Stable fermion bag solitons in the massive Gross-Neveu model: Inverse scattering analysis

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Formation of fermion bag solitons is an important paradigm in the theory of hadron structure. We study this phenomenon nonperturbatively in the $1 + 1$ dimensional Massive Gross-Neveu model, in the large N limit. We find, applying inverse-scattering techniques, that the extremal static bag configurations are reflectionless, as in the massless Gross-Neveu model. This adds to existing results of variational calculations, which used reflectionless bag profiles as trial configurations. Only reflectionless trial configurations which support a single pair of charge-conjugate bound states of the associated Dirac equation were used in those calculations, whereas the results in the present paper hold for bag configurations which support an arbitrary number of such pairs. We compute the masses of these multibound state solitons, and prove that only bag configurations which bear a single pair of bound states are stable. Each one of these configurations gives rise to an $O(2N)$ antisymmetric tensor multiplet of soliton states, as in the massless Gross-Neveu model.

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I. INTRODUCTION

An important dynamical mechanism, by which fundamental particles acquire masses, is through interactions with vacuum condensates. Thus, a massive particle may carve out around itself a spherical region [1] or a shell [2] in which the condensate is suppressed, thus reducing the effective mass of the particle at the expense of volume and gradient energy associated with the condensate. This picture has interesting phenomenological consequences [1,3].

This dynamical distortion of the homogeneous vacuum condensate configuration, namely, formation of fermion bag solitons, was demonstrated explicitly by Dashen, Hasslacher, and Neveu (DHN) [4] many years ago, in their study of semiclassical bound states in the $1 + 1$ dimensional Gross-Neveu (GN) model [5].

Fermion bags in the GN model were discussed in the literature several other times since the work of DHN, using alternative methods [6–8]. For a review on these and related matters (with an emphasis on the relativistic Hartree-Fock approximation) see [9]. For a more recent review of static fermion bags in the GN model (with an emphasis on reflectionless backgrounds and supersymmetric quantum mechanics) see [10]. The large-N semiclassical DHN spectrum of these fermion bags turns out to be essentially correct also for finite N, as analysis of the exact factorizable S matrix of the GN model reveals [11].

A variational calculation of these effects in the $1 + 1$ dimensional massive generalization of the Gross-Neveu model, which we will refer to as MGN, was carried in [12] a few years ago, and more recently in [13]. In this paper we study static fermion bags in the MGN model using inverse-scattering formalism [14], thus avoiding the need to choose a trial variational field configuration. The present work is thus a natural extension of the inverse-scattering analysis carried out by DHN to the massive case.

The MGN model is defined by the action

$$S = \int d^2x\left\{\sum_{a=1}^{N}\bar{\psi}_a(i\partial\!\!\!/ - M)\psi_a + \frac{g^2}{2}\left(\sum_{a=1}^{N}\bar{\psi}_a\psi_a\right)^2\right\}$$
$$:= \int d^2x\left\{\bar{\psi}[i\partial\!\!\!/ - \sigma]\psi - \frac{1}{2g^2}(\sigma^2 - 2M\sigma)\right\}, \qquad (1)$$

describing N self interacting massive Dirac fermions ψ_a carrying a flavor index $a = 1, \ldots, N$, which we promptly suppress.

An obvious symmetry of (1) with its N Dirac spinors is $U(N)$. Actually, (1) is symmetric under the larger group $O(2N)$ [4] (see also Section 1 of [10]). The fact that the symmetry group of (1) is $O(2N)$ rather than $U(N)$, indicates that it is invariant against charge-conjugation. It is easy to see this in a concrete representation for γ matrices, which we choose as the Majorana representation [4,10]

$$\gamma^0 = \sigma_2, \qquad \gamma^1 = i\sigma_3 \quad \text{and} \quad \gamma^5 = -\gamma^0\gamma^1 = \sigma_1. \qquad (2)$$

(Henceforth, in this paper we will use this representation for γ matrices in all explicit calculations.)

In the representation (2), charge-conjugation is realized simply by complex conjugation of the spinor

$$\psi^c(x) = \psi^*(x). \qquad (3)$$

Thus, if $\psi = e^{-i\omega t}u(x)$ is an eigenstate of the Dirac equation

$$[i\partial\!\!\!/ - \sigma(x)]\psi = 0, \qquad (4)$$

with energy ω (assuming time independent $\sigma(x)$), then $\psi^*(x) = e^{i\omega t}u^*(x)$ is an energy eigenstate of (4), with energy $-\omega$. Therefore, the MGN model (1), like the GN model, is invariant against charge-conjugation, and energy eigenstates of (4) come in $\pm\omega$ pairs.

As usual, the theory (1) can be rewritten with the help of the scalar flavor singlet auxiliary field $\sigma(x)$. Also as usual, we take the large N limit holding $\lambda \equiv Ng^2$ fixed. Integrating out the fermions, we obtain the bare effective action

$$S[\sigma] = -\frac{1}{2g^2}\int d^2x(\sigma^2 - 2M\sigma) - iN\,\mathrm{Trlog}\,(i\partial\!\!\!/ - \sigma). \tag{5}$$

Noting that $\gamma_5(i\partial\!\!\!/ - \sigma) = -(i\partial\!\!\!/ + \sigma)\gamma_5$, we can rewrite the $\mathrm{Tr}\log(i\partial\!\!\!/ - \sigma)$ as $\frac{1}{2}\mathrm{Tr}\log[-(i\partial\!\!\!/ - \sigma)\times(i\partial\!\!\!/ + \sigma)]$. In this paper we focus on static soliton configurations. If σ is time independent, the latter expression may be further simplified to $T/2\int d\omega/2\pi[\mathrm{Trlog}(h_+ - \omega^2) + \mathrm{Trlog}(h_- - \omega^2)]$ where

$$h_\pm \equiv -\partial_x^2 + \sigma^2 \pm \sigma', \tag{6}$$

and where T is an infrared temporal regulator.

As it turns out, the two Schrödinger operators $h_\pm$ are isospectral (see Appendix A of [10] and Section 2 of [8]) and thus we obtain

$$S[\sigma] = -\frac{1}{2g^2}\int d^2x(\sigma^2 - 2M\sigma) - iNT\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathrm{Tr}\,\log(h_- - \omega^2). \tag{7}$$

In contrast to the standard massless GN model, the MGN model studied here is not invariant under the Z_2 symmetry $\psi \to \gamma_5\psi$, $\sigma \to -\sigma$, and the physics is correspondingly quite different. As a result of the Z_2 degeneracy of its vacuum, the GN model contains a soliton (the so-called CCGZ kink [4,6,8,10,15]) in which the σ field takes on equal and opposite values at $x = \pm\infty$. The stability of this soliton is obviously guaranteed by topological considerations. With any nonzero M the vacuum value of σ is unique and the CCGZ kink becomes infinitely massive and disappears. If any soliton exists at all, its stability has to depend on the energetics of trapping fermions.

Let us briefly recall the computation of the unique vacuum of the MGN model. We shall follow [12]. For an earlier analysis of the MGN ground state (as well as its thermodynamics), see [16]. Setting σ to a constant we obtain from (7) the renormalized effective potential (per flavor)

$$V(\sigma,\mu) = \frac{\sigma^2}{4\pi}\log\frac{\sigma^2}{e\mu^2} + \frac{1}{\lambda(\mu)}\left[\frac{\sigma^2}{2} - M(\mu)\sigma\right], \tag{8}$$

where μ is a sliding renormalization scale with $\lambda(\mu) = Ng^2(\mu)$ and $M(\mu)$ the running couplings. By equating the coefficient of σ^2 in two versions of V, one defined with μ_1 and the other with μ_2, we find immediately that

$$\frac{1}{\lambda(\mu_1)} - \frac{1}{\lambda(\mu_2)} = \frac{1}{\pi}\log\frac{\mu_1}{\mu_2}, \tag{9}$$

and thus the coupling λ is asymptotically free, just as in the GN model. Furthermore, by equating the coefficient of σ in V we see that the ratio $M(\mu)/\lambda(\mu)$ is a renormalization group invariant. Thus, M and λ have the same scale dependence.

Without loss of generality we assume that $M(\mu) > 0$ and thus the absolute minimum of (8), namely, the vacuum condensate $m = \langle\sigma\rangle$, is the unique (and positive) solution of the gap equation

$$\frac{dV}{d\sigma}\bigg|_{\sigma=m} = m\left[\frac{1}{\pi}\log\frac{m}{\mu} + \frac{1}{\lambda(\mu)}\right] - \frac{M(\mu)}{\lambda(\mu)} = 0. \tag{10}$$

Referring to (1), we see that m is the mass of the fermion. Using (9), we can rewrite the gap equation as $m/\lambda(m) = M(\mu)/\lambda(\mu)$, which shows manifestly that m, an observable physical quantity, is a renormalization group invariant. This equation also implies that $M(m) = m$, which makes sense physically.

Fermion bags correspond to inhomogeneous solutions of the saddle-point equation $\delta S/[\delta\sigma(x,t)] = 0$. In particular, static bags $\sigma(x)$ are the extremal configurations of the energy functional (per flavor)

$$\mathcal{E}[\sigma(x)] = -\frac{S[\sigma(x)]}{NT}, \tag{11}$$

subjected to the boundary condition that $\sigma(x)$ relaxes to its unique vacuum expectation value m at $x = \pm\infty$.

The rest of this paper is organized as follows. In the next section we express $\mathcal{E}[\sigma(x)]$ in terms of the scattering data. DHN have already done most of the work for us, save for one crucial piece, specific to the MGN model: an expression for $\int_{-\infty}^{\infty}(\sigma(x) - m)dx$ in terms of the scattering data. In this paper we derive this relation. The details of this derivation are relegated to the Appendix. We then complete the task of writing down $\mathcal{E}[\sigma(x)]$ in terms of the scattering data. We prove that the static extrema of $\mathcal{E}[\sigma(x)]$ are reflectionless, as in the massless GN model [4], and calculate their masses. Such an extremal configuration, considered as a scalar background in the Dirac equation, typically supports a number K of pairs of charge-conjugate bound states at energies $\pm\omega_n (n = 1, 2, \ldots, K)$ which bind fermions and antifermions. Each one of these ω_ns depends only on the total number ν_n of fermions and antifermions it binds, thus giving rise to an $O(2N)$ rank-ν_n antisymmetric tensor multiplet of soliton states. As it turns out, the mass of such a soliton is a convex function of the ν_ns. In Sec. III, we invoke the convexity of the soliton's mass formula and prove that only solitons which support a single pair of bound states (i.e., $K = 1$) are stable against decaying into lighter solitons. These are precisely the configurations studied variationally in [12,13].

II. STATIC CONFIGURATIONS AND INVERSE-SCATTERING ANALYSIS

A fermion bag is essentially a trap for fermions. Evidently, an apriori specification of a static fermion bag configuration should indicate the list of bound states and enumerate the fermions it traps. To be specific, we shall evaluate in this section the energy functional (11) of a static configuration $\sigma(x)$, obeying the appropriate boundary conditions at spatial infinity, which supports K pairs of bound states of the Dirac equation at energies $\pm\omega_n$, $n = 1, \ldots, K$ (where, of course, $\omega_n^2 < m^2$). The bound states at $\pm\omega_n$ are to be considered together, due to the charge-conjugation invariance of the GN model. Because of Pauli's exclusion principle, we can populate each of the bound states $\pm\omega_n$ with up to N fermions. In such a typical multiparticle state, the negative frequency state is populated by $N - h_n$ fermions and the positive frequency state contains p_n fermions. In the parlance of Dirac's hole theory, this is a many fermion state, with p_n *particles* and h_n *holes* occupying the pair of charge-conjugate bound states at energies $\pm\omega_n$. We shall refer to the total number of particles and antiparticles trapped in the n-th pair of bound states

$$\nu_n = p_n + h_n \tag{12}$$

as the valence, or occupation number of that pair.

From (7) and (11) we obtain the bare energy functional as

$$\mathcal{E}[\sigma(x)] = \frac{1}{2\lambda}\int_{-\infty}^{\infty} dx[V(x) - 2M\sigma(x)] - \int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\,\mathrm{Tr}\,\log[-\partial_x^2 + V(x) - \omega^2], \tag{13}$$

where

$$V(x) = \sigma^2(x) - \sigma'(x). \tag{14}$$

(Here we used $\int_{-\infty}^{\infty} dx\sigma'(x) = 0$ by invoking the boundary conditions $\sigma(x)\rightarrow_{x\rightarrow\pm\infty} m$.) The expression (13) is divergent. We regulate it, as usual, by subtracting from it the divergent contribution of the vacuum configuration $\sigma^2 = m^2$ and by imposing a UV cutoff Λ on ω. Thus, the regulated bare energy functional associated with $\sigma(x)$ is

$$\mathcal{E}^{\rm reg}[\sigma(x)] = \frac{1}{2\lambda}\int_{-\infty}^{\infty} dx[V(x) - m^2 - 2M(\sigma(x) - m)] + i\int_{-\Lambda}^{\Lambda}\frac{d\omega}{2\pi}[\mathrm{Tr}\log(h_- - \omega^2) - \mathrm{Tr}\log(h_{\rm VAC} - \omega^2)], \tag{15}$$

where

$$h_{\rm VAC} = -\partial_x^2 + m^2 \tag{16}$$

is the Hamiltonian corresponding to the vacuum configuration. We are not done yet, as the integrals over ω in (15) still diverge logarithmically with the UV cutoff Λ. However, this logarithmically divergent term is precisely the one that should be added to the bare $1/\lambda$ in the first term in (15) in order to obtain the renormalized coupling appearing in (9) [4,8,10].

Now that we have written the energy-functional (15) of a static configuration, or a fermion bag, our next step is to identify those fermion bags on which (15) is extremal.

The energy functional (15) is, in principle, a complicated and generally unknown functional of $\sigma(x)$ and of its derivatives. Thus, the extremum condition $\delta\mathcal{E}[\sigma]/\delta\sigma(x) = 0$, as a functional equation for $\sigma(x)$, seems intractable. The considerable complexity of the functional equations that determine the extremal $\sigma(x)$ configurations is the source of all difficulties that arise in any attempt to solve the model under consideration.

DHN found a way around this difficulty in the case of the GN model [4]. They have used inverse-scattering techniques [14] to express the energy functional $\mathcal{E}[\sigma]$ (15) in terms of the so-called "scattering data" associated with, e.g., the Hamiltonian h_- in (6) (and thus with $\sigma(x)$), and then solved the extremum condition on (15) with respect to those data.

The scattering data associated with h_- are [14] the reflection amplitude $r(k)$ of h_- at momentum k, the number K of bound states in h_- and their corresponding energies $0 < \omega_n^2 \leq m^2$, $(n = 1, \ldots, K)$, and also additional K parameters $\{c_n\}$, where c_n has to do with the normalization of the nth bound state wave function ψ_n of h_-. More precisely, the nth bound state wave function, with energy ω_n^2, must decay as $\psi_n(x) \sim \mathrm{const.}\exp-\kappa_n x$ as $x \to \infty$, where

$$0 < \kappa_n = \sqrt{m^2 - \omega_n^2}. \tag{17}$$

If we impose that $\psi_n(x)$ be normalized, this will determine the constant coefficient as c_n. (With no loss of generality, we may take $c_n > 0$.) Recall that $r(-k) = r^*(k)$, since the Schrödinger potential $V(x)$ is real. Thus, only the values of $r(k)$ for $k > 0$ enter the scattering data. The scattering data are independent variables, which determine $V(x)$ uniquely, assuming $V(x)$ belongs to a certain class of potentials which fall off fast enough towards infinity.

Since the MGN does not bear topological solitons, neither h_- nor h_+ can have a normalizable zero energy eigenstate [17]. (See also Section A.1.1 in Appendix A of [10].) Thus, all the ω_n are strictly positive.

We can apply directly the results of DHN in order to write down that part of (15) which is common to the MGN and GN models, i.e., (15) with its term proportional to M removed, in terms of the scattering data. In what follows we briefly summarize their results. (See Sections 2, 3, and Appendix B of [4] for details.)

Using the trace identities of the spectral theory of h_-, they were able to show that

$$I_1[r(k),\{\kappa_n\}] \equiv -\frac{1}{2\lambda}\int_{-\infty}^{\infty} dx[V(x)-m^2] = \frac{1}{2\pi\lambda}\int_{-\infty}^{\infty}\log[1-|r(k)|^2]dk + \frac{2}{\lambda}\sum_{n=1}^{K}\kappa_n. \quad (18)$$

This takes care of the first term in (15). Onto the spectral determinants: These encode the contribution of the Fermi fields to the energy of the bag relative to the vacuum. To account for it correctly, we should put our field theory in a big spatial box of length L so as to make the spectrum discrete, which will enable us matching each mode of fermion fluctuations around the vacuum, with its counterpart, obtained as $\sigma(x)$ is turned on adiabatically. We shall take the limit $L \to \infty$ only in the end. Thus, we obtain the Fermi field part of the energy (per flavor) as

$$\mathcal{E}_F = i\int_{-\Lambda}^{\Lambda}\frac{d\omega}{2\pi}[\mathrm{Tr}\log(h_- - \omega^2) - \mathrm{Tr}\log(h_{\mathrm{VAC}} - \omega^2)] = -\sum_\alpha(\omega_\alpha[\sigma(x)] - \omega_\alpha^{\mathrm{VAC}}) - \sum_{n=1}^{K}(\omega_n - m) + \sum_{n=1}^{K}\frac{\nu_n}{N}\omega_n. \quad (19)$$

The first sum runs over all positive-energy scattering states, where $\omega_\alpha[\sigma(x)]$ is the energy of the scattering state to which the Fermi mode energy $\omega_\alpha^{\mathrm{VAC}}$ flows to as the vacuum configuration $\sigma = m$ is deformed adiabatically to $\sigma(x)$. The second sum in (19) accounts for the first K scattering states above the threshold $\omega = m$ which migrate into the gap to become the (positive energy) bound states ω_n as $\sigma(x)$ is switched on. In the limit $L \to \infty$ their energies are indistinguishable from m. Note the minus sign in front of these two sums, as appropriate for fermion zero-point energy. The last sum in (19) is evidently the contribution of *valence* fermions and antifermions trapped inside the bag at the bound states $\pm\omega_n$. By carefully counting scattering modes in the box, DHN arrived at the fairly standard result

$$\sum_\alpha(\omega_\alpha[\sigma(x)] - \omega_\alpha^{\mathrm{VAC}}) = -\frac{1}{\pi}\int_m^{\infty}\delta(\omega)d\omega, \quad (20)$$

where $\delta(\omega)$ is the scattering phase shift. Then, changing to momentum space ($k^2 + m^2 = \omega^2$), and using a dispersion integral representation for $\delta(k)$, DHN derived that

$$I_2[r(k),\{\kappa_n\}] \equiv \int_m^{\infty}\delta(\omega)d\omega = \frac{1}{2\pi}\int_0^{\infty}\frac{kdk}{(k^2+m^2)^{1/2}}P.P.\times\int_{-\infty}^{\infty}\frac{\log[1-|r(q)|^2]}{k-q}dq + 2\int_0^{\Lambda}\frac{kdk}{(k^2+m^2)^{1/2}}\sum_{n=1}^{K}\arctan\frac{\kappa_n}{k}. \quad (21)$$

The second integral in (21) can be calculated explicitly. In the limit $\Lambda/m >> 1$ we obtain

$$\tilde{I}_2 = 2\sum_{n=1}^{K}\left[\kappa_n\log\left(\frac{2\Lambda e}{m}\right) - \frac{\pi m}{2} + \omega_n\arctan\frac{\omega_n}{\kappa_n}\right]. \quad (22)$$

DHN's results, Eqs. (18)–(21), correspond to all terms of (15), save for the term proportional to M, $\int_{-\infty}^{\infty}(\sigma(x) - m)dx$. This integral cannot be expressed in terms of the scattering data based on the trace identities of the Schrödinger operator h_- discussed in Appendix B of [4]. Evidently, new analysis is required to obtain its representation in terms of the scattering data. Happily enough, we were able to obtain such a representation. In the Appendix we prove that

$$\int_{-\infty}^{\infty}(\sigma(x) - m)dx = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log[1-|r(q)|^2]}{im-q}dq + \sum_{n=1}^{K}\log\left(\frac{m-\kappa_n}{m+\kappa_n}\right). \quad (23)$$

Combining (18)–(21) and (23) we obtain the desired expression for the energy functional (15) in terms of the scattering data as

$$\mathcal{E}^{\mathrm{reg}}[\sigma(x)] = -I_1[r(k),\{\kappa_n\}] + \frac{1}{\pi}I_2[r(k),\{\kappa_n\}] - I_3[r(k)] + \sum_{n=1}^{K}\left[\left(\frac{\nu_n}{N} - 1\right)\omega_n + m - \frac{M}{\lambda}\log\left(\frac{m-\kappa_n}{m+\kappa_n}\right)\right], \quad (24)$$

where

$$I_3[r(k)] = \frac{M}{2\pi i\lambda}\int_{-\infty}^{\infty}\frac{\log[1-|r(q)|^2]}{im-q}dq. \quad (25)$$

We shall now extremize (24) with respect to the scattering data, to obtain the self-consistent static fermion bags in the MGN model. Let us vary with respect to $r(k)$ (with $k > 0$) first. As is evident from (18), (21), and (23), $\delta\mathcal{E}^{\mathrm{reg}}[\sigma(x)]/\delta r(k) = F(k)r^*(k)/(1 - |r(k)|^2)$, where $F(k)$ is a calculable function, which does not vanish identically. (For example, it can be shown that $\mathrm{Im}F(k) = -(M/\lambda\pi)k/(m^2 + k^2)$.) Thus, $r(k) \equiv 0$ is the unique solution of the variational equation $\delta\mathcal{E}^{\rm reg}[\sigma(x)]/\delta r(k) = 0$. Static extremal bags in the MGN model are *reflectionless*, as their counterparts in the GN model.

Explicit formulas for reflectionless $\sigma(x)$ configurations with an arbitrary number K of pairs of bound states are displayed in Appendix B of [10]. In particular, the one which supports a single pair of bound states at energies $\pm\omega_b$ ($\kappa = \sqrt{m^2 - \omega_b^2}$), the one originally discovered by DHN, is

$$\sigma(x) = m + \kappa\left[\tanh\left(\kappa x - \frac{1}{4}\log\frac{m+\kappa}{m-\kappa}\right) - \tanh\left(\kappa x + \frac{1}{4}\log\frac{m+\kappa}{m-\kappa}\right)\right]. \tag{26}$$

We see that the formidable problem of finding the extremal $\sigma(x)$ configurations of the energy functional $\mathcal{E}[\sigma]$ (15), is reduced to the simpler problem of extremizing an ordinary function $\mathcal{E}(\omega_n, c_n) = \mathcal{E}[\sigma(x; \omega_n, c_n)]$ with respect to the $2K$ parameters $\{c_n, \omega_n\}$ that determine the reflectionless background $\sigma(x)$. If we solve this ordinary extremum problem, we will be able to calculate the mass of the fermion bag. Let us write down this function explicitly:

$$\mathcal{E}^{\rm reg}(\omega_n) = \sum_{n=1}^{K}\left\{\left(-\frac{2}{\lambda} + \frac{2}{\pi}\log\left(\frac{2\Lambda e}{m}\right)\right)\kappa_n + \left(\frac{\nu_n}{N} - 1 + \frac{2}{\pi}\arctan\frac{\omega_n}{\kappa_n}\right)\omega_n - \frac{M}{\lambda}\log\left(\frac{m-\kappa_n}{m+\kappa_n}\right)\right\}. \tag{27}$$

The logarithmically divergent term in (27) should remind us that this equation is expressed in terms of the bare couplings λ and M. As it turns out, the renormalization procedure of the effective potential for the constant condensate, which we reviewed in the Introduction, applies also in the case of inhomogeneous background $\sigma(x)$: The bare coupling λ in (27) is related to the renormalized one at the energy scale $\mu_2 = m$ via an equation identical to (9), namely,

$$\frac{1}{\lambda(\Lambda)} - \frac{1}{\lambda(m)} = \frac{1}{\pi}\log\frac{2\Lambda}{m}. \tag{28}$$

(Because of the anisotropic cutoff implied in (21), the cutoff scale in (28) is $\mu_1 = 2\Lambda$ rather than just Λ.) In addition, since the ratio M/λ is a renormalization group invariant all the way up to the cutoff scale, we can replace the coefficient of the last term in (27) by the common value of that invariant, namely, $m/\lambda(m)$. With the help of these two relations, we obtain the renormalized form of (27) as

$$\mathcal{E}^{\rm ren}(\omega_n) = \sum_{n=1}^{K}\left\{\left(-\frac{2}{\lambda(m)} + \frac{2}{\pi}\right)\kappa_n + \left(\frac{\nu_n}{N} - 1 + \frac{2}{\pi}\arctan\frac{\omega_n}{\kappa_n}\right)\omega_n - \frac{m}{\lambda(m)}\log\left(\frac{m-\kappa_n}{m+\kappa_n}\right)\right\}. \tag{29}$$

Next, note that the energy functional $\mathcal{E}^{\rm ren}[\sigma]$ evaluated at a reflectionless $\sigma(x; \omega_n, c_n)$, is independent of the normalization coefficients c_n, that do affect the shape of $\sigma(x)$. The c_n's are thus *flat directions* of $\mathcal{E}^{\rm ren}[\sigma]$ in the space of all reflectionless $\sigma(x)$ configurations. In fact, the c_n's (or more precisely, their logarithms) are collective translational coordinates of the fermion bag $\sigma(x)$ (see e.g., Appendix B in [10]). One of these coordinates, corresponds, of course, to global translations of the bag as a whole.

Finally, we are left with the task of extremizing (29) with respect to the bound state energies ω_n of the reflectionless background $\sigma(x)$. To this end, we follow DHN and parametrize these energies as $\omega_n = m\cos\theta_n$, with $0 < \theta_n < \pi/2$. (Note that θ_n cannot attain the endpoint values of its range: $\omega_n = m$ plunges into the continuum, and $\omega_n = 0$ is possible only if $\sigma(x)$ is topologically nontrivial, which is not the case in the MGN model.) From (17) we see that $\kappa_n = m\sin\theta_n$. In terms of the angular variables, we may write (29) as

$$\mathcal{E}^{\rm ren}(\theta_n) = m\sum_{n=1}^{K}\left\{\left(-\frac{2}{\lambda(m)} + \frac{2}{\pi}\right)\sin\theta_n + \left(\frac{\nu_n}{N} - \frac{2\theta_n}{\pi}\right)\cos\theta_n - \gamma\log\left(\frac{1-\sin\theta_n}{1+\sin\theta_n}\right)\right\}, \tag{30}$$

where we have defined the renormalization group invariant

$$\gamma = \frac{1}{\lambda(m)} = \frac{M(\mu)}{m\lambda(\mu)}. \tag{31}$$

Finally, extremizing (30) with respect to θ_n yields

$$\frac{\partial\mathcal{E}^{\rm ren}}{\partial\theta_n} = 2m\left[\left(\frac{\theta_n}{\pi} - \frac{\nu_n}{2N}\right) + \gamma\tan\theta_n\right]\sin\theta_n = 0, \tag{32}$$

thus fixing θ_n as a function of the *filling fraction*

$$x_n = \frac{\nu_n}{N}, \qquad 0 < x_n < 1 \tag{33}$$

according to

$$\frac{\theta_n}{\pi} + \gamma\tan\theta_n = \frac{x_n}{2}. \tag{34}$$

The fact that the extremal value of θ_n is determined by the total number ν_n of particles and holes trapped in the bound states of the Dirac equation at $\pm\omega_n$, and not by the numbers of trapped particles and holes separately, is a manifestation of the underlying $O(2N)$ symmetry, which treats particles and holes symmetrically. As explained in [18] and in Appendix D of [10], this fact indicates that this pair of bound states gives rise to an $O(2N)$ antisymmetric tensor multiplet of rank ν_n of soliton states. The soliton as a whole is therefore the tensor product of all these antisymmetric tensor multiplets. The fermion number operator N_f is one of the generators of $O(2N)$. Its expectation value in the background of the extremal fermion bag in one of its $O(2N)$ states is (see Section 3 of [10])

$$\langle N_f\rangle = \sum_{\substack{n=1\\ \omega_n>0}}^{K}(p_n - h_n), \tag{35}$$

which is simply the sum over the individual valence fermion numbers

$$N_{f,val}^{(n)} = p_n - h_n \tag{36}$$

of each of the states in the individual antisymmetric factors. Evidently, for each of these antisymmetric representations

$$-\nu_n \leq N_{f,val}^{(n)} \leq \nu_n, \tag{37}$$

in accordance with charge-conjugation invariance.

The left-hand side of (34) is a monotonically increasing function. Therefore, (34) has a unique solution in the interval $[0, \pi/2]$. This solution is evidently smaller than $\theta_n^{\rm GN} = \pi\nu_n/2N$, the corresponding value of θ_n in the GN model for the same occupation number. Thus, the corresponding bound state energy $\omega_n = m\cos\theta_n$ in the MGN model is higher than its GN counterpart, and thus less bound.

Substituting the extremal θ's from (34) in (30) we find that the mass $\mathcal{M}$ of our soliton (namely, $N\mathcal{E}^{\rm ren}$ evaluated at the extremal point) is

$$\mathcal{M}(\{\nu_n\}) = Nm\sum_{n=1}^{K}\left(\frac{2}{\pi}\sin\theta_n + \gamma\log\frac{1+\sin\theta_n}{1-\sin\theta_n}\right). \tag{38}$$

In the case of a single pair of bound states, $K = 1$, (32)–(34) and (38) coincide with the corresponding results of variational calculations presented in [12,13], which were based on (26) as a trial configuration. In fact, it was realized in [13] that the trial configuration (26) is an exact solution of the extremum condition $\delta\mathcal{E}[\sigma]/\delta\sigma(x) = 0$, provided (34) is used to fix κ. This choice of trial configuration was very successful indeed!

We should mention that renormalization of the energy functional (15) in the background of a generic *reflectionless* $\sigma(x)$ and its extremization with respect to the θ_ns can be carried out using an alternative method based on the diagonal resolvent of the Dirac operator [10], which is basically a generalization of the calculations in [8,12] to the case of an arbitrary number K of pairs of bound states. As it turns out, there are simple explicit formulas for the diagonal resolvents of the Dirac operator and of h_- in reflectionless $\sigma(x)$ backgrounds, which make these computations possible, and lead to (32)–(38) [19].

III. INVESTIGATING STABILITY OF EXTREMAL STATIC FERMION BAGS

The extremal static soliton multiplets which we encountered in the previous section, correspond, in the limit $N \to \infty$, to exact eigenstates of the Hamiltonian of the MGN model. However, at large but finite N, we expect some of these states to become unstable and thus to acquire small widths, similarly to the behavior of baryons in QCD with a large number of colors [20]. The latter are also solitonic objects and are analogous to the "multiquark" bound states of the MGN and GN models. In particular, Section 9 of [20] offers a sketch of the $1/N$ expansion of two-dimensional QCD in the Coulomb gauge, both in the baryon and meson sectors, which is similar to the corresponding $1/N$ expansion of the GN and MGN models, in the presence of fermion bags. (This expansion is based on the so-called bilocal condensate formalism, which was later developed in [7,21,22].)

Furthermore, we can imagine perturbing the MGN action (1) by a small $O(2N)$ singlet perturbation (e.g., by adding to (1) a term $\Delta S_n = \epsilon\int d^2x\sigma^n$), and ask which of the extremal fermion bags of the previous section are stable against such perturbations. (The perturbations ΔS_1 and ΔS_2 correspond merely to a redefinition of the bare quantities M and g^2, and are thus not interesting. The higher perturbations, with $n > 2$, are nonrenormalizable. However, we could think of the resulting perturbed Lagrangian as an effective one.) Under these circumstances, all possible decay channels of a given unstable soliton multiplet
must conserve energy, momentum and $O(2N)$ quantum numbers.

It turns out that nontrivial results concerning stability may be established without getting into all the details of decomposing $O(2N)$ representations, by imposing a simple necessary condition on the spectrum of the fermion number operator N_f in the multiplets involved in a given decay channel. This way of arguing (as described in detail below) has led, in the case of the GN model (see Section 4 of [10]), to specification of all topologically-trivial stable fermion bags, consistent with the exact results of [11]. Thus, it is reasonable to expect (and conjecture) that applying these stability considerations to the MGN model should lead to the correct list of fermion bags in this model which remain stable at finite N as well. Unfortunately, exact results analogous to [11] and valid for finite N, are not available for the MGN model, so this conjecture has to be verified by explicit calculation of $1/N$ corrections. We shall now make these stability considerations explicit.

As we have learned so far, a given static soliton multiplet in our model is a direct product of $O(2N)$ antisymmetric tensors. The decay products of this soliton also correspond to a direct product of antisymmetric tensors. According to (37), the spectrum of N_f in an antisymmetric tensor representation is symmetric, namely, $-N_f^{\max} \leq N_f \leq N_f^{\max}$. When we compose two such representations D_1, D_2, the spectrum of N_f in the composite representation $D_1 \otimes D_2$, will obviously have the range $-N_f^{\max}(D_1) - N_f^{\max}(D_2) \leq N_f(D_1 \otimes D_2) \leq N_f^{\max}(D_1) + N_f^{\max}(D_2)$. In particular, each of the possible eigenvalues in this range, will appear in at least one irreducible representation in the decomposition of $D_1 \otimes D_2$. More generally, the spectrum of $N_f(D_1 \otimes D_2 \cdots \otimes D_L)$ will have the range $|N_f(D_1 \otimes \cdots \otimes D_L)| \leq N_f^{\max}(D_1) + \cdots N_f^{\max}(D_L)$.

Consider now a decay process, in which a parent static soliton, which belongs to a (possibly reducible) representation D_{parent}, decays into a bunch of other solitons, such that the collection of all irreducible representations associated with the decay products is $\{D_1, \ldots, D_L\}$ (in which a given irreducible representation may occur more than once). By $O(2N)$ symmetry, the representation D_{parent} must occur in the decomposition of $D_1 \otimes D_2 \cdots \otimes D_L$. Thus, according to the discussion in the previous paragraph, if this decay process is allowed, we must have

$$N_f^{\max}(D_{\text{parent}}) \leq N_f^{\max}(D_1) + \cdots + N_f^{\max}(D_L), \qquad (39)$$

which is the necessary condition for $O(2N)$ symmetry we sought for. (Obviously, similar necessary conditions arise for the other $N-1$ components of the highest weight vectors of the representations involved.) The decay process under consideration must respect energy conservation, i.e., it must be exothermic. Thus, we supplement (39) by the requirement

$$\mathcal{M}_{\text{parent}} \geq \sum_{\text{products}} \mathcal{M}_k \qquad (40)$$

on the masses $\mathcal{M}_i$ of the particles involved.

For each of the static soliton multiplets discussed in the previous section, we will scan through all decay channels (into static solitons) and check which of these decay channels are *necessarily* closed, simply by requiring that the two conditions (39) and (40) be mutually contradictory.

In order to complete our argument, we must make the plausible assumption that for any *time dependent* stable soliton with given $O(2N)$ quantum numbers (should such a soliton exist), the static soliton with the same $O(2N)$ quantum numbers is lighter. In the GN this is in fact true for the known time dependent DHN breathers [4].

For all the multiplets with a single pair of bound state, $K = 1$, we will find in this way that *all* decay channels are necessarily closed, thus establishing their stability. That these are stable multiplets is almost obvious to begin with—they are the lightest solitons, given their $O(2N)$ quantum numbers. We cannot establish in this way that all decay channels are necessarily closed for the higher solitons $K > 1$, and they are presumably unstable.

Consider the function

$$\epsilon(x) = \frac{2}{\pi}\sin\theta(x) + \gamma \log\frac{1+\sin\theta(x)}{1-\sin\theta(x)}, \qquad 0 < x < 1, \qquad (41)$$

where $\theta(x)$ is a solution of (34). From (34) and (41) we obtain that

$$\begin{aligned} \frac{d\epsilon(x)}{dx} &= \cos\theta(x) > 0 \\ \frac{d^2\epsilon(x)}{dx^2} &= -\frac{\pi}{2}\frac{\sin\theta(x)}{1+\pi\gamma\sec^2\theta(x)} < 0. \end{aligned} \qquad (42)$$

Thus, $\epsilon(x)$ is a monotonically increasing convex function in the range of interest, satisfying $\epsilon(x_1 + x_2) < \epsilon(x_1) + \epsilon(x_2)$. In terms of (41), we may write the soliton mas (38) simply as

$$\mathcal{M}(x_1, \cdots, x_n) = Nm\sum_{n=1}^{K}\epsilon(x_n). \qquad (43)$$

Now, we are ready to start the stability analysis. Consider the decaying parent soliton to be a static soliton with K pairs of bound states, corresponding to the direct product of K antisymmetric tensor representations of ranks $\tilde{\nu}_1, \ldots, \tilde{\nu}_K$. The mass of this soliton is $\mathcal{M}(\tilde{x}_1, \ldots, \tilde{x}_K) = Nm\sum_{n=1}^{K}\epsilon(\tilde{x}_n)$, and according to (37), the maximal fermion number eigenvalue occurring in this representation is $N_f^{\max}(D_{\text{parent}}) = \sum_{n=1}^{K}\tilde{\nu}_n$.

Following the strategy which we laid above, we shall now scan through all imaginable decay channels of this parent soliton (into final states of purely static solitons), and identify those channels which are necessarily closed. Thus, assume that the parent soliton under consideration decays into a configuration of lighter solitons, with quantum numbers of the direct product of L antisymmetric tensor representations $\nu_1, \ldots, \nu_L$. The way in which these L multiplets are arranged into extremal fermion bags is of no consequence to our discussion. Thus, we discuss all decay channels consistent with these quantum numbers in one sweep.

The necessary conditions (39) and (40) for possible decay imply

$$\sum_{n=1}^{K}\tilde{x}_n \leq \sum_{i=1}^{L}x_i, \qquad \sum_{n=1}^{K}\epsilon(\tilde{x}_n) \geq \sum_{i=1}^{L}\epsilon(x_i), \qquad (44)$$

where all $0 < \tilde{x}_n, x_i < 1$. The two pairs of boundary hypersurfaces in (44) are

$$\begin{aligned} &\Sigma_1, \tilde{\Sigma}_1: x_1 + \cdots + x_L = \tilde{x}_1 + \cdots + \tilde{x}_K \\ &\Sigma_2, \tilde{\Sigma}_2: \epsilon(x_1) + \cdots + \epsilon(x_L) = \epsilon(\tilde{x}_1) + \cdots + \epsilon(\tilde{x}_K) \end{aligned} \qquad (45)$$

where $\Sigma_{1,2}$ are hypersurfaces in x space and $\tilde{\Sigma}_{1,2}$ are the corresponding hypersurfaces in $\tilde{x}$ space.

Consider the behavior of (43) over the hyperplane

$$\tilde{\Sigma}_{r\alpha}: \tilde{x}_1 + \cdots + \tilde{x}_K = r + \alpha, \tag{46}$$

where $0 \leq r \leq K$ is an integer and $0 \leq \alpha < 1$. This hyperplane corresponds, of course, to solitons with a fixed value of the maximal fermion number $N_f^{\max}(D_{\text{parent}})$. We will now prove that $\mathcal{M}(x_1, \cdots, x_n)$ attains its absolute minimum on $\tilde{\Sigma}_{r\alpha}$ in the positive orthant, at the vertices of the intersection of $\tilde{\Sigma}_{r\alpha}$ and the hypercube $[0, 1]^K$, namely, the points

$$\tilde{x}_n^{(v)} = (\delta_{nn_1} + \delta_{nn_2} + \ldots + \delta_{nn_r}) + \alpha \delta_{nn_{r+1}}, \quad (n = 1, \ldots, K), \tag{47}$$

with all possible choices of $r + 1$ coordinates $i_1, \ldots, i_{r+1}$ out of K.

We prove this as follows (see Appendix F of [10]): consider a sequence $0 \leq \tilde{x}_1 \leq \ldots \leq \tilde{x}_K \leq 1$, subjected to (46). Assume that for some i, $0 < \tilde{x}_i \leq \tilde{x}_{i+1} < 1$. We will show that there exists another sequence of $\tilde{x}$'s, with the same sum, but with a lower sum of the ϵs. Thus, let $\delta > 0$ be chosen such that $\tilde{x}_i - \delta > 0$ and $\tilde{x}_{i+1} + \delta < 1$, i.e., $0 \leq \delta \leq \min\{\tilde{x}_i, 1 - \tilde{x}_{i+1}\}$. Modify the sequence under consideration by replacing $\tilde{x}_i$ by $\tilde{x}_i - \delta$ and $\tilde{x}_{i+1}$ by $\tilde{x}_{i+1} + \delta$, keeping the other $K - 2$ terms unaltered. The new sequence thus obtained has the same sum as the original sequence, and thus defines another point on $\tilde{\Sigma}_{r\alpha}$. We must show that $D(\delta) = \mathcal{M}(\text{original sequence}) - \mathcal{M}(\text{new sequence}) > 0$. Indeed, $D(\delta) = \epsilon(\tilde{x}_i) - \epsilon(\tilde{x}_i - \delta) + \epsilon(\tilde{x}_{i+1}) - \epsilon(\tilde{x}_{i+1} + \delta)$. Clearly, $D(0) = 0$, and also $D'(\delta) > 0$ in the relevant range of δ. Thus, $D(\delta)$ increases monotonically with δ, and reaches its maximum at $\delta_{\max} = \min\{\tilde{x}_i, 1 - \tilde{x}_{i+1}\}$, where, depending on the initial condition at $\delta = 0$, either $\tilde{x}_i - \delta_{\max} = 0$ or $\tilde{x}_{i+1} + \delta_{\max} = 1$. Thus, the sequence of $\tilde{x}$'s constrained to $\tilde{\Sigma}_{r\alpha}$, which minimizes (43), cannot have more than one element in the interior of $[0, 1]$. Thus, due to (46), the absolute minimum is the sequence in which the r largest $\tilde{x}$'s are 1, one $\tilde{x}$ is α and the rest are zero, namely, the vertices (47). This is just the statement that the mass function (43), being the sum of the convex functions $\epsilon(x)$, is convex inside the cube $[0, 1]^K$.

Therefore, a parent soliton corresponding to a point in the *interior* of the intersection of $\tilde{\Sigma}_{r\alpha}$ and the hypercube $[0, 1]^K$, can decay into a final state with quantum numbers corresponding to the points (47), i.e., $L = K$ and $x_n = \tilde{x}_n^{(v)}$ in (44). In fact, by continuity, such a parent soliton can decay also at least into the set of final states contained in small pockets above the vertices (47), which correspond to $L = K$ and $x_n = \tilde{x}_n^{(v)} + \epsilon_1$ in (44), where ϵ_1 is some small calculable quantity, or into final states corresponding to $L > K$ in (44), with $x_i = \tilde{x}_i^{(v)}$ for $1 \leq i \leq K$ and $x_i = \epsilon_2$ for $K + 1 \leq i \leq L$, and where ϵ_2 is another small calculable quantity.

On the other hand, the parent soliton which corresponds to the vertices (47) has no open channel to decay through. Hence it is potentially stable. Indeed, if it could decay through a channel corresponding to $x_1, \ldots, x_L$, then, from the requirement that these parameters satisfy (44), we would have

$$\sum_{i=1}^{L} x_i > \alpha + r \qquad \sum_{i=1}^{L} \epsilon(x_i) < r\epsilon(1) + \epsilon(\alpha). \tag{48}$$

Define the hyperplane

$$\Sigma_{r\alpha}: x_1 + \cdots + x_L = \alpha + r. \tag{49}$$

From the analysis above we know that the absolute minimum of $\sum_{i=1}^{L} \epsilon(x_i)$ over the intersection of $\Sigma_{r\alpha}$ and the hypercube $[0, 1]^L$ is $r\epsilon(1) + \epsilon(\alpha)$. Thus, the points which satisfy the second inequality in (48) are bounded by $\sum_{i=1}^{L} x_i < \alpha + r$, in contradiction with the first inequality in (48). This completes the proof that parent solitons which correspond to the vertices (47) are stable.

Each such vertex represents a soliton which cannot decay through the channel under consideration, and is thus potentially stable. More precisely, all these vertices correspond to the same soliton, since the coordinates of these vertices are just permutations of each other, and thus all of them correspond to the same set of parameters, in which

r of the $\tilde{x}$'s are degenerate and equal to 1

one of the $\tilde{x}$'s is equal to α, and (50)

the remaining $K - (r + 1)\tilde{x}$'s are null.

Does such a soliton exist? To answer this question let us recall a few basic facts: The parent soliton under discussion is topologically trivial. As such, it must bind fermions to be stabilized, and none of its bound state energies may vanish. Thus, all the ranks occurring in it must satisfy $0 < \tilde{\nu}_n < N$. Finally, note, that due to the elementary fact, that the spectrum of the one dimensional Schrödinger operator h_- cannot be degenerate, all the ω_n's must be different from each other, and so must be the $\tilde{\nu}_n$'s. Thus, the only physically realizable parent solitons, which are *necessarily* stable against the decay channel in question, correspond to $r = 0$ and $K = r + 1 = 1$. These are, of course, the static solitons studied in [12,13].

A corollary of our analysis so far is that these $K = 1$ solitons are stable against decaying into L_f free fermions or antifermions (i.e., L_f fundamental $O(2N)$ representations) plus $L - L_f$ solitons corresponding to higher antisymmetric tensor representations. In particular, it is stable against complete evaporation into free fermions. (Strictly speaking, this argument is valid only for values of L_f

which are a finite fraction of N, since our mass formula (43) is the leading order in the $1/N$ expansion, while removing a finite number of particles from the parent soliton is a perturbation of the order $1/N$ relative to (43).) Stability of the $K = 1$ solitons was studied in [12] in detail. In particular, the regimes $\gamma << 1$ and $\gamma >> 1$ were investigated. It was noted in [12] that the mass of the most stable soliton, obtained as the filling fraction $x \to 1$, is nonanalytic around $\gamma = 0$ (with leading nonanalytic behavior $\gamma \log\gamma$). It was shown in [12] that this nonanalyticity was related to the enhanced Z_2 symmetry of the GN model at $\gamma = 0$. The opposite regime, $\gamma = M/\lambda m >> 1$ may be attained by making the four-fermi interactions weak. In this regime the theory should be described in terms of quasifree massive fermions of mass m. We thus expect that the binding energy of bags will tend to zero as $\gamma \to \infty$, which was indeed verified in [12].

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APPENDIX: DERIVATION OF (23)

Consider Schrödinger equation

$$(-\partial_x^2 + V(x) - k^2 - m^2)\Psi(x, k) = 0 \qquad \text{(A1)}$$

with $V(x)$ given by (14). Because of the boundary conditions on $\sigma(x)$ at spatial infinity, $V(x) \to_{x\to\pm\infty} m^2$. Let $\phi(x, k)$ be its scattering solution

$$\begin{aligned} t(k)\phi(x, k) &= t(k)e^{ikx} + o(1) \qquad x \to +\infty \\ t(k)\phi(x, k) &= e^{ikx} + r(k)e^{-ikx} + o(1) \qquad x \to -\infty, \end{aligned} \qquad \text{(A2)}$$

where $t(k)$ and $r(k)$ are, respectively, the transmission and reflection amplitudes. As a consequence of analyticity of these amplitudes, the transmission amplitude $t(k)$ is completely determined on the real k axis by $r(k)$ as [14]

$$\begin{aligned} t(k) &= \sqrt{1 - |r(k)|^2}\left(\prod_{l=1}^{K} \frac{k + i\kappa_l}{k - i\kappa_l}\right) \\ &\quad \times \exp\left(\frac{1}{2\pi i}\text{P.P.}\int_{-\infty}^{\infty} \frac{\log[1 - |r(q)|^2]}{q - k} dq\right) \\ &= \left(\prod_{l=1}^{K} \frac{k + i\kappa_l}{k - i\kappa_l}\right)\exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log[1 - |r(q)|^2]}{q - k - i\epsilon} dq\right). \end{aligned} \qquad \text{(A3)}$$

Evidently, the last expression can be extended to the upper-half complex k plane (as usual, $\epsilon \to 0+$), displaying the bound states poles of $t(k)$ in the upper half-plane explicitly. The function

$$\Psi_0(x) = c \exp - \int_0^x \sigma(y) dy \qquad \text{(A4)}$$

solves (A1) for $\omega = 0$, i.e., for $k = im$. Since $\sigma(x) \to_{x\to\pm\infty} m$, it is not normalizable . Moreover, we can always choose the constant c such that $\Psi_0(x) \to_{x\to+\infty} e^{-mx} = e^{i(im)x}$. Thus, with this choice of c,

$$\Psi_0(x) = \phi(x, im). \qquad \text{(A5)}$$

Therefore, from (A2), we obtain that

$$\frac{\phi(L, im)}{\phi(-L, im)} \underset{L\to\infty}{\longrightarrow} t(im)e^{-2mL}. \qquad \text{(A6)}$$

With the help of (A4) we see that (A6) is equivalent to

$$\int_{-\infty}^{\infty} (\sigma(x) - m) dx = -\log t(im). \qquad \text{(A7)}$$

Thus, finally,

$$\begin{aligned} \int_{-\infty}^{\infty} (\sigma(x) - m) dx &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log[1 - |r(q)|^2]}{im - q} dq \\ &\quad + \sum_{n=1}^{K} \log\left(\frac{m - \kappa_n}{m + \kappa_n}\right), \end{aligned} \qquad \text{(A8)}$$

from (A3), thus proving (23). Due to the fact that on the real q axis $r(-q) = r^*(q)$ and $|t(q)|^2 = 1 - |r(q)|^2 \leq 1$, we can rewrite (A8) as

$$\begin{aligned} \int_{-\infty}^{\infty} (\sigma(x) - m) dx &= -\frac{m}{\pi}\int_{0}^{\infty} \frac{\log|t(q)|^2}{q^2 + m^2} dq \\ &\quad + \sum_{n=1}^{K} \log\left(\frac{m - \kappa_n}{m + \kappa_n}\right). \end{aligned} \qquad \text{(A9)}$$

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