Nonlinear realization of pure $\mathcal{N} = 4$, $D = 5$ supergravity

Nejat T. Yilmaz*

Department of Mathematics and Computer Science, Çankaya University, Öğretmenler Cad. No:14, 06530, Balgat, Ankara, Turkey
(Received 15 August 2005; published 18 November 2005)

We perform the nonlinear realization or the coset formulation of the pure $\mathcal{N} = 4$, $D = 5$ supergravity. We derive the Lie superalgebra which parameterizes a coset map whose induced Cartan-Maurer form produces the bosonic field equations of the pure $\mathcal{N} = 4$, $D = 5$ supergravity by canonically satisfying the Cartan-Maurer equation. We also obtain the first-order field equations of the theory as a twisted self-duality condition for the Cartan-Maurer form within the geometrical framework of the coset construction.

DOI: 10.1103/PhysRevD.72.105013 PACS numbers: 04.65.+e, 11.10.Lm, 11.15.−q, 12.60.Jv

I. INTRODUCTION

The method of dualization has been used in [1] to formulate the bosonic sectors of the $D = 11$ supergravity [2] and the maximal supergravities which are obtained from the $D = 11$ supergravity by torodial compactifications as coset models. The first-order formulations of these theories are also embedded in the nonlinear coset formulations as twisted self-duality constraints [3]. The main motivation to study the nonlinear realizations of the $D = 11$ supergravity and its Kaluza-Klein descendant theories is to understand the global symmetries of the $D = 11$ supergravity thus the symmetries of the M-theory.

In general the supergravity theories are the massless sectors or the low energy effective limits of the relative string theories. Thus the symmetries of the supergravity theories have been studied to improve the knowledge of the symmetries and the duality relations of the string theories. In this respect the global symmetries of the supergravity theories gain importance since a restriction of the global symmetry group G of the supergravity theory to the integers is conjectured to be the U-duality symmetry of the relative string theory [4,5]. The nonlinear sigma model or the coset formulation of the supergravities provides an effective tool to study the global symmetries of these theories and thus the symmetries of the relative string theories as mentioned above.

The scalar coset manifolds of the maximal supergravities are based on split real form global symmetry groups [6–10]. In [10] the method of dualization is extended to the nonsplit real form scalar cosets. This enlarged formulation has enabled the coset construction of the matter coupled supergravities which have nonsplit scalar cosets [11–13]. A general dualization and nonlinear realization is also performed for a generic scalar coset which is coupled to matter in [14].

In this work we present the nonlinear realization or the coset construction of the bosonic sector of the pure $\mathcal{N} = 4$, $D = 5$ supergravity [15–18]. Our formulation will be in parallel with the ones in [1,11–13]. We will propose a coset map and we will derive the Lie superalgebra which parameterizes the coset representatives so that the Cartan-Maurer form of the coset map will lead us to the bosonic field equations of the theory by satisfying the Cartan-Maurer equation. Therefore the definition of the coset map and the construction of the algebra which parameterizes this map enables us to formulate the pure $\mathcal{N} = 4$, $D = 5$ supergravity as a nonlinear coset sigma model. The first-order formulation of the theory will also be derived and we will mention its role as a constraint condition in the coset construction.

In section two we will derive the field equations of the pure $\mathcal{N} = 4$, $D = 5$ supergravity. We will also give the locally integrated first-order field equations. In section three after introducing the algebra generators and the coset map we will construct the Lie superalgebra structure of the field generators. We will also show that the first-order field equations can be obtained as a twisted self-duality condition within the coset construction.

II. THE PURE $\mathcal{N} = 4$, $D = 5$ SUPERGRAVITY

The field content of the pure $\mathcal{N} = 4$, $D = 5$ supergravity [15–18] is

$$(e^r_\mu, \psi^i_\mu, \chi^j, v_\mu, V^k_\mu, \phi), \tag{2.1}$$

where e^r_μ is the fünfbein, v_μ is a one-form field in the **1** representation of USp(4), V^k_μ are five one-form fields for $k = 1, \ldots, 5$ in the **5** representation of USp(4), ϕ is a scalar field which is singlet under the action of USp(4), also ψ^i_μ for $i = 1, \ldots, 4$ are four gravitini in the **4** representation of USp(4) and for $j = 1, \ldots, 4$ χ^j are four spin-1/2 fields again in the **4** representation of USp(4). The local symmetries of the pure $\mathcal{N} = 4$, $D = 5$ supergravity are the general coordinate transformations, $\mathcal{N} = 4$ supersymmetry and the $U(1)^6$ gauge invariance whose Abelian gauge fields are the six one-forms. The global symmetry of the theory or the U-duality group is USp(4) × SO(1, 1) which is an electric subgroup of the $D = 4$ Sp(14, **R**) duality group. The R-symmetry group USp(4) is the automorphism group of the $\mathcal{N} = 4$, $D = 5$ supersymmetry algebra and USp(4) is isomorphic to Spin(5) which is the covering

*ntyilmaz@cankaya.edu.tr

 © 2005 The American Physical Society

group of SO(5). For the following formulation in accordance with [18] the signature of the space-time metric will be chosen as

$$\eta_{AB} = \mathrm{diag}(-, +, +, +, +). \tag{2.2}$$

Apart from the gravity sector the bosonic Lagrangian of the pure $\mathcal{N} = 4$, $D = 5$ supergravity can be given as [15–18]

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{[(2/\sqrt{6})\phi]} * dV^i \wedge dV_i \\ &- \frac{1}{2} e^{-[(4/\sqrt{6})\phi]} * dv \wedge dv - \frac{1}{2} dV^i \wedge dV_i \wedge v,\end{aligned} \tag{2.3}$$

where $i = 1, \ldots, 5$ and we should remark that we raise or lower the indices by using the Euclidean metric. If we vary the Lagrangian (2.3) with respect to the fields ϕ, v, V^i we can find the second-order field equations, respectively, as

$$\begin{aligned} d(*d\phi) &= \frac{1}{\sqrt{6}} e^{[(2/\sqrt{6})\phi]} * dV^i \wedge dV_i - \frac{2}{\sqrt{6}} e^{-[(4/\sqrt{6})\phi]} * dv \wedge dv, \\ d(e^{-[(4/\sqrt{6})\phi]} * dv) &= -\frac{1}{2} dV^i \wedge dV_i, \\ d(e^{[(2/\sqrt{6})\phi]} * dV^j &+ dV^j \wedge v) = 0. \end{aligned} \tag{2.4}$$

We will locally integrate the bosonic field Eqs. (2.4) by introducing dual fields and by using the fact that locally a closed form is an exact one. Integration will give us the first-order field equations of the theory. Here by integration we mean to cancel an exterior derivative on both sides of the equations. Thus if we introduce the three-form dual field $\tilde{\phi}$ for the original field ϕ, the five two-form fields $\tilde{V}^i$ for V^i and the two-form field $\tilde{v}$ for v we can express the first-order field equations obtained from the second-order field Eqs. (2.4) as explained above as

$$\begin{aligned} *d\phi &= -\frac{1}{\sqrt{6}} V^i \wedge d\tilde{V}_i + \frac{2}{\sqrt{6}} v \wedge d\tilde{v} - d\tilde{\phi}, \\ e^{[(2/\sqrt{6})\phi]} * dV^i + V^i \wedge dv &= -d\tilde{V}^i, \\ e^{-[(4/\sqrt{6})\phi]} * dv &= -d\tilde{v} - \frac{1}{2} V^i \wedge dV_i. \end{aligned} \tag{2.5}$$

When one applies the exterior derivative on both sides of the equations above one obtains the second-order field equations given in (2.4) which do not contain the dual fields $\tilde{\phi}$, $\tilde{v}$, $\tilde{V}^i$. In the next section when we give the nonlinear coset formulation of the bosonic sector of the pure $\mathcal{N} = 4$, $D = 5$ supergravity we will show that the first-order field equations in (2.5) can be obtained from a twisted self-duality condition [1] which the Cartan-Maurer form of the coset map satisfies.

III. THE COSET FORMULATION

In this section we will present the coset construction or the nonlinear realization of the bosonic sector of the pure $\mathcal{N} = 4$, $D = 5$ supergravity. We will introduce the coset map whose Cartan form realizes the second-order bosonic field Eqs. (2.4) by satisfying the Cartan-Maurer equation. Our main objective will be to derive the Lie superalgebra which parameterizes the coset elements and which leads to the nonlinear coset formulation of the theory. We will follow the method of dualization or the doubled formalism of [1] to define the coset map and to obtain the necessary commutation and the anticommutation relations of the generators of the Lie superalgebra which parameterizes the coset representatives.

Likewise in [1,11–13] as a first task we introduce a dual field for each original bosonic field given in (2.1) namely for the fields (v, V^k, ϕ). The corresponding dual fields are

$$(\tilde{v}, \tilde{V}^k, \tilde{\phi}), \tag{3.1}$$

where $\tilde{\phi}$ is a three-form, for $k = 1, \ldots, 5$ $\tilde{V}^k$ are two-forms also $\tilde{v}$ is a two-form field. It will be clear later when we obtain the first-order equations from the explicit calculation of the Cartan-Maurer form of the constructed coset map that these fields coincide with the fields which we have already introduced in the integrated field equations in (2.5). As a matter of fact these fields are the necessary Langrange multipliers to construct the Bianchi Lagrangians from the Bianchi identities of the corresponding original fields [19]. Next we assign a generator for each bosonic field and its dual field. The original generators are

$$(Y, Z_k, K), \tag{3.2}$$

which in the coset map will be coupled to the fields (v, V^k, ϕ) respectively. Besides, the dual generators which will be coupled to the dual fields $\tilde{v}$, $\tilde{V}^i$, $\tilde{\phi}$ in the coset map will be defined as

$$(\tilde{Y}, \tilde{Z}_i, \tilde{K}), \tag{3.3}$$

respectively. The Lie superalgebra generated by the generators in (3.2) and (3.3) will have the $\mathbb{Z}_2$ grading. The generators will be defined to be odd if the corresponding coupling field is an odd degree differential form and otherwise even [1]. Therefore the generators $\{Z_i, Y, \tilde{K}\}$ are the odd generators and $\{K, \tilde{Z}_i, \tilde{Y}\}$ are the even ones. The coset map which we have mentioned above will be parameterized by a differential graded algebra [1]. This algebra contains the local module of the differential forms and the Lie superalgebra of the original and the dual generators we have introduced above. The differential graded algebra structure has the property that the odd (even) generators behave like odd (even) degree differential forms when they commute with the exterior product. The algebra products of the odd generators obey the anticommutation relations whereas the algebra products of the even ones and the mixed ones obey the commutation relations within the Lie superalgebra structure.

Equipped with the necessary algebraic tools, to start the construction of the coset realization of the pure $\mathcal{N} = 4$, $D = 5$ supergravity we first introduce the coset map

$$\nu' = e^{\phi K} e^{V^i Z_i} e^{\upsilon Y} e^{\tilde{\upsilon}\tilde{Y}} e^{\tilde{V}^i \tilde{Z}_i} e^{\tilde{\phi}\tilde{K}}. \tag{3.4}$$

This parametrization may be considered as a map from the five-dimensional space-time into a symmetry group whose structure will not be the interest of this work. Like in [1,11–13] we will solely focus on the derivation of the Lie superalgebra which generates the coset parametrization (3.4). However certainly this algebra would reflect the properties of the target group at least locally [20]. In a chosen matrix representation of the Lie superalgebra the pullback of the Cartan-Maurer form $\mathcal{G}'$ which is defined on the above mentioned target group G through the map (3.4) can be given as

$$\mathcal{G}' = d\nu' \nu'^{-1}. \tag{3.5}$$

As a result of its definition the Cartan form (3.5) satisfies the Cartan-Maurer equation

$$d\mathcal{G}' - \mathcal{G}' \wedge \mathcal{G}' = 0. \tag{3.6}$$

Our construction will be based on the requirement that the Cartan-Maurer equation should produce the second-order field Eqs. (2.4) when calculated explicitly. At first glance one may observe that to calculate the Cartan-Maurer form thus the Cartan-Maurer equation explicitly one needs the algebra commutators and the anticommutators of the field generators (3.2) and (3.3). Therefore one has to discover the Lie superalgebra structure which would give the correct second-order bosonic field Eqs. (2.4) via the Cartan-Maurer equation. When one reaches the algebra structure one completes the coset construction and succeeds in the nonlinear realization of the bosonic sector of the pure $\mathcal{N} = 4$, $D = 5$ supergravity. The construction of the algebra of the field generators will enable us to express the bosonic field equations as ingredients of a coset structure. The direct but the cumbersome way of solving the algebra structure is to calculate the Cartan-Maurer Eq. (3.6) in terms of the unknown structure constants of the algebra and then to read the structure constants by comparing the result with the second-order equations given in (2.4). To advance in this direction one needs to make use of the matrix identities

$$\begin{aligned} de^X e^{-X} &= dX + \frac{1}{2!}[X, dX] + \frac{1}{3!}[X, [X, dX]] + \ldots, \\ e^X Y e^{-X} &= Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \ldots, \end{aligned} \tag{3.7}$$

repeatedly starting from the definition of the Cartan-Maurer form (3.5) and then one should insert the result in the Cartan-Maurer Eq. (3.6) [1,6]. We should remark one point again, like in [11–13] we assume that we work in a matrix representation of the algebra of the original and the dual generators. We will not lay out the long steps of the calculation discussed above. Thus we will only present the results we have obtained. If one performs the above mentioned calculation one finds that the proper commutation and the anticommutation relations of the original and the dual generators in (3.2) and (3.3) which lead to the second-order field equations of (2.4) in (3.6) are

$$\begin{aligned} &[K, Z_i] = \frac{1}{\sqrt{6}} Z_i, \qquad [K, Y] = -\frac{2}{\sqrt{6}} Y, \\ &[K, \tilde{Y}] = \frac{2}{\sqrt{6}} \tilde{Y}, \qquad [K, \tilde{Z}_i] = -\frac{1}{\sqrt{6}} \tilde{Z}_i, \\ &\{Z_i, Z_j\} = \delta_{ij} \tilde{Y}, \qquad \{Z_i, Y\} = \tilde{Z}_i, \\ &[\tilde{Z}_i, Z_j] = \frac{1}{\sqrt{6}} \delta_{ij} \tilde{K}, \qquad [\tilde{Y}, Y] = -\frac{2}{\sqrt{6}} \tilde{K}. \end{aligned} \tag{3.8}$$

The commutation and the anticommutation relations which are not listed in (3.8) vanish. If O represents the set of the original generators and $\tilde{D}$ the set of the dual generators we observe that the algebra constructed in (3.8) obeys the general structure

$$[O, \tilde{D}\} \subset \tilde{D}, \qquad [\tilde{D}, \tilde{D}\} = 0. \tag{3.9}$$

However unlike the algebras constructed in [11–13] we have

$$[O, O\} \subset O \cup \tilde{D}. \tag{3.10}$$

Now since we have derived the algebra structure of the field generators we can calculate the Cartan-Maurer form $\mathcal{G}' = d\nu' \nu'^{-1}$ starting from the coset map (3.4) exactly. The calculation again needs the use of the identities in (3.7) together with the algebra structure derived in (3.8). The calculation yields

$$\begin{aligned} \mathcal{G}' = d\nu' \nu'^{-1} = {} & d\phi K + e^{[(1/\sqrt{6})\phi]} dV^i Z_i + e^{-[(2/\sqrt{6})\phi]} d\upsilon Y \\ & + \left(\frac{1}{2} e^{[(2/\sqrt{6})\phi]} \delta_{ij} V^i \wedge dV^j + e^{[(2/\sqrt{6})\phi]} d\tilde{\upsilon} \right) \tilde{Y} \\ & + (e^{[-(1/\sqrt{6})\phi]} V^i \wedge d\upsilon + e^{[-(1/\sqrt{6})\phi]} d\tilde{V}^i) \tilde{Z}_i \\ & + \left(d\tilde{\phi} + \frac{1}{\sqrt{6}} V^i \wedge d\tilde{V}_i - \frac{2}{\sqrt{6}} \upsilon \wedge d\tilde{\upsilon} \right) \tilde{K}. \end{aligned} \tag{3.11}$$

The nonlinear realization of the supergravity theories as a result of the dualization of the fields is intimately related to the Langrange multiplier methods which give rise to the first-order formulations of these theories [19]. For this reason our coset formulation also yields the locally integrated first-order field Eqs. (2.5). To obtain the first-order field equations from the Cartan-Maurer form (3.11) we have to introduce a pseudoinvolution S of the Lie superalgebra constructed in (3.8). By following the general scheme introduced in [1] also discussed in [9,10] we can define the action of S on the field generators as

$$SY = \tilde{Y}, \qquad SK = \tilde{K}, \qquad SZ_i = \tilde{Z}_i, \qquad S\tilde{Y} = -Y,$$
$$S\tilde{K} = -K, \qquad S\tilde{Z}_i = -Z_i. \tag{3.12}$$

Now if we require the Cartan-Maurer form in (3.11) to obey the twisted self-duality constraint

$$*\mathcal{G}' = S\mathcal{G}', \tag{3.13}$$

we observe that by equating the coefficients of the linearly independent algebra generators in (3.13) when (3.11) is inserted in it we get exactly the first-order field equations derived in (2.5). Since the equations in (2.5) have been obtained from the second-order field Eqs. (2.4) algebraically owing to abolishing an exterior derivative locally we conclude that the twisted self-duality constraint (3.13) which is reserved on the Cartan-Maurer form (3.11) is a justified condition. Thus our formulation have not only reestablished the bosonic sector of the pure $\mathcal{N} = 4$, $D = 5$ supergravity as a coset model but it also has produced the first-order field equations as a twisted self-duality condition within the coset construction.

IV. CONCLUSION

We have applied the formalism of dualization to construct the nonlinear coset formulation of the bosonic sector of the pure $\mathcal{N} = 4$, $D = 5$ supergravity [15–18]. We have introduced a coset map and constructed the Lie superalgebra which parameterizes this map so that the Cartan-Maurer form induced by the coset map generates the second-order bosonic field equations of the theory in the canonical Cartan-Maurer equation. In this way we have established the necessary algebraic background to interpret the bosonic sector of the pure $\mathcal{N} = 4$, $D = 5$ supergravity as a nonlinear coset sigma model. The bosonic field equations are regained as elements of the geometrical framework of the coset construction. As a consequence of the dualization method which originates from [1] the first-order formulation of the theory is also obtained in the coset formulation. The first-order field equations appear as a twisted self-duality constraint [1,3] which the Cartan-Maurer form satisfies. Therefore we get the first-order field equations as a byproduct of our formulation. This is not a surprise as the method of dualization is another manifestation of the Lagrange multiplier formalism.

Although we have derived the Lie superalgebra which forms the gauge to parameterize the coset representatives in our nonlinear construction we have not inquired the group theoretical structure of the coset. The coset map can be considered to be into a group G which is presumably [1,6] the global symmetry group of the doubled Lagrangian which is obtained as a result of the dualization of all the fields. Thus the study of the coset structure may contribute to the understanding of the global symmetries of the theory. The local symmetries can also be examined in the same way. The Lie superalgebra we have constructed can be considered as a key and a good starting point in the identification of the coset structure [20].

One may couple matter multiplets to the pure $\mathcal{N} = 4$, $D = 5$ supergravity studied in this work [15–18]. The resulting Maxwell-Einstein supergravity can be related to the five-dimensional Kaluza-Klein descendant theory of the ten-dimensional low energy effective heterotic string by the redefinition, truncation and dualization of appropriate fields [16,21]. Therefore the coset formulation of this work which brings an insight in the symmetry structure may also help to understand the symmetries of the heterotic string.

The gravity and the fermionic sectors can also be included in the nonlinear realization analysis. Especially the coset formulation can be extended to include the gravity sector in accordance with [22–24] so that the Kac-Moody symmetries of the pure $\mathcal{N} = 4$, $D = 5$ supergravity can be studied. One may also inspect the comparison of the Lie superalgebra derived here with the ones constructed in [11–13] to draw a general scheme of symmetries.

[1] E. Cremmer, B. Julia, H. Lü, and C. N. Pope, Nucl. Phys. **B535**, 242 (1998).
[2] E. Cremmer, B. Julia, and J. Scherk, Phys. Lett. B **76**, 409 (1978).
[3] B. Julia, LPTENS 00/02 2000.
[4] C. M. Hull and P. K. Townsend, Nucl. Phys. **B438**, 109 (1995).
[5] E. Witten, Nucl. Phys. **B443**, 85 (1995).
[6] E. Cremmer, B. Julia, H. Lü, and C. N. Pope, Nucl. Phys. **B523**, 73 (1998).
[7] A. Keurentjes, Nucl. Phys. **B658**, 303 (2003).
[8] A. Keurentjes, Nucl. Phys. **B658**, 348 (2003).
[9] N. T. Yılmaz, Nucl. Phys. **B664**, 357 (2003).
[10] N. T. Yılmaz, Nucl. Phys. **B675**, 122 (2003).
[11] T. Dereli and N. T. Yılmaz, Nucl. Phys. **B691**, 223 (2004).
[12] N. T. Yılmaz, J. High Energy Phys. 09 (2004) 003.
[13] N. T. Yılmaz, J. High Energy Phys. 06 (2005) 031.
[14] T. Dereli and N. T. Yılmaz, Nucl. Phys. **B705**, 60 (2005).
[15] E. Cremmer, LPTENS 80/17, 1980; *Proceedings of the Nuffield Gravity Workshop, Cambridge, England, 1980* (unpublished).
[16] M. Awada and P. K. Townsend, Nucl. Phys. **B255**, 617 (1985).
[17] G. Dall'Agata, C. Herrmann, and M. Zagermann, Nucl. Phys. **B612**, 123 (2001).
[18] G. Villadoro and F. Zwirner, J. High Energy Phys. 07

(**2004**) 055.
[19] C. N. Pope (unpublished).
[20] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Graduate Studies in Mathematics 34 (American Mathematical Society, Providence, RI, 2001).
[21] H. Lü, C. N. Pope, and K. S. Stelle, Nucl. Phys. **B548**, 87 (1999).
[22] P. West, J. High Energy Phys. 08 (2000) 007.
[23] P. West, Classical Quantum Gravity **18**, 4443 (2001).
[24] I. Schnakenburg and P. West, Phys. Lett. B **517**, 421 (2001).