High energy QCD from Planckian scattering in AdS space and the Froissart bound

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We reanalyze high-energy QCD scattering regimes from scattering in cutoff AdS space via gravity-gauge dualities (a la Polchinski-Strassler). We look at 't Hooft scattering, Regge behavior, and black hole creation in AdS space. Black hole creation in the gravity dual is analyzed via gravitational shockwave collisions. We prove the saturation of the QCD Froissart unitarity bound, corresponding to the creation of black holes of AdS size, as suggested by Giddings.

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I. INTRODUCTION

Gravity-gauge dualities have been an important tool in getting information about nonperturbative Yang-Mills theories, since the original work of Maldacena [1] (see also [2]). But the contact with real high-energy experiments has been lacking, partly because of the absence of a gravity dual of QCD. However, in the work of Polchinski and Strassler [3] (and the later [4] dealing with deep inelastic scattering) it was shown that one can derive a lot of information from just a simple gravity dual model of AdS cutoff in the IR (where the mass gap is related to a modification of the geometry) and maybe in the UV; in other words the two-brane Randall-Sundrum (RS) model [5].

High-energy QCD scattering of colorless objects (e.g. glueballs) is thus related to scattering in the cutoff AdS space via convoluting the scattering amplitude with AdS wave functions. This simple model still allows one to get a lot of information about QCD at high energy, when the IR modifications due to the mass gap are not so important.

Giddings [6] took this proposal further and analyzed the scattering inside AdS space in the extreme inelastic case, when black holes are being formed. He analyzed the black hole formation using a simple model, devised in [7,8] (see also earlier work in [9]), the cross section for black hole formation being equated with the geometric area of the black hole horizon of mass equal to the center of mass energy [$\sigma = \pi r_H^2$, $r_H = r_H(\sqrt{s})$]. At moderate energies, the black holes being formed can be taken to be in flat space, but at higher energies, the size of AdS space becomes important. Giddings argued, by calculating the Newton-like potential (linearized Einstein gravity), that the horizon of a black hole formed on the RS IR brane will be such as to give a saturation of the Froissart unitarity bound for the cross section:

$$h_{00;\mathrm{lin}} \sim G_4\sqrt{s}\frac{e^{-M_1 r}}{r} \Rightarrow r_H \sim \frac{1}{M_1}\ln(G_4\sqrt{s}M_1) \Rightarrow \sigma \simeq \pi r_H^2 \sim \left(\frac{1}{M_1}\ln(\sqrt{s})\right)^2. \quad (1.1)$$

But there are a lot of uncertainties about this calculation. In a previous paper [10], we addressed some of these uncertainties by looking at black hole formation via shockwave collisions (a technique first proposed in [11]) and estimated the maximum impact parameter $b_{\max}$ for which a black hole is being formed and calculated the minimum mass of the formed black hole. The calculation was done in flat $d > 4$, as well as in the background of the one-brane RS model (''alternative to compactification'' [12]), as was appropriate for the various low M_{Pl} scenarios [5,13,14] in which one could detect black holes at accelerators. We have set up the general formalism in a way to be used for the AdS and two-brane RS calculations needed for the QCD dual scattering. We also analyzed the effect of string (α') corrections to the black hole creation. Quantum corrections were also analyzed in a different way in [15,16] and 't Hooft scattering inside AdS space was analyzed using different methods in [17]. The advantage in this semiclassical formalism is that now $\sigma = \pi b_{\max}^2$ is rigorous.

In this paper we will explore the consequences of the calculations in [10] for the high-energy QCD scattering, and we will revisit some of the previous results to see if we can gain more insight. We will try to use consistently the Polchinski-Strassler (PS) setup, in particular finding how to turn the classical scattering with black hole formation, happening at a certain point in the gravity dual, into an integration over the gravity dual. We will use a simple black disk model to turn the classical scattering into an imaginary elastic amplitude for which we can use the PS formalism. The most important piece of information learned in this paper will be that we are able to justify in a more rigorous way the appearance of the Froissart bound. Given the Polchinski-Strassler setup, we will calculate the maximum impact parameter being formed in the scattering of two shockwaves and thus get $\sigma = \pi b_{\max}^2$. The importance of this formalism is that the shockwaves are exact solutions (not linearized ones) giving an advantage over the horizon calculation in [6].

We should note here that we are not attempting to define a general amplitude (nor a S matrix) for a general scattering inside AdS space. We will see in the next two subsections that for our purposes it is enough to assume the existence of a nonperturbative amplitude (and corresponding cross section) when the external legs are situated in a flat 4D slice of AdS space. We will define a simple black disk model for this amplitude that mimics the (as yet unknown and undefined) nonperturbative amplitude by giving the same cross section. For external legs living in a flat 4D slice, one can certainly define amplitudes and cross sections in the usual manner.

The paper is organized as follows: In Sec. II we review the formalism of high-energy scattering in QCD from scattering in cutoff AdS space. In Sec. II A we show how to take a classical scattering in the gravity dual, characterized by a $b_{\max}(s)$, and turn it into a quantum amplitude, which we can relate to QCD. In Sec. III we describe the general formalism for calculating $b_{\max}(s)$ for black hole formation in the scattering of two Aichelburg-Sexl (AS) waves in a background of AdS type. In Sec. IV we apply this formalism for the case of scattering inside AdS space and on the IR brane in the two-brane RS model. Section V is the most important section in which we put together all the pieces of information and analyze the various QCD energy regimes and discuss the saturation of the Froissart bound as well as string corrections in the gravity dual. In Sec. VI we conclude, and in the Appendix we show the details of the calculation of the trapped surfaces being created when two AS shockwaves scatter inside AdS and on the IR brane.

II. HIGH-ENERGY QCD FROM ADS

Polchinski and Strassler [3] have found a simple model for relating high-energy QCD scattering with scattering inside AdS space.

For a conformal field theory, the corresponding near horizon ($r \rightarrow 0$) metric for the brane configuration is

$$ds^2 = \frac{r^2}{R^2} d\vec{x}^2 + \frac{R^2}{r^2} dr^2 + R^2 ds_X^2. \tag{2.1}$$

The global momentum $p_\mu = -i\partial_\mu$ (momentum for gauge theory scattering, for instance) is thus related to the local inertial momentum (of a local inertial observer in AdS space) by

$$\tilde{p}_\mu = \tfrac{R}{r} p_\mu (\tilde{p}_\mu \tilde{p}_\nu \eta^{\mu\nu} = p_\mu p_\nu g^{\mu\nu}). \tag{2.2}$$

Thus high energy is large r and low energy is small r. And then a nonconformal gauge theory like QCD will just be modified at small r (low energy), at high energy remaining conformal.

Since the low energy cutoff for the conformality of the gauge theory should be of the order of the mass of the lightest glueball state Λ, the cutoff on the AdS geometry is

$$r_{\min} \sim R^2 \Lambda. \tag{2.3}$$

Thus in theories with mass gap like QCD the warp factor becomes bounded, and the simplest effective description that we are going to use throughout the paper is just to cut off the integration over r at $r_{\min}$. We can also cut off the theory at high energy, and with UV and IR cutoffs we have the two-brane Randall-Sundrum model [5].

Corresponding to the string tension in AdS space there is also a gauge theory string tension $\hat{\alpha}' = (gN)^{-1/2}\Lambda^{-2}$ and

$$\sqrt{\alpha'}\tilde{p} = \sqrt{\hat{\alpha}'} p \frac{r_{\min}}{r} \rightarrow \sqrt{\alpha'}\tilde{p} \leq \sqrt{\hat{\alpha}'} p. \tag{2.4}$$

A glueball corresponds in AdS space to a state with wave function

$$e^{ipx}\psi(r, \Omega) \tag{2.5}$$

(plane wave in 4D and some wave function for r and X). We assume that scattering of gauge invariant states (e.g. glueballs) within Yang-Mills is equated by AdS-CFT with a scattering inside AdS space of the above states; moreover, assuming that the states scatter locally according to the flat space amplitude, we get

$$\mathcal{A}(p) = \int dr d^5\Omega \sqrt{g} \mathcal{A}_{\text{string}}(\tilde{p}) \prod_i \psi_i. \tag{2.6}$$

Then integrating over r corresponds to integrating over scattering energies in the local frame ($\tilde{p}$). In this picture α' is a constant, as in flat space string theory, but momenta ($\tilde{p}$) "run" as r is varied, as can be seen from (2.4). It is also clear from (2.4) that one can consider $\tilde{p}$ to be fixed and α' to run, but our interpretation makes it clear that we are integrating over string theory momenta. One should also note that although originally in the formula (2.6) $A_{\text{string}}(\tilde{p})$ was meant as a world sheet string amplitude (with vertex operator insertions) that in flat space gives the perturbative spacetime amplitude for scattering particle states (e.g. gravitons); we are now extending its meaning. We will first consider it at a nonperturbative level, thus we will assume that there is a result for the flat space amplitude at the nonperturbative level even if we do not know it. Indeed, we are interested in black hole production, for which we can calculate the cross section, but not the amplitude, thus in the next section we will model the nonperturbative amplitude with a simple black disk that reproduces the cross section. Second, the amplitude that we will extract has external legs defined only in a flat 4D slice of AdS space, not in a general direction in AdS space, which is however sufficient for our purposes of convolution with ψ_i according to (2.6).

Under the assumption that the local string scattering is dominated by the momenta of the order of the string scale, $1/\sqrt{\alpha'}$ (which we will shortly see that it is not as innocent as it looks), we get by the above (2.4) that

$$r_{\text{scatt}} \sim r_{\min}(\sqrt{\hat{\alpha}'} p), \tag{2.7}$$

and if $\sqrt{\hat{\alpha}'} p \gg 1$ (high-energy scattering in the gauge theory) we see that the integral will be concentrated at $r_{\text{scatt}} \gg r_{\min}$, where

$$\psi \sim Cf(r/r_{\min})g(\Omega) \sim C(r/r_{\min})^{-\Delta} g(\Omega). \tag{2.8}$$

Then

$$\mathcal{A}_{\rm QCD}(p) \sim \int drr^3 \bigg(\prod_i \Big(\frac{r_{\min}}{r}\Big)^{\Delta_i}\bigg) \mathcal{A}_{\rm string}(pR/r) \sim \Big(\frac{\Lambda}{p}\Big)^{\sum \Delta_i - 4}, \tag{2.9}$$

as in QCD, and this result was obtained only from conformal invariance. Moreover, for states with spin, we replace Δ with $\tau_i = \Delta_i - \sigma_i$ as in QCD. So the scaling with momenta comes from the large r asymptotics of the wave function, which itself comes from conformal invariance.

But the last scaling relation treats all momenta the same. As we will mostly be interested in the small angle regime, $s \gg t$, let us look what happens for the amplitudes as a function of s and t.

Given that $\mathcal{A}_{\rm string} = \mathcal{A}_{\rm string}(\alpha' \tilde{s}, \alpha' \tilde{t})$ we take $\nu = \alpha' |\tilde{t}| = -\alpha' \tilde{t}$ as integration variable ($r = \nu^{-1/2} r_{\min} \sqrt{\hat{\alpha}' t}$) and since $s/t = \tilde{s}/\tilde{t}$, we have in the new variable ($\Delta \equiv \sum_i \Delta_i$)

$$\mathcal{A}_{\rm QCD} \sim \frac{K}{(\hat{\alpha}'|t|)^{\Delta/2-2}} \int_0^{\nu_{\max}} d\nu \nu^{\Delta/2-3} \mathcal{A}_{\rm string}(\nu s/|t|, \nu), \quad K = \frac{R^{10}\Lambda^4}{2}\Big(\prod_i C_i\Big) \int \sqrt{g_{(5)}} \prod_i g_i(\Omega). \tag{2.10}$$

Note that the main assumption in deriving this was the large r AdS behavior of the wave function (2.8). We will have to remember this as a caveat of the formula, but we will continue nevertheless to use it—it being the simplest model we can have of an AdS-QCD relation.

In this Polchinski-Strassler form for the QCD amplitude, $\nu_{\max} = \hat{\alpha}' t$ and the amplitude to be integrated over is expressed as $\mathcal{A}(\alpha' \tilde{s}, \alpha' \tilde{t})$.

Let us observe how a few possible behaviors of the string amplitude translate into QCD amplitudes, for future use. We will see in the next subsection that these types of amplitudes (power laws plus logarithms) appear from various possible classical cross sections for scattering in the gravity dual, characterized by $b_{\max}(s)$. The various $b_{\max}(s)$ of relevance are analyzed together in Sec. V,

$$\begin{aligned}
&\mathcal{A} \simeq A_1 s^\alpha t^\beta \Rightarrow \mathcal{A}_{\rm QCD} \simeq K A_1 s^\alpha t^\beta \Big(\frac{\hat{\alpha}'}{\alpha'}\Big)^{\alpha+\beta} \frac{1}{\Delta/2 - 2 + \alpha + \beta},\\
&\mathcal{A} \simeq A_2 s^\alpha t^\beta \log(\alpha' t) \quad \Rightarrow \quad \mathcal{A}_{\rm QCD} \simeq K A_2 s^\alpha t^\beta \Big(\frac{\hat{\alpha}'}{\alpha'}\Big)^{\alpha+\beta} \frac{1}{\Delta/2 - 2 + \alpha + \beta}\Big(\log(\hat{\alpha}' t) - \frac{1}{\Delta/2 - 2 + \alpha + \beta}\Big),\\
&\mathcal{A} \simeq A_3 \log(\alpha' s) t^\beta \quad \Rightarrow \quad \mathcal{A}_{\rm QCD} \simeq K A_3 \frac{t^\beta}{\Delta/2 - 2 + \beta}\Big(\frac{\hat{\alpha}'}{\alpha'}\Big)^{\beta}\Big(\log(\hat{\alpha}' s) - \frac{1}{\Delta/2 - 2 + \beta}\Big).
\end{aligned} \tag{2.11}$$

Giddings [6] points out that as one increases the energy of the gauge theory scattering, by (2.4) one increases also the relevant energy in string theory. In (2.4), we have seen that corresponding to the string scale $1/\sqrt{\alpha'}$ there is a gauge theory scale $1/\sqrt{\hat{\alpha}'}$. But there are three further (higher) energy scales (in the case when the string coupling g_s is small but $g_s N$ is large).

The first is the Planck scale

$$M_P \sim g_s^{-1/4}\big(1/\sqrt{\alpha'}\big) = \frac{N^{1/4}}{R} = N^{1/4}\sqrt{\frac{\Lambda}{r_{\min}}}. \tag{2.12}$$

Note that we can rescale 4D coordinates such that $r_{\min} = R$, which, since $\hat{\alpha}' = \Lambda^{-2}(g_s N)^{-1/2}$, translates also into $\hat{\alpha}' = \alpha'$. Then $M_P = N^{1/4}\Lambda$.

In any case, the Planck scale corresponds in gauge theory to

$$\hat{M}_P = g_s^{-1/4}/\sqrt{\hat{\alpha}'} = N^{1/4}\Lambda, \tag{2.13}$$

which is the scale at which (real) black holes start to form.

Giddings [6] proposes that afterwards the black hole production cross section is approximated by $\sigma \simeq \pi r_H^2 \sim E^{2/(d-3)}$ ($E^{2/7}$ if we have approximately 10D flat space). As we can see, this is based on the simple geometrical picture of a static black hole at a given point in AdS space, with the cross section equaling its horizon area. We will try to see whether this picture is valid.

The second scale is the string correspondence principle crossover scale, at which one has to stop talking about string intermediate states and instead use black hole virtual states. That scale is

$$E_c \sim g_s^{-2}/\sqrt{\alpha'} = g_s^{-7/4} M_P = N^{7/4} M_P/(g_s N)^{7/4} = N^2 \sqrt{\frac{\Lambda}{r_{\min}}} \frac{1}{(g_s N)^{7/4}}, \tag{2.14}$$

or in the gauge theory

$$\hat{E}_c \sim \frac{N^2 \Lambda}{(g_s N)^{7/4}}. \tag{2.15}$$

The third energy regime is the most interesting, attained when the size of the black hole r_H that was created reaches the AdS space size R:

$$E \sim M_P^8 r_H^7 (M_P^{d-2} r_H^{d-3}) \rightarrow E_R = M_P^8 R^7, \tag{2.16}$$

or in the gauge theory,

$$\hat{E}_R = N^2\Lambda. \tag{2.17}$$

At that energy, the behavior of r_H with E changes from (2.16) and then so does the cross section. The proposal of [6] is that after this scale we have the onset of the Froissart bound in the gauge theory.

Let us now note that at least after the onset of the Froissart behavior we have black holes being created with Schwarzschild radius greater than the size of AdS space, so one might ask how come we are still using the Polchinski-Strassler formula (2.6) which implies some locality for the scattering in the extra dimensions? In fact, in this regime the PS formula becomes less and less relevant, since the effective scattering region (where the PS integral is concentrated) is small and close to the IR cutoff (as we will see in the following), and the size of the black hole is larger than it. So in this limit, the scattering inside AdS space becomes more and more classical (the fluctuations due to the PS integral become less than the size of the black hole), and the classical cross section calculations in this paper become more directly relevant.

A. Black disk calculations

In the following calculations, we will analyze a classical scattering with black hole creation, out of which we get a classical value for a $b_{\max}(s)$, and correspondingly a cross section for black hole creation, $\sigma = \pi b_{\max}^2$. But in order to use the Polchinski-Strassler formalism and relate it to QCD, we must find a quantum amplitude generating the same AdS cross section. So we will study first how we get a quantum amplitude out of the classical picture.

Ideally, one should do a quantum calculation for the amplitude of the AdS scattering, not just for the cross section (for which we were able to use the formalism of AS shockwave scattering described in the next section) but that would require knowledge of the full nonperturbative quantum gravity. As we do not have that, we will try to make a model for the amplitude that reproduces the classical cross section calculation, $\sigma = \pi b_{\max}^2(s)$.

The simplest thing one can do is to create an eikonal that corresponds to a black disk, that is

$$\begin{aligned} &\mathrm{Re}(\delta(b,s)) = 0, \qquad \mathrm{Im}(\delta(b,s)) = 0, \qquad b > b_{\max}(s);\\ &\mathrm{Im}(\delta(b,s)) = \infty, \qquad b < b_{\max}. \end{aligned} \tag{2.18}$$

Then the eikonal amplitude at a fixed flat 4D slice inside AdS space that reproduces the classical $b_{\max}(s)$ for scattering constrained to lie in the slice in s, t variables is

$$\begin{aligned} \frac{1}{s}\mathcal{A}(s,t) &= -i\int d^2b e^{i\vec{q}\,\vec{b}}(e^{i\delta}-1)\\ &= i\int_0^{b_{\max}(s)} bdb\int_0^{2\pi} d\theta e^{iqb\cos\theta}\\ &= 2\pi i\frac{b_{\max}(s)}{\sqrt{t}}J_1(\sqrt{t}b_{\max}(s)). \end{aligned} \tag{2.19}$$

We will take the amplitude (2.19) to be the amplitude for scattering inside AdS space, parallel to a fixed 4D slice. Here $b_{\max}(s)$ will be determined from black hole production in the scattering of two Aichelburg-Sexl shockwaves inside the curved AdS background.

In general, the notion of scattering inside AdS space and the associated notions of S matrix and in and out states are hard (if not impossible) to define. In particular, the original definition of the AdS-CFT correspondence states that a set of good quantities that can be related to the boundary CFT are the corellators with legs on the AdS boundary, not S matrices. There were several attempts to define S matrices in AdS-CFT by taking the flat space limit of AdS, e.g. [18–20], but none was entirely satisfactory. What we are proposing here is less radical. First of all, unlike these other cases, we are not dealing with global AdS, and we are not relating in and out states defined at the boundary of AdS space (which would be in the Poincare patch at $r = 0$ and $r = \infty$). Rather, we are looking at scattering amplitudes that happen in a 4D slice, at fixed r. Second, when we talk about cross sections we refer to 4D quantities, inside the 4D slice. We are not attempting to define a 5D cross section for scattering inside AdS space. Here the fifth coordinate of AdS space, r, has just the usual role of energy scale (according to the usual UV-IR relation in AdS-CFT), as can be easily seen in (2.6).

We note that if $\sqrt{t}b_{\max}(s) \ll 1$, the result becomes $\pi b_{\max}^2$, so if we take the imaginary part of the forward ($t = 0$) scattering amplitude we get

$$\begin{aligned} \frac{1}{s}\,\mathrm{Im}\mathcal{A}_{\mathrm{elastic}}(k_1,k_2\to k_1,k_2) &= \sigma_{\mathrm{tot}}(k_1,k_2\to \text{anything})\\ &= \pi b_{\max}^2. \end{aligned} \tag{2.20}$$

But we still need to integrate the amplitude over the AdS slice, using the PS formula.

We use the Polchinski-Strassler formula (2.10), i.e.

$$\mathcal{A}_{\mathrm{QCD}} = \frac{K}{(\hat{\alpha}'t)^{\Delta/2-2}}\int_0^{\nu_{\max}=\hat{\alpha}'t} d\nu\nu^{\Delta/2-3}\mathcal{A}\left(\nu\frac{s}{|t|},\nu\right).$$

Upon inserting the eikonal amplitude (2.19) into (2.10), we get

$$\begin{aligned} \mathcal{A}_{\mathrm{QCD}} = \frac{s}{|t|}\frac{\sqrt{\alpha'}2\pi iK}{(\hat{\alpha}'t)^{\Delta/2-2}}\int_0^{\nu_{\max}=\hat{\alpha}'t} d\nu\nu^{(\Delta-5)/2}b_{\max}\\ \times\left(\frac{\nu s}{\alpha'|t|}\right)J_1\left(\sqrt{\frac{\nu}{\alpha'}}b_{\max}\left(\frac{\nu s}{\alpha'|t|}\right)\right). \end{aligned} \tag{2.21}$$

Let us now look at particular cases that will be of interest later on. In all the three cases we will study, the result for $\mathcal{A}_{\mathrm{QCD}}$ will be of the type mass dimension -2 constant (Ka_1, $Ka_2^{2/(1+2\beta)}$, Ka_3^2) $\times$ function of ($\hat{\alpha}'s$, $\hat{\alpha}'t$, Δ, and c). Here c stands for a dimensionless number calculable in the gravity dual (a_1/α', $a_2^{2/(1+2\beta)}/\alpha'$, a_3^2/α'). Here a_1, a_2, a_3 are calculable in the gravity dual and K encodes the details of the transition to

QCD, i.e. it depends heavily of the details of the IR cutoff of the gravity dual, as can be seen from its expression in (2.10).

1. Approximately 4D case: $b_{max}(s) = a_1\sqrt{s}$

Using

$$I(a, b) = \int_0^1 x^b J_1(ax)dx = \frac{a_1F_2(1 + b/2; 2, 2 + b/2; -a^2/4)}{4 + 2b}, \quad (2.22)$$

we get

$$\mathcal{A}_{\rm QCD} = -\frac{2\pi i a_1^2 K}{\Delta}\frac{\hat{\alpha}'^2 s^2}{\alpha'} F_2\Big(\frac{\Delta}{4}; 2, \frac{\Delta+4}{4}, -\frac{a_1^2\hat{\alpha}'^2 st}{4\alpha'^2}\Big) = K a_1 \times {\rm fct}\Big(\frac{a_1}{\alpha'}, \hat{\alpha}'s, \hat{\alpha}'t, \Delta\Big). \quad (2.23)$$

To evaluate the hypergeometric function at large argument (variable), we use that

$$J_1(ax) \simeq -\sqrt{\frac{2}{\pi a x}}\cos(ax + \pi/4), \quad (2.24)$$

and then

$$I(a, b) \simeq \frac{1}{\sqrt{\pi a}}\frac{1}{2b+1}[{}_1F_1(b + 1/2; b + 3/2; ia)(1 + i) - {}_1F_1(b + 1/2; b + 3/2; -ia)(1 - i)] \simeq \frac{2\sqrt{2}}{\sqrt{\pi a}}\frac{\Gamma(b + 3/2)}{2b + 1}\cos(b\pi/2)a^{-(b+1/2)}. \quad (2.25)$$

We get

$$\mathcal{A}_{\rm QCD} \simeq ik\hat{\alpha}'t\Big(\frac{\hat{\alpha}'t}{\alpha'}\Big)^{-[(\Delta-2)/2]}\Big(\frac{s}{t}\Big)^{(8-\Delta)/4} = ik\Big(\frac{\hat{\alpha}'}{\alpha'}\Big)^{-[(\Delta-2)/2]} s^{(8-\Delta)/4}t^{-(\Delta/4)}\hat{\alpha}',$$

$$k = 2\sqrt{2\pi}\Gamma\Big(\frac{\Delta-3}{2}\Big)a_1^{(4-\Delta)/2}\cos\Big(\frac{\pi(\Delta-4)}{4}\Big) \quad (2.26)$$

in the limit $\frac{a_1^2}{\alpha'^2}\hat{\alpha}'^2 st \gg 1$. In the opposite limit, $\frac{a_1^2}{\alpha'^2}\hat{\alpha}'^2 st \ll 1$, we get

$$\mathcal{A}_{\rm QCD} \simeq -\frac{2\pi i a_1^2 K}{\Delta}\frac{\hat{\alpha}'^2 s^2}{\alpha'}. \quad (2.27)$$

2. Flat d dimensional case: $b_{max}(s) = a_2 s^{1/[2(d-3)]} = a_2 s^{\beta}$

Using the same formulas, we get

$$\mathcal{A}_{\rm QCD} = \frac{2\pi i a_2^2 K}{4\beta + \Delta - 2}\Big(\frac{\hat{\alpha}'s}{\alpha'}\Big)^{2\beta}\hat{\alpha}'s\,{}_1F_2\Big(1 + \frac{\Delta-4}{2(2\beta+1)}; 2, 2 + \frac{\Delta-4}{2(2\beta+1)}; -\frac{a_2^2}{4}\frac{\hat{\alpha}'t}{\alpha'}\Big(\frac{\hat{\alpha}'s}{\alpha'}\Big)^{2\beta}\Big) = K a_2^{2/(1+2\beta)} \times {\rm fct}\Big(\frac{a^{2/(1+2\beta)}}{\alpha'}, \hat{\alpha}'s, \hat{\alpha}'t, \Delta\Big). \quad (2.28)$$

It has the limit

$$\mathcal{A}_{\rm QCD} \simeq \bar{K}\Big(\frac{\hat{\alpha}'s}{\alpha'}\Big)^{-[\beta(\Delta-4)]/(2\beta+1)} \times \Big(\frac{\hat{\alpha}'t}{\alpha'}\Big)^{(2-\Delta-4\beta)/[2(2\beta+1)]}\hat{\alpha}'s,$$

$$\bar{K} = iK\frac{2\sqrt{2\pi}}{\beta + 1/2}\frac{\Gamma(\frac{\Delta-4}{2\beta+1} + \frac{1}{2})}{a_2^{(\Delta-4)/(2\beta+1)}}\cos\Big[\frac{\pi}{2}\Big(\frac{\Delta-4}{2\beta+1}\Big)\Big],$$

$$a_2^2\frac{\hat{\alpha}'t}{\alpha'}\Big(\frac{\hat{\alpha}'s}{\alpha'}\Big)^{2\beta} \gg 1. \quad (2.29)$$

If $d = 5$, so $\beta = 1/4$, we get

$$\mathcal{A}_{\rm QCD} \sim k' s^{-[(\Delta-4)]/6}t^{(1-\Delta)/3}\hat{\alpha}'s. \quad (2.30)$$

Another limit is

$$\mathcal{A}_{\rm QCD} \simeq \frac{2\pi i a_2^2 K}{4\beta + \Delta - 2}\Big(\frac{\hat{\alpha}'s}{\alpha'}\Big)^{2\beta}\hat{\alpha}'s, \quad \frac{\hat{\alpha}'t}{\alpha'}\Big(\frac{\hat{\alpha}'s}{\alpha'}\Big)^{2\beta} \ll 1. \quad (2.31)$$

3. Log behavior: $b_{max}(s) = a_3\log(s)$

The log behavior $b_{\max}(s) = a_3\log(s)$ would correspond to the onset of the Froissart bound, at least in the calculation in [6]. We get

$$\mathcal{A}_{\rm QCD} = \sqrt{\alpha'}(\hat{\alpha}'s)\frac{2\pi i a_3 K}{\sqrt{\hat{\alpha}'t}} \times \int_0^1 dy y^{(\Delta-5)/2}[lny + A]J_1[B\sqrt{y}(lny + A)] = K a_3^2 \times {\rm fct}\Big(\frac{a_3^2}{\alpha'}, \hat{\alpha}'s, \hat{\alpha}'t, \Delta\Big), \quad (2.32)$$

$$A = \ln\Big(\frac{\hat{\alpha}'s}{\alpha'}\Big), \qquad B = a_3\sqrt{\frac{\hat{\alpha}'t}{\alpha'}},$$

that unfortunately cannot be algebraically solved exactly.

But in the case of A large, $c = AB$ large, we can do the integral, using the large argument expansion of the Bessel function, and using

$$\int_0^1 dx x^b \ln x(\cos cx - \sin cx) \simeq -\frac{1}{c}\int_0^1 dx x^{b-1}(\sin cx - \cos cx) \simeq -\frac{\Gamma(b)\sqrt{2}}{c^{b+1}}\cos\left(\frac{\pi b}{2}-\frac{\pi}{4}\right), \quad (2.33)$$

and we get

$$\mathcal{A}_{\rm QCD} \simeq 4\sqrt{2\pi} i K\Gamma(\Delta - 9/2)\frac{(a_3\sqrt{\frac{\hat{\alpha}' t}{\alpha'}}\ln(\frac{\hat{\alpha}' s}{\alpha'}))^{4-\Delta}}{\hat{\alpha}' t/\alpha'} \times\left[(\Delta - 9/2)\cos\left(\frac{\pi}{2}(\Delta-3)\right) + 2\frac{\cos(\frac{\pi}{2}(\Delta-5))}{\ln(\frac{\hat{\alpha}' s}{\alpha'})}\right]. \quad (2.34)$$

Also for $A \gg 1$, but $c = AB \ll 1$, we can expand the Bessel function for small argument and do the integration to obtain

$$\mathcal{A}_{\rm QCD} \simeq \hat{\alpha}' s 2\pi i a_3^2 K\left[\frac{(\ln(\frac{\hat{\alpha}' s}{\alpha'}))^2}{\Delta - 2} - \frac{2}{(\Delta-2)^2}\ln\left(\frac{\hat{\alpha}' s}{\alpha'}\right)\right]. \quad (2.35)$$

Note that for all three cases, when we have used the small argument expansion of the Bessel function, we have integrated $\pi b_{\max}(s)^2$ with the PS formula. Correspondingly, in the last case we studied, we obtain $\sigma \sim a_3^2(\ln(s))^2$, as expected.

We should also note that within this section we have used the simplest model of black disk for turning the classical scattering into a quantum scattering amplitude, but for the last behavior (the Froissart bound) there are other choices. For instance, in [21], there are given a few forms for the eikonal which when integrated give the Froissart behavior for the cross section. One of them is of possible relevance here. The eikonal

$$\mathrm{Im}\,\delta(b,s) = \frac{\pi\lambda}{B(s)} s^{\bar{\Delta}} \exp\left\{-\frac{b^2}{B(s)}\right\} \quad (2.36)$$

contains a parameter analogous to the classical maximum impact parameter $b_{\max}(s)$ we have used:

$$B(s) = 2a\ln s + c, \quad (2.37)$$

where however a and $d = \bar{\Delta} + 1$ are related to the Pomeron trajectory $\alpha_P(t) = d + a\,t$. Integrating this eikonal also gives the Froissart behavior for the total cross section

$$\sigma_t(s) \simeq 4\pi a\bar{\Delta}\ln^2 s + o(\ln s). \quad (2.38)$$

We will restrict ourselves to the simplest black disk eikonal as we did in this section, as there is no physical argument on why to choose a particular *nontrivial* eikonal [in the full nonperturbative quantum gravity one should be able to calculate the actual nontrivial eikonal, of course, but in practice there seems to be no reason to prefer a nontrivial phenomenological eikonal like the one in (2.36) over another].

III. BLACK HOLE CREATION IN HIGH-ENERGY SCATTERING; GENERAL FORMALISM

As we saw in the previous section, at sufficiently high energies in the gauge theory, in the gravity dual we will have an inelastic scattering with black hole creation. So we have to be able to analyze the black hole creation in a general background.

In [10] we have extended the formalism in [11] for calculating black hole creation cross sections via analyzing the scattering of two Aichelburg-Sexl shockwaves inside a curved background of AdS type. Here we review the formalism in order to apply it in the next section.

The Aichelburg-Sexl shockwave [22]

$$ds^2 = -dx^+dx^- + (dx^+)^2\delta(x^+)\Phi(x^i) + \sum_{i=1}^{d-2}(dx_i)^2 \quad (3.1)$$

has as a source a massless particle of momentum p ("photon"), with

$$T_{++} = p\delta^{d-2}(x^i)\delta(x^+). \quad (3.2)$$

In flat space, the Einstein equation $R_{++} = 8\pi G T_{++}$ implies

$$\partial_i^2\Phi(x^i) = -16\pi G p\delta^{d-2}(x^i) \quad (3.3)$$

(Φ is harmonic with source).

't Hooft [23,24] has argued that one can describe the scattering of two massless particles at energies close to (but under) the Planck scale ($m_{1,2} \ll M_P, Gs \sim 1$, yet $Gs < 1$) as follows. Particle two creates a massless shockwave of momentum $p_\mu^{(2)}$ and particle one follows a massless geodesic in that metric. He has shown that the S matrix corresponds to a gravitational Rutherford scattering (single graviton being exchanged).

At higher energies ($Gs \gg 1$), one has to consider that both massless particles create AS shockwaves, and this nonlinear process is hard to compute. At most one can compute the metric perturbation away from the interaction point as in [25]. But one can use a formalism to give a lower bound on the size of the black hole being created, and estimate the maximum impact parameter that forms a black hole.

We will be interested in scattering that occurs in a gravitational background of AdS type, maybe with a string-corrected source. So we will use a general Φ and general dimensionality, and a background of the type (the notation is for the one-brane RS model, but it is easy to generalize for the AdS and two-brane RS model cases)

$$ds^2 = e^{-2|y|/l}[-dudv + dx_i^2] + dy^2. \quad (3.4)$$

Let us denote $e^{-2|y|/l} = A$ and let $g_{ij} = A\bar{g}_{ij}$ for the metric in both x and y coordinates (transverse).

We will analyze the collision of two AS waves in this background, one moving in the x^+ (u) and one moving in the x^- (v) direction. The metric in the collision region cannot be calculated exactly even in flat space, but there is an alternative for checking for the presence of a black hole in the future of the collision.

Because of a suggestion made originally by Penrose, one can find a trapped surface at the interaction point $u = v = 0$, that is, a closed $D - 2$ dimensional surface the outer normals of which (in both future-oriented directions) have zero convergence. By a general relativity theorem, we know that there will be a horizon forming outside the trapped surface, therefore of area at least as big as the trapped surface area.

The metric with a single AS wave moving in the $\bar{v}$ direction (at $u = 0$) in the given background is (see [10] for more details)

$$ds^2 = e^{-2|\bar{y}|/l}(-d\bar{u}d\bar{v} + d\vec{x}^2 + \Phi(\vec{x}, y)\delta(\bar{u})d\bar{u}^2) + d\bar{y}^2. \tag{3.5}$$

It is useful to perform a transformation of coordinates to eliminate the delta function singularity in the metric. One finds the transformation

$$\begin{aligned}
\bar{u} &= u,\\
\bar{v} &= v + \Phi\theta(u) + \frac{u\theta(u)}{4}(\partial_i\Phi\partial_j\Phi\bar{g}^{ij} + A(\partial_y\Phi)^2),\\
\bar{x}^i &= x^i + \frac{u\theta(u)}{2}\bar{g}^{ij}\partial_j\Phi,\\
\bar{y} &= y + \frac{u\theta(u)}{2}A\partial_y\Phi
\end{aligned} \tag{3.6}$$

giving

$$\begin{aligned}
ds^2 = {}& A[-dudv + dx_i^2 + u\theta(u)\partial_i\partial_j\Phi dx^i dx^j]\\
&+ dy^2[1 + u\theta(u)A\partial_y^2\Phi] + dydx^i u\theta(u)A\partial_i\partial_y\Phi\\
&+ dydAu\theta(u)\partial_y\Phi + o(u^2),
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
A = e^{-2|\bar{y}|/l} + o(u)^2 \quad &\Rightarrow \quad A|_{u=0} = e^{-2|y|/l};\\
dA|_{u=0} &= -\frac{2}{l}A\left[dy + \frac{A}{2}\partial_y\Phi du\right].
\end{aligned} \tag{3.8}$$

When taking two AS waves, one in $\bar{u}$ and one in $\bar{v}$, the metric for $\bar{u} \le 0$, $\bar{v} \le 0$ (before the collision) contains the linear superposition of the two waves, i.e. $\Phi_1\delta(\bar{u})d\bar{u}^2 + \Phi_2\delta(\bar{v})d\bar{v}^2$, and a trapped surface will form at $\bar{u} = \bar{v} = 0$. The trapped surface in the new coordinates (u, v) is composed of two "disks" S_1 and S_2 glued on to their common boundary C, namely,

"disk" 1: $\{v = -\Psi_1(\vec{x}), u = 0\}$, ($\Psi_1 = 0$ on C);

disk 2: $\{u = -\Psi_2(\vec{x}), v = 0\}$ ($\Psi_2 = 0$ on C).

The null geodesics through the first disk, $\{v = -\Psi(\vec{x}), u = 0\}$, are defined by the tangent vector

$$\xi = \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} + \dot{x}^i\partial_i. \tag{3.9}$$

One finds that (by calculating $\dot{u}, \dot{v}, \dot{x}$ and then lowering indices)

$$\xi = -\frac{1}{4}\bar{g}^{ij}\partial_i\Psi\partial_j\Psi du - dv - \partial_i\Psi dx^i, \tag{3.10}$$

and therefore, the convergence of the null normals defined by ξ, $\theta = g^{ij}D_i\xi_j$ is (see [10] for more details)

$$\theta = -\nabla^2(\Psi_1 - \Phi_1), \tag{3.11}$$

and similarly for the second surface, and

$$\nabla^2 = \frac{1}{A}\nabla_x^2 + \partial_y^2 - \frac{d}{l}sgn(y)\partial_y. \tag{3.12}$$

Here Φ_1, Φ_2 are the profiles of the two waves, whose centers are separated by the impact parameter $\vec{b} = \vec{x}_1 - \vec{x}_2$.

At $b = 0$ and in flat d dimensions ($l \to \infty$) one can choose $\Psi_i = \Phi_i$, and then as we can easily see, the trapped surface $S = S_1US_2$ corresponds as advocated to the interaction point $\bar{u} = \bar{v} = 0$.

The common boundary C is defined then by $\Psi = \Phi = c$ (constant) or rather, one redefines $\Psi \to \Psi - c$ to have $\Psi = 0$ on the boundary. We will keep the constant for convenience, but remember that we should subtract it at the end. The continuity condition for the normal geodesics ξ along C gives then

$$\bar{g}^{ij}\partial_i\Psi\partial_j\Psi = 4 \quad \Rightarrow \quad (\nabla\Psi)^2 + A(\partial_y\Psi)^2 = 4. \tag{3.13}$$

At $b = 0$ and in flat d dimensions, the $\Psi = \Phi = c$ and the continuity condition are compatible with the boundary C being a circle ($r =$ const), and then the continuity condition fixes the radius of the circle.

In a curved background we cannot choose $\Psi = \Phi$ anymore, and the shape of C is fixed by requiring that the two equations are compatible.

Therefore, we write for the trapped surface (surface of zero convergence, $\theta = 0$)

$$\Psi = \Phi + \zeta \Rightarrow \nabla^2\zeta = 0. \tag{3.14}$$

Now the trapped surface $f(r, y) = 0$ is defined by both $\Psi = C$ (const) and by $\bar{g}^{ij}\partial_i\Psi\partial_j\Psi = 4$.

We will find the shape of the surface perturbatively in y, away from the flat 4 dimensions at $y = 0$. We expand ζ near $y = 0$ as

$$\zeta = \zeta_0(r) + \zeta_1(r)y + \frac{y^2}{2}\zeta_2(r). \tag{3.15}$$

Then at $y = 0$ $\nabla^2\zeta = 0$ implies

$$\partial_x^2\zeta_0(r) + \zeta_2(r) - \frac{d}{l}\zeta_1(r) = 0, \tag{3.16}$$

and we do not want to upset the flat space solution, so we will take $\zeta_0 = 0$. Then

$$\Psi = f + ay + \frac{y^2}{2}g + \ldots, \tag{3.17}$$

where $f = \Phi_{|y=0}$, $a = \partial_y \Phi|_{y=0} + \zeta_1$, and $g = \partial_y^2 \Phi|_{y=0} + \frac{d}{l}\zeta_1$.

And one has to match the equations

$$\Psi = C = f + ay + \frac{y^2}{2} g + \ldots, \tag{3.18}$$

and

$$4 = \left(f' + ya' + \frac{y^2}{2} g' + \ldots\right)^2 + \left(1 - \frac{2y}{l} + 2\frac{y^2}{l^2} + \ldots\right)(a + yg + \ldots)^2. \tag{3.19}$$

At nonzero impact parameter b, there are two distinct Φ_1 and Φ_2 (with different centers), and a single curve C, so we cannot take $\Psi_i = \Phi_i$ even in flat d dimensions. The problem of finding C together with the functions Ψ_i is complicated, but we have found instead an approximation scheme. The continuity condition is now $\nabla\Psi_1 \cdot \nabla\Psi_2 = 4$, but we approximate the size of C by putting $\Psi_i = \Psi(\vec{x} - \vec{x}_i)$, with Ψ being the $b = 0$ value. This gives the continuity equation

$$4 = \left(1 - \frac{b^2}{2\rho_c^2}\right)\left[\left(f' + ya' + \frac{y^2}{2} g' + \ldots\right)^2 + \left(1 - \frac{2y}{l} + 2\frac{y^2}{l^2} + \ldots\right)(a + yg + \ldots)^2\right]\Big|_{r=\rho_c}. \tag{3.20}$$

Now one can say that the area of the real trapped surface satisfies

$$S \geq b\sqrt{\rho_c^2 - \frac{b^2}{4}}, \tag{3.21}$$

and more importantly we can estimate a maximum b for which a trapped surface forms.

IV. BLACK HOLE CREATION INSIDE ADS AND ON THE RS IR BRANE

In this section we will apply the formalism of the previous section to the case of AS shockwaves inside AdS and on the IR brane in the two-brane RS model. The details of the calculation of the trapped surface are found in the Appendix, so here we will only show the general features.

Note that the application of the general formalism for scattering on the UV brane of the RS model (as well as for large extra dimensions) and thus for possible applications to accelerators was done in [10]. Here we concentrate on the cases relevant for QCD, namely, (as we will discuss in detail in the next section) scattering inside AdS space and on the IR brane.

We have already derived the form of the AS wave inside AdS space in [10]. The function Φ for an AS wave centered around $r = 0$, $y = y_0$ is

$$\begin{aligned}\Phi(r, y) &= \frac{8G_{d+1}l}{(2\pi)^{(d-4)/2}} p \frac{e^{dy/2l} e^{[(4-d)/2l]y_0}}{r^{(d-4)/2}} \int_0^\infty dq q^{(d-2)/2} J_{(d-4)/2}(qr) K_{d/2}(e^{y/l}lq) I_{d/2}(e^{y_0/l}lq) y > y_0 \\ &= \frac{8G_{d+1}l}{(2\pi)^{(d-4)/2}} p \frac{e^{dy/2l} e^{[(4-d)/2l]y_0}}{r^{(d-4)/2}} \int_0^\infty dq q^{(d-2)/2} J_{(d-4)/2}(qr) I_{d/2}(e^{y/l}lq) K_{d/2}(e^{y_0/l}lq) y < y_0.\end{aligned} \tag{4.1}$$

Let us therefore first derive the AS wave solution inside the two-brane RS metric (the one used by Giddings in his calculation of the Froissart bound) with the AS situated on the IR brane. It will be different from the Emparan [26] solution for the AS wave on the brane in the one-brane RS. We will use the same formalism used for the one-brane RS case by Emparan and for AdS space by [10]. We would still use the AdS metric

$$ds^2 = e^{-2y/l} d\vec{x}^2 + dy^2, \tag{4.2}$$

as in [6], but then $y \in (-\infty, 0)$ and the IR brane is located at $y = 0$ and the UV brane at $y = -|y_{UV}|$. The metric would be valid only in $(-|y_{UV}|, 0)$. For a general metric satisfying the required boundary conditions we will take

$$ds^2 = e^{2|y|/l} d\vec{x}^2 + dy^2. \tag{4.3}$$

It matches with the above for its domain of validity (y negative).

In complete analogy with the calculation in [10,26], we obtain an equation for $h(q, y) = \int d^{d-2}\vec{x} e^{iq\cdot x} \Phi(x, y)$ which is

$$h(q, y)'' + \frac{d}{l} sgn(y) h(q, y)' - q^2 e^{-2|y|/l} h(q, y) = 0, \tag{4.4}$$

with solution

$$A e^{-(d|y|)/2l} (I_{d/2} \text{ or } K_{d/2})(e^{-|y|/l} lq), \tag{4.5}$$

and imposing normalizable behavior at $y = \pm\infty$ (at the UV brane, if that is moved to infinity) we restrict to $I_{d/2}$. Then imposing the jump condition at $y = 0$ (IR brane), that is putting the source of the wave on the IR brane as in [6], we finally get

$$\begin{aligned}\Phi(r, y) &= \frac{4G_{d+1}p}{2\pi} e^{-[(d|y|)/2l]} \int \frac{d^{d-2}\vec{q}}{(2\pi)^{d-2}} e^{i\vec{q}\,\vec{x}} \frac{I_{d/2}(e^{-|y|/l} lq)}{q I_{d/2-1}(lq)} \\ &= \frac{4G_{d+1}p}{(2\pi)^{\frac{d-4}{2}}} \frac{e^{-[(d|y|)/2l]}}{r^{(d-4)/2}} \int_0^\infty dq q^{(d-4)/2} J_{(d-4)/2}(qr) \\ &\quad \times \frac{I_{d/2}(e^{-|y|/l} lq)}{I_{d/2-1}(lq)}\end{aligned} \tag{4.6}$$

(in $d = 4$, $4G_{d+1}p = R_s l$). By comparison, Giddings has a h_{00} (Newton potential) obtained also from a wave equation with sources on the IR brane [6], except his source was a static mass (black hole), whereas for us it is a photon

$$h_{00} \sim e^{dy/2l} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} e^{i\vec{p}\,\vec{x}} \frac{J_{d/2}(iple^{y/l})}{ipJ_{d/2-1}(ipl)}, \tag{4.7}$$

and since $J_\nu(ix) = I_\nu(x)$ and y is negative, we have almost the same solution, except for us the integration is only over $d-2$ transverse coordinates (as for a massless particle), whereas for Giddings the integration is over $d-1$ transverse coordinates, as appropriate for a massive particle.

Let us then analyze the AdS scattering. We are interested in the large r behavior ($e^{y/l}l/r \ll 1$ and $y = y_0$) which is relevant for the Froissart bound that we are after. In that case,

$$\Phi = \frac{\bar{C}l^4}{r^6} e^{\frac{2}{l}(2y+y_0)}. \tag{4.8}$$

We find (see the Appendix for details; here $\epsilon \equiv y - y_0$)

$$\begin{aligned} f &= \Phi|_{\epsilon=0} = \bar{C}\frac{l^4}{r^6} e^{6y_0/l}, \\ a &= \frac{4}{l} f + \zeta_1, \\ g &= \partial_\epsilon^2 \Phi|_{\epsilon=0} + \frac{4}{l}\zeta_1 = \frac{16f}{l^2} + \frac{4}{l}\zeta_1 \\ \Rightarrow \quad a &= -\frac{\alpha}{l} f = -\frac{16}{l} f. \end{aligned} \tag{4.9}$$

As we have explained in the Appendix, we actually get two solutions for the trapped surface. There is one solution for which a is negligible, thus that solution corresponds to what we would have if the scattering was four dimensional (and just Φ was obtained from the 5D equations). But in the $r \gg l$ limit, the AdS warping is very large, and energy scales become larger away from the 4D slice, thus we expect the size of the black hole being created (and thus of the trapped surface) is increased. And so we will take the solution that takes into account the 5D scattering, solution that implies a larger horizon.

The same situation will be encountered in the case of scattering on the IR brane. The space will then be very non-four dimensional in the $r \gg l$ limit (in a sense, it will be anti-four dimensional), thus we expect that the size of the trapped surface will be increased also with respect to pure 4D scattering. This situation is to be contrasted with the scattering on the UV brane we analyzed in [10], in which case for $r \gg l$ the space was approximately four dimensional (the warping is going down away from the brane), and correspondingly we found a solution which had just small corrections to the four-dimensional scattering.

The continuity condition for the larger trapped surface is thus

$$\frac{|\alpha| f}{2l} e^{-y_0/l} = 1, \tag{4.10}$$

so that

$$r = r_{\max} = le^{y_0/l}\left[\frac{3R_s}{le^{y_0/l}}\right]^{1/6}, \tag{4.11}$$

and since in four dimensions $R_s = 2G_4\sqrt{s}$, we have that $r_{\max} \sim s^{1/12}$.

If we introduce a nonzero impact parameter the continuity condition becomes

$$\left(\frac{3R_s l^5 e^{5y_0/l}}{r^6}\right)^2 \left(1 - \frac{b^2}{2r^2}\right) = 1, \tag{4.12}$$

and with $r^2 = x$, $(3R_s l^5 e^{5y_0/l})^2 = a$ we get the equation

$$x^7 - ax + a\frac{b^2}{2} = 0. \tag{4.13}$$

The maximum b for which it has a solution is [using $f(x_0) = 0$ for $f'(x_0) = 0$]

$$\begin{aligned} b_{\max}^2 &= \frac{12}{7}\left[\frac{a}{7}\right]^{1/6} \Rightarrow b_{\max} = 7^{-1/12}\sqrt{\frac{12}{7}} le^{y_0/l}\left[\frac{3R_s}{le^{y_0/l}}\right]^{1/6} \\ &= \frac{1}{7^{1/12}}\sqrt{\frac{12}{7}} r_{\max} \end{aligned} \tag{4.14}$$

so that $b_{\max} \sim as^{1/12}$ as well.

We will now analyze the scattering occurring on the IR brane, the details of which are in the Appendix. The wave profile is

$$\Phi(r, y) = R_s l e^{-2|y|/l} \int_0^\infty dq J_0(qr) \frac{I_2(e^{-|y|/l} lq)}{I_1(lq)}, \tag{4.15}$$

and we find that we can use the contour integration over the complex q plane to calculate

$$\begin{aligned} f = \Phi(r; y = 0) &\simeq R_s\sqrt{\frac{2\pi l}{r}} \sum_n \frac{j_{1,n}^{-1/2} J_2(j_{1,n})}{a_{1,n}} e^{-\bar{q}_n r} \\ &\simeq R_s\sqrt{\frac{2\pi l}{r}} C_1 e^{-\bar{q}_1 r}, \end{aligned} \tag{4.16}$$

where $\bar{q}_n = j_{1,n}/l$ are poles of the integral in the complex momentum plane given by the zeroes of the Bessel function J_1, $j_{1,n}$. In (3.18) from the general formalism, i.e.

$$\Psi = f + ay + g\frac{y^2}{2} + \ldots = C, \tag{4.17}$$

we have [notice the sign difference in g, due to the IR brane metric (4.3) vs the UV brane metric (3.4) used in Sec. III]

$$a = \zeta_1, \qquad g = \partial_y^2 \Phi|_{y=0} - \frac{4}{l}\zeta_1 = -f\bar{q}_1^2 - \frac{4a}{l}. \tag{4.18}$$

Again, as in the AdS case, we find two solutions, one that would be there if we had a 4D scattering, and a larger trapped surface that appears only when we have 5D scattering. In the Appendix we treat in detail the case of the 4D scattering, for completeness. Taking instead the larger trapped surface,

$$a = -3\frac{fr^2}{l^3}, \tag{4.19}$$

we get the continuity condition at $y = 0$

$$\left(\frac{|f|\bar{q}_1}{2}\right)\frac{|\alpha|r}{\bar{q}_1 l^2} = 1 \Rightarrow \frac{R_s\bar{q}_1}{2}\sqrt{\frac{2\pi l}{r}}\frac{|\alpha|C_1}{\bar{q}_1 l}\frac{r}{l}e^{-\bar{q}_1 r} = 1. \tag{4.20}$$

It has a solution that is approximately

$$r_H \simeq \frac{1}{2\bar{q}_1}\ln\left(\frac{\bar{A}}{2\bar{q}_1}\right), \qquad \bar{A} = \left(\frac{R_s\bar{q}_1}{2}\right)^2\frac{2\pi}{l}\left(\frac{3C_1}{\bar{q}_1 l}\right)^2. \tag{4.21}$$

We can see that there is a solution by considering the function

$$\tilde{g}(r) = r - \bar{A}r^2e^{-2\bar{q}_1 r} \Rightarrow \tilde{g}'(r) = 1 + \left(\bar{q}_1 - \frac{1}{r}\right)2\bar{A}r^2e^{-2\bar{q}_1 r}, \tag{4.22}$$

and the solution for r_H is given by $\tilde{g}(r) = 0$. Since $\tilde{g}'(r) > 0$ if $r > 1/\bar{q}_1$ and

$$\tilde{g}\left(\frac{1}{\bar{q}_1}\right) = \frac{1}{\bar{q}_1}\left[1 - \frac{R_s^2}{l^2}\frac{18\pi C_1^2}{4e^2 j_{1,1}}\right] < 0, \tag{4.23}$$

for sufficiently large R_s, there will be a solution.

At nonzero impact parameter, we get the equation

$$\left(\frac{|f|\bar{q}_1}{2}\right)^2\left(\frac{|\alpha|r}{\bar{q}_1 l^2}\right)^2\left(1 - \frac{b^2}{2r^2}\right) = 1. \tag{4.24}$$

Its solution is the zero of the function

$$g(r) = r - \left(1 - \frac{b^2}{2r^2}\right)\bar{A}r^2e^{-2\bar{q}_1 r}. \tag{4.25}$$

And now we have an analysis that is a bit more involved;

$$g'(r) = 1 + 2\bar{q}_1\left(r^2 - \frac{b^2}{2}\right)\bar{A}e^{-2\bar{q}_1 r} - 2r\bar{A}e^{-2\bar{q}_1 r}. \tag{4.26}$$

We are, however, only interested in the maximum value $b_{\max}$ for which $g(r) = 0$ has a solution.

If $r^2 < b^2/2$, then $g(r) > 0$, and $g(0) = b^2\bar{A}/2$, $g(b/\sqrt{2}) = b/\sqrt{2}$. To have a solution of $g(r) = 0$, we need a minimum, $g'(r_1) = 0$, with $g(r_1) < 0$, so necessarily $r_1^2 > b^2/2$.

But if $b^2/2 \ll r^2$, the second term in $g'(r)$ is larger than the third (as $\bar{q}_1 \sim 1/l \gg r$), so $g'(r) > 0$. Thus we need instead

$$\frac{b^2}{2r_1^2} = 1 - \epsilon \simeq 1, \tag{4.27}$$

so if $g(r_1) < 0$ [but close to zero, so that the 1 is negligible in $g'(r)$ and we can use $g'(r_1) = 0$], we get

$$g(r_1) \simeq r_1\left(1 - \frac{\bar{A}}{\bar{q}_1}e^{-2\bar{q}_1 r}\right). \tag{4.28}$$

At $b = b_{\max}$, $g(r_1) = 0$, so

$$r_1 \simeq \frac{1}{2\bar{q}_1}\ln\left(\frac{\bar{A}}{\bar{q}_1}\right), \qquad b^2 \simeq 2r_1^2$$
$$\Rightarrow \quad b_{\max}(s) \simeq \sqrt{2}\frac{1}{2\bar{q}_1}\ln\left(\frac{\bar{A}}{\bar{q}_1}\right) = \frac{\sqrt{2}}{\bar{q}_1}\ln\left[R_s\bar{q}_1\left(\frac{3\sqrt{\pi}}{\sqrt{2}j_{1,1}^{3/2}}\right)\right], \tag{4.29}$$

where $R_s = \sqrt{s}G_4$, $G_4 = 1/(lM_{P,5}^3)$. Thus we get the same formula for $r_H(m)$ as [6] (modulo different constants), and thus the classical cross section in the gravity dual for scattering on the IR brane is of the expected $\ln^2 s$ functional form

$$\sigma = \pi b_{\max}^2(s) \simeq 2\pi\left[\frac{1}{\bar{q}_1}\ln(K_1\sqrt{s}\bar{q}_1 G_4)\right]^2. \tag{4.30}$$

V. QCD SCATTERING REGIMES; THE FROISSART BOUND

Finally, now that we have done all the calculations needed for the scattering in the AdS dual, let us put everything together. In the gravity dual, the scattering happens inside AdS space or, in the Froissart regime, effectively on the IR brane.

In [10], we have analyzed 't Hooft scattering in AdS space, and we have found that we can calculate analytically the scattering amplitude in two limits:

$$\begin{aligned} &\mathcal{A}_{\text{AdS}} \simeq \frac{G_4 l}{2\pi}\frac{s}{\sqrt{t}}e^{(3y-y_0)/2l}e^{-\sqrt{t}l(e^{y/l}-e^{y_0/l})}, \\ &y \neq y_0 < l, \qquad r \ll l, \qquad G_4 s \ll 1, \\ &\mathcal{A}_{\text{AdS}} \propto G_4 l^6 s t^2 \ln(t), \qquad r \gg l, \end{aligned} \tag{5.1}$$

to be compared with the result in flat d dimensions (which can be used in the case $y = y_0$, $r \ll l$ for instance)

$$\mathcal{A}_{\text{AdS}} \sim \frac{G_4 s}{t}. \tag{5.2}$$

All cases still give $\mathcal{A} \sim G_4 s$, only the t behavior is modified, so

$$\frac{d\sigma_{\text{AdS}}}{d^2k} = \frac{4}{s}\frac{d\sigma_{\text{AdS}}}{d\Omega} \sim (G_4 s)^2. \tag{5.3}$$

Since we have a well-defined amplitude, we can use the Polchinski-Strassler formula ([3]) to relate to a QCD amplitude. Of course we have to remember that the formula is valid only for the large r region, away from the IR cutoff, where we can approximate the metric with AdS space and the wave function dependence of r with the power law $r^{-\Delta}$. But unfortunately, in all the cases of interest, studied in (2.11), the main contribution to the integral comes from the region of large ν ($\nu \simeq \nu_{\max} = \hat{\alpha}' t$), corresponding to the region of small r, $r \simeq r_{\min}$.

This is, however, not as bad as it seems, since this IR modification will translate only in the modification of the factorized t behavior (coming from the integration over ν), while the dependence of s comes from the s dependence of the AdS amplitude. We can check this by looking at the formula (2.10). As we will be interested only in the leading s behavior and not the t dependence (be it multiplicative or not), our results will still be valid [see also the discussion further of the original Polchinski-Strassler paper [3] later on, especially around (5.11)].

We also see that in all the cases of interest treated in (2.11), the effect of the AdS integration is only to change the overall normalization, as well as the subleading behavior. The leading behavior is kept the same.

In conclusion, for the case at hand, of the 't Hooft scattering in AdS space, where $\mathcal{A}_{\mathrm{AdS}} \sim G_4 s$, by using the PS formula ([3]) we will still have the same s dependence

$$\mathcal{A}_{\mathrm{QCD}} \sim \hat{G}_4 s \Rightarrow \frac{d\sigma_{\mathrm{QCD}}}{d^2k} = \frac{4}{s}\frac{d\sigma_{\mathrm{QCD}}}{d\Omega} \sim |\mathcal{A}_{\mathrm{QCD}}|^2 \sim (\hat{G}_4 s)^2. \quad (5.4)$$

As we mentioned, the relation between AdS and QCD energy scales is $\sqrt{\alpha' \tilde{s}}|_{\mathrm{AdS}} \le \sqrt{\hat{\alpha}' s}|_{\mathrm{QCD}}$ and $\sqrt{\alpha' \tilde{t}}|_{\mathrm{AdS}} \le \sqrt{\hat{\alpha}' t}|_{\mathrm{QCD}}$. Here

$$\hat{\alpha}' = \frac{1}{\Lambda^2 \sqrt{g_s N}}, \quad (5.5)$$

where Λ is the mass gap (the lightest glueball state). And the minimum r in the cutoff AdS is $r_{\min} = R^2 \Lambda$, $R \equiv l$ in our notation.

We can rescale the 4D coordinates x such as $r_{\min} = R \Rightarrow \hat{\alpha}' = \alpha'$. The Planck scales are

$$M_P(\mathrm{AdS}) = \frac{g_s^{-1/4}}{\sqrt{\alpha'}} \leftrightarrow \hat{M}_P = \frac{g_s^{-1/4}}{\sqrt{\hat{\alpha}'}} = N^{1/4}\Lambda \Rightarrow \frac{\sqrt{\tilde{s}}}{M_P} \le \frac{\sqrt{s}}{\hat{M}_P} \hat{G}_4 \equiv \hat{M}_P^{-2} = \frac{1}{\sqrt{N}\Lambda^2}. \quad (5.6)$$

So from 't Hooft scattering in AdS space we get a Rutherford-type behavior in QCD as well, except with the effective coupling $\hat{G}_4 s$.

This behavior is universal, and if we interpret it as single-particle exchange between the gauge invariant Yang-Mills states (glueballs), we see that the particle being exchanged must also be colorless.

So in the energy regime $\hat{G}_4 s < 1$, but at energies larger than the mass gap, i.e. for

$$\Lambda < \sqrt{s} < \frac{1}{\sqrt{\hat{G}_4}} = \Lambda N^{1/4}, \quad (5.7)$$

colorless states should obey Rutherford scattering behavior, with a universal colorless single-particle exchange, with universal effective coupling $\hat{G}_4 s$. In the case of real QCD ($N = 3$), the energy regime is nonexistent, thus we cannot draw any lessons from experiments, and the interest in this regime is therefore just theoretical.

However, as the universal coupling was obtained from a spin-2 exchange in AdS (graviton), it is natural to assume that the same thing happens in the gauge theory, namely, a universal spin-2 colorless particle is exchanged in this regime that behaves as a graviton. There is a natural candidate for such a composite "particle," namely, the gauge theory dual to the graviton, the energy-momentum tensor. It is not obvious why in this energy regime the energy-momentum tensor should have a gravitonlike universal coupling to all colorless states, nor why graviton exchange in the bulk should be dual to energy-momentum tensor exchange on the boundary, but this is the only plausible candidate.

This is an elastic scattering in AdS space, and it is an elastic cross section in QCD. It could then be possible to detect this elastic σ even at higher energies, when the amplitude is mostly inelastic.

As we go even higher in energies ($\alpha' s \gg 1$), we will start observing the Regge behavior noted by Polchinski and Strassler [3]. Then in AdS space we need to use the string Virasoro-Shapiro amplitude for massless external states ($s + t + u = \sum m_i^2 = 0$)

$$\begin{aligned}\mathcal{A} &= \left[\prod_{x=s,t,u} \frac{\Gamma(-\alpha' \tilde{x}/4)}{\Gamma(1 + \alpha' \tilde{x}/4)}\right] K(\tilde{p}\sqrt{\alpha'}) \\ &= \frac{\Gamma(\alpha' s/4 + \alpha' t/4)}{\Gamma(1 + \alpha' s/4)} \frac{\Gamma(-\alpha' s/4)}{\Gamma(1 - \alpha' s/4 - \alpha' t/4)} \\ &\quad \times \frac{\Gamma(-\alpha' t/4)}{\Gamma(1 + \alpha' t/4)} K(\tilde{p}\sqrt{\alpha'}),\end{aligned} \quad (5.8)$$

becoming in the small angle ($s \gg t$) region of interest

$$\begin{aligned}\mathcal{A} &= (\alpha' s/4)^{\alpha' t/4 - 1} (-\alpha' s/4)^{\alpha' t/4 - 1} \\ &\quad \times \frac{\Gamma(-\alpha' t/4)}{\Gamma(1 + \alpha' t/4)} s^4 (\text{pol. tensors}) \\ &\sim (\alpha' s)^{\alpha' t/2 + 2} \frac{\Gamma(-\alpha' t/4)}{\Gamma(1 + \alpha' t/4)}.\end{aligned} \quad (5.9)$$

Regge behavior in QCD was found to correspond to Regge behavior in AdS space (to the flat space Regge behavior of this VS amplitude, to be exact). The approximation used though is the same approximation that we needed to use for the 't Hooft scattering. Namely, in general the ν integral is dominated by (using stationary phase approximation)

$$\nu_0 = -\frac{\Delta - 4}{\ln(s/|t|)}. \tag{5.10}$$

But Regge behavior is obtained in the case that $\nu_0 > \nu_{\max} = \hat{\alpha}' t$, when the integration is dominated by the upper limit of the integral, corresponding to r close to $r_{\min}$.

But as in the case of 't Hooft scattering, we only need to assume that the maximum of the integral is still outside the region of integration, so that we can approximate the integral with its upper limit. Presumably the existence of the mass gap is enough to satisfy this requirement. Then the factorized t dependence will be modified, but the s behavior still will be of Regge type:

$$\mathcal{A} \sim (\hat{\alpha}' s)^{2+\hat{\alpha}' t/2}. \tag{5.11}$$

Finally, for the elastic amplitude, [3] finds yet another regime. If $\nu_{\max} > \nu_0$ (to be rigorous we would need $\nu_{\max} \gg \nu_0$, corresponding to $r_{\mathrm{scatt}} \gg r_{\min}$), that is if

$$\hat{\alpha}' t > \frac{(\Delta - 4)}{\ln(s/|t|)}, \tag{5.12}$$

one obtains

$$\mathcal{A} \sim s^2 |t|^{-\Delta/2} [\ln(s/|t|)]^{1-\Delta/2}. \tag{5.13}$$

Let us now turn to energies higher than the Planck scale in AdS space, $\tilde{s} > M_P^2 \Rightarrow s > \hat{M}_P^2$, when we will produce black holes in the scattering. There are three dimensionless parameters characterizing the scattering in AdS space, $R_s/l, G_4 s$ and $l M_{P,5}$, so let us first derive their relations to QCD variables.

Since $M_{P,5} = N^{1/4}/l$ in our notation, then

$$R_s = G_4 \sqrt{\tilde{s}} = \frac{\sqrt{\tilde{s}}}{l M_{P,5}^3} \leq \left(\frac{1}{l M_{P,5}^2}\right) \frac{\sqrt{s}}{\hat{M}_P} |_{\mathrm{QCD}}, \tag{5.14}$$

so that

$$\frac{R_s}{l} \leq \frac{1}{\sqrt{N}} \left(\frac{\sqrt{s}}{\hat{M}_P}\right) |_{\mathrm{QCD}}. \tag{5.15}$$

Also, since $G_4 \tilde{s} = R_s^2/G_4$, we get

$$G_4 \tilde{s} \leq N^{-1/4} \left(\frac{\sqrt{s}}{\hat{M}_P}\right)^2 |_{\mathrm{QCD}}. \tag{5.16}$$

Equivalently, in terms of $\hat{\alpha}'$, the two parameters satisfy

$$\frac{R_s}{l} \leq \frac{g_s^{1/4}}{\sqrt{N}} \sqrt{\hat{\alpha}' s} |_{\mathrm{QCD}}, \qquad G_4 \tilde{s} \leq \frac{g_s^{1/2}}{N^{1/4}} (\hat{\alpha}' s) |_{\mathrm{QCD}}. \tag{5.17}$$

Finally, $l M_{P,5} = N^{1/4} \gg 1$, so one first reaches the AdS scale $1/l$, then $M_P \equiv M_{P,5}$, and then the scale $E_R = M_P (l M_P)^{d-3}$, when the black hole size is comparable with the AdS size.

Note that the dimensionality of the gravity dual plays an important role. Most of the calculations in this paper were done assuming that there is only AdS_5 and forgetting about the compact space. In the case of 't Hooft scattering, we argued that the relevant behavior was independent of the dimensionality of the space. In the case of black hole creation, the dependence is more important.

Before the size of AdS space becomes important, black hole formation can be approximated as being in flat space. In that case, we have calculated in [10] a lower limit on the maximum impact parameter that creates a black hole in d dimensions,

$$\begin{aligned} b_{\max}^2 &\leq 2\left[\frac{\alpha}{D-2}\right]^{1/(D-3)} \frac{D-3}{D-2} \\ &= \frac{2(\epsilon r_H)^2}{[D-2]^{(D-2)/(D-3)}} (D-3), \\ r_H &= \left[\frac{16\pi G\sqrt{s}}{(D-2)\Omega_{D-2}}\right]^{1/(D-3)}, \\ \epsilon &= \left[\frac{(D-2)\Omega_{D-2}}{4\Omega_{D-3}}\right]^{1/(D-3)}. \end{aligned} \tag{5.18}$$

In any case, we see that $b_{\max}(s) \simeq a s^{1/[2(D-3)]} = a s^{\beta}$ ($a =$ constant). In Sec. II A we used a simple black disk model to create an imaginary elastic scattering amplitude that was substituted in the Polchinski-Strassler formula (2.10). We derived the forward ($t = 0$) imaginary part of the amplitude, giving us the total QCD scattering cross section in this regime

$$\begin{aligned} \sigma_{\mathrm{QCD}} &= \frac{\pi a^2 K}{2\beta + \Delta/2 - 1} \left(\frac{\hat{\alpha}' s}{\alpha'}\right)^{2\beta} \\ &= \frac{\pi a^2 K}{1/(D-3) + \Delta/2 - 1} \left(\frac{\hat{\alpha}' s}{\alpha'}\right)^{1/(D-3)}. \end{aligned} \tag{5.19}$$

As we have explained in Sec. II A, this is of the type

$$(K a^{2/(1+2\beta)}) \times \mathrm{fct}\left(\frac{a^{2/(1+2\beta)}}{\alpha'}, \hat{\alpha}' s, \hat{\alpha}' t, \Delta\right), \tag{5.20}$$

and here $a^{2/(1+2\beta)}/\alpha'$ is a number depending on the gravity dual (dimension D, Newton constant G), whereas $K a^{2/(1+2\beta)}$ is a dimension -2 constant that also depends strongly on the details of the IR cutoff (through K).

We see that the higher the dimensionality, the smaller the dependence on s. For $d = 5$ we have $\sigma \sim s^{1/2}$, whereas for $d = 10$ we have $\sigma \sim s^{1/7}$.

At even higher energies ($\sqrt{s} > E_R$), we will start feeling the effects of the AdS space size. We have calculated in Sec. IV that the maximum impact parameter for black hole formation is at least equal to

$$b_{\max} = 7^{-1/12} \sqrt{\frac{12}{7}} l e^{y_0/l} \left[\frac{R_s}{l e^{y_0/l}}\right]^{1/6} = \frac{1}{7^{1/12} 2^{2/3}} \sqrt{\frac{12}{7}} r_{\max}, \tag{5.21}$$

where $R_s = 2 G_4 \sqrt{s}$, so that $b_{\max} \simeq a' s^{1/12}$.

Then using the same black disk model to substitute in the PS formula, we get

$$\sigma_{\rm QCD} = \frac{\pi a'^2 K}{2\beta + \Delta/2 - 1}\left(\frac{\hat{\alpha}' s}{\alpha'}\right)^{2\beta} = \frac{\pi a'^2 K}{1/6 + \Delta/2 - 1}\left(\frac{\hat{\alpha}' s}{\alpha'}\right)^{1/6}. \quad (5.22)$$

But as in the previous cases, the PS formula shows that the main integration region is near the cutoff $r_{\rm min}$, which corresponds to the IR brane in the two-brane RS model. But in that case we cannot approximate the scattering as being in AdS space, since the 4D size of the black hole formed is comparable to the AdS scale. As $\rho \gg l$ the horizon will stretch over a size $\Delta y > l$, and if $y \simeq y_{\rm IR}$ it would look as if the black hole is approximately on the IR brane. So the approximation of AS in AdS space will break down and instead the good approximation would be the two AS shockwaves on the IR brane.

So it is not even clear that there is an intermediate regime of the type in (5.22), but it is clear that the cross section will begin flattening out, finally to settle into the final behavior, corresponding to scattering on the IR brane.

In the second part in Sec. IV, we calculated that for AS scattering on the IR brane we get a maximum impact parameter for black hole formation that is at least equal to $b_{\rm max} = \sqrt{2} r_1$, with

$$r_1 = \frac{1}{2\bar{q}_1}\ln\left(\frac{\bar{A}}{\bar{q}_1}\right); \qquad \bar{A} = \left(\frac{R_s \bar{q}_1}{2}\right)^2 \frac{2\pi}{l}\left(\frac{3C_1}{\bar{q}_1 l}\right)^2. \quad (5.23)$$

In that calculation we used the metric

$$ds^2 = e^{2|y|/l} d\vec{x}^2 + dy^2 = \frac{r^2}{l^2}\left(\frac{l^2}{r_{\rm min}^2} d\vec{x}^2\right) + \frac{l^2}{r^2} dr^2, \quad (5.24)$$

so to go back to the real coordinates we substitute

$$\tilde{\rho}({\rm real}) = \frac{l}{r_{\rm min}} \rho({\rm used}) = \frac{\rho({\rm used})}{l\Lambda}. \quad (5.25)$$

Thus

$$b_{\rm max}(s) \simeq \frac{1}{l\Lambda}\sqrt{2}\frac{1}{2\bar{q}_1}\ln\left(\frac{\bar{A}}{\bar{q}_1}\right) = \frac{\sqrt{2}}{\bar{q}_1}\ln\left[R_s \bar{q}_1\left(\frac{3\sqrt{\pi}}{\sqrt{2} j_{1,1}^{3/2}}\right)\right], \quad (5.26)$$

and $R_s = \sqrt{s} G_4$, $G_4 = 1/(l M_{P,5}^3)$. As advocated, we get the same formula for $r_H(m)$ as in [6], modulo different constants.

The gravitational cross section is

$$\sigma \simeq \pi\left[\frac{\sqrt{2}}{l\Lambda}\frac{1}{\bar{q}_1}\ln(K\sqrt{s}\bar{q}_1 G_4)\right]^2, \quad (5.27)$$

and $\sigma_{\rm QCD} \sim \sigma$, as we argued.

Thus as expected, the final behavior of the QCD scattering amplitude corresponds to the Froissart unitarity bound. We have obtained the same behavior that Giddings [6] has proposed, but in a more rigorous setting. Let us compare to the calculation in [6]. There, the "Newton potential" h_{00} was calculated in linearized gravity, obtaining

$$h_{00} \sim e^{dy/2l} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} e^{i\vec{p}\,\vec{x}} \frac{J_{d/2}(iple^{y/l})}{pJ_{d/2-1}(ipl)}, \quad (5.28)$$

which was then used for an estimate of the horizon size by $h_{00} \sim 1 \Rightarrow r = r_H$, and a geometric cross section approximation $\sigma \simeq \pi r_H^2$ was used for the black hole.

In our case, we obtain the exact AS shockwave solution on the IR brane, which is

$$\Phi(r, y) = \frac{R_s l}{2\pi} e^{-[(d|y|)/2l]} \int \frac{d^{d-2}\vec{q}}{(2\pi)^{d-2}} e^{i\vec{q}\,\vec{x}} \frac{I_{d/2}(e^{-|y|/l} lq)}{qI_{d/2-1}(lq)} = \frac{R_s l}{(2\pi)^{(d-4)/2}} \frac{e^{-[(d|y|)/2l]}}{r^{(d-4)/2}} \times \int_0^\infty dq q^{(d-4)/2} J_{(d-4)/2}(qr) \frac{I_{d/2}(e^{-|y|/l} lq)}{I_{d/2-1}(lq)}, \quad (5.29)$$

and since $J_\nu(ix) = I_\nu(x)$ and y is negative, it is very similar to the Giddings case, except that for us the integration is only over $d-2$ transverse coordinates (as for a massless particle), whereas for [6] the integration is over $d-1$ transverse coordinates, as appropriate for a massive particle. The h_{00} in [6] was obtained also from a wave equation with sources on the IR brane, except there the source was a static mass (black hole), whereas for us it is a photon.

At $y = 0$, one obtains similar behaviors,

$$h_{00} \simeq \frac{1}{2\pi r}\sum_n e^{-\bar{q}_n r} \frac{J_2(j_{1,n})}{la_{1,n}} \simeq \frac{1}{2\pi r} e^{-\bar{q}_1 r} \frac{J_2(j_{1,1})}{la_{1,1}} \quad (5.30)$$

versus

$$\Phi(r, y=0) \simeq R_s\sqrt{\frac{2\pi l}{r}}\sum_n \frac{j_{1,n}^{-1/2} J_2(j_{1,n})}{a_{1,n}} e^{-\bar{q}_n r} \simeq R_s\sqrt{\frac{2\pi l}{r}} C_1 e^{-\bar{q}_1 r}, \quad (5.31)$$

and in both cases the fact that allows the logarithmic behavior of r_H is the exponential $e^{-\bar{q}_1 r}$, itself coming from the presence of the pole in the momentum space integrand. In our case, we have the advantage of the scattering picture, in which we can calculate directly the cross section for black hole creation.

Finally, in [10] we have addressed the issue of string corrections to the scattering. This is very relevant for the case of QCD, since string α' and g_s corrections in AdS space translate into $1/N$ and $1/(g_{\rm YM}^2 N)$ corrections in the gauge theory, bringing us closer to the case of QCD. So it is important to realize their effects.

In [10], string corrections were analyzed using a formalism of Amati and Klimcik [27] (as well as using the formalism in [28], based on the action in [29], but we will not describe it) in which one obtains a string-corrected AS metric. These corrections depend the dimensionless ratio (in $d = 4$; here $Y = \alpha' \log(\alpha' s)$)

$$\frac{R_s^2}{Y} = \frac{g^2}{\log(\alpha' s)} \frac{g^2 \alpha' s}{(4\pi)^2}. \tag{5.32}$$

When this parameter is large, the shockwave is approximately AS, with exponentially small corrections, namely,

$$\Phi(b) = -\frac{g^2\sqrt{s}}{4\pi}\alpha'\left(2\log\frac{b}{R_s} - e^{-(b2/4Y)}\left(\frac{b^2}{4Y}\right)^{-1} + \ldots\right)$$
$$= -R_s\left(2\log\frac{b}{R_s} - e^{-(b2/4Y)}\left(\frac{b^2}{4Y}\right)^{-1} + \ldots\right), \tag{5.33}$$

and the maximum impact parameter for black hole creation also increases, but with exponentially small corrections:

$$B_{\max} = \frac{R_s}{\sqrt{2}}(1 + e^{-R_s^2/(8Y)}). \tag{5.34}$$

When $R_s^2/Y \ll 1$, the shockwave is not of AS type

$$\Phi(b) = -2R_s\left(\frac{1}{D-4} - \frac{b^2}{4Y(D-2)} + \ldots\right), \tag{5.35}$$

and we can show that $B_{\max} \sim \sqrt{Y} \gg R_s$ (but we cannot compute the actual value), so one has a huge increase in the cross section.

So string corrections can be quite important, but in the case of $s \to \infty$ (the Froissart unitarity bound), $R_s^2/Y \gg 1$, and the corrections discussed here are exponentially small. Thus the Froissart unitarity bound is unaffected—as expected.

While this calculation was in flat $d = 4$, not in the gravity dual, and string corrections were described using a very simple model, the smallness of the corrections, together with the fact that we do obtain what we expect, namely, the Froissart bound, makes us believe that we are correctly describing a QCD phenomenon.

VI. CONCLUSIONS

In this paper we have analyzed high-energy QCD scattering in the small angle region $s \gg t$. We have applied the high-energy gravitational scattering calculations in [10] to the simple model of QCD gravity dual used in [3]. Namely, for high-energy QCD many of the observed features can be deduced from just an AdS dual cutoff in the IR (and maybe in the UV), giving a two-brane RS model. The gravitational scattering was analyzed using a shockwave analysis. At energies smaller than the Planck scale, the scattering of two massless particles is described by null geodesics propagating in the shockwave background ('t Hooft scattering). At energies higher than the Planck scale, we need to take two AS shockwaves, and black holes are being formed in the future of the collision, for an impact parameter less than a $b_{\max}(s)$.

't Hooft scattering corresponds in the gauge theory to a very restrictive regime ($\Lambda < \sqrt{s} < \Lambda N^{1/4}$) that is too restrictive for real QCD ($N = 3$). For N large though, we obtained a Rutherford-type scattering with effective coupling $\hat{G}_4 s$—implying a universal single-particle exchange. We conjectured that this comes from an exchange of a universal, gravitonlike spin-2 "particle" being exchanged, most likely the energy-momentum tensor (the dual of the graviton). As the cross section is elastic, one could, however, maybe detect this elastic σ even at higher energies in the gauge theory, when the amplitude is mostly inelastic, thus being of possible relevance to real QCD.

At higher energies, Regge behavior sets in, as described by [3]. Regge behavior of the flat space Virasoro-Shapiro amplitude translates directly into Regge behavior in QCD.

At even higher energies, black hole production sets in inside the gravity dual, and it can be described as if happening in an approximately flat d-dimensional space, giving the power law behavior in (5.19), namely, $\sigma_{\rm QCD} \sim s^{1/(d-3)}$.

There is a possible transition region when the size of AdS space becomes important, and the scattering in the gravity dual creates black holes inside AdS space, giving again a power law behavior (5.22), namely, $\sigma_{\rm QCD} \sim s^{1/6}$.

Finally, in the last energy regime most of the scattering happens on the IR brane, and the size of the surrounding AdS space is important. We obtain a shockwave solution that has the same exponential behavior $h \sim e^{-\tilde{q}_1 r}$ as the linearized black hole solution used by Giddings [6]. The scattering of two such shockwaves gives a scattering cross section saturating the Froissart bound (5.27).

We have looked at string corrections to the scattering of two modified shockwaves, and we have found that in the simple Amati-Klimcik model used in [10], and in flat $d = 4$, we obtain exponentially small corrections in the Froissart limit $s \to \infty$. This makes us believe that even in the case of real QCD, when N and $g_{\rm YM}^2 N$ are finite and thus we have large string corrections in the gravity dual, our calculation of the Froissart bound still applies.

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APPENDIX A. TRAPPED SURFACE CALCULATIONS

Let us apply the general formalism to calculate the shape of the trapped surface in AdS space.

The continuity equation at $b = 0$ would be, for $\Psi = \Phi$,

$$(\partial_i\Phi)^2 + e^{-2y/l}(\partial_y\Phi)^2 = 4, \tag{A1}$$

which for $e^{y/l}l/r \gg 1$ gives

$$e^{(3y-y_0)/l}\left(\frac{\bar{C}}{4l}\right)^2\left[\frac{1}{[r^2 + l^2(e^{y/l} - e^{y_0/l})^2]^2} + \frac{9}{4l^2e^{2y/l}[r^2 + l^2(e^{y/l} - e^{y_0/l})^2]}\right] = 1 \tag{A2}$$

(where $\bar{C} = 2R_s l^2$), since

$$\Phi \simeq \frac{\bar{C}e^{(3y-y_0)/2l}}{2l}\frac{1}{\sqrt{r^2 + l^2(e^{y/l} - e^{y_0/l})^2}}. \tag{A3}$$

At $y = y_0$ we get

$$\left(e^{y_0/l}\frac{R_s l}{2r^2}\right)^2\left(1 + \frac{9}{4}\frac{r^2}{l^2e^{2y_0/l}}\right) = 1, \tag{A4}$$

where as we can see the second term [coming from $(\partial_y\Phi)^2$] is a small correction.

For $l \ll r$ we have

$$\Phi = \frac{\bar{C}l^4}{r^6}e^{(2/l)(2y+y_0)}. \tag{A5}$$

That implies

$$\left[\frac{6R_s l^6}{r^7}e^{(2/l)(2y+y_0)}\right]^2\left[1 + \frac{4}{9}\frac{r^2}{l^2e^{2y/l}}\right] = 1. \tag{A6}$$

However, now the second term [coming again from $(\partial_y\Phi)^2$] is dominant.

But we still need to modify Φ: $\Psi = \Phi + \zeta$, as in the RS case. Using the formulas from the RS case described in Sec. III, but remembering that now we expand in $\epsilon = y - y_0$, not in y, we get

$$\Psi = C = f + a\epsilon + \frac{\epsilon^2}{2}g + \dots \tag{A7}$$

where

$$f = \Phi|_{y=y_0}, \qquad a = \zeta_1 + \partial_y\Phi|_{y=y_0}. \tag{A8}$$

Using

$$\Phi = K\frac{e^{d\epsilon/2l}}{r^{(d-4)/2}}\int_0^\infty dq q^{(d-2)/2}J_{(d-4)/2}(qr) \times K_{d/2}(e^{(y_0+\epsilon)/l}lq)I_{d/2}(e^{y_0/l}lq), \tag{A9}$$

and $zK'_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$, we get in $d = 4$

$$\partial_\epsilon\Phi = -\frac{Ke^{2\epsilon/l}}{l}\int_0^\infty dq q J_0(qr) \times (e^{\epsilon/l}e^{y_0/l}lq)K_1(e^{\epsilon/l}e^{y_0/l}lq)I_2(e^{y_0/l}lq). \tag{A10}$$

Then using $I_2(z) = -2I_1(z)/z + I_0(z)$ and

$$\int_0^\infty xJ_0(x)K_1(ax)I_1(ax) = \frac{1 + 2a^2 - \sqrt{1 + 4a^2}}{2a^2\sqrt{1 + 4a^2}},$$
$$\int_0^\infty dx x^2J_0(x)K_1(ax)I_0(ax) = 2a(1 + 4a^2)^{-3/2}, \tag{A11}$$

we get

$$\partial_\epsilon\Phi|_{\epsilon=0} = \frac{Ke^{-2y_0/l}}{l^3}\left[\frac{6e^{4y_0/l}l^4/r^4 + 6e^{2y_0/l}l^2/r^2 + 1}{(1 + 4e^{2y_0/l}l^2/r^2)^{3/2}} - 1\right]. \tag{A12}$$

At $e^{y_0/l}l/r \ll 1$, using $K = \bar{C}e^{2y_0/l} = 2R_s l^2e^{2y_0/l}$, we have

$$\partial_\epsilon\Phi|_{\epsilon=0} \simeq 4\bar{C}e^{6y_0/l}\frac{l^3}{r^6} = 4\frac{\Phi|_{\epsilon=0}}{l}. \tag{A13}$$

Then also

$$\partial_\epsilon^2\Phi|_{\epsilon=0} = \frac{8}{l^2}\Phi|_{\epsilon=0} + Ke^{2y_0/l}\int_0^\infty dq q^3J_0(qr) \times K_0(e^{y_0/l}lq)I_2(e^{y_0/l}lq). \tag{A14}$$

Now, however, we can only do the remaining integral if the argument is small. Namely, one could try using again $I_2(z) = -2I_1(z)/z + I_0(z)$ and

$$\int_0^\infty x^3J_0(x)K_0(ax)I_0(ax) = -4\frac{1 + 2a^2 + 4a^4}{(1 + 4a^2)^{3/2}}, \tag{A15}$$

but the I_1 integral cannot be done. Instead, we expand $K_0(ax)I_2(ax)$ and find

$$\int_0^\infty x^3J_0(x)K_0(ax)I_2(ax) \simeq 8a^2 - 96a^4 + o(a^6), \tag{A16}$$

and thus for $e^{y_0/l}l/r \ll 1$ we have

$$\partial_\epsilon^2\Phi|_{\epsilon=0} \simeq 16 \cdot 2R_s\frac{l^4}{r^6}e^{6y_0/l}. \tag{A17}$$

Thus at $e^{y_0/l}l/r \ll 1$ and $y = y_0$

$$f = \Phi|_{\epsilon=0} = \bar{C}\frac{l^4}{r^6}e^{6y_0/l}, \qquad a = \frac{4}{l}f + \zeta_1,$$
$$g = \partial_\epsilon^2\Phi|_{\epsilon=0} + \frac{4}{l}\zeta_1 = \frac{16f}{l^2} + \frac{4\zeta_1}{l}. \tag{A18}$$

If a is nonzero, we have to match

$$4 = f'^2 + a^2e^{-2y_0/l} + \epsilon\left(2a'f' - 2\frac{a^2}{l}e^{-2y_0/l} + 2age^{-2y_0/l}\right) + \dots \tag{A19}$$

with

$$C = f + a\epsilon + \dots \tag{A20}$$

If we put $\zeta_1 = 0$ we then get

$$1 = \left(R_s \frac{l^6}{r^6} e^{6y_0/l}\right)^2 \frac{16}{l^2} e^{-2y_0/l}\left(1 + \frac{9}{4}\frac{l^2}{r^2} e^{2y_0/l} + 6\frac{\epsilon}{l} + \dots\right)$$
$$= \left(\frac{f}{2}\right)^2 \frac{16}{l^2} e^{-2y_0/l}\left(1 + \frac{9}{4}\frac{l^2}{r^2} e^{2y_0/l} + 6\frac{\epsilon}{l} + \dots\right), \quad \text{(A21)}$$

where now the first term comes from $(\partial_y \Psi)^2$, to be matched with

$$C^2 = f^2\left(1 + \frac{8\epsilon}{l} + \dots\right), \quad \text{(A22)}$$

which does not work. The next try is to put a nonzero ζ_1 that still keeps a nonzero. If it keeps it of the same order as the first term, we will have

$$a = -\frac{\alpha}{l} f, \quad \text{(A23)}$$

which implies we have to match

$$1 = \left(\frac{f}{2}\right)^2 \frac{\alpha^2}{l^2} e^{-2y_0/l}\left(1 + 6\frac{\epsilon}{l} + \dots\right) \quad \text{(A24)}$$

with

$$C^2 = f^2\left(1 - \frac{2\alpha\epsilon}{l} + \dots\right). \quad \text{(A25)}$$

This happens to have a solution, $\alpha = -3$. But let us see if it is unique. We could also have $a = 0$ to this order ($\alpha = 0$). Then let us assume that a is proportional to the next term in the expansion, namely,

$$a = \beta \frac{f}{r} e^{y_0/l} \quad \text{(A26)}$$

so that $\zeta_1 = 4f/l + a$ and so $g = 4a/l + \dots$. Then one finds

$$1 = \left(\frac{f}{2r}\right)^2 (6^2 + \beta^2)\left(1 + \frac{\epsilon}{l}\frac{6\beta^2}{6^2 + \beta^2}\right), \quad \text{(A27)}$$

to be matched with

$$C^2 = f^2\left(1 + 2\epsilon \frac{\beta}{r} e^{y_0/l} + \dots\right), \quad \text{(A28)}$$

and, as we see, the functional dependence is different. For a higher power in the expansion if

$$a = \beta \frac{f}{r} e^{y_0/l}\left(\frac{l}{r} e^{y_0/l}\right)^n, \qquad n \geq 1, \quad \text{(A29)}$$

the first equation will be

$$1 = \left(\frac{6f}{r}\right)^2\left\{1 + \frac{\epsilon}{6l}\left[\beta^2\left(\frac{l}{r} e^{y_0/l}\right)^{2n} + 2\beta(7+n)\left(\frac{l}{r} e^{y_0/l}\right)^{n+1}\right]\right\}, \quad \text{(A30)}$$

whereas the second will be

$$C^2 = f^2\left(1 + 2\epsilon \frac{\beta}{r} e^{y_0/l}\left(\frac{l}{r} e^{y_0/l}\right)^n + \dots\right). \quad \text{(A31)}$$

We see that for $n > 1$ we have a mismatch—we would need $2(n+7)/6 = 2$, which is impossible—whereas for $n = 0$ there is a different functional dependence (and for $n < -1$ we can treat it separately and convince ourselves that the functional dependence is also different). But for $n = 1$ we have another solution; the matching condition is $\beta(\beta + 4) = 0$, so

$$a = -4\frac{fl}{r^2} \quad \text{(A32)}$$

Finally, we cannot have $a = 0$ exactly, since then we need to match

$$4 = f'^2 + y^2(f'g' + g^2 e^{-2y_0/l}) + \dots \quad \text{(A33)}$$

to

$$C = f + \frac{y^2}{2} g + \dots \quad \text{(A34)}$$

and then we have $g = 0$.

So we have two solutions: $a = 3f/l$ and $a = -4fl/r^2$. Which should we choose? A physical argument shows us what happens. In the second solution, a is negligible at $y = y_0$, so we have the same continuity condition that we would have if the shockwave scattering was four dimensional. But now we are in AdS space and the warping is very large, so we are at higher energy away from $y = y_0$, therefore we expect that the black hole formed is larger than what we would have in the four-dimensional case. So whereas the second solution describes a trapped surface that would be there even if the space would be four dimensional, the second trapped surface is larger and is due to the very large warping outside the four-dimensional slice.

We might be worried that there is some theorem stating there should only be a trapped surface, as the trapped surface problem is similar to the Green problem with Neumann boundary conditions in electrostatics. But this is not quite so, since now we must also determine the boundary C from the condition that $(\nabla\Psi)^2 = 4$ matches $\Psi =$ (arbitrary) constant, together with the solution to the Laplace equation $\Delta(\Psi - \Phi) = 0$, so it is not quite the same Green problem.

Thus we take the larger of trapped surfaces, with $a = -\alpha f/l$, in which case the continuity condition becomes

$$\frac{|\alpha| f}{2l} e^{-y_0/l} = 1, \quad \text{(A35)}$$

so that

$$r = r_{\max} = l e^{y_0/l}\left[\frac{3R_s}{l e^{y_0/l}}\right]^{1/6}, \quad \text{(A36)}$$

and since $R_s = 2G_4\sqrt{s}$, we have that $r_{\max} \sim s^{1/12}$.

Now let us do the same for the case of the wave on the RS IR brane.

We have

$$\int d\Omega_{d-3} e^{iqr\cos\theta} = (2\pi)^{(d-2)/2} \frac{J_{(d-4)/2}(qr)}{(qr)^{(d-4)/2}}, \quad \text{(A37)}$$

which we can apply for

$$\int d^{d-2}\vec{q} e^{i\vec{q}\vec{x}} f(q) = \int_0^\infty q^{d-3} dq \int d\Omega_{d-3} e^{iqr\cos\theta} f(q). \quad \text{(A38)}$$

However, in the case of [6], for the calculation of h_{00}, the integral that one has corresponds formally to $d = 5$ in the above, but it is more useful to do the integral in a different way, namely, to write

$$\int_0^\infty q^2 dq \int d\Omega_2 e^{iqr\cos\theta} f(q) = \frac{-2\pi i}{r} \int_0^\infty q dq (e^{iqr} - e^{-iqr}) f(q), \quad \text{(A39)}$$

and if f is even, i.e. $f(q) = f(-q)$, we can rewrite it as

$$\frac{-2\pi i}{r} \int_{-\infty}^{+\infty} q dq f(q) e^{iqr}. \quad \text{(A40)}$$

For [6], on the IR brane, that is at $y = 0$, the function f is

$$f(q) = \frac{J_2(iql)}{iqJ_1(iql)} = \frac{I_2(ql)}{qI_1(ql)} \quad \text{(A41)}$$

$[I_\nu(iz) = i^{-\nu} J_\nu(-z)]$. More precisely,

$$h_{00}(y=0) = \frac{1}{(2\pi)^3} \int d^3\vec{p} e^{i\vec{p}\vec{x}} \frac{J_2(iql)}{iqJ_1(iql)} = \frac{1}{(2\pi)^3} \left(\frac{-2\pi i}{r}\right) \int_{-\infty}^{+\infty} dq q e^{iqr} \frac{I_2(ql)}{qI_1(ql)}. \quad \text{(A42)}$$

But there is a theorem: For a complex function $\bar{f}(z)$ such that $\lim_{z\to\infty} \bar{f}(z) = 0(\mathrm{Im}(z) > 0)$, and a real $\sigma > 0$, we have

$$\int_{-\infty}^{+\infty} \bar{f}(x) e^{i\sigma x} dx = 2\pi i \sum_{\mathrm{Im}(a)>0} \mathrm{Re}z[F, a], \qquad F(z) = \bar{f}(z) e^{i\sigma z}. \quad \text{(A43)}$$

In Giddings's case [6], $\bar{f}(q) = I_2(ql)/I_1(ql)$.

We get

$$h_{00} \simeq \frac{1}{2\pi r} \sum_n e^{-\bar{q}_n r} \frac{J_2(\bar{q}_n l)}{a_{1,n}}, \quad \text{(A44)}$$

where we have defined $q = i\bar{q}$ and the behavior of the Bessel function near a pole is

$$J_1(z) \sim a_{1,n}(z - z_n), \qquad z \to z_n. \quad \text{(A45)}$$

The zeroes of J_1 are called $j_{1,n}$, so $\bar{q}_n = j_{1,n}/l$. Then

$$h_{00} \simeq \frac{1}{2\pi r} \sum_n e^{-\bar{q}_n r} \frac{J_2(j_{1,n})}{l a_{1,n}} \simeq \frac{1}{2\pi r} e^{-\bar{q}_1 r} \frac{J_2(j_{1,1})}{l a_{1,1}}. \quad \text{(A46)}$$

Let us come back to our case of computing $\Phi(r, y)$. We will first try to do the integral exactly, and see that unfortunately we get nonsensical results, and then make an approximation that allows us to do the integral.

The integral we have is

$$\begin{aligned} I &= \int_0^\infty q dq \int_0^{2\pi} d\theta e^{iqr\cos\theta} f(q) \\ &= \int_0^\infty q dq f(q) \int_{-\pi/2}^{+\pi/2} d\theta (e^{iqr\cos\theta} - e^{-iqr\cos\theta}) \\ &= \int_{-\pi/2}^{+\pi/2} d\theta \int_{-\infty}^{+\infty} q dq f(q) e^{iqr\cos\theta}, \end{aligned} \quad \text{(A47)}$$

and we see that in the θ integration regime $1 \geq \cos\theta \geq 0$, and

$$f(q) = \frac{I_2(ql)}{qI_1(ql)} \quad \text{(A48)}$$

as before, so that

$$I = \int_{-\pi/2}^{+\pi/2} d\theta 2\pi i \sum_n e^{-\bar{q}_n r\cos\theta} \frac{J_2(j_{1,n})}{l a_{1,n}}, \quad \text{(A49)}$$

and correspondingly

$$\begin{aligned} \Phi(r, y=0) &= \frac{2\pi i}{(2\pi)^2} R_s \sum_n \frac{J_2(j_{1,n})}{a_{1,n}} \int_{-\pi/2}^{+\pi/2} d\theta e^{-\bar{q}_n r\cos\theta} \\ &= \frac{2\pi i}{(2\pi)^2} R_s \sum_n \frac{J_2(j_{1,n})}{a_{1,n}} \pi (I_0(\bar{q}_n r) - L_0(\bar{q}_n r)). \end{aligned} \quad \text{(A50)}$$

We are interested in the limit $\bar{q}_n r \gg 1$, and

$$L_0(z) \sim I_0(z) \sim \frac{1}{\sqrt{2\pi z}} e^z \quad \text{(A51)}$$

at large z, so that is not very useful. We can instead expand the integral already and get

$$\int_{-\pi/2}^{+\pi/2} d\theta e^{-\bar{q}_n r\cos\theta} = 2 \int_0^{+\pi/2} d\theta e^{-\bar{q}_n r\cos\theta} \simeq 2 \int_0^\epsilon d\bar{\theta} e^{-\bar{q}_n r\bar{\theta}} \simeq \frac{2}{\bar{q}_n r}, \quad \text{(A52)}$$

so it would seem that there is no exponential behavior. However, the approximations used were contradictory, since we have expanded the previous integral about the point where $\cos\theta = 0$, which is exactly the point where the contour integration theorem does not work.

So we must find an approximation regime when we can do the integral.

This time we do the angular integral and obtain

$$I = 2\pi \int_0^\infty dq J_0(qr) \frac{I_2(ql)}{I_1(ql)}. \quad \text{(A53)}$$

Since we want to have $r \gg l$, we can use the large argument expansion of J_0,

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}}\cos(z-\pi/4) = \frac{1}{\sqrt{\pi z}}(\cos z + \sin z), \quad \text{(A54)}$$

thus

$$\begin{aligned} I &\simeq \sqrt{\frac{2}{\pi r}}\int_0^\infty dq q^{-1/2}\frac{e^{iqr-i\pi/4}+e^{-iqr+i\pi/4}}{2}\frac{I_2(ql)}{I_1(ql)} \\ &= e^{-i\pi/4}\sqrt{\frac{1}{2\pi r}}\int_{-\infty}^{+\infty} dq q^{-1/2}e^{iqr}\frac{I_2(ql)}{I_1(ql)} \\ &= e^{-i\pi/4}\sqrt{\frac{1}{2\pi r}}2\pi\sum_{\mathrm{Im}(q)>0}\mathrm{Re}\Big[e^{iqr}q^{-1/2}\frac{I_2(ql)}{I_1(ql)}, q\Big], \end{aligned} \quad \text{(A55)}$$

so that

$$\begin{aligned} \Phi(r, y=0) &\simeq R_s l\sqrt{\frac{2\pi l}{r}}\sum_n \frac{j_{1,n}^{-1/2}J_2(j_{1,n})}{la_{1,n}}e^{-\bar{q}_n r} \\ &\simeq R_s\sqrt{\frac{2\pi l}{r}}C_1 e^{-\bar{q}_1 r}. \end{aligned} \quad \text{(A56)}$$

Let us now calculate the trapped surface shape in the RS background with

$$\Phi(r,y) = R_s l e^{-2|y|/l}\int_0^\infty dq J_0(qr)\frac{I_2(e^{-|y|/l}lq)}{I_1(lq)}. \quad \text{(A57)}$$

The continuity condition is

$$(\nabla\Phi)^2 + e^{2|y|/l}(\partial_y\Phi)^2 = 0 \quad \text{(A58)}$$

(note the different sign in the exponent, due to the two-brane RS background). We have

$$\partial_y\Phi = -R_s l e^{-3|y|/l}\int_0^\infty dq q J_0(qr)\frac{I_1(e^{-|y|/l}lq)}{I_1(lq)} \quad \text{(A59)}$$

[where we have used $zI_\nu'(z) = zI_{\nu-1}(z) - \nu I_\nu(z)$] which means that $\partial_y\Phi|_{y=0} = 0$. And then

$$\begin{aligned} \partial_y^2\Phi|_{y=0} &= \frac{R_s}{l}\int_0^\infty dq(lq)^2 J_0(qr)\frac{I_0(lq)}{I_1(lq)} \\ &= R_s l\int_0^\infty dq q^2 J_0(qr)\frac{I_2(ql)}{I_1(ql)} \end{aligned} \quad \text{(A60)}$$

[where we have used $zI_\nu'(z) = zI_{\nu-1}(z) - \nu I_\nu(z)$ as well as $I_0(x) = 2I_1(x)/x + I_2(x)$].

Again, as before, the integral is zero in perturbation theory, so we must use the contour integral as before, and obtain

$$\partial_y^2\Phi|_{y=0} \simeq \frac{R_s l}{\sqrt{2\pi r}}\frac{2\pi i}{\sqrt{i}}\sum_{\mathrm{Im}(q)>0}\mathrm{Re}z\Big[e^{iqr}q^{-1/2}\frac{q^2 I_2(lq)}{I_1(lq)}, q\Big]. \quad \text{(A61)}$$

We get

$$\begin{aligned} \partial_y^2\Phi|_{y=0} &\simeq -\frac{R_s l}{\sqrt{2\pi r}}\frac{2\pi\sqrt{l}}{l^2}\sum_n j_{1,n}^{-1/2}\frac{j_{1,n}^2 J_2(j_{1,n})}{la_{1,n}}e^{-\bar{q}_n r} \\ &\simeq -\frac{R_s}{l^2}\sqrt{\frac{2\pi l}{r}}\tilde{C}_1 e^{-\bar{q}_1 r} = -\Phi|_{y=0}\bar{q}_1^2 \end{aligned} \quad \text{(A62)}$$

[since $\tilde{C}_1 = C_1(j_{1,1})^2$].

We are now ready to apply the formalism for the trapped surface in curved background. In

$$\Psi = f + ay + g\frac{y^2}{2} + \ldots = C, \quad \text{(A63)}$$

we have

$$a = \zeta_1, \qquad g = \partial_y^2\Phi|_{y=0} - \frac{4}{l}\zeta_1, \qquad f = \Phi|_{y=0}. \quad \text{(A64)}$$

We first try $a = \zeta_1 = 0$ (so that $\Psi = \Phi$). Then we need to match

$$C = f + g\frac{y^2}{2} + \ldots, \qquad 4 = f'^2 + y^2(f'g' + g^2) + \ldots, \quad \text{(A65)}$$

where $g = -f\bar{q}_1^2$ and $f' = -f\bar{q}_1$. We obtain

$$C^2 = f^2(a - y^2\bar{q}_1^2) + \ldots, \qquad 4 = f^2\bar{q}_1^2(1+0) + \ldots, \quad \text{(A66)}$$

which we see do not match. So we need to put a nonzero $a = \zeta_1$.

Then we would need to match

$$\begin{aligned} C &= f + ay + \ldots, \\ 4 &= f'^2 + a^2 + y\Big(2a'f' + 2\frac{a^2}{l} + 2ag\Big) + \ldots, \end{aligned} \quad \text{(A67)}$$

where $f' = -\bar{q}_1 f$ and $g = -\bar{q}_1^2 f - 4a/l$.

We will try first $a = \alpha f/l$, which gives a term in the continuity equation comparable with the leading term. We get

$$\begin{aligned} C^2 &= f^2\Big(1 + 2\frac{\alpha}{l}y\Big) + \ldots\frac{4}{\bar{q}_1^2 + \alpha^2/l^2} \\ &= f^2\Big[1 + \frac{y}{\bar{q}_1^2 + \alpha^2/l^2}\Big(-\frac{6\alpha^2}{l^3}\Big)\Big] + \ldots, \end{aligned} \quad \text{(A68)}$$

and by matching the two equations we get

$$\alpha = \frac{-3 \pm \sqrt{9 - 4l^2\bar{q}_1^2}}{2}. \tag{A69}$$

However, since $j_{1,1} = \bar{q}_1 l \simeq 3.83$, there is no real solution.

Next, we try an order of l/r down from the previous try, namely, $a = \beta f/r$. We get

$$C^2 = f^2\left(1 + 2\frac{\beta}{r}y\right) + \ldots,$$
$$4 = f^2\bar{q}_1^2\left(1 + o\left(\frac{yl}{r^2}\right)\right) + \ldots, \tag{A70}$$

so now even the r dependence does not match. We can see that by adding powers of l^n/r^n we generate again a mismatch of r dependence.

Conversely, we can try to have an $a = \zeta_1$ which is more important than the leading term, namely, $a = \alpha f r/l^2$. Then we get

$$C^2 = f^2\left(1 + 2\alpha\frac{yr}{l^2}\right) + \ldots,$$
$$4 = f^2\frac{\alpha^2 r^2}{l^4}\left(1 - \frac{6y}{l}\right) + \ldots, \tag{A71}$$

so again we have a mismatch of r dependence.

For higher powers of r^n/l^n

$$a = \alpha\frac{fr}{l^2}\left(\frac{r}{l}\right)^n; \tag{A72}$$

we need to match

$$C^2 = f^2\left[1 + 2\alpha\frac{yl}{r^2}\left(\frac{r}{l}\right)^n\right],$$
$$4 = \frac{f^2\alpha^2 r^2}{l^4}\left[1 - \frac{6y}{l}\left(\frac{r}{l}\right)^{2n}\right], \tag{A73}$$

and we see that we get a solution for $n = 1$, $\alpha = -3$, thus

$$a = -3\frac{fr^2}{l^3}. \tag{A74}$$

It would seem that we have exhausted all the possibilities, but this is actually not so. We can still try

$$a = \zeta_1 = \alpha\frac{e^{\beta r}}{r}f, \tag{A75}$$

which gives

$$C^2 = f^2\left(1 + 2\alpha\frac{ye^{\beta r}}{r}\right) + \ldots,$$
$$4 = f^2\bar{q}_1^2\left[1 - 2\alpha\frac{ye^{\beta r}}{r\bar{q}_1^2}\left(\beta\bar{q}_1 + 3\alpha\frac{e^{\beta r}}{lr}\right)\right] + \ldots \tag{A76}$$

We can easily see that $\beta = -\bar{q}_1$ makes the two equations equal, so

$$a = \zeta_1 = \alpha\frac{e^{-\bar{q}_1 r}}{r}f, \tag{A77}$$

and strangely α is arbitrary (but probably it will be fixed in a higher order in y).

In any case, we see that this solution for a does not change the leading order equation at $y = 0$, which is

$$\frac{|f|\bar{q}_1}{2} = 1 \quad \Rightarrow \quad \frac{R_s\bar{q}_1}{2}\sqrt{\frac{2\pi l}{r}}C_1 e^{-\bar{q}_1 r} = 1, \tag{A78}$$

which has a solution that is approximately

$$r_H \simeq \frac{1}{2\bar{q}_1}\ln(2\bar{q}_1 A), \qquad A = \left(\frac{R_s\bar{q}_1}{2}\right)^2 2\pi l C_1^2. \tag{A79}$$

The fact that there is a solution can be easily seen by considering the function

$$\tilde{g}(r) = r - Ae^{-2\bar{q}_1 r} \Rightarrow \tilde{g}'(r) = 1 + 2\bar{q}_1 A e^{-2\bar{q}_1 r} > 0, \tag{A80}$$

and the solution we are looking for is given by $\tilde{g}(r) = 0$. Since $\tilde{g}(0) = -A$ and the function is monotonically increasing, it will have a solution.

At nonzero impact parameter, we get the equation

$$\left(\frac{f\bar{q}_1}{2}\right)^2\left(1 - \frac{b^2}{2r^2}\right) = 1. \tag{A81}$$

Its solution is thus the zero of the function

$$g(r) = r - \left(1 - \frac{b^2}{2r^2}\right)Ae^{-2\bar{q}_1 r}, \tag{A82}$$

but now the analysis of the solution is a bit more involved. Indeed, now

$$g'(r) = 1 + 2\bar{q}_1 A\left(1 - \frac{b^2}{2r^2}\right)e^{-2\bar{q}_1 r} - \frac{b^2}{r^3}Ae^{-2\bar{q}_1 r}. \tag{A83}$$

Luckily, we are only interested in the maximum value $b_{\max}$ for which $g(r) = 0$ has a solution.

If $r^2 < b^2/2$, then $g(r) > 0$. In particular, $g(0) = +\infty$, $g(b/\sqrt{2}) = b/\sqrt{2}$. To have a solution of $g(r) = 0$, we need a minimum, $g'(r_1) = 0$, with $g(r_1) < 0$, so necessarily $r_1^2 > b^2/2$.

But if $1 - b^2/(2r^2) \sim 1$ ($b^2/2$ is significantly lower than r^2), the second term in (A83) is larger than the third, so $g'(r) > 0$ (as $\bar{q}_1 \sim 1/l \gg 1/r$). Thus we need instead

$$\frac{b^2}{2r_1^2} = 1 - \epsilon \simeq 1, \tag{A84}$$

so if $g(r_1) < 0$ we get

$$g(r_1) \simeq r_1 - \frac{A}{\bar{q}_1 r_1}e^{-2\bar{q}_1 r_1}. \tag{A85}$$

At $b = b_{\max}$, $g(r_1) = 0$, so

$$r_1 = \frac{A}{\bar{q}_1 r_1} e^{-2\bar{q}_1 r_1}, \qquad b^2 \simeq 2r_1^2$$

$$\Rightarrow b_{\max}(\sqrt{s}) \simeq \sqrt{2}\frac{1}{2\bar{q}_1}\ln(4\bar{q}_1 A) = \frac{\sqrt{2}}{\bar{q}_1}\ln\left[R_s\bar{q}_1\left(\sqrt{j_{1,1}}C_1\sqrt{\frac{\pi}{2}}\right)\sqrt{2}\right], \quad \text{(A86)}$$

where $R_s = \sqrt{s}G_4$, $G_4 = 1/(lM_{P,5}^3)$.

However, the same physical argument we have used in the AdS case applies. The two solutions we have obtained correspond to the trapped surface that would be there if we had a four dimensional scattering, and only the function Φ would be different (for $a = \alpha e^{-\bar{q}_1 r} f/r$), and the one due to the very large warping outside the 4D IR brane. As the large warping will increase the size of the black hole being formed, we have to take the larger trapped surface, described by

$$a = -3\frac{fr^2}{l^3}, \quad \text{(A87)}$$

which gives the continuity condition at $y = 0$

$$\left(\frac{|f|\bar{q}_1}{2}\right)\frac{|\alpha|r}{\bar{q}_1 l^2} = 1 \Rightarrow \frac{R_s\bar{q}_1}{2}\sqrt{\frac{2\pi l}{r}}\frac{|\alpha|C_1}{\bar{q}_1 l}\frac{r}{l}e^{-\bar{q}_1 r} = 1, \quad \text{(A88)}$$

which has a solution that is approximately

$$r_H \simeq \frac{1}{2\bar{q}_1}\ln\left(\frac{\bar{A}}{2\bar{q}_1}\right), \qquad \bar{A} = \left(\frac{R_s\bar{q}_1}{2}\right)^2\frac{2\pi}{l}\left(\frac{3C_1}{\bar{q}_1 l}\right)^2. \quad \text{(A89)}$$

We can again see that there is a solution by considering the function

$$\tilde{g}(r) = r - \bar{A}r^2 e^{-2\bar{q}_1 r} \Rightarrow \tilde{g}'(r) = 1 + \left(\bar{q}_1 - \frac{1}{r}\right)2\bar{A}r^2 e^{-2\bar{q}_1 r}, \quad \text{(A90)}$$

and the solution for r_H is given by $\tilde{g}(r) = 0$. We can see that $\tilde{g}'(r) > 0$ if $r > 1/\bar{q}_1$ and

$$\tilde{g}\left(\frac{1}{\bar{q}_1}\right) = \frac{1}{\bar{q}_1}\left[1 - \frac{R_s^2}{l^2}\frac{18\pi C_1^2}{4e^2 j_{1,1}}\right] < 0 \quad \text{(A91)}$$

for sufficiently large R_s, so there will be a solution.

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