Counterterm charges generate bulk symmetries

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We further explore the counterterm subtraction definition of charges (e.g., energy) for classical gravitating theories in spacetimes of relevance to gauge/gravity dualities; i.e., in asymptotically anti-de Sitter (AdS) spaces and their kin. In particular, we show in general that charges defined via the counterterm subtraction method generate the desired asymptotic symmetries. As a result, they can differ from any other such charges, such as those defined by bulk spacetime-covariant techniques, only by a function of auxiliary nondynamical structures such as a choice of conformal frame at infinity (i.e., a function of the boundary fields alone). Our argument is based on the Peierls bracket, and in the AdS context allows us to demonstrate the above result even for asymptotic symmetries which generate only conformal symmetries of the boundary (in the chosen conformal frame). We also generalize the counterterm subtraction construction of charges to the case in which additional nonvanishing boundary fields are present.

I. INTRODUCTION

In recent years, the study of gravitational theories in asymptotically anti-de Sitter (AdS) spaces has been of great interest due to the AdS/conformal field theory (CFT) correspondence [1–4], a conjectured equivalence between at least certain such ''bulk'' string theories (which therefore contain gravity) and nongravitational dual theories. In the case of AdS, the nongravitating dual theories are associated with spacetimes that may be considered to form the boundary of the asymptotically anti-de Sitter space. Similar so-called gauge/gravity correspondences also arise for other systems (see e.g. [5–7]) and involve bulk spacetimes with some of the same features as anti-de Sitter space.

As one may expect, the notion of energy (and of other conserved charges) is of significant use in understanding this correspondence. For some time, it has been clear that the dual field theories are closely associated with what is called the ''counterterm subtraction'' definition of energy [8–16] in the bulk. Such ideas are well developed for the case of anti-de Sitter space, and one might expect a suitable generalization to apply to other contexts as well. However, a number of other definitions of energy [17–24] have also

been given for bulk theories in AdS, and these are known to differ from the counterterm subtraction definition.

In particular, these other definitions all assign zero energy to pure AdS space, as is required if the charges are to form a representation of the AdS group. In contrast, in odd dimensions the counterterm subtraction approach assigns a *nonzero* value to AdS space which, moreover, depends on the choice of an auxiliary structure: a conformal frame Ω at infinity. This feature is natural from the point of view of the dual gauge theory (where it is associated with the conformal anomaly [8,9]), but raises the question of the general relationship between the counterterm subtraction energy and other constructions.

A reasonable conjecture is that the difference between these various notions of energy amounts to a ''constant offset'' which might in general depend on the choice of auxiliary conformal frame Ω, but which in no way depends on the dynamical bulk degrees of freedom. If this were so, the difference would be a constant over the phase space of the theory and all notions of energy would generate the same action on observables via the Poisson Bracket. This conjecture is consistent with the interpretation of the ''vacuum energy'' assigned to pure AdS as arising from the Casimir energy in the dual field theory. It is also suggested by numerous calculations (see e.g., [8–16,25], and also [26–28] for cases with slightly weaker asymptotic conditions) of the value of the counterterm energy assigned to particular families of spacetimes (e.g., the Schwarzschild-AdS spacetimes) in a particular conformal frame and also

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by [29]. Under appropriate asymptotically anti-de Sitter asymptotic conditions, this conjecture was recently proven [30] for all solutions and in all conformal frames in $d = 5$ bulk spacetime dimensions. Reference [30] also derives an explicit formula for this difference as a function of the metric on the conformal boundary defined by Ω, and shows under their boundary conditions that the definitions [17–23] also agree with a covariant phase space definition based on techniques of [31–33]. Finally, since the appearance of the first version of the present paper, [34] has extended such arguments to more general asymptotically AdS boundary conditions.

Our purpose here is to demonstrate similar results in all dimensions, and also for a much broader class of asymptotic behaviors. In fact, our arguments below will use only a few basic features associated with the construction of counterterm charges. We state most of the required properties in Sec. II A below, but these properties follow immediately in cases where counterterm subtraction is associated with the conformal boundary of the spacetime manifold. In addition, we will impose a simple causality requirement in Sec. III which naturally occurs whenever the conformal boundary has Lorentz signature. Thus, our results imply those of [30,34] and, in addition, apply equally well to other contexts such as the domain-wall spacetimes renormalized in [35–37] and to the cascading geometries renormalized in [38] (and first studied in [39–41]). Furthermore, if an appropriate set of counterterms can be found, they would also apply to the more general gauge/gravity dualities described in [5].

Our arguments will be based on general properties of the so-called Peierls bracket [42], a manifestly covariant construction which is equivalent to the Poisson bracket on the space of observables (see [43] for extensions of the Peierls bracket to algebras of gauge-dependent quantities and [44,45] for recent related work in quantum field theory). We begin by reviewing both the counterterm subtraction definition of charge and the Peierls bracket in Sec. II. This serves to set a number of conventions, and the counterterm charge discussion provides an opportunity to comment on subtle features associated with the choice of conformal frame Ω used to define the charge associated with a particular asymptotic symmetry ξ. In particular, depending on the choice of conformal frame, a given asymptotic symmetry need not act as a strict symmetry on the collection of boundary fields used to construct the counterterm charges. Instead, it might act only as a conformal symmetry. However, in the special case of appropriate asymptotically anti-de Sitter behavior, one may nevertheless show [8–16] that the difference between the charge evaluated on any two hypersurfaces is determined entirely by the conformal frame Ω and is independent of the bulk dynamics. Thus, even in this context the counterterm definition remains useful. We also take this opportunity to generalize the construction to allow arbitrary tensor and spinor bound-

ary fields.[1] Following this review, we give our main argument in Sec. III and close with a brief discussion of the results.

Since our arguments below will rely only on general properties of the Peierls bracket, they are independent of the details of the bulk dynamics. This is in sharp contrast to the results of [30,34] which also compared various definitions of energy, but which were based on a common technique involving explicit expansion of the Einstein equations in a power series around the boundary of an asymptotically anti-de Sitter space. Our results here are correspondingly more general, but also much less explicit. We remind the reader that [30] was able not only to relate the counterterm energy to the covariant phase space Hamiltonian, but also to show that the covariant phase space Hamiltonian agrees with the constructions of Ashtekar *et al.* based on the electric part of the Weyl tensor [17,18], with the Hamiltonian charge due to Henneaux and Teitelboim [19], and finally with the spinor charge of [21–23] (which guarantees positivity). The Abbott and Deser construction [20] and its extensions [47–49] and the KBL construction [24] (applied to AdS in [50]) were not considered in [30].

II. PRELIMINARIES

In this section, we review the two constructions central to our analysis: the counterterm subtraction definition of conserved charges (Sec. II A) and the Peierls bracket (Sec. II C) between observables.

A. Counterterm subtraction charges

The setting for the counterterm subtraction construction of conserved charges [8,9] is to consider systems associated with a certain sort of variational principle. Now, in general, such a principle specifies a class of variations with respect to which one requires the associated action S to be stationary. Let us suppose that this is done by positing a space of kinematically allowed histories $\mathcal{H}$ ("bulk variables") within which one is allowed to perform an arbitrary variation. There will also be certain features ("boundary values") which are identical for all histories in $\mathcal{H}$ and which are not to be varied. Thus, we in fact consider a family of actions S, each with an associated space of histories $\mathcal{H}$, parametrized by some set of allowed boundary values. Although typically discussed in the context of the conformal completion of some spacetime, the counterterm subtraction construction of conserved charges gener-

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[1]The case of certain scalar fields was considered in [11,12,14,16]. The contribution of gauge fields to the divergence of the stress tensor was considered in [46]. In addition, we understand that the corresponding conserved quantities are also constructed in unpublished work by Kostas Skenderis, with results similar to those presented below.

alizes naturally to a somewhat more abstract setting. We will therefore find it useful to state the minimal axioms for this construction. The reader may readily verify that each axiom holds when the boundary manifold ∂M described below is the conformal boundary of the spacetime M. Though our setting is in principle more abstract, it is convenient to use the term "boundary manifold" and other such terms in our discussion.

The counterterm subtraction construction of conserved charges is relevant when the following conditions hold:

(1) The boundary values can be described by a set of tensor (and perhaps spinor) "boundary fields" on an auxiliary manifold ∂M which is called the "boundary of the spacetime M." This will typically require the introduction of some auxiliary structure, which we call Ω, and which may include, for example, a choice of conformal frame at infinity. The choice of Ω is typically not unique, but is by definition a fixed kinematical structure independent of the bulk state. Given Ω, the boundary fields are determined by the bulk fields.

(2) One of these boundary fields is a metric h_{ab} on ∂M such that $(\partial M, h_{ab})$ is a globally hyperbolic spacetime.

(3) The action S is diffeomorphism invariant in the following sense: Every diffeomorphism ψ_∂ of the boundary manifold ∂M is (not uniquely) associated with a diffeomorphism ψ of the bulk spacetime which i) induces the action of ψ_∂ on the boundary fields through the map that constructs boundary fields from bulk fields, ii) preserves the auxiliary structure Ω, iii) preserves the space $\mathcal{H}$ of histories on which the action S is defined, and iv) is such that S is invariant under the simultaneous action of ψ on the bulk fields, ψ_∂ on the boundary values, and the corresponding transformation on the initial and/or final boundary conditions appropriate to the action S. As a result, the equations of motion are invariant under the action of ψ. We refer to the vector fields generating ψ and ψ_∂ as ξ and ξ_∂. Note that only diffeomorphisms ψ for which ψ_∂ acts as the identity on ∂M are gauge transformtations.

(4) First functional derivatives of the action S with respect to the boundary fields are well-defined and finite when evaluated on the space S of solutions to the equations of motion. This is typically arranged by an appropriate choice of "counterterms," leading to the name counterterm subtraction method.

As a particular example of this construction, one may consider asymptotically anti-de Sitter spacetimes. In this case, one takes ∂M to be the conformal boundary of M defined by the conformal frame Ω. The condition that ψ in requirement (3) above should preserve the conformal frame Ω determines how ψ_∂ is extended from ∂M to M, at least near ∂M.

In addition, we shall further assume:

(5) Given ξ, ξ_∂ as in (3) and any smooth function f on M, there is a smooth function f_∂ on ∂M such that the action on bulk fields of a diffeomorphism along $f\xi$ induces the action of a diffeomorphism on boundary fields along $f_\partial \xi_\partial$.

This latter condition clearly follows when the boundary ∂M is constructed by conformal completion of M, and will be useful in our arguments below.

The above setting is somewhat analogous to consideration of a field theory in the presence of nondynamical background fields. Here, however, the role of the background fields is played only by the boundary fields. As a result, there is an important difference: typically, one may vary background fields independently of dynamical fields, such as when one constructs the stress-energy tensor by varying a background metric for some field theory in curved spacetime. Clearly this is not possible here: since the boundary fields are limiting values of the bulk fields, any variation of the boundary fields necessarily requires a corresponding variation of the bulk fields. This will lead to certain subtleties which must be properly taken into account below.

As a result, the current context will require more reliance on the space of solutions (i.e., "on-shell" techniques) than in the usual background-field setting. In particular, one makes heavy use of the fact that, when evaluated on the space of solutions, variations which preserve both the boundary fields and appropriate boundary conditions in the past and/or future will leave the action invariant. It is this fact which allows property (4) above to hold: as noted above, any variation of the boundary fields must be accompanied by a variation of the bulk fields, and away from the space of solutions the change in the action S depends nontrivially on the choice of bulk variation. However, when evaluated on-shell, the change in S is independent of the choice of bulk variation, so long as it satisfies appropriate boundary conditions in the past and/or future. As a result, one may follow [8–16,46] and define the "boundary stress tensor" τ_{ab} as a function on the space of solutions satisfying

$$\tau_{ab}\boldsymbol{\epsilon} = -2\frac{\delta S}{\delta h^{ab}}, \qquad (2.1)$$

where the functional derivative is computed holding all other boundary fields constant and fixing appropriate boundary conditions in the past and/or future. Here we have used the notation $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{[a_1 a_2 \ldots a_n]}$ for the natural n-form associated with h_{ab}, identified with a density.

The definition (2.1) is sufficient when the metric is the only nontrivial boundary field; i.e., in the context considered by [8,9]. In that context one may show that τ_{ab} is covariantly conserved with respect to the metric h_{ab} on ∂M by following the essential steps through which one would derive covariant conservation of the stress-energy tensor T_{ab} in a curved spacetime. We will describe this argument

below, but we also wish to consider the more general case in which other boundary fields may be nonvanishing. When the extra fields are not scalars, this generalization will require us to introduce a "modified boundary stress tensor" with extra terms representing contributions from these extra boundary fields.

To do so, let us introduce some complete set of bulk fields Φ^I on M, where the I ranges over an appropriate label set to include components of vector and tensor (and perhaps spinor) fields as well as scalars. In particular, Φ^I includes the bulk metric (and any frame fields; see below). We also wish to pick out a complete set of boundary fields. However it turns out that the tensor (or spinor) rank of these fields will affect the detailed form of certain expressions below (including the definition of the charges). As a result, it is convenient at this stage to replace the boundary metric h_{ab} with a set of frame fields $e_a{}^A$ satisfying

$$h_{ab} = \eta_{AB} e_a{}^A e_b{}^B \qquad (2.2)$$

for a fixed metric η_{AB} (perhaps the Minkowski metric). The introduction of the frame fields allows us to write all remaining boundary fields without loss of generality in terms of a set of *scalar* fields, e.g., a tensor field $X_{ab...c}$ is encoded in the set $X_{AB...C} = X_{ab...c} e_A{}^a e_B{}^b \cdots e_C{}^c$ of scalar fields. We denote the collection of scalar fields on ∂M by ϕ_∂^i. Thus, these boundary scalars are just the "tangent space components" of any remaining[2] vector, tensor, or spinor boundary fields. We denote the full set of such boundary scalar fields and the frame fields by

$$\Phi_\partial^I = (\phi^i, e_A{}^a). \qquad (2.3)$$

Having replaced the boundary metric by a set of frame fields, it is natural to introduce the "modified boundary stress tensor"

$$\mathbf{T}^{ab} \boldsymbol{\epsilon} = \frac{\delta S}{\delta e_b{}^A} e^{aA}, \qquad (2.4)$$

where the functional derivative is computed holding fixed the scalars ϕ_∂^i (i.e., the tangent space components of boundary fields).

More specifically, let us introduce the future and past boundaries $\Sigma_\pm$ (perhaps at infinity) of our system in order to keep track of all boundary terms. We shall assume that, as is most common, the action is chosen so that its functional derivatives yield the equations of motion when boundary fields are held fixed together with the fields[3] Φ^I on $\Sigma_\pm$. Thus, a general variation of the action may be

written:

$$\delta S = \int_M \frac{\delta S}{\delta \Phi^I} \delta \Phi^I + \int_{\partial M} \frac{\delta S}{\delta \phi_\partial^i} \delta \phi_\partial^i + \int_{\partial M} \boldsymbol{\epsilon} \mathbf{T}^a{}_A \delta e_a{}^A$$
$$+ \int_{\Sigma_\pm} \pi_I \delta \Phi^I, \qquad (2.5)$$

where $\int_{\Sigma_\pm}$ includes integrals over both Σ_+ and Σ_- and we take the momenta π_I to be defined by this final term. We are then interested in the value of $\mathbf{T}^a{}_A$ on the space of solutions.

In the case where the nontrivial boundary fields are just the metric and some scalars on ∂M, the modified and original boundary stress tensors agree; $\mathbf{T}_{ab} = \tau_{ab}$. However, in the presence of other nontrivial boundary fields, $\mathbf{T}^{ab}$ contains extra contributions from these fields. As usual, we will use the frame fields $e_a{}^A$ and the inverse frames to convert spacetime indices into tangent space indices (and vice versa). In particular, we will make use of $\mathbf{T}^a{}_A$, which is in fact a more fundamental quantity than $\mathbf{T}^{ab}$.

Now, in general, the modified boundary stress tensor $\mathbf{T}^A{}_a$ will fail to be covariantly conserved due to the presence of the other background fields ϕ_∂^i. However, its covariant divergence takes a simple and useful form. This may be demonstrated by considering the simultaneous action of an arbitrary infinitesimal boundary diffeomorphism ψ_∂, which we take to be generated by the vector field ξ_∂^a, and the associated bulk diffeomorphism ψ generated by ξ^a. By property (3) above we then have

$$0 = \int_M \frac{\delta S}{\delta \Phi^I} \mathcal{L}_\xi \Phi^I + \int_{\partial M} \frac{\delta S}{\delta \phi_\partial^i} \mathcal{L}_{\xi_\partial} \phi_\partial^i$$
$$+ \int_{\partial M} \boldsymbol{\epsilon} \mathbf{T}^a{}_A \mathcal{L}_{\xi_\partial} e_a{}^A + \int_{\Sigma_\pm} \pi_I \mathcal{L}_\xi \Phi^I, \qquad (2.6)$$

If we evaluate (2.6) on the space of solutions (so that the bulk equations of motion hold), then the first term vanishes. Considering the second term, the ϕ_∂^i are scalars so that we have $\mathcal{L}_{\xi_\partial} \phi_\partial^i = \xi_\partial^a \nabla_a \phi_\partial^i$, where ∇ is the (torsion-free) covariant derivative on ∂M compatible with the metric h_{ab}. Thus, this term is algebraic in ξ_∂. Finally, turning to the third term, we have

$$\mathcal{L}_{\xi_\partial} e_a{}^A = \xi_\partial^b \nabla_b e_a{}^A + e_b{}^A \nabla_a \xi^b. \qquad (2.7)$$

Thus, we may perform an integration by parts in the third term and use the arbitrariness of ξ_∂^a (and, in particular, the ability to set ξ_∂^a to zero in a neighborhood of $\Sigma_\pm$) to conclude that the covariant divergence of $\mathbf{T}_{ab}$ satisfies[4]

$$\nabla_a \mathbf{T}^{ab} = \sum_i \frac{\delta S}{\delta \phi_\partial^i} \nabla^b \phi_\partial^i + \mathbf{T}^a{}_A \nabla^b e_a{}^A. \qquad (2.8)$$

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[2]The tangent space components of the frame fields are, of course, trivial by definition. These may be included in the set ϕ_∂^i for convenience of notation, but only the set $\{\phi_\partial^i, e_A{}^a\}$ of boundary scalars together with boundary frame fields forms a complete set of boundary fields.

[3]More generally, one might use an action appropriate to fixing various derivatives of Φ^I at $\Sigma_\pm$. It will be clear from the treatment below that our results apply equally well to such cases.

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[4]Some readers may consider it more elegant to introduce another derivative operator D_a on ∂M satisfying $D_a e_b{}^B = 0$. In this case, $D_a \mathbf{T}^{ab}$ is given just by the scalar field term on the right-hand side of (2.8).

We are now in a position to construct the counterterm charges and demonstrate their conservation. To do so, consider a particular choice of boundary values and an infinitesimal diffeomorphism ψ_∂ corresponding to a symmetry of the boundary values. We take ψ_∂ to be generated by the vector field ξ_∂ and the associated bulk diffeomorphism ψ to be generated by ξ. Hence ξ_∂ Lie-derives the boundary fields up to a local gauge transformation

$$\mathcal{L}_{\xi_\partial}e_a{}^A = R^A{}_B e_a{}^B, \qquad \mathcal{L}_{\xi_\partial}\phi_\partial^i = \sum_j R^i{}_j \phi_\partial^j, \qquad (2.9)$$

where $R_{AB} = -R_{BA}$ and $R^i{}_j$ gives the action of the associated frame rotation on the boundary scalars ϕ_∂^i. In fact, as we will see shortly, it is just as easy to allow $(R^A{}_B, R^i{}_j)$ to define an arbitrary infinitesimal transformation $\delta e_a{}^A = R^A{}_B e_a{}^B$, $\delta\phi_\partial^i = \sum_j R^i{}_j \phi_\partial^j$ under which the action S is locally invariant.[5]

We call such a ξ an "asymptotic symmetry compatible with Ω." One then defines the associated "counterterm subtraction charge":

$$Q[\xi] = \int_C \mathbf{T}_{ab} \xi^a ds^b, \qquad (2.11)$$

where C is a Cauchy surface of ∂M, and

$$ds^a = \boldsymbol{\epsilon}^a{}_{b_1 \dots b_{n-1}} dx^{b_1} \cdots dx^{b_{n-1}}$$

is the induced integration element on C. We will refer to C as a "cut" of ∂M in order to avoid confusion with Cauchy surfaces in M. As an example of $Q[\xi]$ in the familiar anti-de Sitter context, one might take the boundary metric to be the Einstein static universe with all other boundary fields vanishing. In this case, one could take ξ to be an asymptotic time translation and the associated $Q[\xi]$ would give the counterterm subtraction definition of energy. Note also that we have defined $Q[\xi]$ only when ξ^a preserves any auxiliary structure (Ω) needed to define the boundary fields. However, in typical examples (e.g., AdS) the result may be applied much more generally: one need only find the boundary symmetry ξ_∂ associated with ξ and then choose another extension ξ' to the bulk which preserves Ω and induces the same action ξ_∂^a on the boundary. One then defines $Q[\xi] := Q[\xi']$.

We wish to prove that $Q[\xi]$ is independent of the choice of cut C. Let us therefore consider some region $V \subset \partial M$ such that the boundary ∂V within ∂M consists of two cuts C_1 and C_2. Let $Q_{C_1}[\xi]$ and $Q_{C_2}[\xi]$ denote the values of $Q[\xi]$ associated with the two cuts, respectively. Then we have

$$Q_{C_1}[\xi] - Q_{C_2}[\xi] = \int_V \boldsymbol{\epsilon} \nabla_a (\mathbf{T}^{ab} \xi_{\partial b}). \qquad (2.12)$$

But we may use (2.7) and (2.8) to express (2.12) as

$$Q_{C_1}[\xi] - Q_{C_2}[\xi] = \int_V \sum_i \frac{\delta S}{\delta \phi_\partial^i} \mathcal{L}_{\xi_\partial} \phi_\partial^i + \int_V \boldsymbol{\epsilon} \mathbf{T}^a{}_A \mathcal{L}_{\xi_\partial} e_a{}^A = \int_V \left(R^A{}_B e^B{}_a \frac{\delta}{\delta e_a{}^A} + \sum_{i,j} R^i{}_j \phi_\partial^j \frac{\delta}{\delta \phi_\partial^i} \right) S = 0, \qquad (2.13)$$

where in the second step we have used the fact that ξ_∂ generates a symmetry of the boundary fields up to a gauge rotation, and where in the final step we have used the fact that S is invariant under such rotations.

Thus, for asymptotic symmetries ξ compatible with Ω, $Q[\xi]$ is indeed independent of the cut C. Note that, as a result, we can weaken the framework to require only that C is homotopic to a Cauchy surface. The result (2.13) generalizes the construction of [8–16,46] to include arbitrary nontrivial (tensor and spinor) boundary fields.

B. Conformal boundary killing fields and asymptotically anti-de Sitter boundary conditions

In [8–16] it was shown that many gravitational theories with asymptotically anti-de Sitter asymptotic behavior satisfy requirements (1)–(5) of Sec. II A. In addition, [8–16] also demonstrate another property associated with the

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[5]By locally invariant, we mean that

$$\int_V \left(R^A{}_B e^B{}_a \frac{\delta}{\delta e_a{}^A} + R^i{}_j \phi_\partial^j \frac{\delta}{\delta \phi_\partial^i} \right) S = 0 \qquad (2.10)$$

for any $V \subset \partial M$. In particular, (2.10) contains no boundary term on ∂V.

conformal invariance of the dual field theory (under the AdS/CFT correspondence). Recall that conformal invariance requires the trace of the stress-energy tensor to be zero. Now, if such a quantum field theory is placed on a generic curved background the trace of the stress tensor might be nonvanishing. This trace—the "anomaly"—is normally given by local curvature terms of the background metric. As a result, the AdS/CFT correspondence suggests that the trace $\tau = h^{ab}\tau_{ab}$ of the boundary stress tensor defined above should depend only on h_{ab} and, in particular, should be a constant on the space of solutions S. That this is the case was shown in [8–16] for their boundary conditions, under which τ^{ab} agrees with our $\mathbf{T}_{ab}$. Indeed, when the metric on ∂M is taken to be the Einstein static universe (and certain other boundary fields vanish), Refs. [8–16] show that τ vanishes.

We may now follow [8–16] and use this observation to generalize the discussion of $Q[\xi]$ to the case where ξ is associated with a vector field ξ_∂ on ∂M which is only a conformal killing field of h_{ab}. Note that in cases where the boundary spacetime $(\partial M, h_{ab})$ is just the conformal boundary of the bulk, any asymptotic symmetry ξ of the bulk should induce such a conformal isometry ξ_∂ of the boundary metric of ∂M so that this procedure will lead to a counterterm charge associated with every conserved quan-

tity that one expects from the symmetries of the bulk system.

In particular, let us suppose that we have a conformal Killing field with

$$\nabla_a \xi_{\partial b} + \nabla_b \xi_{\partial a} = \mathcal{L}_{\xi_\partial} h_{ab} = 2k h_{ab}, \qquad (2.14)$$

for some smooth function k on ∂M, and that

$$\mathcal{L}_{\xi_\partial} \phi_\partial^i = \sum_j K^i{}_j \phi_\partial^j + \sum_j R^i{}_j \phi_\partial^j. \qquad (2.15)$$

Here the coefficients $K^i{}_j$ encode the behavior of the ϕ_∂^i under conformal transformations and the $R^i{}_j$ are as before in Sec. II A. Equation (2.14) implies that $\mathcal{L}_{\xi_\partial} e_a{}^A = k e_a{}^A + R^A{}_B e_a^B$. We now simply repeat the above calculation to see how $Q_C[\xi]$ depends on the cut C. Consider again Eqs. (2.8) and (2.12), but now use Eq. (2.14) to write the right-hand side in the form

$$Q_{C_1}[\xi] - Q_{C_2}[\xi] = \int_V \left(\boldsymbol{\epsilon} \mathbf{T}^a{}_A \mathcal{L}_{\xi_\partial} e^A{}_a + \sum_i \frac{\delta S}{\delta \phi_\partial^i} \mathcal{L}_{\xi_\partial} \phi_\partial^i \right)$$

$$= \int_V \left(k \boldsymbol{\epsilon} \mathbf{T} + \sum_{i,j} K_j{}^i \phi_\partial^j \frac{\delta S}{\delta \phi_\partial^i} \right) \qquad (2.16)$$

$$= \int_V \left(k e^A{}_a \frac{\delta}{\delta e^A{}_a} + \sum_{i,j} K^i{}_j \phi_\partial^j \frac{\delta}{\delta \phi_\partial^i} \right) S, \qquad (2.17)$$

where we have defined $\mathbf{T} := \mathbf{T}^a{}_A e_a{}^A = \mathbf{T}^{ab} h_{ab}$, and in the second line we have used the invariance of S under frame rotations. Thus, assuming that the integrand on the right side is a function of the boundary fields alone and not of the particular solution under consideration (as is the case under the asymptotic conditions of [8–16]), the change in $Q[\xi]$ is a function only of the boundary fields and is otherwise constant over the space of solutions S.

C. The Peierls bracket

Having reviewed (and generalized) the counterterm substraction definition of charges, we now briefly review the other piece of machinery we will need to derive our main result: the Peierls bracket.

The Peierls bracket is an algebraic structure defined on gauge-invariant functions on the space of solutions S associated with an action principle. As shown in the original work [42], this bracket is equivalent to the Poisson bracket under the natural identification of the phase space with the space of solutions. One of the powerful features of the Peierls bracket is that it is manifestly spacetime covariant. Another is that it is defined directly for general gauge invariants A and B whether or not A and B are associated with some common time t. Furthermore, A and B need not be local but can instead be extended over regions of space and time.

These features make the Peierls bracket ideal for studying the boundary stress tensor, which is well-defined only

on the space of solutions and is not a local function in the bulk spacetime.[6] As a result, it will be straightforward to give a Peierls version of a Noether argument to show that the charges $Q[\xi]$ generate the appropriate symmetries when ξ_∂ is a boundary Killing field—or, more generally, a boundary conformal Killing field as discussed in Sec. II B. Since this property is required of any charge defined by Hamiltonian methods, it follows that such charges can differ from $Q[\xi]$ only by a quantity with vanishing Peierls bracket. But all such quantities can depend only on the boundary fields and must otherwise be constants on the space of solutions S.

The Peierls construction considers the effect on one gauge-invariant function (say, B) on the space of histories $\mathcal{H}$ when the action is deformed by a term proportional to the another such function (A). In particular, suppose that the dynamics is determined by an action S. One defines the advanced ($D_A^+ B$) and retarded ($D_A^- B$) effects of A on B by comparing the original system with a new system defined by the action $S_\epsilon = S + \epsilon A$, but associated with the same space of histories. Here ϵ is a real parameter which will soon be taken to be infinitesimal, and the new action is associated with a new space S_ϵ of deformed solutions.

Under retarded (advanced) boundary conditions for which the solutions $s \in S$ and $s_\epsilon \in S_\epsilon$ coincide on Σ_- (Σ^+) of the support of A, the quantity $B_0 = B(s)$ computed using the undeformed solution s will in general differ from $B_\epsilon^\pm = B(s_\epsilon)$ computed using s_ϵ and retarded ($+$) or advanced ($-$) boundary conditions. For small epsilon, the difference between these quantities defines the retarded (advanced) effect $D_A^- B$ ($D_A^+ B$) of A on B through

$$D_A^\pm B = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (B_\epsilon^\pm - B_0), \qquad (2.18)$$

which is a function of the unperturbed solution s. Similarly, one defines $D_B^\pm A$ by reversing the roles of A and B above. Since A, B are gauge invariant, $D_B^\pm A$ is a well-defined (and again gauge-invariant) function on the space S of solutions so long as both A and B are first-differentiable on $\mathcal{H}$ (a requirement which may be subtle when the spacetime supports of A and B extend to Σ_+ or Σ_-).

The Peierls bracket [42] is then defined to be the difference of the advanced and retarded effects:

$$\{A, B\} = D_A^+ B - D_A^- B. \qquad (2.19)$$

One may show that (2.19) depends only on the restriction of A, B to the space of solutions S, so that (2.19) defines an algebra of functions on S, as desired.

The fact that this agrees with the Poisson bracket (supplemented by the equations of motion) was shown in [42], and generalizes the familiar result that the commutator

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[6]For the same reasons, we expect the Peierls bracket to be of use in studying other objects which naturally arise in the AdS/CFT correspondence.

function for a free scalar field is given by the difference between the advanced and retarded Green's functions. In fact, it is enlightening to write the Peierls bracket more generally in terms of such Green's functions. To do so, we again make use of our complete set of (bulk) fields Φ^I (which include the metric and components of bulk tensor and spinor fields) and the associated advanced and retarded Green's functions $G_{IJ}^{\pm}(x, x')$. Note that we have

$$
\begin{aligned}
D_A^+ B &= \int dx dx' \frac{\delta B}{\delta \Phi^I(x)} G_{IJ}^+(x, x') \frac{\delta A}{\delta \Phi^J(x')} \\
&= \int dx dx' \frac{\delta B}{\delta \Phi^J(x')} G_{JI}^-(x', x) \frac{\delta A}{\delta \Phi^J(x)} = D_B^- A.
\end{aligned}
$$

(2.20)

Thus, the Peierls bracket may also be written in the manifestly antisymmetric form

$$
\{A, B\} = D_B^- A - D_A^- B. \tag{2.21}
$$

The expressions (2.20) in terms of $G_{IJ}^+(x, x')$ are also useful in order to verify that the Peierls bracket defines a Lie-Poisson algebra. In particular, the derivation property $\{A, BC\} = \{A, B\}C + \{A, C\}B$ follows immediately from the Leibnitz rule for functional derivatives. The Jacobi identity also follows by a straightforward calculation, making use of the fact that functional derivatives of the action commute (see e.g., [51,52]). If one desires, one may use related Green's function techniques to extend the Peierls bracket to a Lie algebra of gauge-dependent quantities [43].

III. MAIN ARGUMENT

We now use the Peierls bracket to show that the counterterm subtraction charges $Q[\xi]$ generate the appropriate symmetries when ξ_∂ is a boundary Killing field, or, more generally, a boundary conformal Killing field under the conditions of Sec. II B. Since this property is required of any charge defined by Hamiltonian methods, it follows that such charges can differ from $Q[\xi]$ only by a quantity with vanishing Peierls bracket. But any such quantity can be built only from auxiliary structures and must otherwise be constant on the space of solutions S. As in Sec. II, we first address asymptotic symmetries ξ compatible with Ω using the features (1)–(5) of the counterterm subtraction setting as described in Sec. II A, and then proceed to the case where ξ does not preserve Ω so that the associated ξ_∂ acts only as a conformal Killing field on the boundary.

A. Asymptotic symmetries compatible with Ω

The essential point of the argument is that the Peierls bracket allows a simple derivation of Noether's theorem. We will be able to proceed when there is a pair of smooth functions (f, f_∂) on $(M, \partial M)$ satisfying requirement (5) of Sec. II A as well as

    (i) $f = 0$ in a neighborhood of the past boundary Σ_-.

    (ii) $f_\partial = 0$ to the past of some cut C_0 of ∂M.

    (iii) $f = 1$ in a neighborhood of the future boundary Σ_+.

    (iv) $f_\partial = 1$ to the future of some cut C_1 of ∂M.

This is the simple causality requirement mentioned in the introduction. It is naturally satisfied whenever ∂M may be considered as a boundary of M and is of Lorentz signature. In that case we may simply take f_∂ to be defined by limits of f on ∂M.

Let us now consider any asymptotic symmetry ξ compatible with Ω and the associated boundary isometry ξ_∂. Under the action of this symmetry, the bulk and boundary fields transform as

$$
\delta \Phi^I = \mathcal{L}_\xi \Phi^I, \qquad \delta e_a{}^A = \mathcal{L}_{\xi_\partial} e_a{}^A = R^A{}_B e_a{}^B,
$$

and $\quad \delta \phi_\partial^i = \mathcal{L}_{\xi_\partial} \phi_\partial^i = \sum_j R^i{}_j \phi_\partial^j,$ (3.1)

where $(R^A{}_B, R^i{}_j)$ provide an appropriate frame rotation of the boundary fields.

The key point of our argument is to construct a new transformation $\Delta_{f,\xi}$ on the space of fields such that the associated first order change $\Delta_{f,\xi} S$ in the action generates the asymptotic symmetric ξ under the Peierls bracket. We will see that the correct choice is given by $\Delta_{f,\xi} \Phi^I := (\mathcal{L}_{f\xi} - f\mathcal{L}_\xi)\Phi^I$. An important property of this definition is that the change $\Delta_{f,\xi} \Phi^I$ is *algebraic* in Φ^I; i.e., we need not take spacetime derivatives of the fields Φ^I in order to compute $\Delta_{f,\xi} \Phi^I$. Furthermore, $\Delta_{f,\xi} \Phi^I$ is proportional to $\nabla_a f$, and thus vanishes in a neighborhood of Σ_+ and Σ_-. This property guarantees that $\Delta_{f,\xi} S$ is differentiable on the space $\mathcal{H}$ of histories associated with the action S. In particular, solutions to the equations of motion resulting from the deformed action $S + \epsilon \Delta_{f,\xi} S$ are stationary points of $S + \epsilon \Delta_{f,\xi} S$ under *all* variations $\delta \Phi^I$ which preserve the boundary fields $\phi_\partial^i, e_a{}^A$ (up to gauge rotations) and vanish on $\Sigma_\pm$; all boundary terms vanish under arbitrary such variations.

As an additional consequence of the above, we see that (on-shell) the quantity $\Delta_{f,\xi} S$ is gauge invariant: Since the action S is gauge invariant, the quantity $\Delta_{f,\xi} S$ can acquire gauge dependence only through f, ξ. However, the above observation and (2.5) imply that on-shell $\Delta_{f,\xi} S$ depends only on f_∂, ξ_∂. Since gauge transformations have trivial action on ∂M, we conclude that $\Delta_{f,\xi} S$ is gauge invariant on-shell. Thus, we may take the Peierls bracket of $\Delta_{f,\xi} S$ with any other on-shell observable A.

To do so, let us note that if $s \in S$ is a stationary point of the original action S with bulk fields Φ^I and boundary fields $\phi_\partial^i, e_a{}^A$, then to first order in ϵ we see that $s_1 = (1 - \epsilon \Delta_{f,\xi})s$ is a stationary point of $S + \epsilon \Delta_{f,\xi} S$, since to first order this modified action is just $S(\Phi^I + \epsilon \Delta_{f,\xi} \Phi^I)$; i.e., we see that to first order the bulk fields are merely shifted by $-\epsilon \Delta_{f,\xi}$. Since ξ is an asymptotic symmetry

compatible with Ω, property (3) of Sec. II A states that the boundary fields defined by s_1 are also shifted by $-\epsilon\Delta_{f,\xi}$ relative to those of s.

Of course, we desire solutions to the modified equations of motion whose boundary values give the *original* boundary fields of s. However, this can be arranged by making use of another symmetry. Note that, because ξ is an asymptotic symmetry, we may use (2.9) to compute the induced action of $\Delta_{f,\xi}$ on boundary fields as follows:

$$\Delta_{f,\xi}\phi_{\partial}^i = (\mathcal{L}_{f_{\partial}\xi_{\partial}} - f_{\partial}\mathcal{L}_{\xi_{\partial}})\phi_{\partial}^i = \mathcal{L}_{f_{\partial}\xi_{\partial}}\phi^i - f_{\partial}R^i{}_j\phi_{\partial}^j,$$
$$\Delta_{f,\xi}e_a{}^A = (\mathcal{L}_{f_{\partial}\xi_{\partial}} - f_{\partial}\mathcal{L}_{\xi_{\partial}})e_a{}^A = \mathcal{L}_{f_{\partial}\xi_{\partial}}e_a{}^A - f_{\partial}R^A{}_B e_a{}^A.$$
$$(3.2)$$

Thus, the shift of the boundary fields is just given by the a diffeomorphism along the vector field $-f_{\partial}\xi_{\partial}$ and a compensating frame rotation $(R^i{}_j, R^A{}_B)$. In fact, as will shortly be important, the shift $\Delta_{f,\xi}\phi_{\partial}^i$ of the boundary scalars vanishes (and the shift $\Delta_{f,\xi}e_a{}^A$ of the boundary frame fields simplifies dramatically) using (2.9), but for the moment the form (3.2) is more useful. To see why, recall that the equations of motion are invariant under both diffeomorphisms and frame rotations. As a result, if R is a frame rotation on the bulk fields which induces the rotation $(R^i{}_j, R^A{}_B)$ on the boundary, then

$$s_2 = (1 + \epsilon\mathcal{L}_{f\xi} - \epsilon fR)s_1 = (1 + \epsilon f\mathcal{L}_{\xi} - \epsilon fR)s \quad (3.3)$$

with bulk fields

$$\Phi^I|_{s_2} = \Phi^I - \epsilon(\Delta_{f,\xi} - \mathcal{L}_{f\xi})\Phi^I - \epsilon fR^I{}_J\Phi^J$$
$$= \Phi^I + \epsilon f\mathcal{L}_{\xi}\Phi^I - \epsilon fR^I{}_J\Phi^J \quad (3.4)$$

induces the original boundary fields [by (3.2)]

$$\phi_{\partial}^i|_{s_2} = \phi_{\partial}^i|_s, \qquad e_a{}^A|_{s_2} = e_a{}^A|_s, \quad (3.5)$$

and again solves the equations of motion that follow from $S(\Phi^I + \epsilon\Delta_{f,\xi_i}\Phi^I)$.

We may use this observation to compute the advanced and retarded changes $D_{\Delta_{f,\xi}S}^{\pm}A$ of any gauge-invariant quantity A. Let us first consider the retarded change, and let us evaluate this change on a solution s as above. We seek a solution s_{ϵ}^- of the perturbed equations of motion which agrees with s on Σ_-. Since the infinitesimal transformation $f(\mathcal{L}_{\xi} - R)$ vanishes on Σ_-, we see that we may set $s_{\epsilon}^- = s_2$ as defined (3.3) above; i.e. $s_{\epsilon}^- = (1 + \epsilon f[\mathcal{L}_{\xi} - R])s$. Thus, the retarded effect on A is just $D_{\Delta_{f,\xi}S}^-A = f\mathcal{L}_{\xi}A$, where we have used the fact that A must be invariant under local frame rotations.

To compute the advanced effect, we seek a solution s_{ϵ}^+ of the perturbed equations of motion which agrees with s on Σ_+. Consider the history $s_{\epsilon}^+ = (1 - \epsilon[\mathcal{L}_{\xi} - R])s_{\epsilon}^- = (1 + (f - 1)\epsilon[\mathcal{L}_{\xi} - R])s$. Since this differs from s_{ϵ}^- by the action of a symmetry compatible with Ω, it again solves the equations of motion (to first order in ϵ) and induces the

required boundary fields (3.5). In addition, since $f = 1$ on Σ_+, we see that s_{ϵ}^+ and s agree on there. Thus, we may use s_{ϵ}^+ to compute the advanced change in any gauge-invariant A:

$$D_{\Delta_{f,\xi}S}^+A = (f - 1)\mathcal{L}_{\xi}A. \quad (3.6)$$

Finally, we arrive at the Peierls bracket

$$\{\Delta_{f,\xi}S, A\} = D_{\Delta_{f,\xi}S}^+A - D_{\Delta_{f,\xi}S}^-A = -\mathcal{L}_{\xi}A. \quad (3.7)$$

Thus, $-\Delta_{f,\xi}S$ generates a diffeomorphism along the asymptotic symmetry ξ as desired.[7]

Our task is now to relate $\Delta_{f,\xi}S$ to $Q[\xi]$. But this is straightforward. From (2.5), we have

$$\Delta_{f,\xi}S = \int_M \frac{\delta S}{\delta\Phi^I}\Delta_{f,\xi}\Phi^I + \int_{\partial M} \frac{\delta S}{\delta\phi_{\partial}^i}\Delta_{f,\xi}\phi_{\partial}^i$$
$$+ \int_{\partial M} \boldsymbol{\epsilon}\mathbf{T}^a{}_A\Delta_{f,\xi}e_a{}^A + \int_{\Sigma_{\pm}} \pi_I\Delta_{f,\xi}\Phi^I. \quad (3.8)$$

However, $\Delta_{f,\xi}\Phi^I$ vanishes on $\Sigma_{\pm}$ and on the boundary fields we may use (2.7) to find

$$\Delta_{f,\xi}\phi_{\partial}^i = (\mathcal{L}_{f_{\partial}\xi_{\partial}} - f_{\partial}\mathcal{L}_{\xi_{\partial}})\phi_{\partial}^i = 0,$$
$$\Delta_{f,\xi}e_a{}^A = (\mathcal{L}_{f_{\partial}\xi_{\partial}} - f_{\partial}\mathcal{L}_{\xi_{\partial}})e_a{}^A = e_b{}^A\xi_{\partial}^b\nabla_a f_{\partial}. \quad (3.9)$$

Thus, on-shell, only the term containing $\mathbf{T}^a{}_A\nabla_a f_{\partial}$ contributes to (3.8).

Furthermore, since f_{∂} is constant both to the past of C_0 and to the future of C_1, we may replace the integral over ∂M with an integral over the region V between C_0 and C_1. Thus, (3.8) takes the form

$$\Delta_{f,\xi}S = -\int_V \boldsymbol{\epsilon}\mathbf{T}^{ab}(\xi_{\partial})_b\nabla_a f$$
$$= -\int_{C_1} \mathbf{T}_{ab}\xi^b ds^a + \int_V \boldsymbol{\epsilon}f\nabla_a(\mathbf{T}^{ab}(\xi_{\partial})_b)$$
$$= -\int_{C_1} \mathbf{T}_{ab}\xi^b ds^a + \int_V f\left(R^A{}_B e^B{}_a\frac{\delta}{\delta e_a{}^A}\right.$$
$$\left. + \sum_{i,j} R^i{}_j\phi_{\partial}^j\frac{\delta}{\delta\phi_{\partial}^i}\right)S$$
$$= -Q_{C_1}[\xi]. \quad (3.10)$$

In the second line, we have used that ξ is an asymptotic symmetry [see Eqs. (2.12) and (2.13)], and that the action is invariant under frame rotations. In passing from the first to the second line we have used the fact that $f_{\partial} = 0$ on C_0.

Thus, $-\Delta_{f,\xi}S$ agrees (on-shell) with the charge $Q[\xi]$ evaluated on the cut C_1. By the arguments of Sec. II A, this

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[7]The form of $\Delta_{f,\xi}S$ is similar to the Hamilton-Jacobi definition of energy proposed in [53] in the context of asymptotically flat space. As a result, a similar argument might also be used to demonstrate equivalence of such a construction with Hamiltonian methods in that context.

equality also holds on any other cut of ∂M. Consequently, since by Eq. (3.7) the variation $\Delta_{f,\xi}S$ generates the action of the infinitesimal symmetry ξ on observables, it follows that the same must be true for the counterterm charges. Thus,

$$\{Q[\xi], A\} = \mathcal{L}_\xi A, \qquad (3.11)$$

as desired.

B. Asymptotic symmetries not compatible with Ω

In fact, we may apply a similar argument to the case described in Sec. II B, where an asymptotic symmetry ξ is *not* compatible with Ω and is thus associated with a boundary vector field ξ_∂ which is only a conformal Killing field of the chosen boundary fields, see (2.14) and (2.15). As such cases are not addressed by the axioms stated in Sec. II A, we state the corresponding requirements here. We will derive our results when the following additional conditions hold:

(6) Under the action of a diffeomorphism along $f\xi$ on a history (i.e., $h \rightarrow (1 + \epsilon \mathcal{L}_{f\xi}h)$ the boundary fields

transform with additional conformal weights:

$$\delta_{\mathcal{L}_{f\xi}}\phi_\partial^i = \mathcal{L}_{f_\partial\xi_\partial}\phi_\partial^i + f_\partial K^i{}_j\phi^j,$$
$$\delta_{\mathcal{L}_{f\xi}}e_a{}^A = \mathcal{L}_{f_\partial\xi_\partial}e_a{}^A + f_\partial K^A{}_B e_a{}^B. \qquad (3.12)$$

(7) Furthermore, since ξ_∂ is a conformal symmetry of the boundary fields and $K^i{}_j$, $K^A{}_B$ are the associated conformal weights, the right-hand side of (3.12) becomes just a frame rotation when $f = 1$.

The reader may readily check that requirements (6) and (7) above are fulfilled by the usual setting for counterterm subtraction schemes in asymptotically AdS spaces.

As a result of requirements (6) and (7), the histories $s_\epsilon^\pm$ identified in Sec. III A above [see, e.g., (3.3)] again have boundary values identical to those $(\phi_\partial^i, e_a{}^A)$ of s. Thus we may proceed exactly as above to again conclude

$$\{\Delta_{f,\xi}S, A\} = D^+_{\Delta_{f,\xi}S}A - D^-_{\Delta_{f,\xi}S}A = -\mathcal{L}_\xi A. \qquad (3.13)$$

Furthermore, using property (6) we find that, on-shell, we may calculate $\Delta_{f,\xi}S$:

$$\Delta_{f,\xi}S = -\int_V \boldsymbol{\epsilon} \mathbf{T}_{ab}(\xi)^b_\partial \nabla^a f = -\int_{C_1} \mathbf{T}_{ab}\xi_\partial^b ds^a + \int_V f\nabla^a(\mathbf{T}_{ab}\xi^b)$$

$$= -\int_{C_1} \mathbf{T}_{ab}\xi_\partial^b ds^a + \sum_i \int_V f\frac{\delta S}{\delta\phi_\partial^i}\mathcal{L}_{\xi_\partial}\phi_\partial^i + \int_V f\mathbf{T}_a^A\mathcal{L}_{\xi_\partial}e_A^a = -Q_{C_1}[\xi] + \int_V f\left(ke^A{}_a\frac{\delta}{\delta e^A{}_a} + \sum_{i,j}K^i{}_j\phi_\partial^j\frac{\delta}{\delta\phi_\partial^i}\right)S,$$

$$(3.14)$$

where in the last step we have again used property (7) and, as before, the second term in the final line is constant on S. Finally, since we saw in Sec. II B that $Q_C[\xi]$ depends on the cut C only through a term that is constant on S, it follows that we have (3.11) for any cut C. Thus, even when ξ_∂ is only a conformal symmetry of the boundary, $Q_C[\xi]$ can differ from any Hamiltonian generator of the symmetry ξ only through a (possibly cut-dependent) term which is a function *only* of the boundary fields and which is otherwise constant over the space S of solutions.

IV. DISCUSSION

We have used general arguments based on the Peierls bracket to compare the counterterm subtraction charges $Q[\xi]$ of [8–16] with any Hamiltonian charges $H[\xi]$ when ξ is a diffeomorphism which generates a symmetry of an appropriate system. Specifically, when ξ induces a symmetry ξ_∂ of the boundary fields, we have shown that $Q[\xi]$ generates the bulk symmetry associated with ξ via the Peierls bracket. As a result, it can differ from $H[\xi]$ only by a term determined entirely by the boundary fields and which is otherwise constant on the space of solutions. Furthermore, since both $Q[\xi]$ and $H[\xi]$ are conserved, this difference is also independent of the cut of infinity on which it is evaluated.

Our results generalize a conclusion of [30], which was in turn suggested by a number of more specific calculations

(e.g. [8,9,26–29]). Reference [30] showed via direct calculation that $Q[\xi] - H[\xi]$ was a function of boundary fields alone in $d = 5$ spacetime dimensions and under a particular set of asymptotic conditions; indeed, [30] gives an explicit formula for this difference. Reference [30] was also able to show that $H[\xi]$ agrees with a definition of energy in that context due to Ashtekar *et al.* [17,18]. However, from the results of the present paper and the convention that the Hamiltonian charges $H[\xi]$ vanish in AdS space, we may conclude generally that $H[\xi] = Q[\xi] - Q[\xi](AdS)$, where $Q[\xi](AdS)$ is the result obtained by evaluating the counterterm charge in pure AdS space.[8] The present results may also be applied in non-conformal versions of gauge-gravity duality (such as those described in, e.g., [5]) if an appropriate set of counterterms can be identified to implement requirements (1)–(5) of Sec. II. In particular, due to [38] we may apply them directly to certain spacetimes dual to cascading gauge theories and, due to [35–37], to domain-wall spacetimes.

In addition, the work above generalizes the counterterm procedure for constructing conserved charges to the case in which arbitrary (tensor and spinor) nontrivial boundary fields may be present in addition to the boundary metric. The result is simply the replacement of the boundary stress

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[8]After the appearance of the first version of this work, the same result was also derived in [34] under fairly general asymptotically AdS boundary conditions.

tensor with the "modified boundary stress tensor" $\mathbf{T}^{ab}$ of Eq. (2.4), which contains extra terms arising from any nontrivial boundary fields which are not scalars. This modified boundary stress tensor is not covariantly conserved, and even boundary scalar fields contribute to its divergence. Nevertheless, the form of $\nabla_a \mathbf{T}^{ab}$ allows one to show that $Q[\xi]$ is in fact conserved. Furthermore, the Peierls bracket argument again shows that $H[\xi] - Q[\xi]$ is constant on the space of solutions.

We also addressed a special case which arises when the bulk theory is dual to a conformal theory, as in the original anti-de Sitter context. In such cases, the counterterm action changes under a conformal transformation, but only by a function of the boundary fields which is otherwise constant on the space S of solutions. As a result, one may consider the case of a vector field ξ_∂ which acts only as a conformal symmetry on the boundary. The result is again that $Q[\xi]$ generates the action of the bulk symmetry along ξ via the Peierls bracket and thus that $Q[\xi]$ can differ from any Hamiltonian charge $H[\xi]$ only by a term built from the boundary fields (and which is otherwise constant on S). However, in this case the term can depend (through a solution-independent term) on the cut C of the boundary spacetime on which it is evaluated.[9]

Recall that when ∂M is determined through conformal compactification (as in the asymptotically anti-de Sitter

context), any asymptotic symmetry induces a conformal Killing field on the boundary. Thus, in this case one may work with a fixed conformal structure Ω and still construct all conserved quantities via the counterterm subtraction method. Furthermore, Hamiltonian generators which vanish on AdS space itself are given for all asymptotic symmetries ξ by

$$H[\xi] = Q_C[\xi] - Q_C[\xi](AdS), \tag{4.1}$$

where we have once again subtracted off the value $Q_C[\xi](AdS)$ of the counterterm charge evaluated on a corresponding cut C of ∂M in pure anti-de Sitter space. As a result, both $H[\xi]$ and $Q_C[\xi]$ are consistent with the covariant phase space methods of [32], which controls only variations of the Hamiltonian on the space of solutions.

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[9]Note that this dependence vanishes for the special case of asymptotically AdS spaces when the boundary metric is chosen to be the Einstein static universe and all other boundary fields vanish, since in that case $\tau = \tau^{ab} h_{ab} = 0$ [8–16].

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