac{\vec{r}_1+\vec{r}_2}{2} \tag{2}$$

$$\vec{r}=\vec{r}_1-\vec{r}_2 \tag{3}$$

with conjugate momenta

$$\vec{P}=\vec{p}_1+\vec{p}_2 \tag{4}$$

$$\vec{p}=\frac{\vec{p}_1-\vec{p}_2}{2}, \tag{5}$$

where $\vec{p}_i$ is the momentum conjugate to the coordinate $\vec{r}_i$.

Using $\psi_E^{(\mathrm{rel})}(\vec{r}, \vec{R})$ to represent the original wave function expressed in terms of $\vec{r}$ and $\vec{R}$, we can write down the standard equation which it obeys:

$$\left\{-\frac{\nabla_R^2}{4m}-\frac{\nabla_r^2}{m}+V(|\vec{r}|)\right\}\psi_E^{(\mathrm{rel})}(\vec{r},\vec{R})=E\psi_E^{(\mathrm{rel})}(\vec{r},\vec{R}). \tag{6}$$

The periodicity of $\psi(\vec{r}_1, \vec{r}_2)_E^{(\mathrm{rel})}$ under the simultaneous translation $\vec{r}_i \longrightarrow \vec{r}_i + \hat{e}_k L$ (where $\hat{e}_k$ is a unit vector parallel to one of the edges of the box) implies the periodicity of the wave function ψ_r under a translation of R by L. Thus, the conserved total momentum $\vec{P}$ must obey the quantization condition: $\vec{P} = \sum_{k=1}^{3} 2\pi n_k \hat{e}_k/L$ for integer n_k. It is easy to see that, in a direction k for which the integer n_k is even, the allowed component of the relative momentum $p_k = 2\pi n'_k/L$ while if n_k is odd then $p_k = 2\pi(n'_k + \frac{1}{2})/L$, where n'_k is an integer. With this change of coordinates and a specific choice of $\vec{P}$, our two-particle problem reduces to the quantum mechanics of a single particle in an L^3 box obeying either periodic or antiperiodic boundary conditions on each of its three opposing faces.

In infinite volume, this can be viewed as a scattering problem often phrased as a Lippmann-Schwinger integral equation. One defines an energy eigenstate $\psi_{\vec{p}}^{\mathrm{in}}$ whose incoming part (that term with radial dependence e^{-ipr}) is that of a plane wave with momentum $\vec{p}$. Necessarily, the outgoing part of $\psi_{\vec{p}}^{\mathrm{in}}$ is more general, being created by scattering from the potential V. Such a state must have energy $E^{\mathrm{cm}} = p^2/m$. Manipulation of Eq. (6) easily produces the desired integral equation:

$$\psi_{\vec{p}}^{\mathrm{in}}=\phi_{\vec{p}}+\frac{1}{E^{\mathrm{cm}}-H_0+i\epsilon}V\psi_{\vec{p}}^{\mathrm{in}}, \tag{7}$$

where $\phi_{\vec{p}}$ is the plane wave solution $\phi_{\vec{p}}(\vec{r}) = e^{i\vec{p}\cdot\vec{r}}$ of the free Schrödinger equation and $H_0 = -\nabla^2/m$, the free Hamiltonian of a particle with the "reduced mass" $m/2$. Of course the full solution to Eq. (6), $\psi_E^{(\mathrm{rel})}(\vec{r}, \vec{R})$, is a product of a plane-wave depending on the center-of-mass coordinate $\vec{R}$ and the wave function above:

$$\psi_E^{(\mathrm{rel})}(\vec{r},\vec{R})=e^{i\vec{P}\cdot\vec{R}}\psi_{\vec{p}}^{\mathrm{in}}(\vec{r}), \tag{8}$$

and the total energy E is related to the energy in the center-of-mass system by $E = E^{\mathrm{cm}} + \vec{P}^2/4m$.

Examining the asymptotic behavior of Eq. (7), one derives the standard relation between the conventional scattering amplitude $f(\theta)$ and the matrix element of V between the plane wave state $\phi_{\vec{p}'}$ and $\psi_{\vec{p}}^{\mathrm{in}}$:

$$f(\theta)=-2\pi^2m\langle\phi_{\vec{p}'}|V|\psi_{\vec{p}}^{\mathrm{in}}\rangle, \tag{9}$$

where $\vec{p}' \cdot \vec{p} = p^2\cos(\theta)$. We obtain an equation closer to the relativistic Bethe-Salpeter equation by defining the T matrix as

$$\langle\vec{p}'|T|\vec{p}\rangle=\langle\phi_{\vec{p}'}|V|\psi_{\vec{p}}^{\mathrm{in}}\rangle. \tag{10}$$

Here we are using the conventional Dirac bra-ket notation to represent momentum eigenstates: $\phi_{\vec{p}} \equiv |\vec{p}\rangle$. Note, Eq. (10) defines matrix elements of T even when $|\vec{p}'| \neq |\vec{p}|$ and energy is not conserved.

Using the matrix T we can rewrite Eq. (7) in a form that will be useful later, if we multiply by V and transform to momentum space:

$$\langle\vec{p}'|T|\vec{p}\rangle=\langle\vec{p}'|V|\vec{p}\rangle+\int\frac{d^3k}{(2\pi)^3}\langle\vec{p}'|V|\vec{k}\rangle\times\frac{1}{p^2/m-k^2/m+i\epsilon}\langle\vec{k}|T|\vec{p}\rangle. \tag{11}$$

This equation explicitly involves the matrix elements of T between states with different energies. Equation (11) is a component of standard scattering theory and applies only to the case of infinite volume. Since we are interested also in the eigenfunctions and energies for the finite-volume problem, it will be helpful to cast the standard Schrödinger equation, obeyed by an eigenstate, $\psi_n(\vec{r})$ with the discrete energy $E_n^{\rm cm}$,

$$(H_0 + V)\psi_n = E_n^{\rm cm}\psi_n, \qquad (12)$$

into a similar form:

$$\langle \vec{p}'|V|\psi_n\rangle = \langle \vec{p}'|V\frac{1}{E_n^{\rm cm} - H_0}V|\psi_n\rangle = \int \frac{d^3k}{(2\pi)^3}\langle \vec{p}'|V|\vec{k}\rangle \frac{1}{E^{\rm cm} - k^2/m}\langle \vec{k}|T|\vec{p}\rangle. \qquad (13)$$

This equation demonstrates that the state $V\psi_{E_n}$ solves the homogenous Lippmann-Schwinger equation if the energy argument in the denominator, $E^{\rm cm} = \vec{p}^2/m$ is continued to that state's actual energy $E^{\rm cm} = E_n^{\rm cm}$. Equation (13) can be used in infinite volume to determine the energy of a possible bound state. It also can be applied to the case of finite volume if one makes a simple replacement of the integral over the relative momentum $\vec{k}$ by an appropriate discrete sum.

In Eq. (11) we have followed the standard procedure, exploiting the separation of relative and center-of-mass variables permitted by Eq. (6), and written that integral equation as an equation obeyed by functions of a single, three-dimensional, relative momentum $\vec{k}$ or $\vec{p}$. The three-momentum of the center-of-mass $\vec{P}$ disappears from the problem once we use the energy in the center-of-mass system $E^{\rm cm}$. However, we could create a more explicit analogy with the relativistic discussion to follow if we viewed the states $|\vec{p}\rangle$ and $|\vec{k}\rangle$ as functions of the four-momentum of the center-of-mass as well: $P(\vec{p}) = [(\vec{P}^2/4m) + \vec{p}^2/m, \vec{P}]$ and $P(\vec{k}) = [(\vec{P}^2/4m) + \vec{k}^2/m, \vec{P}]$, respectively. Had we done this, Eq. (11) would be an equation obeyed by functions of seven variables. Each factor in this equation would be explicitly diagonal in $\vec{P}$ and the difference of these total energy variables $P_0(\vec{p})$ and $P_0(\vec{k})$ would replace the present denominator in that equation. [A similar remark applies to Eq. (13) as well.]

The final step to be reviewed in this section is the connection between the infinite-volume scattering problem, defined by Eq. (11), and the finite-volume energy eigenvalues of the original Schrödinger equation, Eq. (6) or Eq. (13). This is the problem solved by Lüscher in Refs. [4,5]. Recall that the scattering amplitude $f(\theta)$ can be written as a sum over partial waves as

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)\frac{e^{i2\delta_l} - 1}{2ip}P_l(\cos(\theta)), \qquad (14)$$

where the δ_l are the standard scattering phase shifts and the $P_l(\cos(\theta))$ the usual Legendre polynomials. In Refs. [4,5], Lüscher examines the case of a potential of finite range, $V(r) = 0$ for $r > R_{\rm max}$, and for the case $L > 2R_{\rm max}$ derives a relation between the allowed energies $E^{\rm cm} = p^2/m$ in the finite box and the phase shifts δ_l. For the simplest case where all $\delta_l \approx 0$ for $l > 0$, this finite-volume quantization condition becomes

$$n\pi - \delta_0(p) = \phi(q), \qquad (15)$$

where n is an integer, $p = \sqrt{mE}$, $q = pL/2\pi$, and the function $\phi(q)$ is a known kinematic function given by

$$\tan\phi(q) = -\frac{\pi^{3/2}q}{\mathcal{Z}_{00}(1;q^2)}, \qquad \phi(0) = 0, \qquad (16)$$

with the zeta function $\mathcal{Z}_{00}(s;q^2)$ defined by

$$\mathcal{Z}_{00}(s;q^2) = \frac{1}{\sqrt{4\pi}}\sum_{n\in\mathbb{Z}^3}(n^2 - q^2)^{-s}. \qquad (17)$$

The zeta function defined above applies to the case that the integers appearing in the center-of-mass momentum $\vec{P}$ are even. For the case that one or more is odd, the finite-volume problem will obey antiperiodic boundary conditions in those directions and an appropriate offset of $1/2$ must be added in the summation in Eq. (17), as discussed in Ref. [6]. We conclude that the total energy of a finite-volume system with total momenta $\vec{P}$, is given by $E = \vec{P}^2/4m + E^{\rm cm}$, where $p = \sqrt{mE^{\rm cm}}$ obeys Eq. (15).

Thus, Lüscher's relation for nonrelativistic quantum mechanics between the scattering phase shifts and the energy eigenvalues for that same system in a finite volume is straightforward to generalize to the case that the two-particle system carries nonzero total momentum. We now consider the generalization of the next step to nonzero total momentum: the connection between the Bethe-Salpeter equation of relativistic quantum field theory and Eq. (11) above, obeyed by the nonrelativistic T matrix. This represents the new result of this paper.

III. FINITE-VOLUME RELATIVISTIC TWO-PARTICLE STATES

In this section we will generalize to the case of nonzero total momentum the procedure introduced by Lüscher to reduce the Bethe-Salpeter equation of relativistic field theory to an equation whose form is identical to the nonrelativistic Eqs. (11) and (13). This will permit the quantization condition described in Eqs. (15)–(17) to be applied to quantum field theory, and QCD, in particular. As discussed in Refs. [4,5], we must do this by carefully distinguishing effects of finite volume which fall as powers or exponentially in the system size. Note: our objective is to derive an equation which is both similar in form to Eqs. (11) and (13) and also accurate for both finite and infinite volume so that it can be used to relate the finite-volume spectrum and the infinite-volume scattering amplitude.

We begin with the standard Bethe-Salpeter equation, which connects the amputated four-point function $T(p'_1, p'_2; p_1, p_2)$, the two-particle irreducible kernel $K(p'_1, p'_2; p_1, p_2)$, and the single-particle propagator $\Delta(k^2)$:

$$T(p'_1, p'_2; p_1, p_2) = K(p'_1, p'_2; p_1, p_2) + \int \frac{d^4\bar{k}}{(2\pi)^4} K(p'_1, p'_2; P/2 + \bar{k}, P/2 - \bar{k})\Delta((P/2 + \bar{k})^2)\Delta((P/2 - \bar{k})^2) \times T(P/2 + \bar{k}, P/2 - \bar{k}, p_1, p_2), \tag{18}$$

where $P = p_1 + p_2 = p'_1 + p'_2$ is the total four-momentum. For simplicity, we have also removed an overall δ-function for the conservation of total four-momentum from $T(p'_1, p'_2; p_1, p_2)$, $K(p'_1, p'_2; p_1, p_2)$, and $\Delta(k^2)$. For example, $T(p'_1, p'_2; p_1, p_2)$ is defined by

$$(2\pi)^4\delta^4(p'_1 + p'_2 - p_1 - p_2)T(p'_1, p'_2; p_1, p_2) = \prod_i \left\{(p_i'^2 - m^2)(p_i^2 - m^2) \int d^4x'_i e^{ip'_ix'_i} \int d^4x_i e^{-ip_ix_i}\right\} \times \langle 0|\phi(x'_1)\phi(x'_2)\phi(x_1)\phi(x_2)|0\rangle_{\rm conn}, \tag{19}$$

where $\phi(x)$ is the quantized scalar field used in this example (normalized so that the single-particle pole in the 2-point function has unit residue), $\langle\ldots\rangle_{\rm conn}$ indicates that only connected diagrams are to be included, and we are following the conventions of Peskin and Schroeder [8].

For simplicity we write the sum over the internal four-momentum $\bar{k}$ in Eq. (18) as an integral. However, this equation applies equally well to a finite-volume system if the spatial part of this continuous integral is replaced by an appropriate discrete sum. Specifically, in finite volume the total momentum operator $\vec{P}_{\rm op}$ is conserved and, given periodic spatial boundary conditions, has components which are quantized in units of $2\pi/L$. The single-particle three momenta, $\vec{p}_1, \vec{p}'_1, \vec{P}/2 + \vec{\bar{k}}$, etc. correspond to indices in a finite volume Fourier series and are also quantized in units of $2\pi/L$.

Equation (18) is more general than needed, constraining the off-shell, two-particle scattering amplitude, a matrix acting on a space of functions of eight momentum components. We must specialize this equation, reducing the number of momentum variables from eight to seven. The choice of this seven-dimensional restriction of Eq. (18) is the key step in the desired generalization to nonzero total three-momentum, $\vec{P} \neq 0$. Once that has been done, the resulting equation will still be quite different from our nonrelativistic target, Eq. (11). However, the final steps connecting Eq. (18) with Eqs. (11) and (13) proceed in a fashion very similar to those in Lüscher's original derivation: We extract from the second term in Eq. (18) particular pieces which are regular functions of the total energy P_0. By a rearrangement procedure, these terms are incorporated in a modified kernel $\tilde{K}(p'_1, p'_2; p_1, p_2)$. After this step, our rearranged equation will have the same form as the nonrelativistic Eqs. (11) and (13).

In contrast to the nonrelativistic case, our new equation will have a volume dependent potential. However, if we remain below the four-particle threshold and have introduced into our rearranged kernel $\tilde{K}$ only regular factors, the difference between the finite and infinite-volume kernels will vanish exponentially in the box size. Similar exponentially small errors will come from the failure of the resulting potential to have a truly finite range. Thus, we will be assured that the quantization condition in Eq. (15) will apply to the relativistic case with $\vec{P} \neq 0$ up to exponentially small corrections.

We begin by restricting the general Bethe-Salpeter equation of Eq. (18) to a carefully defined seven-dimensional momentum surface, making it similar to the target Eq. (11). [As discussed earlier, Eq. (11) can also be viewed as involving functions of seven dimensions provided the trivial dependence on the total four-momentum is included.]

First, express the pairs of two-particle momenta p'_1, p'_2 and p_1, p_2 using relative and total momenta:

$$p_{1,2} = P/2 \pm k \qquad p'_{1,2} = P/2 \pm k'. \tag{20}$$

In Eq. (18) we have already imposed four-momentum conservation, removing an overall momentum conserving delta function from both T and K. Next we impose a further condition on the four-momenta k and k' requiring that $k_0 = \beta k_{\|}$, where $\beta = |\vec{P}|/P_0$ and $k_{\|}$ is the spatial component of the four-vector k in the direction of $\vec{P}$. A similar condition defines a restricted value for k' and the integration variable $\bar{k}$ in Eq. (18). This condition is not immediately useful since, while we can consistently impose it on the external k and k' variables in Eq. (18), the integral (or sum) over $\bar{k}$ does not obey any such restriction. Note, the restriction $k_0 = \beta k_{\|}$ can be applied equally well in finite or infinite volume since, while $k_{\|}$ will be discrete for the finite-volume case, the time component, k_0, is always continuous and can be chosen to obey such a relation.

To make progress we must remove from the integral over $\bar{k}$ some terms which are not singular as P_0 approaches an allowed two-particle energy, $P_0 \to \omega_+ + \omega_-$, where the single-particle energies are given by $\omega_\pm = \sqrt{(\vec{P}/2 \pm \vec{k})^2 + m^2}$. This can be done by generalizing a discussion in Ref. [4] and arguing that the portion of the product $\Delta((P/2 + \bar{k})^2)\Delta((P/2 - \bar{k})^2)$ which is a singular function of P_0 comes from the product of the single-particle singularities in each of the Δ factors. This singular term, arising when these poles pinch the $\bar{k}_0$ contour, can be viewed as a distribution in $\bar{k}_0$, allowing us to write this product as

$$\begin{aligned}\Delta((P/2 + k)^2)\Delta((P/2 - k)^2) &= \frac{-i\pi}{P^2/4 + k^2 - m^2 + i\epsilon}\delta(P \cdot k) + R(P, k)\\ &\equiv S(P, k) + R(P, k),\end{aligned} \tag{21}$$

where the function $R(P, \bar{k})$ is a regular function of P_0 in the interval below the four-pion threshold:

$$2\sqrt{\frac{\vec{P}^2}{4} + m^2} \le P_0 \le 4\sqrt{\frac{\vec{P}^2}{16} + m^2}, \tag{22}$$

and the singular part $S(P, k)$ is defined by Eq. (21). Equation (21) is derived in the appendix and is a generalization of Lüscher's Eq. 3.16 from Ref. [4] to the case of $\vec{P} \neq 0$.

The next step absorbs the contribution from the regular function $R(P, k)$ as follows. First rewrite Eq. (18) in a more symbolic form exploiting the decomposition in Eq. (21):

$$T = K + K(S + R)T. \tag{23}$$

The second term on the right-hand side can then be moved to the left-hand side:

$$(1 - KR)T = K + KST. \tag{24}$$

Finally we divide by the factor $(1 - KR)$ and define

$$\tilde{K} = \frac{1}{1 - KR}K. \tag{25}$$

The resulting equation can then be written

$$\begin{aligned}T(k'; k) = \tilde{K}(k'; k) + \int \frac{d^4\bar{k}}{(2\pi)^4}\tilde{K}(k'; \bar{k})\\ \times \frac{-i\pi\delta(\bar{k} \cdot P)}{\frac{P^2}{4} + \bar{k}^2 - m^2}T(\bar{k}; k).\end{aligned} \tag{26}$$

Here we have replaced the variables p_i and p_i' with the total and relative four-momenta P, k', and k and suppressed the variable P.

It is now easy to see that Eq. (26) has a form identical to the original nonrelativistic Lippmann-Schwinger equation, Eq. (11). First we observe that the delta function $\delta(\bar{k} \cdot P)$ forces the integration four-momentum $\bar{k}$ to obey our restriction:

$$0 = \bar{k} \cdot P = \bar{k}_0 P_0 - \bar{k}_\parallel |\vec{P}| \quad \text{or} \quad \bar{k}_0 = \beta \bar{k}_\parallel. \tag{27}$$

This permits us to impose this relation between the time and parallel components of the relative four-momenta everywhere in this equation, effectively reducing it to a three-dimensional integral equation as is the case for the nonrelativistic problem. (As in that case, this equation is diagonal in the total four-momentum, P, the remaining four of our seven-dimensional momentum variables.)

Second, we observe that the denominator has a nonrelativistic form if we rescale the axis parallel to $\vec{P}$ by a factor of $\gamma = 1/\sqrt{1 - \beta^2}$:

$$\begin{aligned}\frac{P^2}{4} + \bar{k}^2 - m^2 &= \frac{P_0^2}{4} - \frac{\vec{P}^2}{4} + \bar{k}_0^2 - \vec{\bar{k}}^2 - m^2\\ &= \frac{P_0^2}{4} - \frac{\vec{P}^2}{4} - \frac{1}{\gamma^2}\bar{k}_\parallel^2 - \vec{\bar{k}}_\perp^2 - m^2.\end{aligned} \tag{28}$$

Thus, if we change variables from $\bar{k}_\parallel$ to

$$\tilde{k}_\parallel = \frac{1}{\gamma}\tilde{k}_\parallel, \tag{29}$$

the denominator has the normal Laplacian form. We obtain a complete match between the denominators in Eqs. (11) and (13) and that in Eq. (26) if we remove a factor of m from the denominator of Eq. (26) and identify the nonrelativistic energy $E^{\rm cm}$ with $(P_0^2 - \vec{P}^2)/4m - m$.

Here we should recall the standard connection between the Bethe-Salpeter equation, Eq. (18), and the version of the Schrödinger equation given by the homogenous Eq. (13), which determines the discrete, finite-volume energies. Since the Bethe-Salpeter equation, e.g. Eq. (26), holds in finite volume, the energy eigenvalues, E_n for the finite-volume, relativistic system will correspond to poles in the 2-particle scattering amplitude T obeying that equation. As one approaches the singularity at $P_0 \to E_n$ the inhomogeneous term, proportional to the kernel K, is not singular and can be dropped from the equation leaving a homogenous equation identical in form to Eq. (13).

Finally, we must investigate the rotational symmetry of the kernel, $\tilde{K}(k'; k)$. Since this function also depends on the four-vector P, there is possible rotationally asymmetric dependence on arguments of the form $\vec{P} \cdot \vec{k}'$ and $\vec{P} \cdot \vec{k}$ in addition to the acceptable dependence on $(\vec{k}')^2$, $\vec{k}^2$, and $\vec{k}' \cdot \vec{k}$. Fortunately, if the components of $\vec{k}'$ and $\vec{k}$ parallel to $\vec{P}$ are rescaled as described in Eq. (29), it is easy to see that the resulting function $\tilde{K}(k'; k)$ becomes rotationally symmetric. This can be demonstrated by exploiting the Lorentz invariance of the function $\tilde{K}(k'; k)$ which is assured by the covariant separation of regular and singular parts used in Eq. (21). We use this Lorentz symmetry to equate $\tilde{K}(k'; k)$

to its value in the rest system:

$$\tilde{K}(P;k';k) = \tilde{K}(P_{\rm cm};k'_{\rm cm};k_{\rm cm}), \tag{30}$$

where for clarity we now also display the dependence on the total four-momentum P.

Since in the center-of-mass frame $\vec{P}_{\rm cm} = 0$, the right-hand side of Eq. (30) is a manifestly rotationally invariant function of the center-of-mass variables $\vec{k}'_{\rm cm}$ and $\vec{k}_{\rm cm}$. However we can easily express these variables in terms of the original $\vec{k}'$ and $\vec{k}$. For example,

$$(\vec{k}_{\rm cm})_\perp = \vec{k}_\perp \tag{31}$$

$$(k_{\rm cm})_\parallel = \gamma(k_\parallel - \beta k_0) = \frac{1}{\gamma} k_\parallel \tag{32}$$

$$(k_{\rm cm})_0 = \gamma(k_0 - \beta k_\parallel) = 0, \tag{33}$$

where the second and third lines follow from the constraint obeyed by k_0, imposed when we specialized to this three-dimensional equation. The third equation for $(k_{\rm cm})_0$ is necessary to ensure that rotational asymmetry is not introduced through a nonsymmetric dependence of this variable on the original laboratory variables.

Note that the rescaled variables on which $\tilde{K}(P;k';k)$ depends symmetrically, $(k_{\rm cm})_\parallel = (1/\gamma)k_\parallel$ and $(k'_{\rm cm})_\parallel = (1/\gamma)k'_\parallel$, are precisely the variables in terms of which the denominator in Eq. (26) contains the standard Laplacian.

For clarity, we now rewrite the resulting three-dimensional integral equation, Eq. (26), in terms of these translated momentum variables, labeled suggestively $\vec{k}_{\rm cm}$:

$$\langle \vec{k}'_{\rm cm}|T|\vec{k}_{\rm cm}\rangle = \langle \vec{k}'_{\rm cm}|\tilde{K}|\vec{k}_{\rm cm}\rangle + \int \frac{d^3\bar{k}_{\rm cm}}{(2\pi)^3}\langle \vec{k}'_{\rm cm}|\tilde{K}|\vec{\bar{k}}_{\rm cm}\rangle \times \frac{-i}{2P_0}\frac{1}{(P^2/4 - m^2 - \vec{\bar{k}}^2_{\rm cm})}\langle \vec{\bar{k}}_{\rm cm}|T|\vec{k}_{\rm cm}\rangle. \tag{34}$$

While the variables $\vec{k}'_{\rm cm}$, $\vec{\bar{k}}_{\rm cm}$, and $\vec{k}_{\rm cm}$ may appear to be Lorentz transformed, center-of-mass variables, in fact, they are simply the original variables defined in the laboratory system except for a transformation of scale for the single component parallel to $\vec{P}$. This rescaling is equally well defined if the corresponding variable is continuous or discrete. The only explicit use of Lorentz symmetry is to constrain the possible dependence of the function $\tilde{K}(P,k',k)$ on its arguments. Thus, for the case of finite volume, Eq. (34) is obtained from Eq. (26) by a simple relabeling. For the case of infinite volume, Eq. (34) is obtained from Eq. (26) by both a change of integration variable and a change of normalization for the states. These two changes introduce compensating factors of γ.

Thus, we have demonstrated that our original, field-theoretic Bethe-Salpeter equation can be rewritten as a nonrelativistic Lippmann-Schwinger equation if we change to rescaled variables $(k^{\rm cm})_\parallel = (1/\gamma)k_\parallel$. We must now ask if such a rescaled set of momenta corresponds to an actual finite-volume problem. Recall that we began by examining a two-particle problem in a finite, cubic box of side L. The transformation to the coordinates $\vec{R}$ and $\vec{r}$ of Eqs. (2) and (3) and the choice of total momenta $\vec{P}$ then requires that the relative momenta $\vec{k}$ or $\vec{k}'$ that appear in the Bethe-Salpeter Eq. (26) have the form $2\pi(n_1,n_2,n_3)/L$ where n_i is an integer or half-integer depending on whether $LP_i/2\pi$ is an even or odd integer. Now, we have recognized that this problem is equivalent to a nonrelativistic problem with transformed momenta given by Eqs. (31) and (32). If these $k^{\rm cm}$ momenta correspond to those for a finite-volume problem, then we have succeeded in casting the original relativistic problem into a nonrelativistic problem which can be solved using the techniques developed by Lüscher in Refs. [4,5].

That this is in fact the case can be seen by examining two simple examples. In the first example $\vec{P} = 2\pi(0,0,1)/L$. In this case the transformed variables $k_i^{\rm cm}$ take on a very simple form:

$$k_1^{\rm cm} = k_1 = \frac{2\pi}{L} n_1 \tag{35}$$

$$k_2^{\rm cm} = k_2 = \frac{2\pi}{L} n_2 \tag{36}$$

$$k_3^{\rm cm} = \gamma(k_3 - \beta k_0) = \frac{1}{\gamma}k_3 = \frac{2\pi}{\gamma L} n_3, \tag{37}$$

where n_1, n_2, and $n_3 + \frac{1}{2}$ are integers. These quantized momenta correspond to a simple finite volume of length L in the 1- and 2-directions and expanded length γL in the 3-direction. If periodic boundary conditions are imposed in the 1- and 2-directions and antiperiodic conditions imposed in the 3-direction, the resulting quantized momenta will correspond precisely with those in Eqs. (35)–(37). Thus, after generalizing Lüscher's nonrelativistic technique to this sort of asymmetric box, we will obtain the desired relation between the infinite-volume scattering phase shifts and the discrete finite-volume energies in this asymmetric box. Of course, our analysis above has determined that these same energies will be found in the original relativistic problem with total momentum $\vec{P} = 2\pi(0,0,1)/L$ and cubic box of side L. This agrees with the result obtained Rummukainen and Gottlieb [6] and corresponds to a case of immediate practical interest.

Now let us examine a second case where $\vec{P} = (1,1,0)2\pi/L$. Using Eqs. (31) and (32) the rescaled momentum $\vec{k}^{\rm cm}$ is given by

$$\vec{k}^{\rm cm} = \vec{k} - \frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2}\vec{P} + \frac{1}{\gamma}\frac{\vec{k}\cdot\vec{P}}{|\vec{P}|^2}\vec{P} \tag{38}$$

where the first two terms correspond to the untransformed perpendicular component while the third term is the transformed parallel piece. Written in terms of the individual components this equation becomes

$$k_1^{\rm cm} = \left\{\frac{n_1 - n_2}{2} + \frac{n_1 + n_2}{2\gamma}\right\}\frac{2\pi}{L} \tag{39}$$

$$k_2^{\rm cm} = \left\{\frac{n_2 - n_1}{2} + \frac{n_1 + n_2}{2\gamma}\right\}\frac{2\pi}{L} \tag{40}$$

$$k_3^{\rm cm} = k_3 = n_3\frac{2\pi}{L}. \tag{41}$$

These quantization conditions can be easily realized if we impose the following (anti)periodicity conditions on the wave function $\psi^{\rm cm}(\vec{r})$ on which the operators in the integral equation, Eq. (34), act:

$$\psi^{\rm cm}(\vec{r}) = -\psi^{\rm cm}(\vec{r} + \vec{D}_i), \qquad \text{for } i = 1, 2 \tag{42}$$

$$\psi^{\rm cm}(\vec{r}) = +\psi^{\rm cm}(\vec{r} + \hat{e}_3 L) \tag{43}$$

where the displacement vector D_i is chosen to pick out the integer n_i from the dot product $\vec{D}_i \cdot \vec{k}^{\rm cm} = 2\pi n_i L$:

$$D_1 = \hat{e}_1\frac{\gamma + 1}{2}L + \hat{e}_2\frac{\gamma - 1}{2}L \tag{44}$$

$$D_2 = \hat{e}_1\frac{\gamma - 1}{2}L + \hat{e}_2\frac{\gamma + 1}{2}L. \tag{45}$$

The quantization condition given in Eqs. (42) and (43) is equivalent to requiring that the wave function $\psi^{\rm cm}(\vec{r})$ obeys antiperiodic boundary conditions on the faces of a rhombus whose sides are parallel to the vectors D_i and whose diagonals have length $|D_1 + D_2| = \sqrt{2}\gamma L$ and $|D_1 - D_2| = \sqrt{2}L$, precisely the earlier result of Rummukainen and Gottlieb [6]. From these two examples, it is clear that the case of general total momentum $\vec{P} = (n_1, n_2, n_3)2\pi/L$ can also be realized by imposing (anti)periodic boundary conditions on an appropriately distorted volume.

The results of this section can be summarized by returning to our first example, the case of a symmetrical L^3 box and a two-pion state with total momentum oriented in the 3-direction: $\vec{P} = (2\pi/L)\hat{e}_3$. Under these circumstances, the Bethe-Salpeter equation obeyed by the two-particle, off-shell scattering amplitude has been shown to be equivalent to a Schrödinger-like wave equation obeyed in an asymmetrical box with sides $L \times L \times \gamma L$ with the longer γL side parallel to the 3-direction. Thus, up to exponentially small corrections, the energy eigenvalues of the original 2-particle, L^3 system can be predicted using this $L \times L \times \gamma L$, Schrödinger-like system. As a result, the allowed energies E of this original system must obey a quantization condition similar to that given in Eq. (15) where the function $\phi(k)$ in that equation must be modified to describe the antiperiodic boundary conditions and expanded length in the 3-direction:

$$n\pi - \delta_0(k_{\rm cm}) = \phi(q) \tag{46}$$

$$\tan\phi(q) = -\frac{\gamma\pi^{3/2}q}{Z_{00}(1; q^2; \gamma)}, \qquad \phi(0) = 0 \tag{47}$$

$$Z_{00}(s; q^2; \gamma) = \frac{1}{\sqrt{4\pi}}\sum_{n\in\mathbb{Z}^3}\left(n_1^2 + n_2^2 + \frac{1}{\gamma^2}\left(n_3 + \frac{1}{2}\right)^2 - q^2\right)^{-s}, \tag{48}$$

where n is an integer, $k_{\rm cm} = \frac{1}{2}\sqrt{P^2 - (2m)^2}$ and $q = k_{\rm cm}L/2\pi$. This is the original result of Rummukainen and Gottlieb [6].

IV. TWO-PARTICLE DECAY OF STATES WITH NONZERO MOMENTUM

The last part of this discussion is a generalization of the arguments of Lellouch and Lüscher in Ref. [2] to the case of nonzero total momenta. Fortunately, this is very straightforward because the methods employed in that paper work equally well for $\vec{P} \neq 0$ once the formula relating energy levels and scattering phase shifts has been generalized to this nonzero momentum case.

We begin by reviewing the Lellouch-Lüscher approach, which is somewhat indirect. For our present purposes we will describe this method for the case of nonzero center-of-mass momentum, $\vec{P} \neq 0$. One considers a finite-volume system with both a K^0 meson and a degenerate $\pi^+\pi^-$ state, each with total four-momentum P. Here the finite volume has been adjusted to ensure that $E_{\pi\pi} = \sqrt{m_K^2 + \vec{P}^2}$, including the effects of the $\pi - \pi$ interaction. Next, the effects of the weak interaction Hamiltonian, H_W, mixing these two states, are then examined in perturbation theory. To zeroth order, the K^0 and $\pi^+\pi^-$ states are degenerate and noninteracting. To first order these states mix and their energies can be determined in first order, degenerate perturbation theory:

$$P_0 = \sqrt{m_K^2 + \vec{P}^2} \tag{49}$$

$$\to P_0 + \Delta P_0 \tag{50}$$

$$= P_0 \pm \langle\pi^+\pi^-|H_W|K^0\rangle \tag{51}$$

$$\equiv P_0 \pm M, \tag{52}$$

where the states appearing in this formula are finite-volume states with nonzero total momentum, normalized to unit probability.

The next step relates the finite-volume amplitude M, which can be computed directly in a lattice QCD calculation, with the infinite-volume matrix element of H_W that determines the physical partial width. This step uses

Eq. (46) to relate the infinite-volume phase shift, computed at the quantized, finite-volume energy $P_0 + \Delta P_0$ with that energy shift related to M by Eqs. (50)–(52). Since the effect of H_W on the scattering phase shift can be determined analytically in terms of the infinite-volume matrix elements of H_W, this equation will then relate the known, finite-volume matrix element of H_W with the desired, infinite-volume matrix element.

Thus, we must compute the variation in the $\pi - \pi$ scattering phase shift caused by the resonant scattering into the K meson state. This infinite-volume calculation is most easily done in the $\pi - \pi$ center-of-mass system and follows easily from the single $\pi - \pi$ scattering diagram with a K meson intermediate state shown in Fig. 1 giving

$$\Delta\delta_0(k_{\rm cm}) = -\frac{k_{\rm cm}|A|^2}{32\pi m_K^2 \Delta(P_0)_{\rm cm}}. \tag{53}$$

Here we have evaluated the addition to the $\pi - \pi$ scattering phase shift coming from the resonant scattering into the K meson state at a center-of-mass energy which corresponds to the laboratory energy $P_0 + \Delta P_0$ determined by Eqs. (52):

$$\Delta(P_0)_{\rm cm} = \frac{\partial\sqrt{P_0^2 - \vec{P}^2}}{\partial P_0}\Delta P_0 = \gamma\Delta P_0 = \pm\gamma|M|. \tag{54}$$

The infinite-volume decay amplitude A appearing in Eq. (53) is normalized following the conventions of Lellouch and Lüscher so that the corresponding decay width is given by

$$\Gamma_{K\to\pi\pi} = \frac{k_{\rm cm}}{16\pi m_K^2}|A|^2. \tag{55}$$

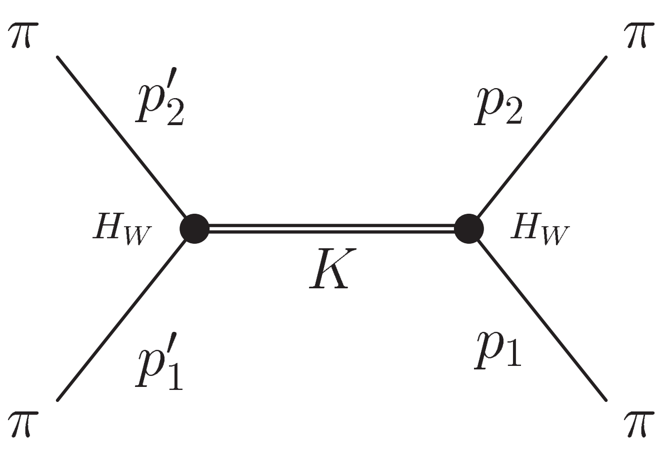

FIG. 1. The contribution of H_W to $\pi - \pi$ scattering involving resonant production of a K meson. Because of the singular K-meson propagator, the amplitude corresponding to this graph will be first order in H_W when evaluated at the center-of-mass energies $m_K \pm \gamma|M|$ as required for our application.

Finally, these results are used to evaluate the terms in the quantization condition, Eq. (46), for the original finite volume $\vec{P} \neq 0$, $\pi - \pi$ system. The terms in this equation which are first order in the matrix elements of H_W are

$$-\Delta k_{\rm cm}\left\{\frac{\partial\delta_0(k)}{\partial k}\right\}_{k=k_{\rm cm}} + \frac{k_{\rm cm}|A|^2}{32\pi m_K^2\Delta(P_{\rm cm})_0} = \Delta k_{\rm cm}\left\{\frac{\partial\phi(q)}{\partial k}\right\}_{k=k_{\rm cm}}, \tag{56}$$

where $q = k_{\rm cm}L/2\pi$ and the relation between $\Delta k_{\rm cm}$ and ΔP_0 is determined by $k_{\rm cm} = \sqrt{P^2/4 - m^2}$ which implies

$$\Delta k_{\rm cm} = \frac{P_0\Delta P_0}{4k_{\rm cm}}. \tag{57}$$

Combining Eqs. (52) and (57) then gives the desired connection between A and $M = \langle\pi^+\pi^-|H_W|K^0\rangle$:

$$\mp\frac{P_0|M|}{4k_{\rm cm}}\left\{\frac{\partial\delta_0(k)}{\partial k}\right\}_{k=k_{\rm cm}} \pm \frac{k_{\rm cm}|A|^2}{32\pi m_K^2\gamma|M|} = \pm\frac{P_0|M|}{4k_{\rm cm}}\left\{\frac{\partial\phi(q)}{\partial k}\right\}_{k=k_{\rm cm}}. \tag{58}$$

If this equation is solved for $|A|^2$ we obtain the desired generalization of the original Lellouch-Lüscher condition to the case of $\vec{P} \neq 0$:

$$|A|^2 = 8\pi\frac{m_K^3}{k_{\rm cm}^3}\gamma^2\left\{k\frac{\partial\delta}{\partial k} + q\frac{\partial\phi}{\partial q}\right\}|M|^2. \tag{59}$$

This formula differs from that of Ref. [2] by the presence of the factor of γ^2 and the more complex, γ-dependent definition of the function $\phi(q)$ given in Eqs. (47) and (48).

V. CONCLUSION

In the preceding sections we have examined the case of two interacting particles confined in a finite spatial volume and carrying nonzero total momentum. We have determined a relation between the quantized energies of these finite-volume states and the two-particle scattering phase shifts. This result, first obtained by Rummukainen and Gottlieb in Ref. [6], is here obtained from the Bethe-Salpeter equation of relativistic quantum field theory using an extension of the methods that Lüscher applied to the case of zero total momentum in Refs. [4,5]. We then exploit these finite-volume results to analyze two-particle decays. The result, an extension of earlier work of Lellouch and Lüscher, Ref. [2], to the case of nonzero total momentum, provides an explicit formula that relates finite-volume decay matrix elements computed using lattice gauge theory techniques and the infinite-volume quantities that enter physical decay rates.

The ability to work with states with nonzero total momentum when computing such decay matrix elements offers two potentially important benefits to the study of

$K \to \pi\pi$ decay. First, by a proper choice of the total momentum, the corresponding two-pion state with lowest energy can be arranged to have an energy equal to that of the K meson, permitting the direct calculation of a physical, on-shell, decay matrix element. Second, by working with a K meson with nonzero momentum, we ensure that the unphysical vacuum decay amplitude, normally dominant in such a Euclidean matrix element calculation, will vanish because of momentum conservation. We are now exploring this approach numerically.

APPENDIX: SINGULAR PART OF TWO-PROPAGATOR PRODUCT

The singular contribution to the integral in Eq. (18) will come as the energy P_0 is adjusted to cause singularities present in both of the single-particle propagator factors, $\Delta((P/2 \pm \bar{k})^2)$, to pinch the $\bar{k}_0$ contour. Thus, we can obtain the singular part of the integral over $\bar{k}_0$ in Eq. (18) by replacing both propagators by their singular part:

$$\Delta((P/2 \pm k)^2) \to \frac{i}{(P/2 \pm k)^2 - m^2}. \tag{A1}$$

Equation (21) is based on the following expression for the singular part of the product of two free scalar propagators:

$$\frac{i}{(P/2 + k)^2 - m^2 + i\epsilon} \frac{i}{(P/2 - k)^2 - m^2 + i\epsilon} \equiv \frac{-i\pi\delta(P \cdot k)}{P^2/4 + k^2 - m^2 + i\epsilon}, \tag{A2}$$

where these two quantities are equivalent in the sense that they have the same pole in the total energy P_0 at the point $P_0 = \omega_+ + \omega_-$

Following Lüscher, this formula is interpreted as relating two distributions in the variable k_0 over the space of test functions $f(k_0)$ analytic in k_0 within a band around the imaginary k_0 axis. Their equivalence can be demonstrated by multiplying by such a test function and then integrating k_0 along the imaginary axis. The left-hand side of Eq. (A2) is evaluated by moving the k_0 contour past one of the two pinching poles, $k_0 = P_0/2 - \omega_-$ or $k_0 = -P_0/2 + \omega_+$, and keeping the contribution of that pole given by Cauchy's theorem. Following this procedure the left-hand side of Eq. (A2) becomes

$$-i\pi \frac{f((\omega_+ - \omega_-)/2)}{2\omega_+\omega_-(P_0 - \omega_+ - \omega_-)}, \tag{A3}$$

where we have simplified this expression by replacing P_0 in the residue of the $P_0 = \omega_+ + \omega_-$ pole by its value at that pole. Next we evaluate the right-hand side of Eq. (A2) by using the delta function to evaluate the integral over k_0. We obtain

$$-i\pi \frac{f(\frac{\vec{P}\cdot\vec{k}}{P_0})}{P_0\{P_0^2/4 + (\frac{\vec{P}\cdot\vec{k}}{P_0})^2 - \vec{P}^2/4 - \vec{k}^2 - m^2 + i\epsilon\}}, \tag{A4}$$

where for clarity the P_0 arguments have not been simplified. The final step requires computing the location and residue of the pole in P_0 found in the expression in Eq. (A4). The result agrees precisely with that shown in Eq. (A3) provided one is careful to include the effects of the $(\vec{P} \cdot \vec{k}/P_0)^2$ term when computing the residue and then replaces P_0 in the residue with $\omega_+ + \omega_-$, its value at the pole. We forego a discussion of the specific region of analyticity for the test functions above and its consistency with the analytic properties of the amplitudes in the Bethe-Salpeter equation, Eq. (18), since these kinematics should be a direct Lorentz transform of those discussed by Lüscher in Ref. [4].

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