# Recursive method to obtain the parametric representation of a generic Feynman diagram

Iván González* and Iván Schmidt†

*Departamento de Física, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile*

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A recursive algebraic method which allows one to obtain the Feynman or Schwinger parametric representation of a generic $L$-loops and $(E + 1)$ external lines diagram, in a scalar $\phi^3 \oplus \phi^4$ theory, is presented. The representation is obtained starting from an initial parameters matrix, which relates the scalar products between internal and external momenta, and which appears directly when this parametrization is applied to the momentum space representation of the graph. The final product is an algebraic formula that shows explicitly the external momenta dependence and also an algorithm that can be easily programmed, either in a computer programming language (C/C++, Fortran, ...) or in a symbolic calculation package (MAPLE, MATHEMATICA, ...).

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## I. INTRODUCTION

An important mathematical problem in elementary particle physics is the evaluation of Feynman integrals, which usually appear in the perturbative treatment of quantum field theory amplitudes. Besides the intrinsic difficulty in solving the integrals associated with a specific graph, in general the number of diagrams grows rapidly when the number of loops is increased, which makes it necessary to develop methods that allow for the automatization both in the generation and the evaluation of such integrals. The first problem that we face in dealing with a Feynman diagram is to decide which integral representation is the most convenient in order to start the process for finding a solution. Among the different alternatives we have the parametric representations, in particular, the Feynman parametrization [1–3] and the Schwinger parametrization ($\alpha$-parameters) [1,3,4], which allow for the transformation of the loop integrals into scalar multidimensional integrals. These representations also permit, using dimensional regularization, a clear and direct analysis of the convergence problem, and furthermore the property of Lorentz invariance is also explicit in these representations. Recently very efficient analytical and numerical methods for evaluating loop integrals have been proposed, which use as a starting point a scalar representation. In particular the *Mellin-Barnes* [5] representation allows for analytical solutions of complicated diagrams, starting from a Feynman parameters integral. In numerical calculations an excellent technique is the so-called *sector decomposition* [5–7], which allows one to find the Laurent series of the diagram in terms of the dimensional regulator ($\epsilon$), systematically separating by integration sectors the divergences in the Feynman parameters integral. From this point of view, it would be convenient to find an accessible way for obtaining the above-mentioned parametric representations. Although at present there are in the literature algorithms of topological nature [8], its implementation is quite complicated from the point of view of the automatization.

We will consider here a scalar theory. The final result is a simple algorithm which allows one to find the parametric representation of any loop integral, and which can be easily programmed computationally. The basis of this formalism is a generalization of the completion of the squares procedure used in mathematics structures denominated *quadratic forms* [9–11], which are precisely those that appear when applying a scalar parametrization to the momentum space integral representations. In essence, the expression of the form $\mathbf{Q}^t\mathbf{M}\mathbf{Q}$ is a quadratic form, where $\mathbf{Q}$ is an $(L + E)$-vector that contains all the independent internal and external momenta of the graph and $\mathbf{M}$ is a matrix denominated initial parameters matrix (IPM). The end result of the process is a recurrence equation that is the support of the algorithm.

This study is developed making emphasis on the differences that exist in the way of finding the parametric representation and the resultant mathematical structure, between the usual method and the one proposed here. We also find that the parametric representation can be expressed in two equivalent and directly related ways, the first one in terms of matrix elements generated by recursion starting from the IPM and the second in terms of determinants of submatrices of the IPM. The relationship between both representations is demonstrated in Appendices A and B. Finally, two detailed examples are presented, illustrating the procedure for obtaining the parametric representation of a Feynman diagram, and which allow one to compare in practical terms the usual method and the one proposed here. We also add the explicit code to generate the recursive elements of the scalar representation, in the symbolic calculation package MAPLE.

## II. THE FORMALISM

Let us consider a generic topology $G$ that represents a Feynman diagram in a scalar theory, and suppose that this graph is composed of $N$ propagators or internal lines, $L$

*Electronic address: ivan.gonzalez@usm.cl
†Electronic address: ivan.schmidt@usm.cl

 

loops (associated to independent internal momenta $\underline{q} = \{q_1, \ldots, q_L\}$, and $E$ independent external momenta $\underline{p} = \{p_1, \ldots, p_E\}$. Each propagator or internal line is characterized by an arbitrary and in general different mass, $\underline{m} = \{m_1, \ldots, m_N\}$.

Using the prescription of dimensional regularization, we can write the momentum space integral expression that represents the diagram in $D = 4 - 2\epsilon$ dimensions as

$$\begin{aligned} G &= G(\underline{p}, \underline{m}) \\ &= \int \frac{d^D q_1}{i\pi^{D/2}} \cdots \frac{d^D q_L}{i\pi^{D/2}} \\ &\quad \times \frac{1}{(B_1^2 - m_1^2 + i0)^{\nu_1}} \cdots \frac{1}{(B_N^2 - m_N^2 + i0)^{\nu_N}}. \end{aligned} \tag{1}$$

In this expression the symbol $B_j$ represents the momentum of the $j$ propagator or internal line, which in general depends on a linear combination of external $\{\underline{p}\}$ and internal $\{\underline{q}\}$ momenta: $B_j = B_j(\underline{q}, \underline{p})$.

We also define the set $\underline{\nu} = \{\nu_1, \ldots, \nu_N\}$ as the set of powers of the propagators, which in general can take arbitrary values.

Here we will study two well-known parametric representations: the Feynman parametrization and the Schwinger parametrization. In the next sections we will show how to express Eq. (1) in terms of these two scalar representations. The technique consists in transforming the product of denominators in (1) into a sum through the use of an integral identity.

## A. Momentum representation and his scalar parametrization

### *1. Feynman parametrization*

Using the identity

$$\begin{aligned} \frac{1}{\prod_{j=1}^N A_j^{\nu_j}} &= \frac{\Gamma(\nu_1 + \cdots + \nu_N)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 dx_1 \ldots dx_N \, \delta\Big(1 - \sum_{j=1}^N x_j\Big) \\ &\quad \times \frac{\prod_{j=1}^N x_j^{\nu_j - 1}}{[\sum_{j=1}^N x_j A_j]^{\nu_1 + \cdots + \nu_N}} \end{aligned} \tag{2}$$

and after defining $A_j = (B_j^2 - m_j^2)$, we can replace (2) into Eq. (1) and thus obtain the following generic result:

$$\begin{aligned} G &= \frac{\Gamma(N_\nu)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 d\vec{x} \, \delta\Big(1 - \sum_{j=1}^N x_j\Big) \\ &\quad \times \int \frac{\prod_{j=1}^L d^D q_j}{(i\pi^{D/2})^L} \frac{1}{[\sum_{j=1}^N x_j B_j^2 - \sum_{j=1}^N x_j m_j^2]^{N_\nu}}. \end{aligned} \tag{3}$$

For simplicity, from now on we use the following notation: $d\vec{x} = dx_1 \ldots dx_N \prod_{j=1}^N x_j^{\nu_j - 1}$ and $N_\nu = (\nu_1 + \cdots + \nu_N)$.

### *2. Schwinger parametrization*

The fundamental identity for this specific parametrization is given by the equation

$$\frac{1}{A_j^{\nu_j}} = \int_0^\infty dx_j x_j^{\nu_j - 1} \exp(-x_j A_j), \tag{4}$$

which allows, after replacing $A_j = (B_j^2 - m_j^2)$, to express Eq. (1) in the following general form:

$$\begin{aligned} G &= \frac{1}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^\infty d\vec{x} \exp\Big(\sum_{j=1}^N x_j m_j^2\Big) \int \frac{\prod_{j=1}^L d^D q_j}{(i\pi^{D/2})^L} \\ &\quad \times \exp\Big(-\sum_{j=1}^N x_j B_j^2\Big). \end{aligned} \tag{5}$$

The next step is integrating (3) and (5) with respect to the internal momenta, obtaining in this way the corresponding scalar parametrization.

## B. Loop momenta integration and parametric representation (usual method)

The usual way to integrate over internal momenta consists in expanding the sum $\sum_{j=1}^N x_j B_j^2$ and reorder it in the following manner:

$$\sum_{j=1}^N x_j B_j^2 = \sum_{i=1}^L \sum_{j=1}^L q_i A_{ij} q_j - 2 \sum_{i=1}^L k_i q_i + J, \tag{6}$$

or expressed more compactly in matrix form:

$$\sum_{j=1}^N x_j B_j^2 = \mathbf{q}^t \mathbf{A} \mathbf{q} - 2\mathbf{k}^t \mathbf{q} + J, \tag{7}$$

where the following quantities have been defined:

- **A** Symmetric matrix of dimension $L \times L$, whose elements are functions of the parameters $\underline{x}$ only: $\mathbf{A} = \mathbf{A}(\underline{x})$.
- **q** $L$-vector, whose components are the loop or internal 4-vector momenta: $\mathbf{q} = [q_1 \ldots q_L]^t$.
- **k** $L$-vector, whose components are linear combinations of external momenta, with coefficients that are functions of the parameters $\underline{x}$ only, so $\mathbf{k} = \mathbf{k}(\underline{x}, \underline{p})$.
- $J$ Scalar term, which is a linear combination of scalar products of external momenta, with coefficients that depend on the parameters $\underline{x}$ only, $J = J(\underline{x}, \underline{p})$.

Evidently the specific form of each of these quantities depends on the topology of the corresponding diagram, and is made explicit once the parametrization formula is applied to Eq. (1). With the reordering presented in (7), we can write both parametrizations and their respective solutions after performing the momenta integrations.

#### 1. Feynman parametrization

$$G = \frac{\Gamma(N_\nu)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 d\vec{x}\delta\Big(1-\sum_{j=1}^N x_j\Big)\int\frac{\prod_{j=1}^L d^Dq_j}{(i\pi^{D/2})^L} \times \frac{1}{[\mathbf{q}^t\mathbf{A}\mathbf{q}-2\mathbf{k}^t\mathbf{q}+J-\sum_{j=1}^N x_j m_j^2]^{N_\nu}}, \tag{8}$$

which, once the loop momenta integrals are performed, gives finally the Feynman parametric representation:

$$G = \frac{(-1)^{N_\nu}\Gamma(N_\nu-\frac{LD}{2})}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 d\vec{x}\delta\Big(1-\sum_{j=1}^N x_j\Big) \times \frac{[\det\mathbf{A}]^{N_\nu-(L+1)(D/2)}}{[\det\mathbf{A}(\sum_{j=1}^N x_j m_j^2 - J+\mathbf{k}^t\mathbf{A}^{-1}\mathbf{k})]^{N_\nu-[(LD)/2]}}. \tag{9}$$

#### 2. Schwinger parametrization

$$G = \frac{1}{\Gamma(\nu_1)\dots\Gamma(\nu_N)}\int_0^\infty d\vec{x}\exp\Big(\sum_{j=1}^N x_j m_j^2 - J\Big) \times\int\frac{\prod_{j=1}^L d^Dq_j}{(i\pi^{D/2})^L}\exp(-\mathbf{q}^t\mathbf{A}\mathbf{q}+2\mathbf{k}^t\mathbf{q}). \tag{10}$$

In an analogous way, after integration over internal momenta, we obtain Schwinger's parametrization of $G$:

$$G = \frac{(-1)^{(LD)/2}}{\Gamma(\nu_1)\dots\Gamma(\nu_N)}\int_0^\infty d\vec{x}[\det\mathbf{A}]^{-D/2} \times\exp\Big(\sum_{j=1}^N x_j m_j^2 - J+\mathbf{k}^t\mathbf{A}^{-1}\mathbf{k}\Big). \tag{11}$$

The techniques for solving the momenta integrals in (8) and (10) can be found in detail in the literature, both for the Feynman parametrization case [1,2], as well as for the Schwinger [1] case. This last one is usually solved using products of $D$-dimensional Gaussian integrals, in Minkowski or Euclidean spaces.

Notice that in both parametrizations [Eqs. (9) and (11)] it is necessary to evaluate a matrix product that involves an inverse matrix calculation.

### C. Alternative procedure for obtaining the parametric representation (I)

Starting from Eqs. (3) and (5), we can choose to represent the term $\sum_{j=1}^N x_j B_j^2$ as a function of both internal and external momenta scalar products, related through the symmetric matrix $\mathbf{M}^{(1)}$, which we will call IPM. The dimension of this matrix is therefore $(L+E)\times(L+E)$.

For convenience, let us define the momentum:

$$Q_j = \begin{cases} q_j & \text{if } L\geq j\geq 1 \\ p_{j-L} & \text{if } E\geq j > L, \end{cases} \tag{12}$$

and the $(L+E)$-vector $\mathbf{Q} = [\,Q_1 \quad Q_2 \quad \dots Q_{(L+E)}\,]^t$.

Using this definition we can reorder the sum $\sum_{j=1}^N x_j B_j^2$, and rewrite it as

$$\sum_{j=1}^N x_j B_j^2 = \sum_{i=1}^{L+E}\sum_{j=1}^{L+E} Q_i M_{ij}^{(1)} Q_j = \mathbf{Q}^t\mathbf{M}^{(1)}\mathbf{Q}, \tag{13}$$

where $\mathbf{M}^{(1)}$ is clearly a matrix that only depends on parameters.

The difference in the matrix structure, with respect to the usual method of finding the parametric representation, is that here we include both external and internal momenta in the same quadratic representation, and not only the internal ones as in the usual case we presented above [see Eq. (7)], which produces matrix $\mathbf{A}$. In fact, matrix $\mathbf{A}$ is a submatrix of $\mathbf{M}^{(1)}$, which already shows an important difference in the parametrization starting point, with respect to the usual method. More explicitly we have that

$$A = \begin{pmatrix} a_{11} & \dots & a_{1L} \\ \vdots & & \vdots \\ a_{L1} & \dots & a_{LL} \end{pmatrix}, \tag{14}$$

whereas the initial parameters matrix is given by

$$M^{(1)} = \begin{pmatrix} a_{11} & \dots & a_{1L} & & \dots & M^{(1)}_{1(L+E)} \\ \vdots & & \vdots & & & \vdots \\ a_{L1} & \dots & a_{LL} & & & \\ \vdots & & & \ddots & & \\ M^{(1)}_{(L+E)1} & & \dots & & & M^{(1)}_{(L+E)(L+E)} \end{pmatrix}, \tag{15}$$

$$\text{with } M_{ij}^{(1)} = \begin{cases} a_{ij} & \text{if } L\geq i\geq 1, L\geq j\geq 1 \\ M_{ij}^{(1)} & \text{in other cases.} \end{cases} \tag{16}$$

In Appendix A we show a generalization of the *square completion method* for diagonalizing quadratic forms, which is what appears when we parametrize the loop integrals. Looking at the definition of $\mathbf{Q}$ in (12), and since we have to integrate only the first $L$ momenta, only the main $L\times L$ submatrix has to be diagonalized; that is, we need to perform a change of variables in the first $L$ momenta of the $(L+E)$-vector $\mathbf{Q}$. This can be summarized in the following expression:

$$\mathbf{Q}^{\,t}\mathbf{M}^{(1)}\mathbf{Q} = \sum_{j=1}^L M_{jj}^{(j)}\tilde{Q}_j^2 + \sum_{i=L+1}^{L+E}\sum_{j=L+1}^{L+E} Q_i M_{ij}^{(L+1)} Q_j. \tag{17}$$

Using the definition (12), the double sum can be expressed in terms of the external momenta:

$$\sum_{i=L+1}^{L+E}\sum_{j=L+1}^{L+E} Q_i M_{ij}^{(L+1)} Q_j = \sum_{i=1}^E\sum_{j=1}^E M_{(L+i)(L+j)}^{(L+1)} p_i\cdot p_j. \tag{18}$$

Thus the quadratic form of the momenta $\mathbf{Q}$ can be written as

$$\mathbf{Q}^t\mathbf{M}^{(1)}\mathbf{Q} = \sum_{j=1}^{L} M_{jj}^{(j)}\tilde{Q}_j^2 + \sum_{i=1}^{E}\sum_{j=1}^{E} M_{(L+i)(L+j)}^{(L+1)} p_i \cdot p_j. \tag{19}$$

When the square completion procedure is performed to the quadratic form (13), the linear transformation for each internal momentum is given in general by an expression of the form

$$\tilde{Q}_j = Q_j + f(\underline{x}, Q_{j+1}, \ldots, Q_{L+E}) \quad \text{with } j = 1, \ldots, L, \tag{20}$$

whose Jacobian is equal to unity. The matrix elements of the type $M_{ij}^{(k)}$ are defined through the following recursion relation (see Appendix A):

$$M_{ij}^{(k+1)} = \begin{cases} 0 & \text{if } i < (k+1) \vee j < (k+1) \\ M_{ij}^{(k)} - \dfrac{M_{ik}^{(k)} M_{kj}^{(k)}}{M_{kk}^{(k)}} & \text{in other cases.} \end{cases} \tag{21}$$

Therefore in a generic way the first $L$ momenta of the vector $\mathbf{Q}$ have been diagonalized, using the square completion method. Once this has been done, we are in a position to obtain the desired parametric representation.

#### 1. Feynman parametrization in terms of the matrix elements $M_{ij}^{(k)}$

Using Eq. (13), the identity (3) can be written in terms of the vector $\mathbf{Q}$, and thus we get the following equality:

$$G = \frac{\Gamma(N_\nu)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 d\vec{x}\,\delta\Big(1 - \sum_{j=1}^{N} x_j\Big) \times \int \frac{\prod_{j=1}^{L} d^D Q_j}{(i\pi^{D/2})^L} \frac{1}{[\mathbf{Q}^t\mathbf{M}^{(1)}\mathbf{Q} - \sum_{j=1}^{N} x_j m_j^2]^{N_\nu}}. \tag{22}$$

Then we expand the denominator of the previous equation, using equality (19), and therefore we obtain a more explicit expression with respect to the integration variables $\tilde{Q}_j$:

$$G = \frac{\Gamma(N_\nu)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 d\vec{x}\,\delta\Big(1 - \sum_{j=1}^{N} x_j\Big) \int \frac{\prod_{j=1}^{L} d^D \tilde{Q}_j}{(i\pi^{D/2})^L} \times \frac{1}{[\sum_{j=1}^{L} M_{jj}^{(j)} \tilde{Q}_j^2 - \Delta]^{N_\nu}}, \tag{23}$$

where it has been defined

$$\Delta = \sum_{j=1}^{N} x_j m_j^2 - \sum_{i,j=1}^{E} M_{(L+i)(L+j)}^{(L+1)} p_i \cdot p_j. \tag{24}$$

If we now make a second change of variables, such that

$$\tilde{\tilde{Q}}_j = [M_{jj}^{(j)}]^{1/2}\tilde{Q}_j \Rightarrow \tilde{Q}_j = [M_{jj}^{(j)}]^{-1/2}\tilde{\tilde{Q}}_j, \tag{25}$$

then

$$d^D\tilde{Q}_j = d^D([M_{jj}^{(j)}]^{-1/2}\tilde{\tilde{Q}}_j) = [M_{jj}^{(j)}]^{-D/2} d^D\tilde{\tilde{Q}}_j$$

with $j = 1, \ldots, L$ (26)

and, replacing this in Eq. (23), we will have the following transformed loop momenta integral:

$$G = \frac{\Gamma(N_\nu)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 d\vec{x}\,\delta\Big(1 - \sum_{j=1}^{N} x_j\Big) \int \frac{\prod_{j=1}^{L} d^D \tilde{\tilde{Q}}_j}{(i\pi^{D/2})^L} \times \frac{[M_{11}^{(1)}\ldots M_{LL}^{(L)}]^{-D/2}}{[\sum_{j=1}^{L} \tilde{\tilde{Q}}_j^2 - \Delta]^{N_\nu}}. \tag{27}$$

In order to perform the integral with respect to the variables $\tilde{\tilde{\mathbf{Q}}}_j$, let us define now the *hypermomentun* $\mathbf{R}$ of $(LD)$ components in Minkowski space, such that

$$R^2 = \sum_{j=1}^{L} \tilde{\tilde{Q}}_j^2 \tag{28}$$

$$d^D\tilde{\tilde{Q}}_1 \ldots d^D\tilde{\tilde{Q}}_L = d^{LD}R. \tag{29}$$

Then the expression (27) is reduced to

$$G = \frac{\Gamma(N_\nu)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 d\vec{x}\,\delta\Big(1 - \sum_{j=1}^{N} x_j\Big) \times \int \frac{d^{LD}R}{(i\pi^{D/2})^L} \frac{[M_{11}^{(1)}\ldots M_{LL}^{(L)}]^{-D/2}}{(R^2 - \Delta)^{N_\nu}}. \tag{30}$$

The solution of this integral, with respect to the hypermomentum $\mathbf{R}$, can be found using the following identity:

$$\int \frac{d^{LD}R}{(i\pi^{D/2})^L} \frac{1}{[R^2 - \Delta]^{N_\nu}} = (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{LD}{2})}{\Gamma(N_\nu)} \times \frac{1}{\Delta^{N_\nu - [(LD)/2]}}, \tag{31}$$

and which finally applied to Eq. (30) gives us the scalar integral, that is the Feynman parametric representation of $G$,

$$G = \frac{(-1)^{N_\nu}\Gamma(N_\nu - \frac{LD}{2})}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 d\vec{x}\,\delta\Big(1 - \sum_{j=1}^{N} x_j\Big) \times \frac{[M_{11}^{(1)}\ldots M_{LL}^{(L)}]^{-D/2}}{[\sum_{j=1}^{N} x_j m_j^2 - \sum_{i,j=1}^{E} M_{(L+i)(L+j)}^{(L+1)} p_i \cdot p_j]^{N_\nu - [(LD)/2]}}, \tag{32}$$

where the matrix elements $M_{(L+i)(L+j)}^{(L+1)}$ can be easily obtained from the IPM using the recursion formula:

$$M_{(L+i)(L+j)}^{(L+1)} = M_{(L+i)(L+j)}^{(L)} - \frac{M_{(L+i)L}^{(L)} M_{L(L+j)}^{(L)}}{M_{LL}^{(L)}}. \tag{33}$$

#### 2. Schwinger parametrization in terms of the matrix elements $M_{ij}^{(k)}$

Analogously, using Eq. (13), the identity (5) can be written in terms of the vector $\mathbf{Q}$ as

$$G = \frac{1}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^\infty d\vec{x} \exp\left(\sum_{j=1}^N x_j m_j^2\right) \int \frac{\prod_{j=1}^L d^D Q_j}{(i\pi^{D/2})^L} \times \exp\left(-\mathbf{Q}^t \mathbf{M}^{(1)} \mathbf{Q}\right), \quad (34)$$

or equivalently, using the expansion given in Eq. (19), we get

$$G = \frac{1}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^\infty d\vec{x} \exp(\Delta) \int \frac{\prod_{j=1}^L d^D \tilde{Q}_j}{(i\pi^{D/2})^L} \times \exp\left(-\sum_{j=1}^L M_{jj}^{(j)} \tilde{Q}_j^2\right), \quad (35)$$

where again we have defined

$$\Delta = \sum_{j=1}^N x_j m_j^2 - \sum_{i,j=1}^E M_{(L+i)(L+j)}^{(L+1)} p_i \cdot p_j. \quad (36)$$

Now we can solve the momentum integral:

$$\int \frac{\prod_{j=1}^L d^D \tilde{Q}_j}{(i\pi^{D/2})^L} \exp\left(-\sum_{j=1}^L M_{jj}^{(j)} \tilde{Q}_j^2\right) = \int \frac{d^D \tilde{Q}_1}{i\pi^{D/2}} \exp(-M_{11}^{(1)} \tilde{Q}_1^2) \ldots \int \frac{d^D \tilde{Q}_L}{i\pi^{D/2}} \exp(-M_{LL}^{(L)} \tilde{Q}_L^2). \quad (37)$$

In order to find a solution of this integral we make use of the Minkowski space identity:

$$\int \frac{d^D \tilde{Q}_j}{(i\pi^{D/2})} \exp(-M_{jj}^{(j)} \tilde{Q}_j^2) = \frac{(-1)^{D/2}}{[M_{jj}^{(j)}]^{D/2}}, \quad (38)$$

which will allow to evaluate (37). Replacing afterwards this result in (35), we obtain finally the Schwinger parametric representation for the generic graph $G$:

$$G = \frac{(-1)^{(LD)/2}}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^\infty d\vec{x} [M_{11}^{(1)} \ldots M_{LL}^{(L)}]^{-D/2} \times \exp\left(\sum_{j=1}^N x_j m_j^2 - \sum_{i,j=1}^E M_{(L+i)(L+j)}^{(L+1)} p_i \cdot p_j\right), \quad (39)$$

in terms again of the matrix elements $M_{(L+i)(L+j)}^{(L+1)}$, which as we have said before can be readily obtained from the IPM using the recursion formula equation (33).

### D. Alternative procedure for obtaining the parametric representation (II)

There exists a direct relation between the matrix elements $M_{ij}^{(k)}$ and the determinants of submatrices of the IPM. Such a relation can be expressed by the identity

$$M_{ij}^{(k+1)} = \frac{\Delta_{ij}^{(k+1)}}{\Delta_{kk}^{(k)}}, \quad (40)$$

where $\Delta_{ij}^{(k+1)}$ is a determinant which in general is defined by the equation

$$\Delta_{ij}^{(k+1)} = \begin{vmatrix} M_{11}^{(1)} & \cdots & M_{1k}^{(1)} & M_{1j}^{(1)} \\ \vdots & & \vdots & \vdots \\ M_{k1}^{(1)} & \cdots & M_{kk}^{(1)} & M_{kj}^{(1)} \\ M_{i1}^{(1)} & \cdots & M_{ik}^{(1)} & M_{ij}^{(1)} \end{vmatrix}. \quad (41)$$

This result is shown in Appendix B. There we also present several relations that are fulfilled by these determinants and the matrices $\mathbf{M}^{(k)}$, and furthermore show how it is possible to evaluate them directly in terms of the matrix element $M_{ij}^{(k)}$. Meanwhile, let us express the results we have found in (32) and (39), in terms of determinants, using for such a purpose the identity (40). Then, by direct replacement, we find the following final expressions for both parametrizations.

#### 1. Feynman parametrization

After replacing the matrix elements $M_{ij}^{(k)}$ for the result given defined in (40), Eq. (32) is written as

$$G = \frac{(-1)^{N_\nu}\Gamma(N_\nu - \frac{LD}{2})}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 d\vec{x}\, \delta\left(1 - \sum_{j=1}^N x_j\right) \times \frac{[\frac{\Delta_{11}^{(1)}}{\Delta_{00}^{(0)}} \frac{\Delta_{22}^{(2)}}{\Delta_{11}^{(1)}} \cdots \frac{\Delta_{LL}^{(L)}}{\Delta_{(L-1)(L-1)}^{(L-1)}}]^{-D/2}}{[\sum_{j=1}^N x_j m_j^2 - \sum_{i,j=1}^E \frac{\Delta_{(L+i)(L+j)}^{(L+1)}}{\Delta_{LL}^{(L)}} p_i \cdot p_j]^{N_\nu - (LD)/2}}. \quad (42)$$

After a little algebra, we get the final Feynman parametric representation:

$$G = \frac{(-1)^{N_\nu}\Gamma(N_\nu - \frac{LD}{2})}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 d\vec{x}\, \delta\left(1 - \sum_{j=1}^N x_j\right) \times \frac{[\Delta_{LL}^{(L)}]^{N_\nu - (L+1)(D/2)}}{[\Delta_{LL}^{(L)} \sum_{j=1}^N x_j m_j^2 - \sum_{i,j=1}^E \Delta_{(L+i)(L+j)}^{(L+1)} p_i \cdot p_j]^{N_\nu - (LD)/2}}. \quad (43)$$

#### 2. Schwinger parametrization

In an analogous way, applying identity (40) to (39), we obtain

$$G = \frac{(-1)^{(LD)/2}}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \times \int_0^\infty d\vec{x}\left[\frac{\Delta_{11}^{(1)}}{\Delta_{00}^{(0)}}\frac{\Delta_{22}^{(2)}}{\Delta_{11}^{(1)}}\cdots\frac{\Delta_{LL}^{(L)}}{\Delta_{(L-1)(L-1)}^{(L-1)}}\right]^{-D/2} \times \exp\left(\sum_{j=1}^{N} x_j m_j^2 - \sum_{i,j=1}^{E}\frac{\Delta_{(L+i)(L+j)}^{(L+1)}}{\Delta_{LL}^{(L)}} p_i \cdot p_j\right), \quad (44)$$

or simply

$$G = \frac{(-1)^{(LD)/2}}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)}\int_0^\infty d\vec{x}[\Delta_{LL}^{(L)}]^{-D/2} \times \exp\left(\frac{\Delta_{LL}^{(L)}\sum_{j=1}^{N} x_j m_j^2 - \sum_{i,j=1}^{E}\Delta_{(L+i)(L+j)}^{(L+1)} p_i \cdot p_j}{\Delta_{LL}^{(L)}}\right), \quad (45)$$

which corresponds to Schwinger's parametric representation. In Appendix B it is shown that these determinants can be evaluated from the matrix elements obtained using a recursion relation starting from the IPM, using the following rule:

$$\Delta_{ij}^{(k+1)} = M_{11}^{(1)}\ldots M_{kk}^{(k)} M_{ij}^{(k+1)}. \quad (46)$$

This identity is important since it allows one to evaluate the determinants that appear in the parametric representations obtained in (43) and (45). We should point out that the matrix $\mathbf{A}$, defined in Sec. II B, has a determinant which is equal to $\Delta_{LL}^{(L)}$, that is,

$$\det\mathbf{A} = \Delta_{LL}^{(L)}, \quad (47)$$

a very useful identity for comparing more rigorously the different methods for finding parametric representations of Feynman diagrams.

## III. THE COMPUTATIONAL CODE

The fundamental equation, which allows one to evaluate the matrices $\mathbf{M}^{(k)}$ starting from the initial parameters matrix is given by the recursive relation:

$$M_{ij}^{(k+1)} = M_{ij}^{(k)} - \frac{M_{ik}^{(k)} M_{kj}^{(k)}}{M_{kk}^{(k)}}, \quad (48)$$

or equivalently

$$M_{ij}^{(k)} = M_{ij}^{(k-1)} - \frac{M_{i(k-1)}^{(k-1)} M_{(k-1)j}^{(k-1)}}{M_{(k-1)(k-1)}^{(k-1)}}, \quad (49)$$

which can be easily programmed in any computer language, and also in a *CAS* (*Computer Algebra System*). The codification of this equation gives rise to a simple recursive procedure, which we present here in MAPLE:

```
>R:=proc(m,k,i,j) local val:
>if k=1 then val:=m[i,j]:
>else val:=simplify(R(m,k-1,i,j)-R(m,k-1,i,k-1)
*R(m,k-1,k-1,j)/R(m,k-1,k-1,k-1) ):
>fi:end:
```

In this procedure we have codified the recursive function $R(m, k, i, j)$, where the input parameters are given by the following definitions:

- $m$ Corresponds to the IPM, which is obtained at the beginning of the parametrization process. The matrix that relates internal and external momenta in the quadratic form $\mathbf{Q}^t\mathbf{M}^{(1)}\mathbf{Q}$ ($\mathbf{Q} = [q_1 \ldots q_L p_1 \ldots p_E]$ and $m = \mathbf{M}^{(1)}$).
- $k$ Corresponds to the order of recursion of the matrix. The case $k = 1$ represents the IPM, and the cases $k > 1$ correspond to matrices obtained by recursion starting from the IPM.
- $i, j$ The matrix element to be evaluated.

The algorithm is very simple. It is only necessary to parametrize the loop integral and *recognize* the matrix $\mathbf{M}^{(1)}$. Then we make $m = \mathbf{M}^{(1)}$. Finally if we want to evaluate any matrix element $M_{ij}^{(k)}$, we just execute the following command or instruction in MAPLE: **>R(m,k,i,j);**.

## IV. APPLYING THE ALGORITHM, SIMPLE EXAMPLES

### A. Example I

Now we will compare in actual calculations the usual form and the one presented here for finding the parametric representation in terms of Feynman parameters. For that purpose let us consider the following diagram, Fig. 1, where the masses associated at each propagator are taken as different.

First we write the momentum representation of the graph:

$$G = \int \frac{d^D q_1}{i\pi^{D/2}}\frac{d^D q_2}{i\pi^{D/2}}\frac{1}{(B_1^2 - m_1^2)}\frac{1}{(B_2^2 - m_2^2)}\frac{1}{(B_3^2 - m_3^2)}, \quad (50)$$

where the branch momenta $B_j$ are in this case defined as

$$B_1 = q_1, \qquad B_2 = q_1 + q_2, \qquad B_3 = p_1 + q_2. \quad (51)$$

Applying Feynman parametrization we obtain the follow-

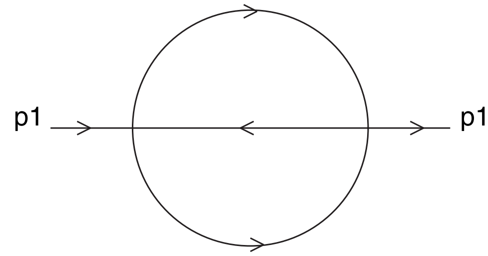


FIG. 1. Sunset diagram.

ing integral:

$$G = \Gamma(3)\int_0^1 dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3) \times \int \frac{d^D q_1}{i\pi^{D/2}} \frac{d^D q_2}{i\pi^{D/2}} \frac{1}{\Omega^3}, \tag{52}$$

where we define

$$\Omega = \sum_{j=1}^{3} x_j B_j^2 - \sum_{j=1}^{3} x_j m_j^2. \tag{53}$$

Then, expanding the previous sum and factorizing the result in terms of internal momenta, we get a quadratic form in these momenta, which reads

$$\Omega = (x_1 + x_2)q_1^2 + 2x_2 q_1 \cdot q_2 + (x_2 + x_3)q_2^2 + 2x_3 p_1 \cdot q_2 + x_3 p_1^2 - \sum_{j=1}^{3} x_j m_j^2. \tag{54}$$

### 1. *Usual method of finding the parametric representation*

According to the previous formulation [see Eq. (7)], we can identify the necessary basic elements for finding the parametric representation. These are

$$\mathbf{A} = \begin{pmatrix} x_1 + x_2 & x_2 \\ x_2 & x_2 + x_3 \end{pmatrix}, \quad \mathbf{k} = (0 \ \ -x_3 p_1)^t, \quad J = x_3 p_1^2. \tag{55}$$

We start from the general result that we found in Eq. (9) for Feynman's parametrization:

$$G = \frac{(-1)^{N_\nu}\Gamma(N_\nu - \frac{LD}{2})}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int d\vec{x}\,\delta\Big(1 - \sum_{j=1}^{N} x_j\Big) \times \frac{[\det\mathbf{A}]^{N_\nu - (L+1)D/2}}{[\det\mathbf{A}(\sum_{j=1}^{N} x_j m_j^2 - J + \mathbf{k}^t\mathbf{A}^{-1}\mathbf{k})]^{N_\nu - L(D/2)}}. \tag{56}$$

In the present case this gives

$$G = (-1)^3\Gamma(3 - D)\int dx_1 \ldots dx_3 \delta\Big(1 - \sum_{j=1}^{3} x_j\Big) \times \frac{[\det\mathbf{A}]^{3-3(D/2)}}{[\det\mathbf{A}(\sum_{j=1}^{N} x_j m_j^2 - J + \mathbf{k}^t\mathbf{A}^{-1}\mathbf{k})]^{3-D}}. \tag{57}$$

Evaluating the terms that are involved here we get

$$\begin{aligned} \det\mathbf{A} &= x_1x_2 + x_1x_3 + x_2x_3, \\ \mathbf{A}^{-1} &= \frac{1}{x_1x_2 + x_1x_3 + x_2x_3} \times \begin{pmatrix} x_1 + x_2 & -x_2 \\ -x_2 & x_2 + x_3 \end{pmatrix}, \\ \mathbf{k}^t\mathbf{A}^{-1}\mathbf{k} &= \frac{x_3^2(x_1 + x_2)p_1^2}{x_1x_2 + x_1x_3 + x_2x_3}, \\ \det\mathbf{A}(-J + \mathbf{k}^t\mathbf{A}^{-1}\mathbf{k}) &= -(x_1x_2x_3)p_1^2, \end{aligned} \tag{58}$$

and considering also the fact that $D = 4 - 2\epsilon$, one finally obtains the Feynman parametric representation:

$$G = -\Gamma(-1 + 2\epsilon)\int_0^1 d\vec{x}\,\delta\Big(1 - \sum_{j=1}^{3} x_j\Big) \times \frac{[x_1x_2 + x_1x_3 + x_2x_3]^{-3+3\epsilon}}{[(x_1x_2 + x_1x_3 + x_2x_3)\sum_{j=1}^{3} x_j m_j^2 - (x_1x_2x_3)p_1^2]^{-1+2\epsilon}}, \tag{59}$$

with $d\vec{x} = dx_1dx_2dx_3$.

### 2. *Obtaining the scalar representation by recursion*

Remembering the general formula that is used in this method for the parametric representation:

$$G = \frac{(-1)^{N_\nu}\Gamma(N_\nu - \frac{LD}{2})}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int d\vec{x}\,\delta\Big(1 - \sum_{j=1}^{N} x_j\Big) \times \frac{[M_{11}^{(1)} \ldots M_{LL}^{(L)}]^{-D/2}}{[\sum_{j=1}^{N} x_j m_j^2 - \sum_{i,j=1}^{E} M_{(L+i)(L+j)}^{(L+1)} p_i \cdot p_j]^{N_\nu - [(LD)/2]}}, \tag{60}$$

which in the present case gets reduced to the following:

$$G = -\Gamma(-1 + 2\epsilon)\int dx_1dx_2dx_3 \delta\Big(1 - \sum_{j=1}^{3} x_j\Big) \times \frac{[M_{11}^{(1)} M_{22}^{(2)}]^{-2+\epsilon}}{[\sum_{j=1}^{3} x_j m_j^2 - M_{33}^{(3)} p^2]^{-1+2\epsilon}}. \tag{61}$$

From Eq. (54) one can find immediately the IPM:

$$\mathbf{M}^{(1)} = \mathbf{M} = \begin{pmatrix} x_1 + x_2 & x_2 & 0 \\ x_2 & x_2 + x_3 & x_3 \\ 0 & x_3 & x_3 \end{pmatrix}. \tag{62}$$

It is now only necessary to calculate the matrix elements using the recursive function described in Sec. III. Basically we need to evaluate the following identities:

$$M_{11}^{(1)} = R(M, 1, 1, 1), \qquad M_{22}^{(2)} = R(M, 2, 2, 2), \qquad M_{33}^{(3)} = R(M, 3, 3, 3). \tag{63}$$

The results, after writing the commands in MAPLE are respectively:

*>R(M,1,1,1);*

$$x_1 + x_2$$

*>R(M,2,2,2);*

$$\frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1 + x_2}$$

*>R(M,3,3,3);*

$$\frac{x_1x_2x_3}{x_1x_2 + x_1x_3 + x_2x_3}.$$

Thus replacing these expressions into Eq. (61), we obtain

$$G = -\Gamma(-1 + 2\epsilon)\int d\vec{x}\delta\Big(1 - \sum_{j=1}^{3} x_j\Big) \times \frac{[x_1x_2 + x_1x_3 + x_2x_3]^{-2+\epsilon}}{[\sum_{j=1}^{3} x_j m_j^2 - \frac{x_1x_2x_3}{x_1x_2+x_1x_3+x_2x_3} p^2]^{-1+2\epsilon}}, \quad (64)$$

and then we have the same scalar representation as found before in (59).

### B. Example II

Let us consider now the following diagram (Fig. 2).

The loop integral is given in this case by

$$G = \int \frac{d^D q_1}{i\pi^{D/2}} \frac{1}{(B_1^2 - m_1^2)} \frac{1}{(B_2^2 - m_2^2)} \frac{1}{(B_3^2 - m_3^2)}, \quad (65)$$

where the branch momenta $B_j$ have been defined in the following way:

$$B_1 = q_1, \qquad B_2 = p_1 + q_1, \qquad B_3 = p_1 + p_2 + q_1. \quad (66)$$

The next step is to apply Feynman's parametrization, obtaining the following integral:

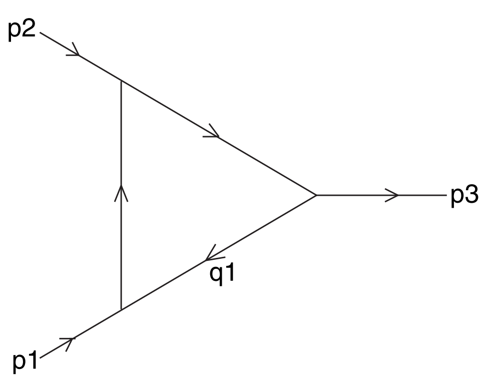


FIG. 2. Triangle diagram.

$$G = \Gamma(3)\int_0^1 dx_1dx_2dx_3\delta\Big(1 - \sum_{j=1}^{3} x_j\Big)\int \frac{d^D q_1}{i\pi^{D/2}}\frac{1}{\Omega^3}, \quad (67)$$

where the denominator $\Omega$ is given in terms of the internal momenta by

$$\Omega = (x_1 + x_2 + x_3)q_1^2 + 2[(x_2 + x_3)p_1 + x_3p_2]\cdot q_1 + (x_2 + x_3)p_1^2 + 2x_3p_1\cdot p_2 + x_3p_2^2 - \sum_{j=1}^{3} x_j m_j^2. \quad (68)$$

#### *1. Usual method of finding the parametric representation*

Starting from Eq. (68) we can recognize right away the basic necessary elements for finding the parametric representation. These are

$$\mathbf{A} = (x_1+x_2 + x_3), \qquad \mathbf{k} = -(x_2 + x_3)p_1 - x_3p_2, \qquad J = (x_2 + x_3)p_1^2 + 2x_3p_1.p_2 + x_3p_2^2, \quad (69)$$

and therefore the resulting scalar integral will be given in this case by the expression

$$G = (-1)^3\Gamma\Big(3 - \frac{D}{2}\Big)\int dx_1\dots dx_3\delta\Big(1 - \sum_{j=1}^{3} x_j\Big) \times \frac{[\det\mathbf{A}]^{3-D}}{[\det\mathbf{A}(\sum_{j=1}^{N} x_jm_j^2 - J + \mathbf{k}^t\mathbf{A}^{-1}\mathbf{k})]^{3-D/2}}. \quad (70)$$

Evaluating each term, we obtain

$$\det\mathbf{A} = x_1 + x_2 + x_3, \qquad \mathbf{A}^{-1} = \frac{1}{x_1 + x_2 + x_3}, \qquad \mathbf{k}^t\mathbf{A}^{-1}\mathbf{k} = \frac{[(x_2 + x_3)p_1 + x_3p_2]^2}{x_1 + x_2 + x_3} \quad (71)$$

and

$$\begin{aligned}\det\mathbf{A}(-J + \mathbf{k}^t\mathbf{A}^{-1}\mathbf{k}) &= -(x_1x_2 + x_1x_3)p_1^2 - 2x_1x_3p_1\cdot p_2 \\ &\quad - (x_1x_3 + x_2x_3)p_2^2 \\ &= -x_1x_2p_1^2 - x_2x_3p_2^2 \\ &\quad - x_1x_3(p_1 + p_2)^2 \\ &= -x_1x_2p_1^2 - x_2x_3p_2^2 - x_1x_3p_3^2, \quad (72)\end{aligned}$$

where we have used the condition $(p_1 + p_2)^2 = p_3^2$, and then put $D = 4 - 2\epsilon$. Thus we finally arrive at Feynman's parametric representation:

$$G = -\Gamma(1+\epsilon)\int_0^1 d\vec{x}\,\delta\Big(1-\sum_{j=1}^{3}x_j\Big)\frac{[x_1+x_2+x_3]^{-1+2\epsilon}}{[(x_1+x_2+x_3)\sum_{j=1}^{3}x_jm_j^2-(x_1x_2)p_1^2-(x_2x_3)p_2^2-(x_1x_3)p_3^2]^{1+\epsilon}}, \tag{73}$$

with $d\vec{x} = dx_1dx_2dx_3$.

### *2. Obtaining the scalar representation by recursion*

In this method the general formula for the parametric representation is

$$G = \frac{(-1)^{N_\nu}\Gamma(N_\nu - \frac{LD}{2})}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)}\int d\vec{x}\,\delta\Big(1-\sum_{j=1}^{N}x_j\Big)\frac{[M_{11}^{(1)}\ldots M_{LL}^{(L)}]^{-D/2}}{[\sum_{j=1}^{N}x_jm_j^2-\sum_{i,j=1}^{E}M_{(L+i)(L+j)}^{(L+1)}p_i\cdot p_j]^{N_\nu-(LD)/2}}, \tag{74}$$

which in our case is reduced to the following in $D = 4-2\epsilon$ dimensions:

$$G = -\Gamma(1+\epsilon)\int dx_1dx_2dx_3\,\delta\Big(1-\sum_{j=1}^{3}x_j\Big)\frac{[M_{11}^{(1)}]^{-2+\epsilon}}{[\sum_{j=1}^{3}x_jm_j^2-M_{22}^{(2)}p_1^2-M_{23}^{(2)}p_1\cdot p_2-M_{32}^{(2)}p_2\cdot p_1-M_{33}^{(2)}p_2^2]^{1+\epsilon}}. \tag{75}$$

The next step consists in the evaluation of the matrix elements of $M_{ij}^{(k)}$. In order to do this, and starting from Eq. (68) we find the IPM:

$$\mathbf{M}^{(1)} = \mathbf{M} = \begin{pmatrix} x_1+x_2+x_3 & x_2+x_3 & x_3 \\ x_2+x_3 & x_2+x_3 & x_3 \\ x_3 & x_3 & x_3 \end{pmatrix}. \tag{76}$$

Using the recursive routine proposed in Sec. III, the necessary matrix elements $M_{ij}^{(k)}$ are evaluated:

$$M_{11}^{(1)} = R(M,1,1,1), \qquad M_{22}^{(2)} = R(M,2,2,2),$$
$$M_{23}^{(2)} = M_{32}^{(2)} = R(M,2,2,3), \qquad M_{33}^{(2)} = R(M,2,3,3), \tag{77}$$

and executing the MAPLE commands, we get the following results:

*>R(M,1,1,1);*

$$x_1+x_2+x_3$$

*>R(M,2,2,2);*

$$\frac{x_1x_2+x_1x_3}{x_1+x_2+x_3}$$

*>R(M,2,2,3);*

$$\frac{x_1x_3}{x_1+x_2+x_3}$$

*>R(M,2,3,3);*

$$\frac{x_1x_3+x_2x_3}{x_1+x_2+x_3}.$$

For the sum $\sum_{i,j=1}^{2}M_{(L+i)(L+j)}^{(L+1)}p_i\cdot p_j$, we get

$$\sum_{i,j=1}^{2}M_{(L+i)(L+j)}^{(L+1)}p_i\cdot p_j = \frac{(x_1x_2+x_1x_3)p_1^2+2x_1x_3p_1\cdot p_2+(x_1x_3+x_2x_3)p_2^2}{x_1+x_2+x_3} = \frac{(x_1x_2)p_1^2+(x_2x_3)p_2^2+(x_1x_3)p_3^2}{x_1+x_2+x_3}. \tag{78}$$

Thus, replacing these quantities in (75), we obtain

$$G = -\Gamma(1+\epsilon)\int dx_1dx_2dx_3\,\delta\Big(1-\sum_{j=1}^{3}x_j\Big)\times\frac{[x_1+x_2+x_3]^{-2+\epsilon}}{[\sum_{j=1}^{3}x_jm_j^2-\frac{(x_1x_2)p_1^2+(x_2x_3)p_2^2+(x_1x_3)p_3^2}{x_1+x_2+x_3}]^{1+\epsilon}}, \tag{79}$$

which finally is reduced to the same parametric representation deduced before in (73)

$$G = -\Gamma(1+\epsilon)\int d\vec{x}\,\delta\Big(1-\sum_{j=1}^{3}x_j\Big)\frac{[x_1+x_2+x_3]^{-1+2\epsilon}}{[(x_1+x_2+x_3)\sum_{j=1}^{3}x_jm_j^2-(x_1x_2)p_1^2-(x_2x_3)p_2^2-(x_1x_3)p_3^2]^{1+\epsilon}}, \tag{80}$$

where $d\vec{x} = dx_1dx_2dx_3$.

## V. CONCLUSIONS

There are two main aspects that need to be emphasized in the present work. The first is the simplicity of the method, both in the actual calculation and in its application to a particular topology. From the point of view of the mathematical structure of the final scalar representation, there is a remarkable difference with the usual method. In the usual parametric form of a loop integral, it is necessary to evaluate a scalar term and a matrix product that involves an inverse matrix calculation. The method proposed in this work is based on a simple change in the initial procedure in the search for a parametric representation of the momentum integral, so that both the scalar term and the matrix product with inverse matrix are included in an *explicit expansion* of internal products of external momenta, in which the coefficients of such expansion are determinants of submatrices of the matrix that relates internal and external momenta (IPM). Moreover, the most important aspect is that such determinants can be in turn calculated from matrix elements obtained using a recursion relation starting from the IPM, in a simple and straightforward way.

The second relevant aspect is that this method can be easily implemented computationally. This allows for a fast automatization of Feynman diagram generation, obtaining simply and directly the parametric representation as a step towards a complete numerical or analytical evaluation whenever possible.

## APPENDIX A: QUADRATIC FORMS AND ITS DIAGONALIZATION BY SQUARE COMPLETION

A quadratic form in $n$ variables is an expression which can be written in matrix form as the product $\mathbf{x}^t\mathbf{M}\mathbf{x}$, where $\mathbf{x}$ is an n-dimensional vector, given by $\mathbf{x} = [x_1, \ldots, x_n]^t$, and $\mathbf{M}$ is a generic $n \times n$ dimensional matrix. That is

$$\mathbf{x}^t\mathbf{M}\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i M_{ij} x_j. \tag{A1}$$

Let $\mathbf{D}$ be an $n \times n$ diagonal matrix. The expressions $\mathbf{x}^t\mathbf{M}\mathbf{x}$ and $\bar{\mathbf{y}}^t\mathbf{D}\mathbf{y}$ are equivalent if there exists a linear transformation $\mathbf{y} = \mathbf{P}\mathbf{x}$ and $\bar{\mathbf{y}} = \bar{\mathbf{P}}\mathbf{x}$ such that $\mathbf{x}^t\mathbf{M}\mathbf{x} = \bar{\mathbf{y}}^t\mathbf{D}\mathbf{y}$, that is,

$$\mathbf{M} = \bar{\mathbf{P}}^t\mathbf{D}\mathbf{P}. \tag{A2}$$

The quadratic form is then transformed into a sum of $n$ linear terms of the type $\bar{y}_i y_j \delta_{ij}$.

### 1. Square Completion Procedure

#### *a. Completing the square for $x_1$*

Every quadratic form can be diagonalized using the *square completion* procedure, which generates the required linear transformation. First we define a matrix $\mathbf{M} = \mathbf{M}^{(1)}$, and then we expand the matrix product $\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x}$ in order to complete the square associated to the parameter $x_1$. Then we arrive at the following result:

$$\begin{aligned}
\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x} &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i M_{ij}^{(1)} x_j = M_{11}^{(1)}x_1^2 + x_1\left(\sum_{j=2}^{n} M_{1j}^{(1)}x_j\right) + \left(\sum_{i=2}^{n} x_i M_{i1}^{(1)}\right)x_1 + \sum_{i,j=2}^{n} x_i M_{ij}^{(1)} x_j \\
&= M_{11}^{(1)}\left[x_1^2 + x_1\left(\sum_{j=2}^{n}\frac{M_{1j}^{(1)}}{M_{11}^{(1)}}x_j\right) + \left(\sum_{i=2}^{n} x_i\frac{M_{i1}^{(1)}}{M_{11}^{(1)}}\right)x_1\right] + \sum_{i,j=2}^{n} x_i M_{ij}^{(1)} x_j \\
&= M_{11}^{(1)}\left[\left(x_1 + \sum_{i=2}^{n} x_i\frac{M_{i1}^{(1)}}{M_{11}^{(1)}}\right)\left(x_1 + \sum_{j=2}^{n}\frac{M_{1j}^{(1)}}{M_{11}^{(1)}}x_j\right)\right] + \sum_{i,j=2}^{n} x_i\left(M_{ij}^{(1)} - \frac{M_{i1}^{(1)}M_{1j}^{(1)}}{M_{11}^{(1)}}\right)x_j.
\end{aligned} \tag{A3}$$

Let us define now the new variables

$$\bar{y}_1 = x_1 + \sum_{i=2}^{n} x_i\frac{M_{i1}^{(1)}}{M_{11}^{(1)}}, \qquad y_1 = x_1 + \sum_{j=2}^{n}\frac{M_{1j}^{(1)}}{M_{11}^{(1)}}x_j, \tag{A4}$$

and also the matrix $\mathbf{M}^{(2)} = \{M_{ij}^{(2)}\}$, such that the elements of this be given by the relation

$$M_{ij}^{(2)} = M_{ij}^{(1)} - \frac{M_{i1}^{(1)}M_{1j}^{(1)}}{M_{11}^{(1)}}. \tag{A5}$$

Therefore we can rewrite the quadratic form (A1), with the first parameter already diagonalized in the following way:

$$\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x} = M_{11}^{(1)}\bar{y}_1 y_1 + \sum_{i=2}^{n}\sum_{j=2}^{n} x_i M_{ij}^{(2)} x_j. \tag{A6}$$

The second term in the right-hand side can be simplified, since from Eq. (A5) one obtains that $M_{j1}^{(2)} = M_{1j}^{(2)} = 0$, for $1 \le j \le n$. Thus we can write

$$\sum_{i=2}^{n}\sum_{j=2}^{n} x_i M_{ij}^{(2)} x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i M_{ij}^{(2)} x_j = \mathbf{x}^t\mathbf{M}^{(2)}\mathbf{x}. \tag{A7}$$

In summary, in the quadratic expansion the first term has been already diagonalized, a fact that can be described by the following expression:

$$\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x} = M_{11}^{(1)}\bar{y}_1 y_1 + \mathbf{x}^t\mathbf{M}^{(2)}\mathbf{x}. \tag{A8}$$

#### *b. Completing the square for $x_2$*

Now we take the second term of (A8), and the same procedure followed above is repeated in order to complete the square for the parameter $x_2$, which gives

$$\begin{aligned}\mathbf{x}^t\mathbf{M}^{(2)}\mathbf{x} &= \sum_{i=2}^{n}\sum_{j=2}^{n} x_i M_{ij}^{(2)} x_j \\ &= M_{22}^{(2)}x_2^2 + x_2\Big(\sum_{j=3}^{n} M_{2j}^{(2)}x_j\Big) + \Big(\sum_{i=3}^{n} x_i M_{i2}^{(2)}\Big)x_2 \\ &\quad + \sum_{i,j=3}^{n} x_i M_{ij}^{(2)} x_j \\ &= M_{22}^{(2)}\Big[\Big(x_2 + \sum_{i=3}^{n} x_i \frac{M_{i2}^{(2)}}{M_{22}^{(2)}}\Big)\Big(x_2 + \sum_{j=3}^{n}\frac{M_{2j}^{(2)}}{M_{22}^{(2)}}x_j\Big)\Big] \\ &\quad + \sum_{i,j=3}^{n} x_i\Big(M_{ij}^{(2)} - \frac{M_{i2}^{(2)}M_{2j}^{(2)}}{M_{22}^{(2)}}\Big)x_j. \end{aligned} \tag{A9}$$

Let us define, analogously to Eq. (A4), the new variables:

$$\bar{y}_2 = x_2 + \sum_{i=3}^{n} x_i \frac{M_{i2}^{(2)}}{M_{22}^{(2)}}, \qquad y_2 = x_2 + \sum_{j=3}^{n} \frac{M_{2j}^{(2)}}{M_{22}^{(2)}} x_j, \tag{A10}$$

and the matrix $\mathbf{M}^{(3)} = \{M_{ij}^{(3)}\}$, where we set $M_{ij}^{(3)} = M_{ij}^{(2)} - (M_{i2}^{(2)}M_{2j}^{(2)})/M_{22}^{(2)}$. Then we obtain

$$\mathbf{x}^t\mathbf{M}^{(2)}\mathbf{x} = M_{22}^{(2)}\bar{y}_2 y_2 + \sum_{i=3}^{n}\sum_{j=3}^{n} x_i M_{ij}^{(3)} x_j. \tag{A11}$$

The expression for the matrix element $M_{ij}^{(3)}$ implies that $M_{2j}^{(3)} = M_{j2}^{(3)} = 0$, with $2 \le j \le n$, and since we also had that $M_{1j}^{(2)} = M_{j1}^{(2)} = 0$, then $M_{1j}^{(3)} = M_{j1}^{(3)} = 0$, where $1 \le j \le n$. In this way we can write the second term as

$$\sum_{i=3}^{n}\sum_{j=3}^{n} x_i M_{ij}^{(3)} x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i M_{ij}^{(3)} x_j = \mathbf{x}^t\mathbf{M}^{(3)}\mathbf{x}, \tag{A12}$$

and therefore now the first two components of $\mathbf{x}$ have been diagonalized:

$$\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x} = M_{11}^{(1)}\bar{y}_1 y_1 + M_{22}^{(2)}\bar{y}_2 y_2 + \mathbf{x}^t\mathbf{M}^{(3)}\mathbf{x}. \tag{A13}$$

#### *c. Generalization of the square completion procedure for $x_j$*

Notice that the last term in (A13) is another quadratic form, which then will allow us to complete the square for the parameter $x_3$. The procedure can be repeated successively for $x_3, \ldots, x_n$, and therefore the following relations are determined by induction:

$$\bar{y}_l = x_l + \sum_{i=l+1}^{n} x_i \frac{M_{il}^{(l)}}{M_{ll}^{(l)}}, \qquad y_l = x_l + \sum_{j=l+1}^{n} \frac{M_{lj}^{(l)}}{M_{ll}^{(l)}} x_j,$$
$$M_{ij}^{(l+1)} = \begin{cases} 0 & \text{if } i < (l+1) \lor j < (l+1) \\ M_{ij}^{(l)} - \frac{M_{il}^{(l)}M_{lj}^{(l)}}{M_{ll}^{(l)}} & \text{in other cases.} \end{cases} \tag{A14}$$

Here the matrix $\mathbf{M}^{(k)}$ (with $1 \le k \le n$) has the following generic structure:

$$\mathbf{M}^{(k)} = \begin{pmatrix} 0 & & \cdots & & & 0 \\ & & & & & \vdots \\ \vdots & & 0 & \cdots & & 0 \\ & & \vdots & M_{kk}^{(k)} & \cdots & M_{kn}^{(k)} \\ & & & \vdots & & \vdots \\ 0 & \cdots & 0 & M_{nk}^{(k)} & \cdots & M_{nn}^{(k)} \end{pmatrix}. \tag{A15}$$

In general, the procedure of square completion of the $k$-element of $\mathbf{x}$, for $1 \le k < n$, transforms the initial quadratic form into

$$\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x} = M_{11}^{(1)}\bar{y}_1 y_1 + \cdots + M_{kk}^{(k)}\bar{y}_k y_k + \mathbf{x}^t\mathbf{M}^{(k+1)}\mathbf{x}. \tag{A16}$$

The complete process, that is after $n$ square completions, diagonalizes the quadratic form $\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x}$ and transforms it into a diagonal bilineal structure, of the form

$$\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x} = M_{11}^{(1)}\bar{y}_1 y_1 + \cdots + M_{nn}^{(n)}\bar{y}_n y_n = \bar{\mathbf{y}}^t\mathbf{D}\mathbf{y}, \tag{A17}$$

where we identify

$$D = \mathrm{diag}[M_{11}^{(1)}, M_{22}^{(2)}, \ldots, M_{nn}^{(n)}], \qquad \bar{\mathbf{y}} = [\overline{y_1}, \ldots, \bar{y}_n]^t,$$
$$\mathbf{y} = [y_1, \ldots, y_n]^t. \tag{A18}$$

### 2. Some properties

(1) The relation between the vectors $\bar{y}, y$ and $x$, is defined by Eq. (A14), and from it we can identify the transformation matrices that fulfill the equations:

$$\mathbf{y} = \mathbf{P}\mathbf{x} \land \bar{\mathbf{y}} = \bar{\mathbf{P}}\mathbf{x}. \tag{A19}$$

Specifically, it is possible to determine $\mathbf{P}$ and $\bar{\mathbf{P}}$, given by

$$P = \begin{pmatrix} 1 & \frac{M_{12}^{(1)}}{M_{11}^{(1)}} & \frac{M_{13}^{(1)}}{M_{11}^{(1)}} & \cdots & \frac{M_{1n}^{(1)}}{M_{11}^{(1)}} \\ 0 & 1 & \frac{M_{23}^{(2)}}{M_{22}^{(2)}} & \cdots & \frac{M_{2n}^{(2)}}{M_{22}^{(2)}} \\ \vdots & 0 & 1 & & \vdots \\ & & \ddots & \ddots & \\ 0 & \cdots & & 0 & 1 \end{pmatrix} \tag{A20}$$

$$\overline{P} = \begin{pmatrix} 1 & \frac{M_{21}^{(1)}}{M_{11}^{(1)}} & \frac{M_{31}^{(1)}}{M_{11}^{(1)}} & \cdots & \frac{M_{n1}^{(1)}}{M_{11}^{(1)}} \\ 0 & 1 & \frac{M_{32}^{(2)}}{M_{22}^{(2)}} & \cdots & \frac{M_{n2}^{(2)}}{M_{22}^{(2)}} \\ \vdots & 0 & 1 & & \vdots \\ & & \ddots & \ddots & \\ 0 & \cdots & & 0 & 1 \end{pmatrix}. \quad \text{(A21)}$$

(2) From the equations in (A14), we find that

$$\overline{y}_n = y_n = x_n, \qquad \mathbf{M}^{(n+1)} = \{0\}. \quad \text{(A22)}$$

(3) The transformation matrices $\mathbf{P}$ and $\overline{\mathbf{P}}$ have the following property:

$$\det\mathbf{P} = \det\mathbf{P}^t = 1, \qquad \det\overline{\mathbf{P}} = \det\overline{\mathbf{P}}^t = 1. \quad \text{(A23)}$$

(4) If $\mathbf{M}^{(1)} = [\mathbf{M}^{(1)}]^t$ (symmetric case), then the following identities hold:

$$\begin{aligned} &\overline{\mathbf{y}} = \mathbf{y}, \qquad \overline{\mathbf{P}} = \mathbf{P}, \qquad \mathbf{M}^{(k)} = [\mathbf{M}^{(k)}]^t, \\ &\mathbf{x}^t\mathbf{M}^{(1)}\mathbf{x} = \mathbf{y}^t\mathbf{D}\mathbf{y}, \qquad \mathbf{M}^{(1)} = \mathbf{P}^t\mathbf{D}\mathbf{P},, \end{aligned} \quad \text{(A24)}$$

where the matrix $\mathbf{D}$ is the diagonal matrix given by

$$\mathbf{D} = \mathrm{diag}[M_{11}^{(1)}, M_{22}^{(2)}, \ldots, M_{nn}^{(n)}]. \quad \text{(A25)}$$

### 3. Evaluation of the determinant of $\mathbf{M}^{(1)}$

From the previous results, the determinant of $\mathbf{M}^{(1)}$ is given by

$$\begin{aligned} \det\mathbf{M}^{(1)} &= \det\overline{\mathbf{P}}^t\mathbf{D}\mathbf{P} = \det\overline{\mathbf{P}}^t \cdot \det\mathbf{D} \cdot \det\mathbf{P} \\ &= M_{11}^{(1)} M_{22}^{(2)} \ldots M_{nn}^{(n)}. \end{aligned} \quad \text{(A26)}$$

The conditions for evaluating this determinant are given in Appendix B.

## APPENDIX B: MATRICES $\mathbf{M}^{(k)}$

### 1. Generalization of the matrices $\mathbf{M}^{(k)}$

It is possible to generalize the $n \times n$ dimensional matrices $\mathbf{M}^{(k)}$ starting from the recurrence equation

$$M_{ij}^{(k+1)} = M_{ij}^{(k)} - \frac{M_{ik}^{(k)} M_{kj}^{(k)}}{M_{kk}^{(k)}}. \quad \text{(B1)}$$

As an example let us consider a generic matrix $\mathbf{A}_{n\times n} = \{a_{ij}\}$, and define an input matrix $\mathbf{M}^{(1)} \equiv \mathbf{A}_{n\times n}$.

#### *a. Generating* $\mathbf{M}^{(2)}$

Let us evaluate the particular cases of the first row and first column. That is,

$$M_{1j}^{(2)} = M_{1j}^{(1)} - \frac{M_{11}^{(1)} M_{1j}^{(1)}}{M_{11}^{(1)}} = 0, \qquad (j = 1, \ldots, n) \quad \text{(B2)}$$

$$M_{i1}^{(2)} = M_{i1}^{(1)} - \frac{M_{i1}^{(1)} M_{11}^{(1)}}{M_{11}^{(1)}} = 0, \qquad (i = 1, \ldots, n). \quad \text{(B3)}$$

The other matrix elements do not present a particular interest, and are evaluated using the recursion relation (B1). Then the matrix $\mathbf{M}^{(2)}$ gets structured in the following manner:

$$\mathbf{M}^{(2)} = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & M_{22}^{(2)} & \cdots & M_{2n}^{(2)} \\ & \vdots & & \vdots \\ 0 & M_{n2}^{(2)} & \cdots & M_{nn}^{(2)} \end{pmatrix}. \quad \text{(B4)}$$

Notice that $\mathbf{M}^{(2)}$ is computable only if $M_{11}^{(1)} \neq 0$.

#### *b. Generating* $\mathbf{M}^{(3)}$

Having $\mathbf{M}^{(2)}$ already evaluated, one can construct $\mathbf{M}^{(3)}$. Let us analyze the first and second row. For the first row we have that

$$\begin{aligned} &M_{1j}^{(3)} = M_{1j}^{(2)} - \frac{M_{12}^{(2)} M_{2j}^{(2)}}{M_{22}^{(2)}} = 0, \\ &\text{since } M_{1j}^{(2)} = 0 \qquad (j = 1, \ldots, n), \end{aligned} \quad \text{(B5)}$$

while for the second row

$$M_{2j}^{(3)} = M_{2j}^{(2)} - \frac{M_{22}^{(2)} M_{2j}^{(2)}}{M_{22}^{(2)}} = 0. \quad \text{(B6)}$$

Analogously, for the first and second column we have the following values, respectively:

$$\begin{aligned} &M_{i1}^{(3)} = M_{i1}^{(2)} - \frac{M_{i2}^{(2)} M_{21}^{(2)}}{M_{22}^{(2)}} = 0, \\ &\text{since } M_{i1}^{(2)} = 0 \qquad (i = 1, \ldots, n) \end{aligned} \quad \text{(B7)}$$

and

$$M_{i2}^{(3)} = M_{i2}^{(2)} - \frac{M_{i2}^{(2)} M_{22}^{(2)}}{M_{22}^{(2)}} = 0. \quad \text{(B8)}$$

The other elements have values according to (B1). Finally the matrix $\mathbf{M}^{(3)}$ gets the following form:

$$\mathbf{M}^{(3)} = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & 0 & \cdots & & 0 \\ & \vdots & M_{33}^{(3)} & \cdots & M_{3n}^{(3)} \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & M_{n3}^{(3)} & \cdots & M_{nn}^{(3)} \end{pmatrix}. \quad \text{(B9)}$$

The matrix is defined only if the matrix element $M_{22}^{(2)} \neq 0$. The procedure can be repeated successively for the rest of the matrices generated by recursion, thus finding that for $k \in [1, \ldots, n]$ one gets

$$\mathbf{M}^{(k)} = \begin{pmatrix} 0 & \cdots & & & \cdots & 0 \\ \vdots & & & & & \vdots \\ & & 0 & \cdots & & 0 \\ & & \vdots & M_{kk}^{(k)} & \cdots & M_{kn}^{(k)} \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & 0 & M_{nk}^{(k)} & \cdots & M_{nn}^{(k)} \end{pmatrix}, \tag{B10}$$

with the condition that $\mathbf{M}^{(k)}$ is defined only if $M_{kk}^{(k)} \neq 0$ or $k = 1, 2, \ldots, n-1$.

### 2. Elements $M_{ij}^{(k)}$

From the previous results we can find the relation that exists between the matrix elements generated by recursion and the input matrix elements $\mathbf{M}^{(1)} = \mathbf{A}_{n\times n} = \{a_{ij}\}$. For $M_{ij}^{(2)}$,

$$M_{ij}^{(2)} = M_{ij}^{(1)} - \frac{M_{i1}^{(1)} M_{1j}^{(1)}}{M_{11}^{(1)}} = \frac{M_{11}^{(1)} M_{ij}^{(1)} - M_{i1}^{(1)} M_{i1}^{(1)}}{M_{11}^{(1)}} = \frac{\begin{vmatrix} M_{11}^{(1)} & M_{i1}^{(1)} \\ M_{i1}^{(1)} & M_{ij}^{(1)} \end{vmatrix}}{|M_{11}^{(1)}|}, \tag{B11}$$

or equivalently

$$M_{ij}^{(2)} = \frac{\begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix}}{|a_{11}|}. \tag{B12}$$

For $M_{ij}^{(3)}$ we have that

$$M_{ij}^{(3)} = M_{ij}^{(2)} - \frac{M_{i2}^{(2)} M_{2j}^{(2)}}{M_{22}^{(2)}} = \frac{\begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix}}{|a_{11}|} - \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{i1} & a_{i2} \end{vmatrix} \begin{vmatrix} a_{11} & a_{1j} \\ a_{21} & a_{2j} \end{vmatrix}}{|a_{11}| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}. \tag{B13}$$

Some simple algebra gives the following result:

$$M_{ij}^{(3)} = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{1j} \\ a_{21} & a_{22} & a_{2j} \\ a_{i1} & a_{i2} & a_{ij} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}. \tag{B14}$$

Let us now define the determinant $\Delta_{ij}^{(k+1)}$, such that it corresponds to the determinant of a submatrix of the input matrix $\mathbf{M}^{(1)} = \mathbf{A}_{n\times n}$, whose dimension is $(k+1) \times (k+1)$, and which is given by the following identity:

$$\Delta_{ij}^{(k+1)} = \begin{vmatrix} a_{11} & \cdots & a_{1k} & a_{1j} \\ \vdots & & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk} & a_{kj} \\ a_{i1} & \cdots & a_{ik} & a_{ij} \end{vmatrix}. \tag{B15}$$

Let us see the following examples.

#### *a. Example I*

$$\Delta_{34}^{(2)} = \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{34} \end{vmatrix}. \tag{B16}$$

#### *b. Example II*

$$\Delta_{33}^{(3)} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{B17}$$

Applying this definition in Eqs. (B12) and (B14), we obtain

$$M_{ij}^{(2)} = \frac{\Delta_{ij}^{(2)}}{\Delta_{11}^{(1)}}, \tag{B18}$$

$$M_{ij}^{(3)} = \frac{\Delta_{ij}^{(3)}}{\Delta_{22}^{(2)}}. \tag{B19}$$

Through an induction process we can directly generalize the relation that exists between the matrix elements of $M_{ij}^{(k)}$ and the input matrix $\mathbf{M}^{(1)} = \mathbf{A}_{n\times n}$. In general, one gets

$$\mathbf{M}_{ij}^{(k+1)} = \frac{\Delta_{ij}^{(k+1)}}{\Delta_{kk}^{(k)}}. \tag{B20}$$

### 3. The matrix $\mathbf{M}^{(k)}$ in terms of determinants of submatrices of $\mathbf{M}^{(1)}$

In Appendix A it was previously shown that the determinant of the input matrix $\mathbf{M}^{(1)} = \mathbf{A}_{n\times n}$ is given by the expression

$$\det\mathbf{A} = \det\mathbf{M}^{(1)} = M_{11}^{(1)} M_{22}^{(2)} \ldots M_{nn}^{(n)}. \tag{B21}$$

Using Eq. (B20) we can write the matrix elements $M_{kk}^{(k)}$ as ratios of determinants of submatrices of $\mathbf{M}^{(1)}$. Then we have that

$$\det\mathbf{A}_{n\times n} = \det\mathbf{M}^{(1)} = \frac{\Delta_{11}^{(1)}}{\Delta_{00}^{(0)}} \frac{\Delta_{22}^{(2)}}{\Delta_{11}^{(1)}} \cdots \frac{\Delta_{(n-1)(n-1)}^{(n-1)}}{\Delta_{(n-2)(n-2)}^{(n-2)}} \frac{\Delta_{nn}^{(n)}}{\Delta_{(n-1)(n-1)}^{(n-1)}} = \frac{\Delta_{nn}^{(n)}}{\Delta_{00}^{(0)}}. \tag{B22}$$

Here $\Delta_{00}^{(0)} = 1$, which can be shown by calculating the determinant of a scalar. Let us evaluate the determinant of the matrix $\mathbf{M}^{(1)} = \mathbf{A}_{1\times 1} = (a_{11})$, that is,

$$\det\mathbf{A}_{1\times 1} = \det\mathbf{M}^{(1)} = a_{11}.$$

On the other hand, we have that

$$\det\mathbf{A}_{1\times 1} = \det\mathbf{M}^{(1)} = M_{11}^{(1)} = \frac{\triangle_{11}^{(1)}}{\triangle_{00}^{(0)}}, \tag{B23}$$

and applying Eq. (B15) one obtains that $\triangle_{11}^{(1)} = a_{11}$, which by comparison gives

$$\triangle_{00}^{(0)} = 1. \tag{B24}$$

Finally it is shown that

$$\det\mathbf{A}_{\mathbf{n}\times\mathbf{n}} = \det\mathbf{M}^{(1)} = \Delta_{nn}^{(n)} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}, \tag{B25}$$

a result that is evidently correct. In summary we can rewrite the matrix $\mathbf{M}^{(k)}$ in terms of subdeterminants of $\mathbf{M}^{(1)} = \mathbf{A}_{\mathbf{n}\times\mathbf{n}}$, that is,

$$\mathbf{M}^{(k)} = \frac{1}{\Delta_{(k-1)(k-1)}^{(k-1)}} \begin{pmatrix} 0 & \cdots & & & \cdots & 0 \\ \vdots & & & & & \vdots \\ & & 0 & \cdots & & 0 \\ & & \vdots & \Delta_{kk}^{(k)} & \cdots & \Delta_{kn}^{(k)} \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & 0 & \Delta_{nk}^{(k)} & \cdots & \Delta_{nn}^{(k)} \end{pmatrix}. \tag{B26}$$

Notice that the relation of the recursive matrix elements with the ratios of determinants provides the condition for evaluating the matrix $\mathbf{M}^{(k)}$. This is that the determinants $\Delta_{(k-1)(k-1)}^{(k-1)}$ (principal minors) be nonvanishing, a condition that is evident in identity (B26).

#### 4. Evaluation of determinants $\Delta_{ij}^{(l)}$ in terms of the matrix elements of $\mathbf{M}^{(k)}$

The relation that we found for the recursive matrix elements in terms of a ratio of determinants is given by the equation

$$M_{ij}^{(k+1)} = \frac{\Delta_{ij}^{(k+1)}}{\Delta_{kk}^{(k)}}. \tag{B27}$$

We can reorder this such that

$$\Delta_{ij}^{(k+1)} = \Delta_{kk}^{(k)} M_{ij}^{(k+1)}, \tag{B28}$$

where $\Delta_{kk}^{(k)}$ corresponds to a determinant called the principal minor of order $k \times k$, which can be expressed directly in terms of recursive matrix elements, such that

$$\Delta_{kk}^{(k)} = M_{11}^{(1)} \ldots M_{kk}^{(k)} \tag{B29}$$

and therefore we obtain the identity

$$\Delta_{ij}^{(k+1)} = M_{11}^{(1)} \ldots M_{kk}^{(k)} M_{ij}^{(k+1)},$$

which allows for the possibility of evaluating any subdeterminant of the matrix $\mathbf{M}^{(1)}$.


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