Cross sections of muon and charged pion pair production in electron-positron annihilation near the threshold

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(Received 17 June 2005; published 27 December 2005)

The processes of muon (tau) and charged pion pair production at electron-positron annihilation with $\mathcal{O}(\alpha)$ radiative corrections are considered. The calculation results are presented assuming the energies of final particles (center of mass system implied) to be not far significantly from threshold production. The invariant mass distributions for the muon (tau) and pion pairs are obtained both for the initial and final state radiation. Some analytical calculations are illustrated numerically. The pions were assumed to be pointlike objects and scalar QED was applied for calculation. The QED radiative corrections related to the final state radiation, in addition to the well-known Coulomb factor, are treated near threshold region exactly.

DOI: 10.1103/PhysRevD.72.114019 PACS numbers: 13.66.De, 13.66.Bc

I. INTRODUCTION

The current precision of the evaluation of a hadron's contribution to the anomalous magnetic moment of muon is mainly driven by the systematic error of the cross section of pion pair production at the region where the total center of mass system energy of pair does not exceed threshold value significantly. Therefore, the lowest order radiative corrections and effects due to the Coulomb interaction in the final state become essential.

The cases with a hard additional photon must be calculated in frames of perturbation theory for the purposes of comparisons with experimental data. One of the motivations of this paper is to calculate these contributions within the traditional QED approach by assigning to the photon a small mass and calculate the virtual, soft, and hard photon contributions separately. The main characteristics of photon emission at annihilation e^+e^- to a pair of charged particles has been investigated [1]. The spectra and total cross sections were obtained in papers published in 1983 and 1985 [2,3], but the calculation method was too complicated.

Below we repeat in part those calculations and obtain the explicit expressions for the spectra distributions on the effective mass of pair and the corresponding contributions to the total cross sections due to photon radiation by initial or final particles using the invariant integration method. We do not consider the interference of these amplitudes assuming the experimental setup to be charge blind. In this case the interference contribution to the total cross section is zero.

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In Secs. II and III we consider the final state and initial state radiation of virtual and real photons in a muon pair production process. In Sec. IV similar calculations for the charged pion pair production (assuming the pion to be a pointlike object) are done. The results presented in Secs. II–IV are in agreement with the ones obtained in previous papers [2–4] but have a more convenient form for different applications. Some of them concerning initial state radiation are new ones. In Sec. IV we also discuss some possibilities of experimental separation of contribution of initial and final state radiation. In Sec. V the discussion of accuracy of results obtained is given. Whenever possible, the analytical results are used as a cross-check with ultrarelativistic limit.

II. FINAL STATE RADIATION (FSR) IN MUON PAIR PRODUCTION

Since we are interested in muon effective mass spectrum let us put the cross section in the following form:

$$d\sigma = \frac{1}{8s}\int \sum_{\text{spins}} |M|^2 d\Gamma.$$

The summed over spin states matrix element squared can be put in the form (for notations see Fig. 1):

$$\sum |M|^2 = -(4\pi\alpha)^2 \frac{1}{s^2} L_{\mu\nu} T^{\mu\nu}, \qquad s = (p_+ + p_-)^2,$$
$$L_{\mu\nu} = \mathrm{Tr}[\hat{p}_- \gamma_\mu \hat{p}_+ \gamma_\nu], \tag{1}$$
$$T_{\mu\nu} = \mathrm{Tr}[(\hat{q}_- + M) O_{\mu\eta} (\hat{q}_+ - M) \tilde{O}_{\nu\eta}],$$

$$O_{\mu\nu} = \gamma_\nu \frac{\hat{q}_- + \hat{k} + M}{\chi_-} \gamma_\mu + \gamma_\mu \frac{-\hat{q}_+ - \hat{k} + M}{\chi_+} \gamma_\nu, \tag{2}$$

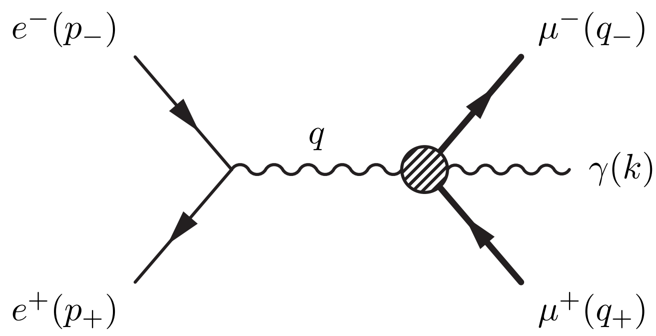

FIG. 1. Final state radiation corrections to $e^+e^- \to \mu^+\mu^-$ process.

where $\chi_\pm = 2kq_\pm$, $p_- + p_+ = q = q_- + q_+ + k$, $q_\pm^2 = M^2$, $p_\pm^2 = m^2$, and $k^2 = 0$. m, M are electron and muon masses, respectively. Introducing the energy fractions of final particles we have

$$\nu_\pm = \frac{2qq_\pm}{s}; \qquad \nu = \frac{2qk}{s}, \qquad \nu + \nu_+ + \nu_- = 2,$$

$$\int d\Gamma = \int \frac{1}{(2\pi)^5} \frac{d^3q_-}{2E_-} \frac{d^3q_+}{2E_+} \frac{d^3k}{2\omega} \times \delta^4(p_+ + p_- - q_+ - q_- - k) = \frac{s}{2^7\pi^3} \int_\Delta^{\beta^2} d\nu \int_{\nu_1}^{\nu_2} d\nu_+,$$

$$\nu_{1,2} = \frac{1}{2}(2-\nu) \pm \frac{\nu}{2}R(\nu); \quad (1-\nu)(1-\nu_-)(1-\nu_+) > \sigma\nu^2,$$

$$R(\nu) = \sqrt{1 - \frac{4\sigma}{1-\nu}} = \sqrt{\frac{\beta^2 - \nu}{1-\nu}}, \quad \beta^2 = 1 - 4\sigma, \quad \sigma = \frac{M^2}{s}.$$

Because of gauge invariance of tensor $T^{\mu\nu}$ we can write down the following:

$$\int d\Gamma T_{\mu\nu} = \frac{1}{3}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \int d\Gamma T_\eta^\eta.$$

Further simplification follows from gauge invariance of initial leptons tensor $L^{\mu\nu} q_\mu = 0$. Simple calculation gives

$$\sum T_\eta^\eta = 4\left[\frac{A}{(1-\nu_+)^2} + \frac{B}{1-\nu_+} + C + (\nu_+ \to \nu_-)\right], \tag{3}$$

$$A = -\frac{1}{2}(3-\beta^2)(1-\beta^2), \qquad C = -2;$$
$$B = \frac{1}{\nu}(3-\beta^2)(1+\beta^2) - 2(3-\beta^2) + 2\nu.$$

Integration on the muon energy fraction can be performed using the expressions

$$\int_{\nu_1}^{\nu_2} d\nu_+ \left[\frac{1}{(1-\nu_+)^2}; \frac{1}{1-\nu_+}; 1\right] = \left[\frac{1-\nu}{\nu\sigma} R(\nu); \ln\frac{1+R(\nu)}{1-R(\nu)}; \nu R(\nu)\right]. \tag{4}$$

Distribution on the invariant mass square of muons $m_{\mu\mu}^2 = (q_+ + q_-)^2 = s(1-\nu)$ for the case when the energy of hard photon exceeds some value $\omega > \sqrt{s}\Delta/2$, $\Delta \ll 1$ has a form (see Fig. 2)

$$\frac{d\sigma_{\rm FSR}^h}{d\nu} = \frac{2\alpha^3}{3s}\left[\left[\frac{(1+\beta^2)(3-\beta^2)}{\nu} - 2(3-\beta^2) + 2\nu\right] \times \ln\frac{1+R(\nu)}{1-R(\nu)} - 2\left[\frac{3-\beta^2}{\nu}(1-\nu) + \nu\right]R(\nu)\right]. \tag{5}$$

Contribution to the total cross section can be obtained by performing the integration on invariant muon mass. We use the following set of integrals:

$$\int_\Delta^{\beta^2} R(\nu)\left[\frac{1}{\nu}; 1; \nu\right] d\nu = \left[-L_\beta + \beta \ln\frac{4\beta^2}{(1-\beta^2)\Delta}; \beta - \frac{1-\beta^2}{2} L_\beta; \beta\frac{3-\beta^2}{4} - \frac{(3+\beta^2)(1-\beta^2)}{8} L_\beta\right] + O(\Delta), \tag{6}$$

$$\int_\Delta^{\beta^2} \ln\frac{1+R(\nu)}{1-R(\nu)}\left[\frac{1}{\nu}; 1; \nu\right] d\nu = \left[L_\beta \ln\frac{1}{\Delta} + 2\Phi(\beta); -\beta + \frac{1}{2}(1+\beta^2)L_\beta; \frac{1}{16}(3 + 2\beta^2 + 3\beta^4)L_\beta - \frac{3}{8}\beta(1+\beta^2)\right] + O(\Delta), \tag{7}$$

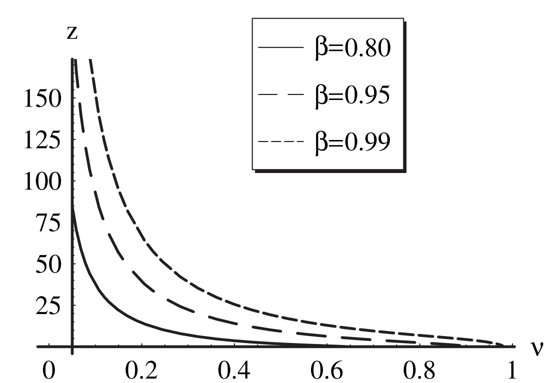

FIG. 2. Distribution on ν for FSR of muons, i.e. the value $z = (2\alpha^3/3s)^{-1}(d\sigma_{\rm FSR}^h/d\nu)$ [see (5)] is shown.

with

$$L_\beta = \ln\frac{1+\beta}{1-\beta};$$
$$\Phi(\beta) = \mathrm{Li}_2(1-\beta) - \mathrm{Li}_2(1+\beta) - \mathrm{Li}_2\Big(\frac{1-\beta}{2}\Big) + \mathrm{Li}_2\Big(\frac{1+\beta}{2}\Big). \tag{8}$$

The result is

$$\sigma_h^{e^+e^-\to\mu^+\mu^-\gamma} = \frac{2\alpha}{\pi}\sigma_B(s)\Big[\Big(\frac{1+\beta^2}{2\beta}L_\beta - 1\Big)\ln\frac{1}{\Delta} + \frac{7}{4} - \ln\frac{4\beta^2}{1-\beta^2} - \frac{3(1+\beta^2)}{8(3-\beta^2)} + \frac{9-2\beta^2+\beta^4}{16\beta(3-\beta^2)}L_\beta + \frac{1+\beta^2}{\beta}\Phi(\beta)\Big], \tag{9}$$

where $\sigma_B(s) = 2\pi\alpha^2\beta(3-\beta^2)/(3s)$ is the cross section of muon pair production in Born approximation. In the ultrarelativistic limit we have

$$\sigma_h^{e^+e^-\to\mu^+\mu^-\gamma}|_{\beta\to 1} = \frac{4\pi\alpha^2}{3s}\frac{2\alpha}{\pi}\Big[(l_\mu - 1)\ln\frac{1}{\Delta} - \frac{3}{4}l_\mu + \frac{11}{8} - \zeta_2\Big], \tag{10}$$

where $l_\mu = \ln(s/M^2)$, $\zeta_2 = \pi^2/6$. This result differs from the one given in [5]. The contribution of soft real photons emissions with energy $\omega = \sqrt{k^2+\lambda^2} < \sqrt{s}\Delta/2$, where λ is "photon mass," is given by

$$\sigma_{\rm FSR}^s = \sigma_B(s)\Big(-\frac{\alpha}{4\pi^2}\Big)\int\frac{d^3k}{\omega}\Big(\frac{q_-}{q_-k} - \frac{q_+}{q_+k}\Big)^2,$$

and performing the standard calculations can be written in the form [6]

$$\sigma_{\rm FSR}^s = \frac{2\alpha}{\pi}\sigma_B(s)\Big[\Big(\frac{1+\beta^2}{2\beta}L_\beta - 1\Big)\Big(\ln\frac{M}{\lambda} + \ln\Delta\Big) + \frac{1+\beta^2}{2\beta}\Big[\frac{1}{4}L_\beta^2 - \mathrm{Li}_2(\beta) + \mathrm{Li}_2(-\beta) - \mathrm{Li}_2\Big(\frac{1-\beta}{2}\Big) - \ln\Big(\frac{1+\beta}{2}\Big)\ln(1-\beta) + \frac{1}{2}\ln^2(1+\beta) + \mathrm{Li}_2\Big(\frac{1}{2}\Big) + L_\beta\ln\frac{2}{1+\beta}\Big] + \ln\Big(\frac{1+\beta}{2}\Big) + \frac{1-\beta}{2\beta}L_\beta\Big]. \tag{11}$$

The correction of virtual photon emission includes the Dirac and Pauli form factors of muon. It has a form [7]

$$\sigma_{\rm FSR}^v = \frac{2\alpha}{\pi}\sigma_B(s)\Big[\Big(1 - \frac{1+\beta^2}{2\beta}L_\beta\Big)\ln\frac{M}{\lambda} - 1 + \Big(\frac{1+\beta^2}{2\beta} - \frac{1}{4\beta}\Big)L_\beta + \frac{1+\beta^2}{2\beta}\Big[2\zeta_2 - \frac{1}{4}L_\beta^2 - L_\beta\ln\frac{2\beta}{1+\beta} + \mathrm{Li}_2\Big(\frac{1-\beta}{1+\beta}\Big)\Big] - \frac{3(1-\beta^2)}{4\beta(3-\beta^2)}L_\beta\Big]. \tag{12}$$

The sum of contributions from virtual and soft real photons reads

$$\sigma_{\rm FSR}^{v+s} = \frac{2\alpha}{\pi}\sigma_B(s)\Big[\Big(\frac{1+\beta^2}{2\beta}L_\beta - 1\Big)\ln\Delta - 1 + \ln\frac{1+\beta}{2} + \Big(\frac{3-2\beta+2\beta^2}{4\beta} - \frac{3(1-\beta^2)}{4\beta(3-\beta^2)}\Big)L_\beta + \frac{1+\beta^2}{2\beta}\Big(-2\mathrm{Li}_2(\beta) + 2\mathrm{Li}_2(-\beta) + \mathrm{Li}_2\Big(\frac{1+\beta}{2}\Big) - \mathrm{Li}_2\Big(\frac{1-\beta}{2}\Big) + 3\zeta_2\Big)\Big]. \tag{13}$$

In the ultrarelativistic limit we have

$$\sigma_{\rm FSR}^{v+s}|_{\beta\to 1} = \frac{2\alpha}{\pi}\sigma_B(s)\Big[(l_\mu - 1)\ln\Delta - 1 + \frac{3}{4}l_\mu + \zeta_2\Big]. \tag{14}$$

The total sum of contributions from virtual, soft, and hard real photons does not contain photon mass λ and the separation parameter Δ:

$$\sigma_{\rm FSR}^{e^+e^-\to\mu^+\mu^-\gamma} = \frac{2\alpha}{\pi}\sigma_B(s)\Delta_{\rm FSR}^{\mu^+\mu^-}(\beta), \tag{15}$$

where

$$\Delta_{\rm FSR}^{\mu^+\mu^-}(\beta) = \frac{3(5-3\beta^2)}{8(3-\beta^2)} + \frac{(1-\beta)(33-39\beta-17\beta^2+7\beta^3)}{16\beta(3-\beta^2)}L_\beta + 3\ln\Big(\frac{1+\beta}{2}\Big) - 2\ln\beta + \frac{1+\beta^2}{2\beta}F(\beta), \tag{16}$$

$$F(\beta) = -2\mathrm{Li}_2(\beta) + 2\mathrm{Li}_2(-\beta) - 2\mathrm{Li}_2(1+\beta) + 2\mathrm{Li}_2(1-\beta) + 3\mathrm{Li}_2\Big(\frac{1+\beta}{2}\Big) - 3\mathrm{Li}_2\Big(\frac{1-\beta}{2}\Big) + 3\zeta_2. \tag{17}$$

The quantity $\Delta_{\rm FSR}^{\mu^+\mu^-}(\beta)$ agrees with the result obtained in

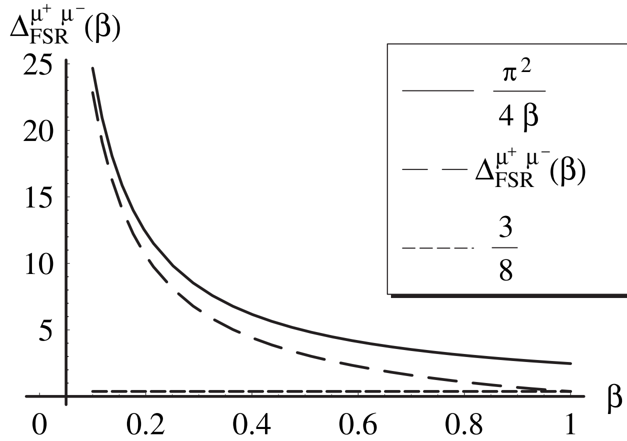

FIG. 3. The dependence of quantity $\Delta_{\mathrm{FSR}}^{\mu^+\mu^-}(\beta)$ as a function of β for FSR of muons. See formula (16) and its asymptotic behavior.

[8] and presented in Fig. 3 as a function of β. This correction in the ultrarelativistic limit tends to the value 3/8;

$$\sigma_{\mathrm{FSR}}^{e^+e^-\to\mu^+\mu^-\gamma}|_{\beta\to 1} = \frac{4\pi\alpha^2}{3s}\frac{2\alpha}{\pi}\frac{3}{8} = \frac{\alpha^3}{s}.$$

Cancellation of "large" logarithms $l_\mu = \ln(s/M^2)$ is the consequence of the Kinoshita-Lee-Nauenberg theorem [9].

III. INITIAL STATE RADIATION (ISR) IN MUON PAIR PRODUCTION

The matrix element of the process of muon pair production with hard photon radiated from initial state has a form (for notations see Fig. 4)

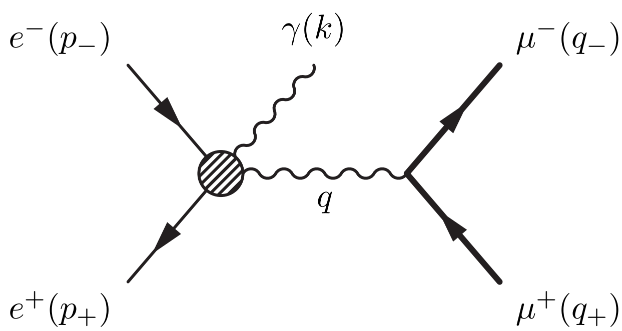

FIG. 4. Initial state radiation corrections to $e^+e^- \to \mu^+\mu^-$ process.

$$M_{\mathrm{ISR}} = \frac{(4\pi\alpha)^{3/2}}{s(1-\nu)}\bar{v}(p_+)\bigg[\hat{Q}\frac{\hat{p}_- - \hat{k} + m}{-2kp_-}\hat{e}(k) + \hat{e}(k)\frac{-\hat{p}_+ + \hat{k} + m}{-2kp_+}\hat{Q}\bigg]u(p_-), \tag{18}$$

where $Q_\eta = \bar{u}(q_-)\gamma_\eta v(q_+)$ is the muon current. Using the gauge condition for muon current $q^\eta Q_\eta = 0$, $q = q_+ + q_- = p_+ + p_- - k$ we have

$$\sum \int Q_\mu (Q_\nu)^* \frac{d^3q_+}{2E_+}\frac{d^3q_-}{2E_-}\delta^4(q - q_+ - q_-) = D\Big(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\Big), \tag{19}$$

$$D = -\frac{2\pi s}{3}\bigg[\frac{3-\beta^2}{2} - \nu\bigg]R(\nu), \qquad q^2 = s(1-\nu),$$

with notations given above. Using these relations, the calculation of the summed upon spin states of matrix element squared is straightforward. Performing the angular integrations (assuming $m^2 \ll s$)

$$\int_{-1}^{1} dc\bigg[\frac{1}{1-\beta_e c}; \frac{4m^2}{s(1-\beta_e c)^2}; 1\bigg] = [l_e; 2; 2], \quad l_e = \ln\frac{s}{m^2}, \qquad \beta_e = \sqrt{1 - \frac{4m^2}{s}}, \tag{20}$$

we obtain the distribution on the muons invariant mass (see Fig. 5):

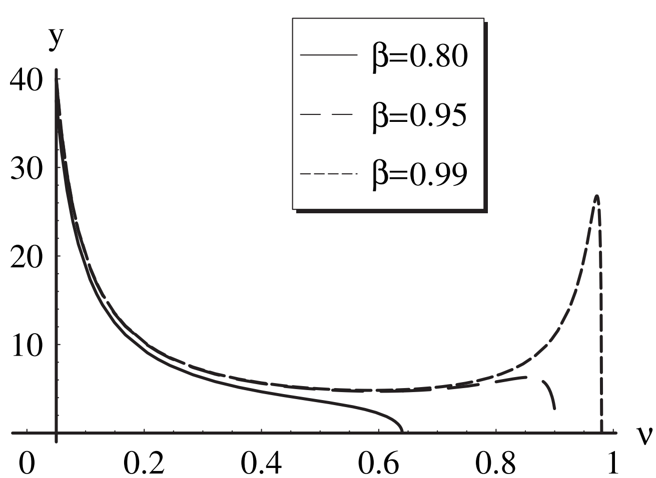

FIG. 5. Distribution of muon pairs as a function on ν for ISR. Vertical axis represents the quantity $y = [(4\alpha^3/3s)(l_e - 1)]^{-1} \times (d\sigma_{\mathrm{ISR}}^h/d\nu)$ [see (21)] and horizontal axis represents the fraction of radiated photon energy ν.

$$\frac{d\sigma_{\rm ISR}^h}{d\nu} = \frac{4\alpha^3}{3s\nu(1-\nu)^2}[1+(1-\nu)^2](l_e-1) \times \left(\frac{3-\beta^2}{2}-\nu\right)R(\nu), \qquad \nu > \Delta. \tag{21}$$

Further integration on the photon energy fraction ν can be performed using the set of integrals given above and two additional ones:

$$\int_0^{\beta^2} R(\nu)\left[\frac{1}{(1-\nu)^2};\frac{1}{1-\nu}\right]d\nu = \left[\frac{2\beta^3}{3(1-\beta^2)};-2\beta+L_\beta\right].$$

As a result, we obtain the cross section due to radiation of a hard photon from ISR:

$$\sigma_{\rm ISR}^h = \frac{2\alpha}{\pi}\sigma_B(s)(l_e-1)\left[\ln\frac{1}{\Delta}-\frac{1-3\beta+\beta^3}{\beta(3-\beta^2)}L_\beta - \frac{4}{3}+2\ln\frac{2\beta}{1+\beta}\right]. \tag{22}$$

The contribution to the cross section taking into account the virtual and soft photons to the initial state is given by

$$\sigma_{\rm ISR}^{s+v} = \frac{2\alpha}{\pi}\sigma_B(s)\left[(l_e-1)\ln\Delta+\frac{3}{4}l_e-1+\zeta_2\right]. \tag{23}$$

Let us note that the spectral distribution on invariant mass of a final system has a form consistent with renormalization group prescriptions, namely, one can recognize the kernel of evolution equation contribution [see (21) and (23)].

Now, we can collect all the terms mentioned above and write out the expression for the total cross section due to ISR:

$$\sigma_{\rm ISR}^{s+v+h} = \frac{2\alpha}{\pi}\sigma_B(s)\Delta_{\rm ISR}^{\mu^+\mu^-}(\beta), \tag{24}$$

$$\Delta_{\rm ISR}^{\mu^+\mu^-}(\beta) = (l_e-1)\left[-\frac{1-3\beta+\beta^3}{\beta(3-\beta^2)}L_\beta-\frac{4}{3} + 2\ln\frac{2\beta}{1+\beta}\right]+\frac{3}{4}l_e-1+\zeta_2. \tag{25}$$

The dependence of this quantity on a muon's velocity β is shown in Fig. 6. In the ultrarelativistic limit we have

$$\sigma_{\rm ISR+FSR}^{s+v+h}|_{\beta\to1} = \frac{8\alpha^3}{3s}\left[\frac{1}{2}l_el_\mu-\frac{1}{2}l_\mu-\frac{7}{12}l_e+\zeta_2+\frac{17}{24}\right], \tag{26}$$

which is in agreement with [2,3]. The leading term $\sim l_el_\mu$ is in agreement with the result [5].

The total cross section contains the so-called double-logarithmical terms ($\sim l_el_\mu$), which already contradict the renormalization group predictions (single logarithmic).

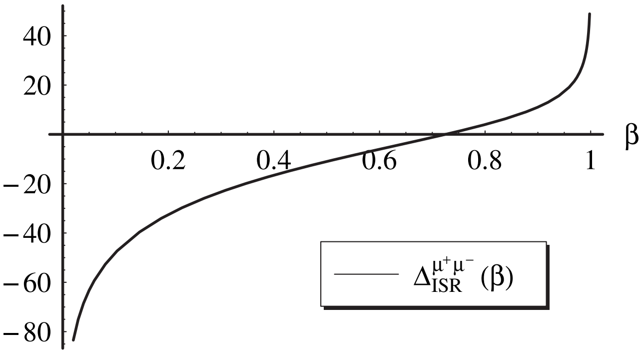

FIG. 6. Distribution of muon pairs on β for ISR. See formula (25) for the quantity $\Delta_{\rm ISR}^{\mu^+\mu^-}(\beta)$.

IV. THE FINAL STATE RADIATION IN PION PAIR PRODUCTION

It is worth a reminder here that the total cross section $\sigma(e^+e^-\to\pi^+\pi^-)$ with $\mathcal{O}(\alpha)$ corrections is required in many subjects of particle physics. It is required especially to determine with a better accuracy the precision of the evaluation of vacuum polarization effects in a photon propagator. The other well-known application is the calculation of the hadronic contribution to the anomalous magnetic moment of muon $a_\mu^{\rm hadr}$ [10]:

$$a_\mu^{\rm hadr} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\int_{4M_\pi^2}^\infty ds\frac{R(s)K(s)}{s}, \qquad R(s) = \frac{\sigma^{e^+e^-\to\pi^+\pi^-}(s)}{\sigma^{e^+e^-\to\mu^+\mu^-}(s)}. \tag{27}$$

A contribution to this integral coming from a high energy region can be calculated within the QCD framework, while for the low energy range the experimental values R's must be taken as an input [11]. A numerical evaluation of this integral in relative unities gives the value of ~70 ppm.

The goal of the new experiment at Brookhaven National Laboratory (E969) is to measure the anomalous magnetic moment of muon with the relative accuracy of about ~0.25 ppm and to improve the previous result [12] by a factor of 2. Consequently the value $a_\mu^{\rm hadr}$ should be calculated as precisely as possible. In this context the required theoretical precision of the cross sections with radiative corrections as well as the calculation accuracy of the vacuum polarization effects should not be worse than ~0.2% as it follows from the estimation: 70 ppm × 0.2% ~ 0.14 ppm. This short observation shows why high precision calculation of the hadronic cross sections is extremely important.

A. Final state radiation

As well as it was done for the muons, the contributions with one photon radiation in the final state can be divided into three separate parts: virtual, soft, and hard. The expression for the virtual photon emission from final state can be found in [13] and is given by

$$\sigma_v = \frac{\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\bigg[2\ln\frac{M_\pi}{\lambda}\bigg(1-\frac{1+\beta^2}{2\beta}L_\beta\bigg)-2 + \frac{1+\beta^2}{\beta}L_\beta + \frac{1+\beta^2}{\beta}\bigg(-\frac{1}{4}L_\beta^2 + L_\beta\ln\frac{1+\beta}{2\beta} + 2\zeta_2 + \mathrm{Li}_2\bigg(\frac{1-\beta}{1+\beta}\bigg)\bigg)\bigg]. \tag{28}$$

Here L_β, λ, β were defined above, β is a pion velocity in c.m. frame, $\sigma_B^{\pi^+\pi^-}(s) = (\pi\alpha^2\beta^3)/(3s)|F_\pi(s)|^2$ is the cross section production of charged pion pair in the Born approximation, and $F_\pi(s)$ is the pion strong interaction form factor. The cross section is due to emission of a soft photon when its energy does not exceed $\Delta\varepsilon$ and is given by

$$\sigma_{\mathrm{FSR}}^s = \frac{\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\bigg[2\ln\bigg(\frac{2\Delta\varepsilon}{\lambda}\bigg)\bigg(\frac{1+\beta^2}{2\beta}L_\beta - 1\bigg) + \frac{1}{\beta}L_\beta + \frac{1+\beta^2}{\beta}\bigg(-\frac{1}{4}L_\beta^2 + L_\beta\ln\frac{1+\beta}{2\beta} - \zeta_2 + \mathrm{Li}_2\bigg(\frac{1-\beta}{1+\beta}\bigg)\bigg)\bigg], \qquad \Delta\varepsilon \ll \varepsilon = \frac{\sqrt{s}}{2}. \tag{29}$$

The sum of the contributions from virtual and soft photons can be presented in a convenient way as

$$\sigma_{\mathrm{FSR}}^{v+s} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\bigg[\bigg(\frac{1+\beta^2}{2\beta}L_\beta - 1\bigg)\ln\Delta + b(s)\bigg], \tag{30}$$

where

$$b(s) = -1 + \frac{1-\beta}{2\beta}\rho + \frac{2+\beta 2}{\beta}\ln\frac{1+\beta}{2} + \frac{1+\beta 2}{2\beta}\times\bigg[\rho + \zeta 2 + L\beta\ln\frac{1+\beta}{2\beta 2} + 2Li2\bigg(\frac{1-\beta}{1+\beta}\bigg)\bigg],$$
$$\rho = \ln\frac{4}{1-\beta 2}, \qquad \Delta = \frac{\Delta\varepsilon}{\varepsilon}.$$

Calculations similar to the ones given above for muons FSR lead to the pion pair invariant mass distribution [$m_{\pi\pi}^2 = s(1-\nu)$, see Fig. 7]:

$$\frac{\sigma_{\mathrm{FSR}}^h}{d\nu} = \frac{2\alpha^3\beta^2}{3s}\bigg[\bigg(\frac{\nu}{\beta^2} - \frac{1-\nu}{\nu}\bigg)R(\nu) + \bigg(\frac{1+\beta^2}{2\nu} - 1\bigg)\times\ln\frac{1+R(\nu)}{1-R(\nu)}\bigg]|F_\pi(s)|^2. \tag{31}$$

Contribution to the total cross section can be obtained performing the integration on an invariant pion pair mass. In agreement with [4,14] the relevant contribution has the following form:

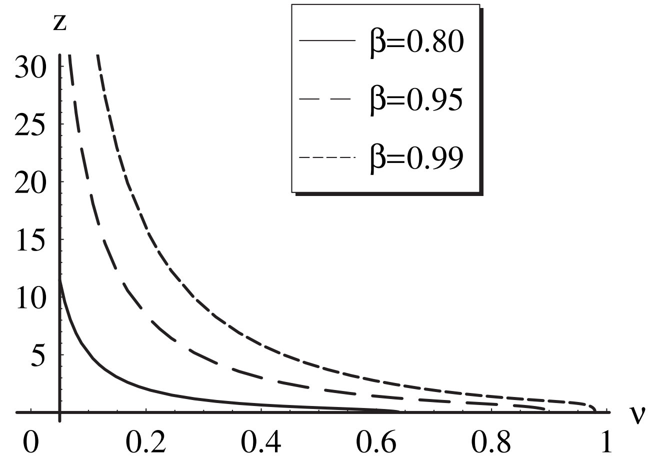

FIG. 7. The pion invariant mass distribution on ν for FSR. The vertical and horizontal axes represent the value $z = [(2\alpha^3/3s)]^{-1}(d\sigma_{\mathrm{FSR}}^h/d\nu)$ [see (31)] and fraction of photon energy, respectively.

$$\sigma_{FSR}^h = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)[\ln\frac{1}{\Delta}(\frac{1+\beta^2}{2\beta}L_\beta - 1) + 2 + \frac{3-\beta^2}{4\beta^2} - \frac{(3+\beta^2)(1-\beta^2)}{8\beta^3}L_\beta - \ln\frac{4\beta^2}{1-\beta^2} + \frac{1+\beta^2}{\beta}\Phi(\beta)], \tag{32}$$

with $\Phi(\beta)$ defined in (8). Now we can write down the complete expression for the total cross section:

$$\sigma_{\mathrm{FSR}}^{e^+e^-\to\pi^+\pi^-\gamma} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta), \tag{33}$$

$$\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta) = \frac{3(1+\beta^2)}{4\beta^2} - 2\ln\beta + 3\ln\frac{1+\beta}{2} + \frac{1+\beta^2}{2\beta}F(\beta) + \frac{(1-\beta)(-3-3\beta+7\beta^2-5\beta^3)}{8\beta^3}L_\beta, \tag{34}$$

(see Fig. 8) with the same expression for $F(\beta)$ as in the muon case (17). The factor $\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}$ represents the correction to the Born cross section caused by final state radiation. In a low β limit $\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta) \approx \pi^2/4\beta$, which is the manifestation of Coulomb interaction of pions. $\Delta_{\mathrm{FSR}}^{\mu^+\mu^-}(\beta)$ reveals exactly the same behavior as in the case of muons [see (16)]. It is well known that in the limit $\alpha/\beta \geq 1$ the perturbative analysis is not valid. The relevant modifications of formulas will be given in the conclusion.

In the ultrarelativistic limit we have $\Delta_{\mathrm{FSR}}^{\pi^+\pi^-}(\beta \to 1) = 3/2$. One can see again that all large logarithms cancelled out in accordance with the Kinoshita-Lee-Nauenberg theo-

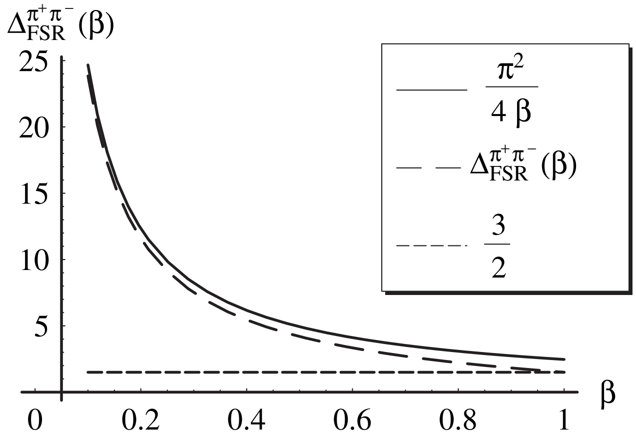

FIG. 8. The dependence of quantity $\Delta_{FSR}^{\pi^+\pi^-}(\beta)$ on β for FSR of pions. See formula (33) and its asymptotic behavior.

rem. The expression for $\Delta_{FSR}^{\pi^+\pi^-}(\beta)$ coincides with the one obtained in [4,15]. It is worth noting that in papers [4,14,15] the quantity $\Delta_{FSR}^{\pi^+\pi^-}(\beta)$ was presented without the separator Δ between soft and hard photons. But for some applications it can be useful to have these two parts separately.

In order to check experimentally the validity of pointlike pions assumption, which is used in the paper, it is necessary to separate out the FSR events. Unfortunately, we should notice that ISR events are 10 times more probable then the FSR ones. Nevertheless, there are at least two ways to select FSR events and to suppress the ISR background.

First, we may consider the region of ρ-meson peak left slope, i.e. $\sqrt{s} < 770$ MeV. In that case the resonance returning mechanism does not take place and the ratio of FSR events increases.

Second, we can throw out the events with pions acollinearity larger than some predefined angle, for instance 0.25 rad [16].

Figure 9 shows the result of modelling of value $\sigma_{ISR+FSR}/\sigma_{ISR}$ with the application of both FSR separation methods described above. The different curves correspond to different energy thresholds of emitted photons ($\omega > 10 - 170$ MeV). One can see that the energy range from 720 to 780 MeV is preferable for our purpose—if photon energy exceeds 150 MeV then the ratio $\sigma_{ISR+FSR}/\sigma_{ISR}$ is about 5; this means that the relative admixture of ISR events is about 20% only.

It is worth noting that the form of spectrum at high photon energies is just the subject of interest. The comparison of the simulated spectrum with the experimental one can elucidate the discussed problem.

B. Initial state radiation

Let us consider now the initial state radiation (ISR) effects in pion pair production. Performing the calculations

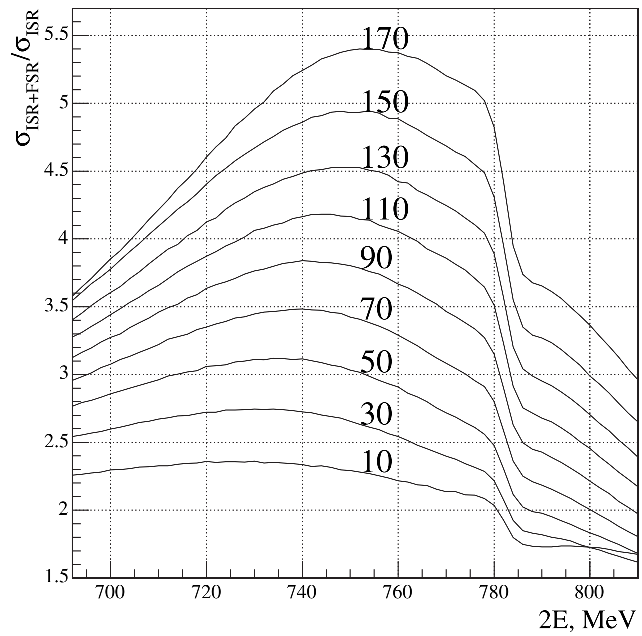

FIG. 9. The ratio of the cross sections with ISR + FSR divided on the cross section with ISR as a function of energy in c.m. frame. The different smooth curves represent this ratio vs threshold photon energy (in MeV) to be detected.

similar to the case of muon pair production we have

$$\frac{d\sigma_{ISR}^{e^+e^-\to\pi^+\pi^-\gamma}}{d\nu} = \frac{\alpha^3}{3s}\frac{1+(1-\nu)^2}{(1-\nu)^2\nu}(l_e-1)(\beta^2-\nu) \times \sqrt{\frac{\beta^2-\nu}{1-\nu}}|F_\pi(s(1-\nu))|^2, \qquad (35)$$

where $q^2 = (q_+ + q_-)^2 = s(1-\nu)$. The calculation results are shown in Fig. 10 with $F_\pi = 1$. Using integrals

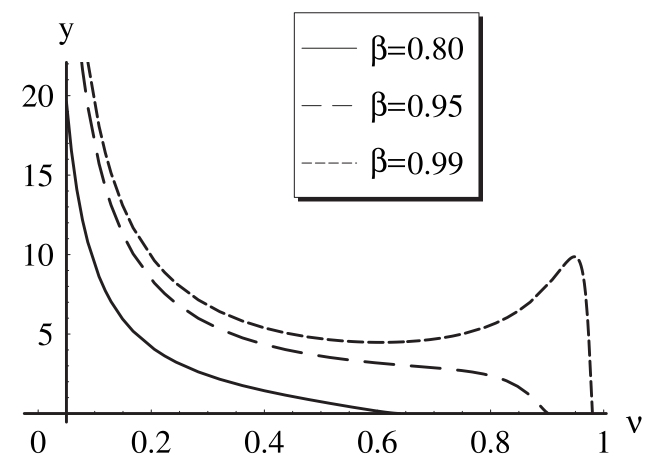

FIG. 10. The distribution pion pairs as a function on ν for ISR. The vertical axis represents the quantity $y = (\alpha^3/3s(l_e - 1))^{-1} d\sigma_{ISR}^{e^+e^-\to\pi^+\pi^-\gamma}/d\nu$ [see (35)] and the horizontal axis represents the fraction of radiated photon energy.

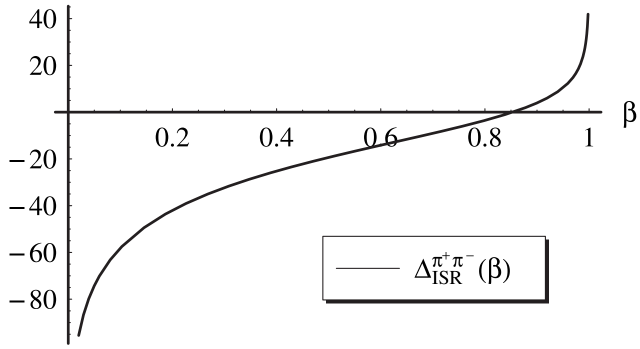

FIG. 11. The dependence of the quantity $\Delta_{\rm ISR}^{\pi^+\pi^-}(\beta)$ [see (39)] on β for ISR.

presented above we can obtain the following expression for the cross section with hard photon radiation:

$$\sigma_{\rm ISR}^{h} = \frac{2\alpha^3\beta^3}{3s}(l_e - 1)\left\{\ln\frac{1}{\Delta} + 2\ln\left(\frac{2\beta}{1+\beta}\right) - \frac{4}{3} - \frac{1}{2\beta^2} + \frac{1-3\beta^2+4\beta^3}{4\beta^3}L_\beta\right\}, \quad (36)$$

where $l_e = \ln(s/m^2)$. Here we had assumed the pions to be pointlike, i.e $F_\pi = 1$. The sum of the contributions of virtual and soft photon emission has a form

$$\sigma_{\rm ISR}^{v+s} = \frac{2\alpha}{\pi}\sigma_B^{\pi^+\pi^-}(s)\left\{(l_e - 1)\ln\Delta + \frac{3}{4}l_e - 1 + \zeta_2\right\}. \quad (37)$$

The total cross section accounted for initial state radiation can be presented as

$$\sigma_{\rm ISR}^{e^+e^-\to\pi^+\pi^-\gamma} = \frac{2\alpha^3\beta^3}{3s}\Delta_{\rm ISR}^{\pi^+\pi^-}(\beta), \quad (38)$$

$$\Delta_{\rm ISR}^{\pi^+\pi^-}(\beta) = (l_e - 1)\left[2\ln\frac{2\beta}{1+\beta} - \frac{4}{3} - \frac{1}{2\beta^2} + \frac{1-3\beta^2+4\beta^3}{4\beta^3}L_\beta\right] + \frac{3}{4}l_e - 1 + \zeta_2. \quad (39)$$

The quantity $\Delta_{\rm ISR}^{\pi^+\pi^-}(\beta)$ as a function of β is shown in Fig. 11. In the ultrarelativistic limit in pointlike approximation for pions we have

$$\sigma_{\rm ISR+FSR}^{e^+e^-\to\pi^+\pi^-\gamma}|_{\beta\to 1} = \frac{2\alpha^3}{3s}\left\{\frac{1}{2}l_e l_\pi - \frac{1}{2}l_\pi + \frac{3}{2}l_e + \frac{1}{6} + \zeta_2\right\}, \quad (40)$$

where $l_\pi = \ln(s/M_\pi^2)$.

V. ACCURACY ESTIMATION

The theoretical uncertainties of the cross sections with $\mathcal{O}(\alpha)$ corrections given above are defined by the nonaccounted higher order corrections and they are estimated to be at $\sim 0.2\%$ level. Below are the main sources of uncertainties which were omitted in the current formulas:

(i) Weak interactions are not considered here arising from replacement of virtual photon Green function by a Z-boson one. It results in

$$d\sigma \to d\sigma\left[1 + \mathcal{O}\left(\left(\frac{s}{M_Z^2}\right)^2, \frac{M_\mu^2}{M_Z^2}\right)\right], \quad (41)$$

which for $\sqrt{s} \le 10$ GeV is about or lesser than 0.1% in a charge-blind experimental setup, when we can omit the $\gamma - Z$ interference contribution.

(ii) Here we systematically omit the terms of order $(m/M_\mu)^2$ compared to 1

$$\mathcal{O}\left(\frac{m^2}{M_\mu^2}\right) \le 0.1\%. \quad (42)$$

(iii) The higher orders contributions (not considered here) can be separated by two classes. The first class, leading by large logarithm $l_e = \ln(s/m^2)$, is connected with ISR:

$$d\sigma \to d\sigma[1 + \mathcal{O}((\alpha/\pi)^2 l_e^2, (\alpha/\pi)^2 l_e)], \quad (43)$$

$$\mathcal{O}((\alpha/\pi)^2 l_e^2) \sim 0.2\%,$$
$$\mathcal{O}((\alpha/\pi)^2 l_e) \sim 0.01\%.$$

These kinds of contributions can be taken into account by structure function approach as it was done in [17].

(iv) The second class is the higher orders of contributions connected with FSR which give

$$d\sigma\left[1 + \mathcal{O}\left(\left(\frac{\alpha}{\pi}l_\beta\right)^2\right)\right], \qquad \mathcal{O}\left(\left(\frac{\alpha}{\pi}l_\beta\right)^2\right) \sim 0.05\%. \quad (44)$$

In the ultrarelativistic limit $l_\beta \to \ln(s/M_\mu^2)$ they as well can be taken into account by the structure function method.

Considering the uncertainty sources mentioned above as independent, we can conclude that the total systematic error of the cross sections with $\mathcal{O}(\alpha)$ radiative corrections is less than 0.22%. However we remind here that taking into account higher order contributions connected with ISR using the structure function approach [17] allows one to decrease the total error down to level 0.05%. $\Delta_{\rm FSR}^{(\mu,\pi)}$ and $\Delta_{\rm ISR}^{(\mu,\pi)}$ are drawn in the figures and one can see that corrections to the Born cross sections $(2\alpha/\pi)\Delta$ can reach several percents near threshold.

VI. CONCLUSION

One possible application of the formulas given above is for them to be used for normalization purposes at MC simulation. Our results can be used also for improvement of the calculation accuracy of vacuum polarization effects in the virtual photon propagator at low energies not far significantly from threshold production. This calculation, in its turn, is required to improve the precision of the theoretical prediction for an anomalous magnetic moment of muon.

The expressions for the cross sections of $\tau^+\tau^-$ and K^+K^- production are similar to those for muons and pions. The muon and kaon masses as well as the pion form factor should be replaced in the above expressions by the tau and kaon ones, respectively. The cross section being multiplied by the exact Coulomb factor will interpolate the energy dependence of the cross section from the threshold production to the relativistic region.

We do not consider C-odd interference in real and virtual photons emission—it gives zero contribution to the total cross section. Neither do we consider the effects of virtual photon polarization operator insertion; it can be found in the literature [13,18].

$\Delta_{\rm FSR}^{(\mu,\pi)}$ and $\Delta_{\rm ISR}^{(\mu,\pi)}$ are drawn in the figures and one can see that corrections to Born cross sections $(2\alpha/\pi)\Delta$ can reach several percents near threshold.

In regions where $\beta \sim \alpha$ formulas must be modified [8]. Taking into account that $\Delta^{(i)}(\beta) \sim \pi^2/4\beta,\ \beta \to 0$, we must replace

$$1 + \frac{2\alpha}{\pi}\Delta^{(i)}(\beta) \to \left(1 + \frac{2\alpha}{\pi}\left(\Delta^{(i)}(\beta) - \frac{\pi^2}{4\beta}\right)\right)f(z),$$

where $f(z) = z/(1 - e^{-z})$ is the Sommerfeld-Sakharov factor, $z = (\pi\alpha/\beta)$. In the region where $\beta \ll \alpha$ the formulas must be modified according to [8,19].

ACKNOWLEDGMENTS

We are grateful to A. Arbuzov, V. Bytev, and V. Tayursky for valuable discussions. We also are grateful for the support of the RFBR Grant No. 03-02-17077 and INTAS Grant No. 0366.

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