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Minkowski vacuum stress tensor fluctuations

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We study the fluctuations of the stress tensor for a massless scalar field in two- and four-dimensional Minkowski spacetime in the vacuum state. Covariant expressions for the stress tensor correlation function are obtained as sums of derivatives of a scalar function. These expressions allow one to express spacetime averages of the correlation function as finite integrals. We also study the correlation between measurements of the energy density along a world line. We find that these measurements may be either positively correlated or anticorrelated. The anticorrelated measurements can be interpreted as telling us that, if one measurement yields one sign for the averaged energy density, a successive measurement with a suitable time delay is likely to yield a result with the opposite sign.

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I. INTRODUCTION

Because physically realizable states in quantum field theory are not eigenstates of the stress tensor operator, quantum stress tensor fluctuations are a universal feature of quantum fields. These fluctuations can have physical effects, including Casimir force fluctuations [1–4], radiation pressure fluctuations [5], and passive fluctuations of the gravitational field [6–24]. Passive fluctuations of gravity are those driven by fluctuations of the matter field stress tensor, as opposed to the active fluctuations due to the quantum nature of gravity itself. The quantum stress tensor correlation function is singular in the limit of coincident points. However, this does not prevent us from obtaining physically meaningful results for observable quantities, such as the luminosity fluctuations of a distant source seen through the fluctuating spacetime [20]. These observables are expressed as spacetime integrals of the correlation function, which can be defined by an integration by parts procedure. Alternatively, one could use other approaches, such as dimensional regularization [25].

In general, the stress tensor correlation function can be decomposed into three terms: a ''fully normal-ordered'' term which is state dependent, but free of singularities, a vacuum term which is singular, but state independent, and a ''cross term'' which is both singular and state dependent. In many situations, one is interested in state dependent effects, so the vacuum term can be ignored. For example, radiation pressure fluctuations in a coherent state arise solely from the cross term [5]. However, this does not mean that the vacuum term is devoid of any physical content.

The main purpose of this paper is the search for such content. Here we will be concerned with a free, massless scalar field in Minkowski spacetime, and its stress tensor correlation function in the Minkowski vacuum state. In a previous paper [26], we studied the subtle stress tensor correlations in nonvacuum states created by moving mirrors in two-dimensional flat spacetime. One of the key results of the present paper will be the derivation of a covariant expression for the correlation function as a sum of total derivative terms. This expression will be given in Sec. II A for two dimensions and in Sec. III A for four dimensions, with the details of the derivations presented in Appendices A and B, respectively. We will discuss spacetime averages of the energy density correlation function in Secs. II B and III B, and averages along a world line in Secs. II C and III C. The results will be summarized and discussed in Sec. IV. Units in which $\hbar = c = 1$ and a spacelike metric signature will be used throughout this paper.

II. TWO DIMENSIONS

A. Covariant stress tensor correlation function

We will be concerned with the stress tensor correlation function

$$C_{\mu\nu\alpha\beta}(x, x') = \langle :T_{\mu\nu}(x)::T_{\alpha\beta}(x'):\rangle \tag{1}$$

for a massless, minimally coupled scalar field in two-dimensional Minkowski spacetime in the vacuum state. Here $:T_{\mu\nu}(x):$ is the normal-ordered stress tensor operator, so $\langle :T_{\mu\nu}(x):\rangle = 0$. We especially seek an expression for $C_{\mu\nu\alpha\beta}(x, x')$ as a sum of terms, each of which is a total derivative of a function with at most logarithmic singularities as $x' \to x$. This will allow us to define integrals of the correlation function by integration by parts.

Such a form is derived in Appendix A, where it is shown that

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$$C_{\mu\nu\alpha\beta}(x,x') = \frac{1}{384\pi^2}[-8\partial_\mu\partial_\nu\partial_\alpha\partial_\beta f_1 - 2g_{\mu\nu}g_{\alpha\beta}\Box\Box f_2 + (g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\Box\Box f_2 + 2(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta)\Box f_2 - (g_{\alpha\nu}\partial_\mu\partial_\beta + g_{\alpha\mu}\partial_\nu\partial_\beta + g_{\beta\nu}\partial_\mu\partial_\alpha + g_{\beta\mu}\partial_\nu\partial_\alpha)\Box f_2], \quad (2)$$

where

$$f_1 = \ln(\Delta x^2/\ell^2), \quad (3)$$

and

$$f_2 = \ln^2(\Delta x^2/\ell^2), \quad (4)$$

where ℓ is an arbitrary constant with dimensions of length. The correlation function is independent of the choice of ℓ. Here $\Box = \partial^\mu\partial_\mu$ is the wave operator, and $\Delta x^2 = (x^\mu - x'^\mu)(x_\mu - x'_\mu)$. Because $\partial_\mu f_1 = \partial f_1/\partial x^\mu = -\partial_{\mu'} f_1 = \partial f_1/\partial x'^\mu$, the correlation function, Eq. (2), can be written in several equivalent forms.

The energy density correlation function becomes

$$C(x,x') = C_{ttt't'} = -\frac{1}{48\pi^2}\partial_t^4 f_1 = -\frac{1}{48\pi^2}\partial_t^2\partial_{t'}^2 f_1. \quad (5)$$

Note that none of the f_2 terms contribute in this case. This expression allows us to compute the mean squared average energy density. Let $g(t)$ be a time sampling function, and $h(x)$ be a spatial sampling function. Then we define the averaged energy density operator as

$$\bar{\rho} = \int dt\, g(t) \int dx\, h(x) :T_{tt}:. \quad (6)$$

The mean square of this operator is

$$\hat{C} = \langle\bar{\rho}^2\rangle = \int dt\, g(t) \int dx\, h(x) \int dt'\, g(t') \int dx'\, h(x') C(x,x'). \quad (7)$$

If we insert Eq. (5) into the above expression, and then integrate by parts, we can write

$$\hat{C} = -\frac{1}{48\pi^2}\int dt\, \ddot{g}(t) \int dt'\, \ddot{g}(t') \int dx\, h(x) \times \int dx'\, h(x') f_1. \quad (8)$$

In the limit that the width of the spatial sampling function goes to zero, $h(x) \to \delta(x)$ and we obtain

$$\hat{C} = -\frac{1}{48\pi^2}\int dt\, \ddot{g}(t) \int dt'\, \ddot{g}(t') \ln[(\Delta t)^2/\ell^2]. \quad (9)$$

B. Averaging over space and time—2D

Rather than using Eq. (9), in some cases we can also directly evaluate the integral in Eq. (7) using contour integration methods. For the explicit examples to be treated in this paper, the latter approach is more convenient. The energy density correlation function, Eq. (5), can be expressed as

$$C(x,x') = \frac{(\Delta t^2 + \Delta x^2)^2 + 4\Delta t^2\Delta x^2}{4\pi^2(\Delta t^2 - \Delta x^2)^4}, \quad (10)$$

where $\Delta t = t - t'$, and $\Delta x = x - x'$. In this subsection we will sample this correlation function in both space and time with Lorentzian functions of width α in t and t', and β in x and x'. Further, let the spatial sampling functions coincide, but let the temporal ones be displaced by t_0.

Let

$$\hat{C}(t_0) = \int_{-\infty}^{\infty} dt\, g_L(\alpha, t + t_0) \int_{-\infty}^{\infty} dt'\, g_L(\alpha, t') \times \int_{-\infty}^{\infty} dx\, g_L(\beta, x) \int_{-\infty}^{\infty} dx'\, g_L(\beta, x') C(x,x') \quad (11)$$

where

$$g_L(\alpha, t) = \frac{\alpha}{\pi(t^2 + \alpha^2)}, \quad (12)$$

and

$$\int_{-\infty}^{\infty} dt\, g_L(\alpha, t) = 1. \quad (13)$$

Now let $t \to t - t_0$, so that we have

$$\hat{C}(t_0) = \int_{-\infty}^{\infty} dt\, g_L(\alpha, t) \int_{-\infty}^{\infty} dt'\, g_L(\alpha, t') \times \int_{-\infty}^{\infty} dx\, g_L(\beta, x) \times \int_{-\infty}^{\infty} dx'\, g_L(\beta, x') C(t - t' - t_0, x - x') = \int_{-\infty}^{\infty} d\tau\, g_L(a, \tau) \int_{-\infty}^{\infty} d\rho\, g_L(b, \rho) C(\tau - t_0, \rho), \quad (14)$$

where $a = 2\alpha$, $b = 2\beta$, $\tau = t - t'$ and $\rho = x - x'$. In the last step, we used the identity

$$\int_{-\infty}^{\infty} dt\, g_L(\alpha, t) \int_{-\infty}^{\infty} dt'\, g_L(\alpha, t') F(t - t') = \int_{-\infty}^{\infty} d\tau\, g_L(a, \tau) F(\tau). \quad (15)$$

We may do the integral on ρ first, by contour integration. The integrand has simple poles at $\rho = \pm ib$ and fourth order poles at $\rho = \pm(\tau - t_0)$. We choose a contour in the upper half-plane which avoids the fourth order poles, the contour C_1 in Fig. 1. In fact, we could use other contours such as C_2 and still obtain the same answer. Even if we chose a contour which enclosed either of the fourth order poles, our answer for the real part of the

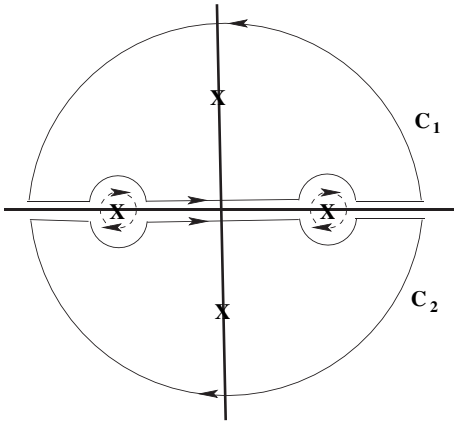

FIG. 1. Some possible integration contours for Eq. (16) are illustrated. There are two simple poles on the imaginary axis, and two higher order poles on the real axis. Both types of poles are denoted by the letter **X**. The contours C_1 and C_2 both yield the same result for the integral. Integration around either of the poles on the real axis (dashed line circles) gives an imaginary result, so the real part of the integral is independent of whether these poles are enclosed or not.

integral would still be the same. This is because the contribution of either of these poles, the result of integrating around the closed circular paths, is pure imaginary. Note that the straight segments and the semicircular segments of C_1 each contain real terms which diverge as the radii of the semicircles go to zero. However, these terms cancel when the straight and semicircular contributions are added. The divergent terms on the straight segments are the boundary terms that would arise from integrating by parts along these segments only. Thus integration by parts along the straight segments and discarding the boundary terms produces the same result as integration along the complete contour.

In any case, using the residue theorem we obtain

$$\int_{-\infty}^{\infty} d\rho\, g_L(b,\rho) C(\tau - t_0, \rho) = \frac{[(\tau - t_0)^2 - b^2]^2 - 4b^2(\tau - t_0)^2}{4\pi^2[(\tau - t_0)^2 + b^2]^4}. \tag{16}$$

The subsequent τ-integration was performed and yields

$$\hat{C}(t_0) = \frac{[t_0(t_0 + 2a + 2b) - (a+b)^2][t_0(t_0 - 2a - 2b) - (a+b)^2]}{4\pi^2[t_0^2 + (a+b)^2]^4}. \tag{17}$$

(This and several other calculations in this paper were done using the public domain algebraic manipulation program MAXIMA.) In the special case when $t_0 = 0$, we simply have

$$\hat{C}(0) = \frac{1}{4\pi^2(a+b)^4}. \tag{18}$$

Let us define

$$K(t_0, a, b) = \frac{\hat{C}(t_0)}{\hat{C}(0)}. \tag{19}$$

In general, we have that $\hat{C}(t_0) = \hat{C}(-t_0)$. From Eqs. (17) and (19), we find that

$$\int_{-\infty}^{\infty} K(t_0, a, b)\, dt_0 = 0, \tag{20}$$

and similarly

$$\int_{0}^{\infty} K(t_0, a, b)\, dt_0 = 0. \tag{21}$$

This result tells us that positively correlated regions ($K > 0$), and anticorrelated regions ($K < 0$) have equal weight.

C. Sampling along a world line—2D

In this subsection, we shall specialize to the case of sampling along a world line, i.e., we will effectively set the width of the spatial sampling function to zero. Define a normal-ordered smeared stress tensor operator by

$$S(t_0) = \int_{-\infty}^{\infty} dt\, g(t, t_0) {:}T_{tt}(t){:}, \tag{22}$$

where $g(t, t_0)$ is a sampling function whose peak is at $t = t_0$. Although $\langle S \rangle = 0$ in the vacuum state, $\langle S^2 \rangle \neq 0$. From Eq. (9), we have that

$$\langle S^2 \rangle = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, g(t, t_0) g(t', t_0) C(t, t') = -\frac{1}{48\pi^2} \int_{-\infty}^{\infty} dt\, \ddot{g}(t, t_0) \int_{-\infty}^{\infty} dt'\, \ddot{g}(t', t_0) \ln[\Delta t^2/\ell^2]. \tag{23}$$

The case we want to consider is two regions of time-sampled energy density which are allowed to initially coincide but which are then gradually separated from one another. One sampling function has its peak at $t' = 0$ and the other at $t = t_0$. We want to imagine sliding these regions away from one another (see Fig. 2), and examine the behavior of the vacuum correlation function as we vary t_0.

With Eq. (22), we can write

$$\langle S(t_0) S(0) \rangle = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, g(t, t_0) g(t', 0) C(t, t') \tag{24}$$

$$= -\frac{1}{48\pi^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \ddot{g}(t, t_0) \ddot{g}(t', 0) \ln[\Delta t^2/\ell^2]. \tag{25}$$

This represents the smeared energy density correlation function for two displaced regions along a world line. We

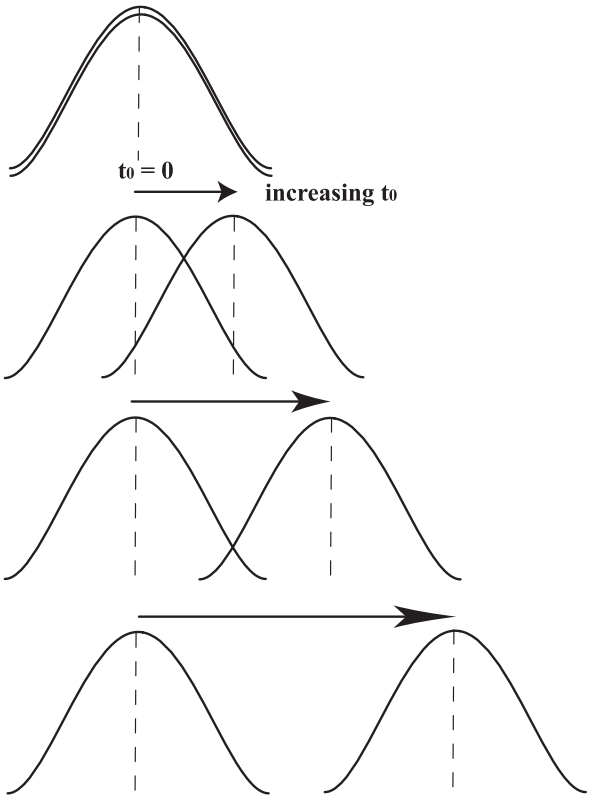

FIG. 2. Two sampling functions which initially coincide and are then gradually separated.

can normalize this quantity by defining

$$K(t_0) = \frac{\langle S(t_0)S(0)\rangle}{\langle S^2(0)\rangle}. \tag{26}$$

As an example, we take the sampling function to be a Lorentzian. If we set $b = 0$ and $a = 1$ in Eq. (17), we find

$$K(t_0) = \frac{(t_0^2 - 2t_0 - 1)(t_0^2 + 2t_0 - 1)}{(t_0^2 + 1)^4} = \frac{(1 - 6t_0^2 + t_0^4)}{(1 + t_0^2)^4}. \tag{27}$$

The choice of $b = 0$ corresponds to sampling in time only, with displaced sampling functions. A plot of this function appears in Fig. 3(a). The plot is somewhat deceiving because there is actually a second positive peak which, on the scale of the plot, is too small to be seen. However, it must be there since $\hat{C}(t_0) \sim 1/(4\pi^2 t_0^4)$, as $t_0 \to \infty$, and hence $K(t_0)$ has to approach 0 from above for large t_0. The magnified view in Fig. 3(b) reveals the second positive peak. We can also see this by computing the extrema of $K(t_0)$ using

$$K'(t_0) = -\frac{4t_0(t_0^4 - 10t_0^2 + 5)}{(t_0^2 + 1)^5}. \tag{28}$$

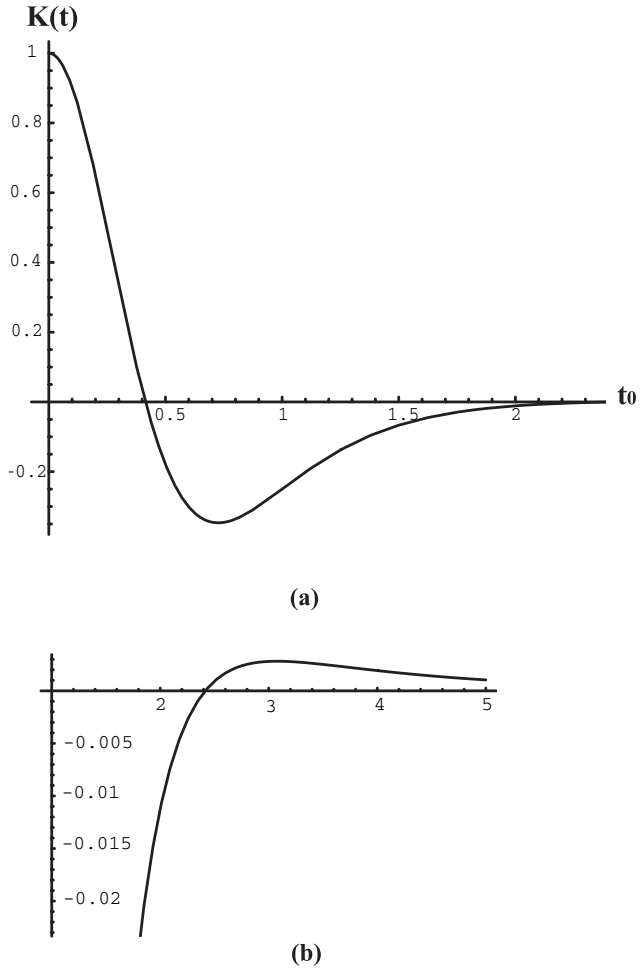

FIG. 3. The graph of $K(t_0)$ versus t_0 for a Lorentzian sampling function, in units with $a = 1$. Here we have chosen $b = 0$, so the sampling is in time only. Here (a) shows the overall form of $K(t_0)$, but on a scale which does not reveal the final maximum. This peak is revealed on a smaller scale graph, (b).

One finds that $K'(t_0) = 0$ at $t_0 = 0$ (first maximum), $t_0 \approx 0.73$ (minimum), and $t_0 \approx 3.1$ (second maximum).

As a second example, consider a compactly supported sampling function of width a with $g = \dot{g} = 0$ at $t = t_0 \pm a/2$. A simple choice of function which has this form is

$$g(t, t_0) = g(t - t_0) = \frac{30}{a^5}(t - t_0 - a/2)^2(t - t_0 + a/2)^2. \tag{29}$$

The second derivative of this function is

$$\ddot{g}(t - t_0) = \frac{30[12(t - t_0)^2 - a^2]}{a^5}, \tag{30}$$

and

$$\langle S^2(0)\rangle = \frac{25}{2\pi^2 a^4}. \tag{31}$$

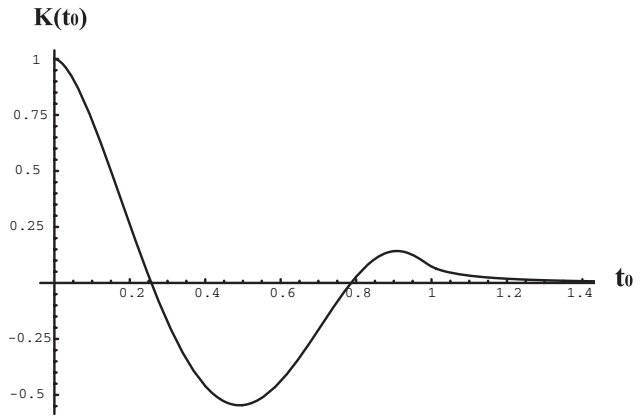

FIG. 4. The graph of $K(t_0)$ versus t_0 for the compactly supported sampling function given by Eq. (29), in units where $a = 1$.

Using Eqs. (25), (26), (30), and (31), one can evaluate $K(t_0)$, which is plotted as a function of t_0 in Fig. 4. Note that the number of maxima and minima of $K(t_0)$ for the compactly supported sampling function, given in Eq. (29), is the same as for the Lorentzian sampling function shown earlier. However, for the compactly supported sampling function case, the second maximum is much more pronounced. A calculation also shows that, for both the Lorentzian and the compactly supported sampling functions, we have that

$$\int_0^\infty K(t_0)dt_0 = 0. \tag{32}$$

We will show in Appendix C that this is true for arbitrary smooth sampling functions. In this appendix, we also prove that

$$\langle S^2(0)\rangle > 0. \tag{33}$$

This establishes that the behavior illustrated in Fig. 4 is independent of the details of the sampling function. The fact that $\langle S^2(0)\rangle > 0$ implies that nearly overlapping regions are positively correlated with one another. As t_0 increases, the correlation is replaced by anticorrelation, as shown by the negative minimum in $K(t_0)$. This anticorrelation implies that, if we measure positive energy in a given region, there must be negative energy found nearby. Finally, when the regions are sufficiently separated, the positive correlation returns, as evidenced by the final positive peak in Fig. 4. One can understand why disjoint regions must be positively correlated from the fact that $C(x, x') > 0$. When $x \neq x'$ everywhere in the range of integration, then the integral for $\hat{C}$ is well defined as an ordinary integral, and must be positive. On the other hand, when we must integrate through points where $x = x'$, then $C(x, x')$ becomes defined only as a distribution, and the integration by parts procedure can produce a negative result.

III. FOUR DIMENSIONS

A. Covariant stress tensor correlation function

In this section, we consider the vacuum stress tensor correlation function in four dimensions. The covariant form of this function is derived in Appendix B, with the result

$$\begin{aligned} C_{\mu\nu\alpha\beta}(x,x') = -\frac{1}{61\,440\pi^4}[&8\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\Box\Box f_2 \\ &+ 6g_{\mu\nu}g_{\alpha\beta}\Box^4 f_2 + (g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\Box^4 f_2 \\ &- 6(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta)\Box^3 f_2 - (g_{\alpha\nu}\partial_\mu\partial_\beta \\ &+ g_{\alpha\mu}\partial_\nu\partial_\beta + g_{\beta\nu}\partial_\mu\partial_\alpha + g_{\beta\mu}\partial_\nu\partial_\alpha)\Box^3 f_2]. \end{aligned} \tag{34}$$

Note that only the function f_2, defined in Eq. (4), appears here, in contrast to the two-dimensional result. The energy density correlation function in four dimensions is given by

$$\begin{aligned} C(x, x') = C_{ttt't'} &= -\frac{1}{7680\pi^4}(\nabla^2)^2\Box^2 f_2 \\ &= -\frac{1}{7680\pi^4}\nabla^2\nabla'^2\Box\Box' f_2, \end{aligned} \tag{35}$$

where $\nabla^2 = \Box + \partial_t^2$ is the three-dimensional Laplacian operator. This form may be used to compute the mean squared average energy density over a spacetime region defined by a sampling function $F(x)$. If we define

$$\bar{\rho} = \int d^4x\, F(x) {:}T_{tt}{:}, \tag{36}$$

then

$$\hat{C} = \langle\bar{\rho}^2\rangle = \int d^4x\, F(x)\int d^4x'\, F(x')C(x, x'). \tag{37}$$

After an integration by parts, this may be expressed as

$$\begin{aligned} \hat{C} = -\frac{1}{7680\pi^4}&\int d^4x\,\nabla^2\Box F(x) \\ &\times\int d^4x'\,\nabla'^2\Box' F(x')f_2(x - x'). \end{aligned} \tag{38}$$

At first sight, the process of obtaining finite spacetime averages of the correlation function may seem mysterious. We start with an expression for $C_{\mu\nu\alpha\beta}(x, x')$ which diverges as $(x - x')^{-8}$ as $x' \to x$, which seems to be a nonintegrable singularity. Yet we nonetheless obtain finite integrals of this expression. The reason that this is possible is that although $C_{\mu\nu\alpha\beta}(x, x')$ is singular as a function, it is a well-defined distribution. This is shown by the existence of the expression, Eq. (34), where $C_{\mu\nu\alpha\beta}(x, x')$ is expressed as

a sum of derivatives of a function with no more than logarithmic singularities. An alternative treatment of the singularities of stress tensor correlation functions was given in Ref. [25]. There dimensional regularization was used to render the correlation functions finite. In the limit in which $n \to 4$, where n is the spacetime dimension, time-ordered stress tensor correlation functions possess a pole term, which can be absorbed in a renormalization involving R^2 and $R_{\mu\nu}R^{\mu\nu}$ counterterms in the gravitational action. However, the correlation functions without time ordering, such as $C_{\mu\nu\alpha\beta}(x, x')$, have no pole term and are hence finite in dimensional regularization in the $n \to 4$ limit. This is another way to understand why $C_{\mu\nu\alpha\beta}(x, x')$ is a well-defined distribution, and why the integration by parts method yields finite results.

B. Averaging over space and time—4D

Here we will perform a calculation analogous to that in Sec. II B, except involving averaging over space and time in four dimensions. The energy density correlation function, Eq. (35), may be expressed as

$$C(x, x') = \frac{(\tau^2 + 3r^2)(3\tau^2 + r^2)}{2\pi^4(\tau^2 - r^2)^6}, \tag{39}$$

where $\tau = t - t'$ and $r = |\mathbf{x} - \mathbf{x}'|$. As before, we use Lorentzian sampling functions of width α in t and in t'. The time-averaged correlation function is

$$\begin{aligned}\hat{C}_T &= \int_{-\infty}^{\infty} dt\, g_L(\alpha, t) \int_{-\infty}^{\infty} dt'\, g_L(\alpha, t') C(x, x') \\ &= \int_{-\infty}^{\infty} d\tau\, g_L(a, \tau) C(x, x'),\end{aligned} \tag{40}$$

where $a = 2\alpha$. The integrand in the τ integral has first order poles at $\tau = \pm ia$ and sixth order poles at $\tau = \pm r$. The integral may be performed by contour integration in a way analogous to the integral in Eq. (16). The result is

$$\hat{C}_T = \frac{(3r^2 - a^2)(r^2 - 3a^2)}{2\pi^4(r^2 + a^2)^6}. \tag{41}$$

Next we wish to average $\hat{C}_T$ over the spatial directions. Here it will be convenient to use a Gaussian sampling function

$$g_G(\beta, x) = \frac{1}{\sqrt{\pi}\beta} e^{-x^2/\beta^2}, \tag{42}$$

in each of the Cartesian space coordinates, x, y, z, x', y', z', and define the spacetime average as

$$\begin{aligned}\hat{C} = &\int_{-\infty}^{\infty} dx\, g_G(\beta, x) \int_{-\infty}^{\infty} dy\, g_G(\beta, y) \int_{-\infty}^{\infty} dz\, g_G(\beta, z) \\ &\times \int_{-\infty}^{\infty} dx'\, g_G(\beta, x') \int_{-\infty}^{\infty} dy'\, g_G(\beta, y') \\ &\times \int_{-\infty}^{\infty} dz'\, g_G(\beta, z') \hat{C}_T.\end{aligned} \tag{43}$$

We may use the fact that

$$\begin{aligned}&\int_{-\infty}^{\infty} dx\, g_G(\beta, x) \int_{-\infty}^{\infty} dx'\, g_G(\beta, x') f(x - x') \\ &\quad = \int_{-\infty}^{\infty} d\Delta x\, g_G(b, \Delta x) f(\Delta x),\end{aligned} \tag{44}$$

where $\Delta x = x - x'$ and $b = \sqrt{2}\beta$. This leads to

$$\begin{aligned}\hat{C} &= \int_{-\infty}^{\infty} d\Delta x\, g_G(b, \Delta x) \int_{-\infty}^{\infty} d\Delta y\, g_G(b, \Delta y) \\ &\quad \times \int_{-\infty}^{\infty} d\Delta z\, g_G(b, \Delta z) \hat{C}_T \\ &= \frac{4}{\sqrt{\pi} b^3} \int_0^{\infty} dr\, r^2 e^{-r^2/\beta^2} \hat{C}_T,\end{aligned} \tag{45}$$

where $r^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$. If we use Eq. (41), then we can write the spacetime averaged correlation function as

$$\hat{C} = \frac{2}{\pi^{9/2} b^3} \int_0^{\infty} dr\, r^2 \frac{(3r^2 - a^2)(r^2 - 3a^2)}{(r^2 + a^2)^6} e^{-r^2/\beta^2}. \tag{46}$$

The integral in the above expression may evaluated in terms of the error function, erf, as

$$\begin{aligned}\hat{C} = \frac{1}{15\pi^4 a b^{13}} \Big\{ &\sqrt{\pi} \Big[1 - \mathrm{erf}\Big(\frac{a}{b}\Big) \Big] e^{a^2/b^2} (15b^6 + 90a^2 b^4 \\ &+ 60a^4 b^2 + 8a^6) - 2ab(3b^2 + 2a^2)(11b^2 + 2a^2) \Big\}.\end{aligned} \tag{47}$$

Now we wish to discuss the limits in which one sampling length scale is small compared to the other. First consider the case of a small spatial scale, $b \ll a$. The exponential factor in Eq. (46) guarantees that only values of $r \lesssim b$ contribute. Thus we can assume that $r \ll a$ in the integrand and write

$$\frac{(3r^2 - a^2)(r^2 - 3a^2)}{(r^2 + a^2)^6} \approx \frac{3}{a^8}. \tag{48}$$

Then we have

$$\hat{C} \approx \frac{3}{2\pi^4 a^8} \tag{49}$$

when $b \ll a$. This shows that only temporal sampling is necessary in order for $\hat{C}$ to be finite. Equation (49) may also be derived from the explicit form, Eq. (47), by use of the asymptotic form of the error function for large argument.

Next we consider the opposite limit, where $a \ll b$. However, $\hat{C} \to \infty$ as $a \to 0$ for fixed, nonzero b. This may be seen from the integral, Eq. (46), which becomes proportional to $\int_0^{\infty} dr r^{-6} e^{-r^2/\beta^2}$ as $a \to 0$. Alternatively, we can expand Eq. (47) for small a and show that

$$\hat{C} \sim \frac{1}{\pi^{7/2} a b^7}, \qquad \text{as } a \to 0. \tag{50}$$

Thus in four dimensions, averaging over space alone is not sufficient to lead to a finite mean squared energy density. This result was obtained previously by Guth [27] and by Roura [28].

C. Sampling along a world line—4D

In the previous subsection, we found that it is possible to take the limit of a vanishing spatial sampling scale, so that one is sampling along a world line. Here we will consider that limit for displaced temporal sampling functions. First consider Lorentzian sampling functions and let

$$\begin{aligned}\hat{C}(t_0, r) &= \int_{-\infty}^{\infty} dt\, g_L(\alpha, t + t_0) \int_{-\infty}^{\infty} dt'\, g_L(\alpha, t') C(\tau, r) \\ &= \int_{-\infty}^{\infty} d\tau\, g_L(a, \tau - t_0) C(\tau, r),\end{aligned} \tag{51}$$

where $C(\tau, r)$ is given by Eq. (39), and $a = 2\alpha$. If we were to sample in space with a function whose width is small compared to a, the result is the same as setting $r = 0$ in the above expression. More precisely, we perform the integral for nonzero r, using the same method as used to obtain Eq. (16), and then take the $r \to 0$ limit. The result is

$$\hat{C}(t_0, 0) = \frac{(t_0^4 - 4a t_0^3 - 6a^2 t_0^2 + 4a^3 t_0 + a^4)(t_0^4 + 4a t_0^3 - 6a^2 t_0^2 - 4a^3 t_0 + a^4)}{\pi^4 (t_0^2 + a^2)^8}. \tag{52}$$

This function has a form similar to that illustrated in Fig. 3, except that it has three maxima and two minima. It is somewhat difficult to graph because the relative sizes of the extrema decrease very rapidly.

In the limit that $r = 0$, we may write the four-dimensional correlation function as

$$C(t, t') = \frac{3}{2\pi^4 (t - t')^8} = -\frac{1}{6720\pi^4} \partial_t^4 \partial_{t'}^4 \ln[(t - t')^2/\ell^2]. \tag{53}$$

We can sample the energy density with arbitrary displaced sampling functions and write

$$\langle S(t_0) S(0) \rangle = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, g(t - t_0) g(t') C(t, t') \tag{54}$$

$$\begin{aligned}&= -\frac{1}{6720\pi^4} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' [\partial_t^4 g(t - t_0)] \\ &\quad \times [\partial_{t'}^4 g(t')] \ln[\Delta t^2/\ell^2].\end{aligned} \tag{55}$$

It should be noted that here we did not use the form of the energy density correlation function, Eq. (35), which follows from the covariant form. Instead, we let $r \to 0$, and then expressed the result in terms of time derivatives of a logarithmic function. A more rigorous approach would be to average Eq. (35) over both space and time, and then let the widths of the spatial sampling functions go to zero. However, this is difficult to do explicitly with general sampling functions. The equivalence of the two approaches needs to be studied more carefully.

Let us next consider a compactly supported sampling function given by

$$g(t) = \frac{630}{a^9} (t - a/2)^4 (t + a/2)^4, \tag{56}$$

for $|t| \leq a/2$, and $g(t) = 0$ for $|t| \geq a/2$. Note that $g(t)$ and its first three derivatives vanish at $t = \pm \frac{1}{2} a$, so all surface terms vanish when we integrated by parts in Eq. (55) to obtain the second form for $\langle S(t_0) S(0) \rangle$. We may again define $K(t_0)$ by Eq. (26) and evaluate it numerically. The result is plotted in Fig. 5.

As in two dimensions, there are regions of correlation and of anticorrelation as t_0 increases. However, the behavior in four dimensions is more complicated, with three maxima and two minima. This appears to be due to the greater number of derivatives of the sampling function in Eq. (55), as compared to Eq. (25).

In Appendix C, we show that in two and four dimensions

$$\int_0^{\infty} K(t_0) dt_0 = 0, \tag{57}$$

and that

$$\langle S^2(0) \rangle > 0, \tag{58}$$

for a general $g(t)$. From Eq. (52), we can also explicitly

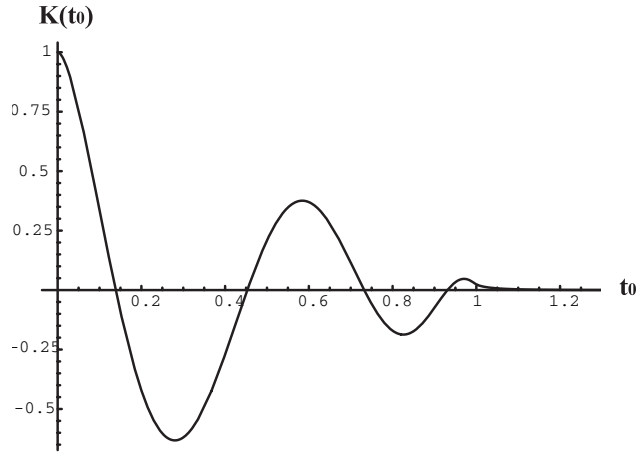

FIG. 5. The graph of $K(t_0)$ in four dimensions as a function of t_0 for the compactly supported sampling function given by Eq. (56), in units where $a = 1$.

verify that $\int_0^\infty K(t_0)dt_0 = 0$ for the Lorentzian sampling function.

IV. SUMMARY

In this paper, we have presented covariant expressions for the Minkowski vacuum stress tensor correlation function in two dimensions, Eq. (2), and in four dimensions, Eq. (34). These expressions are of the form of a sum of terms, each of which is a derivative of a scalar function with logarithmic singularities in the coincidence limit. These expressions allow one to write spacetime averages of the correlation function as finite integrals. We explicitly evaluated such averages of the energy density in two dimensions using Lorentzian sampling functions in both space and time. The resulting expression, Eq. (17), is symmetric in the spatial and temporal sampling widths, and is finite as either width goes to zero with the other width fixed at a nonzero value.

We next studied the correlations of the sampled 2D energy density along a world line using displaced sampling functions. This reveals the correlation and anticorrelation of measurements of the energy density in overlapping intervals. The result is illustrated in Fig. 4 for a compactly supported sampling function. When the intervals nearly overlap, the two measurements are positively correlated, as expected. When the overlap has decreased somewhat, the two measurements become anticorrelated. This can be interpreted as telling us that, if we find energy density of one sign on the first measurement, we should find the opposite sign on the next measurement. Finally, as the intervals become disjoint, the measurements are again positively correlated. Furthermore, we show that, for an arbitrary sampling function, the net area under the correlation graph, e.g., the one depicted in Fig. 4, is equal to zero. It is hoped that further investigation will elucidate this interesting behavior.

The analogous calculation in four dimensions yields similar results. However, in this case there are two regions of anticorrelation and three of positive correlation. The fluctuations in the averaged energy density remain finite in the limit that the spatial width vanishes, but not in the limit that the temporal width goes to zero. Thus, in four dimensions, the averaged energy density correlation function requires averaging in time to be finite.

There is a vaguely analogous result concerning quantum inequalities on the averaged expectation value of the stress tensor in an arbitrary state. There are finite lower bounds on the expectation value of the energy density averaged on a world line in both 2D and 4D, and on the spatial average in 2D. However, the spatial average in 4D has no lower bound [29]. The search for a deeper link between quantum inequalities and the vacuum stress tensor correlation function is a topic for future research.

Another question which needs to be explored further is that of the physical effects of the passive metric fluctuations driven by vacuum stress tensor fluctuations. One approach is that adopted in Ref. [20] where the Raychaudhuri equation was used as a Langevin equation to study the luminosity fluctuations and angular blurring of a distant source produced by passive metric fluctuations. The case of the Minkowski vacuum was briefly discussed in Ref. [20], where it was found that the natural quantum uncertainty in the test particles used to probe the fluctuating geometry tends to hide the effects of the metric fluctuations. However, this does not necessarily mean that these fluctuations are in principle unobservable. This is another question for further study.

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APPENDIX A

In this appendix, we give the derivation of the stress tensor correlation function in two-dimensional spacetime in the Minkowski vacuum state. We first start with the form of the stress tensor for a massless, minimally coupled scalar field:

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi^{,\rho}\phi_{,\rho}. \tag{A1}$$

From this expression we find the correlation function

$$\begin{aligned} C_{\mu\nu\alpha\beta}(x,x') &= \langle{:}T_{\mu\nu}(x){::}T_{\alpha\beta}(x'){:}\rangle \\ &= \langle{:}\partial_\mu\phi\partial_\nu\phi{::}\partial_{\alpha'}\phi\partial_{\beta'}\phi{:}\rangle \\ &\quad - \frac{1}{2}g_{\mu\nu}\langle{:}\partial^\rho\phi\partial_\rho\phi{::}\partial_{\alpha'}\phi\partial_{\beta'}\phi{:}\rangle \\ &\quad - \frac{1}{2}g_{\alpha\beta}\langle{:}\partial_\mu\phi\partial_\nu\phi{::}\partial^{\sigma'}\phi\partial_{\sigma'}\phi{:}\rangle \\ &\quad - \frac{1}{4}g_{\mu\nu}g_{\alpha\beta}\langle{:}\partial^\rho\phi\partial_\rho\phi{::}\partial^{\sigma'}\phi\partial_{\sigma'}\phi{:}\rangle. \end{aligned} \tag{A2}$$

Here unprimed indices refer to the point x and primed indices to x'. Next we use the identity

$$\langle{:}\phi_1\phi_2{::}\phi_3\phi_4{:}\rangle = \langle\phi_1\phi_3\rangle\langle\phi_2\phi_4\rangle + \langle\phi_1\phi_4\rangle\langle\phi_2\phi_3\rangle, \tag{A3}$$

where the ϕ_i are quantum fields or derivatives of quantum fields. From this identity, we can show that

$$\begin{aligned} \langle{:}\partial_\mu\phi\partial_\nu\phi{::}\partial_{\alpha'}\phi\partial_{\beta'}\phi{:}\rangle &= (\partial_\mu\partial_{\alpha'}D)(\partial_\nu\partial_{\beta'}D) \\ &\quad + (\partial_\mu\partial_{\beta'}D)(\partial_\nu\partial_{\alpha'}D), \end{aligned} \tag{A4}$$

where

$$D = D(x, x') = \langle \phi(x)\phi(x') \rangle \tag{A5}$$

is the two-point function. We can express the correlation function in terms of derivatives of D as

$$\begin{aligned} C_{\mu\nu\alpha\beta}(x, x') = {} & (\partial_\mu \partial_{\alpha'} D)(\partial_\nu \partial_{\beta'} D) + (\partial_\mu \partial_{\beta'} D)(\partial_\nu \partial_{\alpha'} D) \\ & - g_{\mu\nu}(\partial^\rho \partial_{\alpha'} D)(\partial_\rho \partial_{\beta'} D) \\ & - g_{\alpha\beta}(\partial_\mu \partial^{\sigma'} D)(\partial_\nu \partial_{\sigma'} D) \\ & - \frac{1}{2} g_{\mu\nu} g_{\alpha\beta}(\partial^\rho \partial^{\sigma'} D)(\partial_\rho \partial_{\sigma'} D). \end{aligned} \tag{A6}$$

An equivalent expression for the case of a massive, nonminimal scalar field has been given by Martin and Verdaguer. [See Eq. (3.42) in Ref. [13].] The analogous expression for the electromagnetic field is given in Ref. [18].

Up to this point, our treatment applies to spacetimes of any dimensionality. Now we specialize to two-dimensional Minkowski spacetime. There is an infrared divergence in the two-point function for a massless scalar field in the Minkowski vacuum state in two dimensions. Thus, the field must either have a nonzero mass, or else the only physically allowed states are ones which break Lorentz invariance [30]. Fortunately, the details of either approach have no effect on our results. If we let the scalar field have a small mass m, then the two-point function is given by

$$D = -\frac{1}{4\pi} \ln(cm^2 \Delta x^2) \tag{A7}$$

in the limit that $m^{-2} \gg \Delta x^2$. Here c is a dimensionless constant and $\Delta x^2 = (x^\mu - x'^\mu)(x_\mu - x'_\mu)$. Because the stress tensor correlation function depends only upon derivatives of D, it is independent of c and m. The second derivative of D is

$$\partial_\mu \partial_{\alpha'} D = -\frac{2\Delta x_\mu \Delta x_\alpha - g_{\mu\alpha} \Delta x^2}{2\pi (\Delta x^2)^2}, \tag{A8}$$

where $\Delta x_\mu = x_\mu - x'_\mu$. We can now combine Eqs. (A6) and (A8) to obtain an explicit expression for the stress tensor correlation function in two dimensions:

$$\begin{aligned} C_{\mu\nu\alpha\beta}(x, x') = \frac{1}{4\pi^2} \Bigg[& \frac{8}{(\Delta x^2)^4} \Delta x_\mu \Delta x_\nu \Delta x_\alpha \Delta x_\beta \\ & - \frac{2}{(\Delta x^2)^3} (g_{\mu\alpha} \Delta x_\nu \Delta x_\beta + g_{\mu\beta} \Delta x_\nu \Delta x_\alpha \\ & + g_{\nu\alpha} \Delta x_\mu \Delta x_\beta + g_{\nu\beta} \Delta x_\mu \Delta x_\alpha) \\ & + \frac{1}{(\Delta x^2)^2} (g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - g_{\mu\nu} g_{\alpha\beta}) \Bigg]. \end{aligned} \tag{A9}$$

We next wish to express $C_{\mu\nu\alpha\beta}(x, x')$ as a sum of derivatives of scalar functions. Lorentz symmetry suggests that these be functions of Δx^2. Let $f = f(\Delta x^2)$. Then the derivatives of f are

$$\partial_\mu f = 2\Delta x_\mu f', \tag{A10}$$

$$\partial_\mu \partial_\nu f = 2 g_{\mu\nu} f' + 4 \Delta x_\mu \Delta x_\nu f'', \tag{A11}$$

$$\begin{aligned} \partial_\mu \partial_\nu \partial_\alpha f = {} & 4(g_{\mu\nu} \Delta x_\alpha + g_{\mu\alpha} \Delta x_\nu + g_{\nu\alpha} \Delta x_\mu) f'' \\ & + 8 \Delta x_\mu \Delta x_\nu \Delta x_\alpha f''', \end{aligned} \tag{A12}$$

and

$$\begin{aligned} \partial_\mu \partial_\nu \partial_\alpha \partial_\beta f = {} & 4(g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} + g_{\mu\nu} g_{\alpha\beta}) f'' \\ & + 8(g_{\mu\alpha} \Delta x_\nu \Delta x_\beta + g_{\mu\beta} \Delta x_\nu \Delta x_\alpha \\ & + g_{\nu\alpha} \Delta x_\mu \Delta x_\beta + g_{\nu\beta} \Delta x_\mu \Delta x_\alpha \\ & + g_{\mu\nu} \Delta x_\alpha \Delta x_\beta + g_{\alpha\beta} \Delta x_\mu \Delta x_\nu) f''' \\ & + 16 \Delta x_\mu \Delta x_\nu \Delta x_\alpha \Delta x_\beta f''''. \end{aligned} \tag{A13}$$

Here primes denote derivatives of f with respect to its argument. We will also need some expressions involving the wave operator:

$$\Box f = \partial_\mu \partial^\mu f = 2n f' + 4 \Delta x^2 f'', \tag{A14}$$

$$\begin{aligned} \Box \Box f = {} & 4n(n+2) f'' + 16(n+2) \Delta x^2 f''' \\ & + 16 (\Delta x^2)^2 f'''', \end{aligned} \tag{A15}$$

and

$$\begin{aligned} \partial_\mu \partial_\nu \Box f = {} & 4(n+2) g_{\mu\nu} f'' + 8[g_{\mu\nu} \Delta x^2 \\ & + (n+4) \Delta x_\mu \Delta x_\nu] f''' \\ & + 16 \Delta x_\mu \Delta x_\nu \Delta x^2 f''', \end{aligned} \tag{A16}$$

where n is the dimension of the spacetime.

Our goal is to express $C_{\mu\nu\alpha\beta}(x, x')$ as a sum of derivatives acting on one or more choices of f. Because $\partial_\mu f_1 = -\partial_{\mu'} f_1$, we can write our results in several equivalent forms, but here and in Appendix B we will use derivatives with unprimed indices. If f is dimensionless, then in two dimensions we will need four derivatives in each term in order that $C_{\mu\nu\alpha\beta}(x, x')$ has dimensions of length^{-4}. There are five fourth-rank tensors that we can form which have the correct dimensions and symmetry properties:

$$\partial_\mu \partial_\nu \partial_\alpha \partial_\beta f, \tag{A17}$$

$$(g_{\alpha\beta} \partial_\mu \partial_\nu + g_{\mu\nu} \partial_\alpha \partial_\beta) \Box f, \tag{A18}$$

$$(g_{\alpha\nu} \partial_\mu \partial_\beta + g_{\alpha\mu} \partial_\nu \partial_\beta + g_{\beta\nu} \partial_\mu \partial_\alpha + g_{\beta\mu} \partial_\nu \partial_\alpha) \Box f, \tag{A19}$$

$$g_{\mu\nu} g_{\alpha\beta} \Box \Box f, \tag{A20}$$

and

$$(g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha}) \Box \Box f. \tag{A21}$$

We would like f to have an integrable singularity at $\Delta x^2 = 0$, so a natural choice is a power of a logarithmic function. First consider

$$f_1 = \ln(\Delta x^2/\ell^2), \tag{A22}$$

where ℓ is an arbitrary constant with dimensions of length. However, $\Box f_1 = 0$ in two dimensions, so the only nonzero tensor from the above list which can be formed from f_1 is

$$\begin{aligned}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta f_1 = &-\frac{96}{(\Delta x^2)^4}\Delta x_\mu\Delta x_\nu\Delta x_\alpha\Delta x_\beta \\ &+\frac{16}{(\Delta x^2)^3}(g_{\mu\alpha}\Delta x_\nu\Delta x_\beta + g_{\mu\beta}\Delta x_\nu\Delta x_\alpha \\ &+ g_{\nu\alpha}\Delta x_\mu\Delta x_\beta + g_{\nu\beta}\Delta x_\mu\Delta x_\alpha \\ &+ g_{\mu\nu}\Delta x_\alpha\Delta x_\beta + g_{\alpha\beta}\Delta x_\mu\Delta x_\nu) \\ &-\frac{4}{(\Delta x^2)^2}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} + g_{\mu\nu}g_{\alpha\beta}).\end{aligned} \tag{A23}$$

This is not sufficient to form $C_{\mu\nu\alpha\beta}(x, x')$, so we need another choice of f, which we take to be

$$f_2 = \ln^2(\Delta x^2/\ell^2). \tag{A24}$$

From Eqs. (A15) and (A16) with $n = 2$, we find

$$\Box f_2 = \frac{8}{\Delta x^2} \tag{A25}$$

and

$$\Box\Box f_2 = \frac{32}{(\Delta x^2)^2}. \tag{A26}$$

This allows us to form four tensors from f_2 with the correct symmetry properties and dimension:

$$g_{\mu\nu}g_{\alpha\beta}\Box\Box f_2 = 32 g_{\mu\nu}g_{\alpha\beta}\frac{1}{(\Delta x^2)^2}, \tag{A27}$$

$$\begin{aligned}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\Box\Box f_2 = &\, 32(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}) \\ &\times\frac{1}{(\Delta x^2)^2},\end{aligned} \tag{A28}$$

$$\begin{aligned}&(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta)\Box f_2 \\ &\quad = -32\frac{g_{\mu\nu}g_{\alpha\beta}}{(\Delta x^2)^2} + 64\frac{g_{\mu\nu}\Delta x_\alpha\Delta x_\beta + g_{\alpha\beta}\Delta x_\mu\Delta x_\nu}{(\Delta x^2)^3},\end{aligned} \tag{A29}$$

and

$$\begin{aligned}&(g_{\alpha\nu}\partial_\mu\partial_\beta + g_{\alpha\mu}\partial_\nu\partial_\beta + g_{\beta\nu}\partial_\mu\partial_\alpha + g_{\beta\mu}\partial_\nu\partial_\alpha)\Box f_2 \\ &\quad = -32\frac{g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}}{(\Delta x^2)^2} + \frac{64}{(\Delta x^2)^3}(g_{\mu\alpha}\Delta x_\nu\Delta x_\beta \\ &\qquad + g_{\mu\beta}\Delta x_\nu\Delta x_\alpha + g_{\nu\alpha}\Delta x_\mu\Delta x_\beta + g_{\nu\beta}\Delta x_\mu\Delta x_\alpha).\end{aligned} \tag{A30}$$

Note that $\partial_\mu\partial_\nu\partial_\alpha\partial_\beta f_2$ is not a suitable term because it contains logarithmic pieces that do not appear in $C_{\mu\nu\alpha\beta}(x, x')$ and which cannot be canceled by any other terms. This leaves us with five tensors from which to form the stress tensor correlation function.

Let

$$\begin{aligned}C_{\mu\nu\alpha\beta}(x, x') = &\frac{1}{384\pi^2}[c_1\partial_\mu\partial_\nu\partial_\alpha\partial_\beta f_1 + c_2 g_{\mu\nu}g_{\alpha\beta}\Box\Box f_2 \\ &+ c_3(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\Box\Box f_2 \\ &+ c_4(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta)\Box f_2 \\ &+ c_5(g_{\alpha\nu}\partial_\mu\partial_\beta + g_{\alpha\mu}\partial_\nu\partial_\beta \\ &+ g_{\beta\nu}\partial_\mu\partial_\alpha + g_{\beta\mu}\partial_\nu\partial_\alpha)\Box f_2].\end{aligned} \tag{A31}$$

If we insert Eq. (A23) and Eqs. (27)–(30) into this expression and compare with Eq. (A9), we find five conditions on the five coefficients. The unique solution of these conditions gives

$$\begin{aligned}&c_1 = -8, \qquad c_2 = -c_4 = -2, \\ &\text{and} \quad c_3 = -c_5 = 1.\end{aligned} \tag{A32}$$

As a check, the correlation function may be shown explicitly to satisfy the conservation law

$$\partial^\mu C_{\mu\nu\alpha\beta}(x, x') = \partial^{\alpha'} C_{\mu\nu\alpha\beta}(x, x') = 0. \tag{A33}$$

APPENDIX B

Here we repeat the derivation in the previous appendix for the case of four-dimensional Minkowski spacetime. The general form, Eq. (A6), for the correlation function still holds, but the two-point function for a massless scalar field is now

$$D = \frac{1}{4\pi^2\Delta x^2}. \tag{B1}$$

If we insert this form into Eq. (A6), we find the four-dimensional analog of Eq. (A9):

$$C_{\mu\nu\alpha\beta}(x,x') = \frac{1}{4\pi^4}\Bigg[\frac{32}{(\Delta x^2)^6}\Delta x_\mu \Delta x_\nu \Delta x_\alpha \Delta x_\beta - \frac{4}{(\Delta x^2)^5}(g_{\mu\alpha}\Delta x_\nu \Delta x_\beta + g_{\mu\beta}\Delta x_\nu \Delta x_\alpha + g_{\nu\alpha}\Delta x_\mu \Delta x_\beta + g_{\nu\beta}\Delta x_\mu \Delta x_\alpha) - \frac{8}{(\Delta x^2)^5}(g_{\mu\nu}\Delta x_\alpha \Delta x_\beta + g_{\alpha\beta}\Delta x_\mu \Delta x_\nu) + \frac{1}{(\Delta x^2)^4}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} + 4g_{\mu\nu}g_{\alpha\beta})\Bigg]. \tag{B2}$$

In four dimensions, the correlation function has dimensions of $1/\text{length}^8$. Thus any expression involving derivatives on a dimensionless function will require eight derivatives. Because there are only four free indices, there will have to be at least two wave operators. This eliminates the logarithm function f_1, Eq. (A22), because in four dimensions

$$\Box\Box f_1 = 0. \tag{B3}$$

However, the squared logarithm function f_2 may be used to form the following five tensors with the correct dimensions and symmetry:

$$\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\Box\Box f_2, \tag{B4}$$

$$(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta)\Box^3 f_2, \tag{B5}$$

$$(g_{\alpha\nu}\partial_\mu\partial_\beta + g_{\alpha\mu}\partial_\nu\partial_\beta + g_{\beta\nu}\partial_\mu\partial_\alpha + g_{\beta\mu}\partial_\nu\partial_\alpha)\Box^3 f_2, \tag{B6}$$

$$g_{\mu\nu}g_{\alpha\beta}\Box^4 f_2, \tag{B7}$$

and

$$(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\Box^4 f_2. \tag{B8}$$

We may repeatedly use Eqs. (A14) and (A15) with $n = 4$ to demonstrate that, in four dimensions,

$$\Box\Box f_2 = -\frac{32}{(\Delta x^2)^2}, \tag{B9}$$

$$\Box^3 f_2 = -\frac{256}{(\Delta x^2)^3}, \tag{B10}$$

and

$$\Box^4 f_2 = -\frac{6144}{(\Delta x^2)^4}. \tag{B11}$$

From these expressions, we may show that

$$\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\Box\Box f_2 = -\frac{61\,440}{(\Delta x^2)^6}\Delta x_\mu \Delta x_\nu \Delta x_\alpha \Delta x_\beta + \frac{6144}{(\Delta x^2)^5}(g_{\mu\alpha}\Delta x_\nu \Delta x_\beta + g_{\mu\beta}\Delta x_\nu \Delta x_\alpha + g_{\nu\alpha}\Delta x_\mu \Delta x_\beta + g_{\nu\beta}\Delta x_\mu \Delta x_\alpha + g_{\mu\nu}\Delta x_\alpha \Delta x_\beta + g_{\alpha\beta}\Delta x_\mu \Delta x_\nu) - \frac{768}{(\Delta x^2)^4}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} + g_{\mu\nu}g_{\alpha\beta}), \tag{B12}$$

$$(g_{\alpha\nu}\partial_\mu\partial_\beta + g_{\alpha\mu}\partial_\nu\partial_\beta + g_{\beta\nu}\partial_\mu\partial_\alpha + g_{\beta\mu}\partial_\nu\partial_\alpha)\Box^3 f_2 = \frac{3072}{(\Delta x^2)^4}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}) - \frac{12\,288}{(\Delta x^2)^5}(g_{\mu\alpha}\Delta x_\nu \Delta x_\beta + g_{\mu\beta}\Delta x_\nu \Delta x_\alpha + g_{\nu\alpha}\Delta x_\mu \Delta x_\beta + g_{\nu\beta}\Delta x_\mu \Delta x_\alpha), \tag{B13}$$

and

$$(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta)\Box^3 f_2 = 3072\Bigg[\frac{g_{\mu\nu}g_{\alpha\beta}}{(\Delta x^2)^4} - 4\frac{g_{\mu\nu}\Delta x_\alpha \Delta x_\beta + g_{\alpha\beta}\Delta x_\mu \Delta x_\nu}{(\Delta x^2)^5}\Bigg]. \tag{B14}$$

We now express the correlation function as a sum of the tensors formed from f_2 as

$$C_{\mu\nu\alpha\beta}(x,x') = \frac{1}{61\,440\pi^4}[c_1\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\Box\Box f_2 + c_2 g_{\mu\nu}g_{\alpha\beta}\Box^4 f_2 + c_3(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\Box^4 f_2 + c_4(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta)\Box^3 f_2 + c_5(g_{\alpha\nu}\partial_\mu\partial_\beta + g_{\alpha\mu}\partial_\nu\partial_\beta + g_{\beta\nu}\partial_\mu\partial_\alpha + g_{\beta\mu}\partial_\nu\partial_\alpha)\Box^3 f_2]. \tag{B15}$$

If we insert the explicit forms for these tensors and compare with Eq. (B2), we again find five conditions on the five coefficients, leading to the solution

$$c_1 = -8, \qquad c_2 = -c_4 = -6, \quad \text{and} \quad c_3 = -c_5 = -1. \tag{B16}$$

As required, the correlation function has a vanishing divergence on any index.

APPENDIX C

In this appendix, we will prove Eqs. (32) and (33) in both two and four dimensions. We will proceed by first showing that

$$\int_{-\infty}^{\infty} K(t_0)dt_0 = 0. \tag{C1}$$

Then we will prove that $K(t_0)$ is a symmetric function, and hence show that

$$\int_0^\infty K(t_0)dt_0 = 0, \tag{C2}$$

as well.

Let $g(t)$ be an arbitrary smooth sampling function. From Eq. (24) or (54) and Eq. (26), we have

$$K(t_0) = \frac{1}{\langle S^2(0)\rangle}\int_{-\infty}^{\infty} dt\, g(t-t_0)\int_{-\infty}^{\infty} dt'\, g(t')C(t-t'). \tag{C3}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} K(t_0)dt_0 &= \frac{1}{\langle S^2(0)\rangle}\int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} dt_0\, g(t-t_0)\\ &\quad\times\int_{-\infty}^{\infty} dt'\, g(t')C(t-t')\\ &= \frac{1}{\langle S^2(0)\rangle}\int_{-\infty}^{\infty} dt'\, g(t')\int_{-\infty}^{\infty} dt\, C(t-t'),\end{aligned} \tag{C4}$$

where we have interchanged the order of integrations, and used the fact that, for $y = t - t_0$,

$$\int_{-\infty}^{\infty} dt_0\, g(t-t_0) = -\int_{\infty}^{-\infty} dy\, g(y) = \int_{-\infty}^{\infty} dy\, g(y) = 1. \tag{C5}$$

However, if we can write $C(t-t') = \partial F(t-t')/\partial t$, where $F(t-t') \to 0$ as $t \to \pm\infty$, then

$$\int_{-\infty}^{\infty} dt\, C(t-t') = [F(t-t')]_{t=-\infty}^{t=+\infty} = 0, \tag{C6}$$

which in turn implies that

$$\int_{-\infty}^{\infty} K(t_0)dt_0 = 0. \tag{C7}$$

Recall that in two dimensions, the world line vacuum correlation function is $C(t-t') = 1/[4\pi^2(t-t')^4]$, and in four dimensions it is $C(t-t') = 3/[2\pi^4(t-t')^8]$, so in both cases the condition Eq. (C6) is satisfied. Note that in four dimensions, it is necessary to assume that we set the spatial separation r in Eq. (39) to zero and then average over time, as discussed in Sec. III C.

We now show that $K(t_0) = K(-t_0)$. Let us write

$$\begin{aligned}\langle S^2(0)\rangle K(t_0) &= \int_{-\infty}^{\infty} dt\, g(t-t_0)\int_{-\infty}^{\infty} dt'\, g(t')C(t-t')\\ &= \int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t})\int_{-\infty}^{\infty} dt'\, g(t')C(\bar{t}+t_0-t')\\ &= \int_{-\infty}^{\infty} d\bar{t}\, g(\bar{t})\int_{-\infty}^{\infty} d\bar{t}'\, g(\bar{t}'+t_0)C(\bar{t}-\bar{t}'),\end{aligned} \tag{C8}$$

where we have let $\bar{t} = t - t_0$, so $t = \bar{t} + t_0$, and $\bar{t}' = t' - t_0$. If we now let $\bar{t}' \to t$, $\bar{t} \to t'$, we have

$$\begin{aligned}\langle S^2(0)\rangle K(t_0) &= \int_{-\infty}^{\infty} dt\, g(t+t_0)\int_{-\infty}^{\infty} dt'\, g(t')C(t'-t)\\ &= \langle S^2(0)\rangle K(-t_0),\end{aligned} \tag{C9}$$

where we have used the fact $C(t'-t) = C(t-t')$. Note that the symmetry of $K(t_0)$ depends only on that of C and does not assume that the sampling function $g(t)$ is symmetric. Thus since

$$\int_{-\infty}^{\infty} K(t_0)\, dt_0 = 0, \tag{C10}$$

and $K(t_0)$ is symmetric, it also follows that

$$\int_0^{\infty} K(t_0)\, dt_0 = 0. \tag{C11}$$

In order to determine whether a fluctuation is correlated or anticorrelated with itself, we must determine the sign of $\langle S^2(0)\rangle$ in the general case. We would expect that a fluctuation should be correlated with itself, and thus that $\langle S^2(0)\rangle > 0$. This can be proven from the fact that $S(0)$, as defined by Eq. (22), is a self-adjoint operator [31]. Let $|\psi\rangle$ be the state under consideration, which in our case is the Minkowski vacuum. Then

$$|\Psi\rangle = S(0)|\psi\rangle \tag{C12}$$

is a well-defined state vector with positive norm. Thus we have

$$\|\Psi\|^2 = \langle\psi|S^\dagger(0)S(0)|\psi\rangle = \langle S^2(0)\rangle > 0. \tag{C13}$$

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