Comment on "Self-dual teleparallel formulation of general relativity and the positive energy theorem"

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We give a correct tensor proof of the positive energy problem for the case including momentum on the basis of conditions of the existence of the two-to-one correspondence between the Sen-Witten spinor field and the Sen-Witten orthonormal frame. These conditions were obtained in our previous publications, but the true significance of our works was not estimated properly by Chee, and these were not correctly quoted in his publication. On other hand, the main result of our work is a substantial argument in favor of the geometrical nature of the Sen-Witten spinor field.

Ever since Witten developed the spinor method to prove the positive energy theorem (PET) for gravity, the problem of comparing this method with the tensor methods has been a subject of continuing interest. Goldberg's initial categorical negation of the possibility that connections exist between these two methods [1] ("For the first time spinors have an intrinsic role for which tetrads cannot be substituted") was partially disproved by Dimakis and Müller-Hoissen [2], and later by Frauendiener [3]. Dimakis and Müller-Hoissen supposed that the spinor field could be "replaced" by some orthonormal frame field, so that the existence of a global solution to the Sen-Witten equation would imply the existence of globally defined orthonormal frames on the Cauchy surface. But, in general, the solution to the Sen-Witten equation will have zeros; from this Dimakis and Müller-Hoissen concluded that each orthonormal frame field, as well as Nester's special orthonormal frame field (SOF, triad) on a spacelike hypersurface in an asymptotically Minkowskian manifold, can exist almost everywhere [2,4].

Frauendiener established that correspondence may exist between the spinor field λ^A, which satisfies on a spacelike hypersurface Σ the Sen-Witten equation (SWE)

$$\mathcal{D}^A{}_B \lambda^B = 0,$$

and a triad, which satisfies on Σ a certain gauge condition, and noted that this gauge is closely related to Nester's. But this Frauendiener result is valid only under the additional assumption that the Sen-Witten spinor field has no zeros.

Nester's SOF consists of the variables that describe the physical degrees of freedom in general relativity. Analogously, the preferred lapse $N = \lambda_A \lambda^{+A} \equiv \lambda$ and shift $N^a = -\sqrt{2}i\lambda^{+(A}\lambda^{B)}$, constructed by Ashtekar and Horowitz [5] from the Witten spinor, give an especially simple form of gravitational Hamiltonian. Nevertheless, degeneracy of Nester's SOF or Ashtekar and Horowitz preferred time variables, which is due to the existence of

zeros of the spinor field, and may occur on subsets of dimensions lower that 3 on the Cauchy hypersurface, puts the physical sense of these two constructions in doubt. Taking this degeneracy into account, Nester [6] had supposed that a SOF exists at least for geometries in a neighborhood of Euclidean space. Chee in his paper [7] states that the Nester gauge condition can be derived from Witten's equation without any additional conditions for all geometries, even on nonmaximal hypersurfaces. Below we prove that this statement is not valid without additional assumptions and give a corrected proof of the PET for the case including momentum.

Indeed, the correspondence between the spinor field, which satisfies the Sen-Witten equation, and a triad, which satisfies a certain gauge condition, is correctly defined by the Sommers transformation [8]

$$\theta^1 = \frac{\sqrt{2}}{2\lambda}(L + \overline{L}), \qquad \theta^2 = \frac{\sqrt{2}}{2\lambda i}(L - \overline{L}), \qquad \theta^3 = \tilde{L},$$

(1)

where θ^a is a coframe basis, $L = -\lambda_A \lambda_B$, $\lambda = \lambda_A \lambda^{A+}$, and $\tilde{L} = |L|^{-1} *(L \wedge \overline{L})$ if and only if the spinor field λ_A vanishes nowhere on Σ. This follows from the fact that the bilinear form

$$\frac{1}{\sqrt{2}} n^{A\dot{A}} \lambda_A \overline{\lambda}_{\dot{A}} = \lambda_A \lambda^{A+} \equiv \lambda,$$

where n is the unit normal one-form to Σ, is Hermitian positive definite, and λ does not vanish at a point on Σ if the solution λ_A does not have a zero at this point. But λ_A is the solution of the SWE, which is of elliptic type; zeros of solutions to such equations not only may, but must exist, and these have a clear physical meaning: for example, zeros of solution to the equation for vibrations of a flat membrane are the node lines of standing waves.

In Chee's work, the possible existence of node manifolds for the SWE is not excluded but it is ignored completely—there even is no mention of the assumption $\chi^2 \equiv \lambda \equiv \lambda_A \lambda^{A+} \neq 0$. As a result, the Sommers trans-

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formation (1), written by Chee as formula (47), does not exist on node manifolds, and, consequently his conditions (48), which are Nester's conditions, are not fulfilled. Then equation (51) for the boundary term

$$\oint_S \tilde{B}^{(AB)} dS_{AB}$$

is not fulfilled, the choice $N = \lambda_A \lambda^{A+}$ is not possible, and this means that the last formula (53) of publication [7],

$$\oint_S \tilde{B}^{(AB)} dS_{AB} = 2 \int_\Sigma \sigma (\nabla^{(BC)} \lambda^A)^+ (\nabla_{(BC)} \lambda_A) dV$$
$$+ 4\pi G \int_\Sigma \sigma \lambda^{+A} (T_{00} \lambda_A$$
$$+ \sqrt{2} T_{0AB} \lambda^B) dV$$

in general is not correct.[1]

We now give the corrected proof of the PET for the case including momentum on the basis of conditions for the existence of the correspondence between Nester's gauge and the SWE, obtained by us in publications [10–12].

Definition 1.—A point where the solution for an elliptic system of equations is equal to zero is called a node point of the solution.

From the general theory of elliptic differential equations it is known that nontrivial solutions cannot vanish on an open subdomain, but they can become zero on subsets of lower dimensions $k, k = 0, 1, \ldots n - 1$, where n is the dimension of the domain.

Definition 2.—A node submanifold of dimension $s, s = 1, 2, \ldots n - 1$, is a maximal connected subset of dimension s consisting of node points of the solution.

In the case of a single self-adjoint elliptic equation in V^3, the node submanifolds can only be surfaces that divide the domain, but in the case of a system of equations the topology of node submanifolds has greater variety: it can be also that of lines or of points.

Let us consider first the case when the Cauchy hypersurface is maximal.

Theorem 1.—[10] Let λ^C satisfy Reula's condition [13] and be a solution of the SWE with an asymptotically flat initial data set, satisfying the dominant energy condition. Then on a maximal hypersurface Σ, the solution λ^C is everywhere free from the node point.

On the basis of this theorem we obtain the following.

Theorem 2.—Let an initial data set $(h_{\mu\nu}, \mathcal{K}_{\pi\rho})$ on a maximal hypersurface Σ be asymptotically flat and satisfy the dominant energy condition. Then everywhere on Σ the Sen-Witten equation with Reula conditions for the spinor field [13] and Nester's gauge are equivalent (up to the sign of the spinor).

Therefore, in this simple case of maximal hypersurface, the Chee proof is correct, if the hypersurface is asymptotically flat and the dominant energy condition is fulfilled.

To investigate the node manifolds of the SWE on nonmaximal hypersurfaces in [11], we had developed an oscillation theory for general double-covariant systems[2] of elliptic equations of 2nd order in $\mathbb{R}^3$. Applying it in the same work to the SWE for solutions of the form $\lambda^C = \lambda_\infty^C + \beta^C$, where λ_∞^C is an asymptotically covariant constant spinor field on Σ, and β^C is an element of the Hilbert space $\mathcal{H}$, defined in [13] (we call these conditions for solution the Reula conditions for the spinor field), we had obtained the following theorem.

Theorem 3.—Let:
(a) the initial data set be asymptotically flat;
(b) (b) the matrix of the spinorial tensor

$$C_A^{\ B} := \frac{\sqrt{2}}{4} D_A^{\ B} \mathcal{K}$$
$$+ \frac{1}{4} \varepsilon_A^{\ B} \left(\mathcal{K}^2 + \frac{1}{2} \mathcal{K}_{\pi\rho} \mathcal{K}^{\pi\rho} + \mu \right)$$

have everywhere on Σ at least one non-negative eigenvalue, for definiteness C_0;
(c) $\mathrm{Re}\lambda_\infty^0$ or $\mathrm{Im}\lambda_\infty^0$ be asymptotically nowhere equal to zero.

Then the asymptotically constant nontrivial solution λ^C of the SWE does not have node points on Σ .

This theorem allowed us to prove in Theorem 4 the existence everywhere on Σ of a certain class of orthonormal three-frames, which generalize Nester's special three-frame (Sen-Witten orthonormal three-frame, SWOF). This class of SWOF satisfies the gauge conditions

$$\varepsilon^{abc} \omega_{abc} \equiv *q = 0, \qquad \omega^a_{\ 1a} \equiv -\tilde{q}_1 = F_1,$$
$$\omega^a_{\ 2a} = -\tilde{q}_2 = F_2, \qquad \omega^a_{\ 3a} = -\tilde{q}_3 = \mathcal{K} + F_3, \tag{2}$$

where ω_{abc} are the connection one-form coefficients, and $F = d \ln \lambda$; conditions (2) coincide with Nester's gauge if and only if the one-form $\mathcal{K}\lambda^{+(A}\lambda^{B)}$ is exact.

Theorem 4.—Let conditions of Theorem 3 be fulfilled. Then a two-to-one correspondence between the Sen-Witten spinor and the Sen-Witten orthonormal frame exists everywhere on Σ.

That is why in the case of nonmaximal hypersurfaces the tensor proof of the PET for the case including momentum is valid only if conditions (a) and (b) of our Theorem 3 are fulfilled.

I wish to thank D. Brill for a helpful discussion.

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[1] For an absolutely correct proof, it is also necessary to abandon the application of a three-dimensional truncation of the four-dimensional Gauss theorem [9].

[2] We call a system of equations double-covariant if it is covariant under arbitrary transformations of coordinates in V^3, and covariant under local SU(2) transformations in a local space that is isomorphic to the complexified tangent space in every point to V^3.

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