

**Hamiltonian boundary term and quasilocal energy flux**

Chiang-Mei Chen\*

*Department of Physics, National Central University, Chungli 32054, Taiwan*James M. Nester<sup>†</sup>*Department of Physics and Institute of Astronomy, National Central University, Chungli 32054, Taiwan*Roh-Suan Tung<sup>‡</sup>*Center for Astrophysics, Shanghai Normal University, 100 Guilin Road, Shanghai 200234, China*

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The Hamiltonian for a gravitating region includes a boundary term which determines not only the quasilocal values but also, via the boundary variation principle, the boundary conditions. Using our covariant Hamiltonian formalism, we found four particular quasilocal energy-momentum boundary term expressions; each corresponds to a physically distinct and geometrically clear boundary condition. Here, from a consideration of the asymptotics, we show how a fundamental Hamiltonian identity naturally leads to the associated quasilocal energy flux expressions. For electromagnetism one of the four is distinguished: the only one which is gauge invariant; it gives the familiar energy density and Poynting flux. For Einstein's general relativity two different boundary condition choices correspond to quasilocal expressions which asymptotically give the ADM energy, the Trautman-Bondi energy and, moreover, an associated energy flux (both outgoing and incoming). Again there is a distinguished expression: the one which is covariant.

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**I. INTRODUCTION**

The Hamiltonian that generates the dynamical evolution of any physical system within a (finite or infinite) spatial region necessarily includes, in addition to an integral over the spatial volume, an integral over the boundary of the region. Our concern is with this Hamiltonian boundary term: we wish to understand both its role and proper form.

The Hamiltonian is, of course, related to energy. Every physical system carries mass-energy and consequently inevitably generates gravity, hence gravity plays the major role in our analysis. We consider only geometric gravity theories, and here more specifically only Einstein's general theory of relativity (GR).

The essential necessity for, and the role of, the Hamiltonian boundary term for asymptotically flat spaces in GR was first clearly discussed in a seminal work of Regge and Teitelboim [1]. They argued that in order for the Hamiltonian to be functionally differentiable on the phase space of asymptotically flat spatial metrics, certain boundary integrals over the 2-sphere at spatial infinity must be included. These turned out to be in fact just the ADM [2] asymptotic expressions for the conserved total quantities: energy, momentum, and angular momentum (plus an additional expression for the center of mass). Subsequently certain improvements were made in the analysis by Beig and Ó Murchadha [3]; more recently Szabados has made some further refinements [4].

Although the total conserved quantities are well defined for asymptotically flat spaces, as is well known, there are no well-defined local densities for these quantities. This can be understood in terms of the equivalence principle, which precludes the detection of gravity at a point (see, e.g. [5] Sec. 20.4). The modern idea is that one should have quasilocal quantities, i.e., quantities associated with a closed 2-surface (for a nice review of this topic see [6]).

For finite regions, we found that the value of the Hamiltonian boundary term can determine the quasilocal quantities. Using essentially the same principle as used by Regge-Teitelboim (this principle will play the major role in our subsequent discussion) we found that the Hamiltonian boundary term also determines the boundary conditions. Using this idea along with our covariant Hamiltonian formalism, for each dynamic field we found four types of quasilocal energy-momentum boundary term expressions; each corresponds to a physically distinct and geometrically clear boundary condition [7–9].

However a consideration of the phase space asymptotics necessary for a well-defined Hamiltonian naturally leads to the recognition that the conditions *are not met* in the radiating regime. This apparent difficulty provides the opportunity for our main new result. We show here how a fundamental Hamiltonian identity naturally leads, for each Hamiltonian boundary expression, to an associated quasilocal energy flux expression.

We note that gravitational energy flux (also known as balance) expressions have a long history, going back to Trautman [10], Bondi [11], and Sachs [12]. Nevertheless there still is considerable interest in this topic and some significant progress has been made recently; see in par-

\*Electronic address: cmchen@phy.ncu.edu.tw

<sup>†</sup>Electronic address: nester@phy.ncu.edu.tw<sup>‡</sup>Electronic address: tung@shnu.edu.cn

ticular [13–19]. We include here some applications of our new natural energy flux expressions.

An application of our formalism with its quasilocal energy and energy flux expressions to the electromagnetic field, where we can readily interpret the expressions and boundary conditions, is included to illustrate the ideas in a familiar setting. For the electromagnetic case one of our four expressions is distinguished: the only one which is gauge invariant; it gives the usual energy density and Poynting flux.

When applied to Einstein’s general relativity two different boundary condition choices correspond to quasilocal expressions which asymptotically give the ADM energy, the Trautman-Bondi energy and, moreover, an associated energy flux (both outgoing and incoming). However, once again there is a distinguished expression: in this case it is the one which is covariant.

The plan of this work is as follows: In Sec. II we discuss the first order Lagrangian formalism. In Sec. III we consider local translational invariance. The basic Hamiltonian formulation is considered in Sec. IV. In Sec. V we consider refined boundary terms. There follows a discussion of the phase space and the asymptotics in Sec. VI. Our new flux expressions are presented in Sec. VII. In Sec. VIII we consider the application of these ideas to electromagnetism. The application to Einstein’s gravity theory is covered in Sec. IX. A discussion forms the concluding section.

## II. THE FIRST ORDER LAGRANGIAN FORMALISM

Our formalism is intended to be applicable to general types of fields. Technically we find it convenient to represent dynamic fields in terms of differential forms. (More precisely to accommodate all the fields found in nature we would use a collection of tensor and spinor valued forms including Dirac spinors and gauge potential one-forms along with the spacetime orthonormal coframe and connection one-forms.) Here we develop the representative case of a  $k$ -form field  $\varphi$  (the field may take its values in some vector space and may thus carry some indices which are here suppressed; the generalization to include several fields, possibly of different grades, is straightforward).

We proceed from the action principle. It can be shown that any action principle can be rewritten in an equivalent form, which (following e.g. [2,20]) we refer to as *first order*; this is the most convenient form for our purposes. A *first order Lagrangian 4-form* for a  $k$ -form field  $\varphi$  has the form

$$\mathcal{L} = d\varphi \wedge p - \Lambda(\varphi, p). \quad (1)$$

Its variation has the form

$$\delta \mathcal{L} = d(\delta\varphi \wedge p) + \delta\varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \delta p, \quad (2)$$

implicitly defining a pair of first order Euler-Lagrange

expressions, which are explicitly given by

$$\frac{\delta \mathcal{L}}{\delta p} := d\varphi - \partial_p \Lambda, \quad \frac{\delta \mathcal{L}}{\delta \varphi} := -sdp - \partial_\varphi \Lambda, \quad (3)$$

where  $s := (-1)^k$ . The integral of  $\mathcal{L}$  associates an action with any spacetime region. The variation of this action is given by the integral of  $\delta \mathcal{L}$ . The total differential term in (2) then leads to an integral over the boundary of the region. Hamilton’s principle—that the action should be extreme with  $\delta\varphi$  vanishing on the boundary—yields the field equations: the vanishing of the Euler-Lagrange expressions (3).

## III. LOCAL TRANSLATION INVARIANCE

The action should not depend on the particular way points are labeled. Thus it should be invariant under diffeomorphisms, including infinitesimal diffeomorphisms—which correspond to a displacement along some vector field  $N$ . From a gauge theory perspective such displacements are regarded as a “local translation.” Under a local translation quantities change according to the Lie derivative. Hence, for a diffeomorphism invariant action the relation (2) should be identically satisfied when the variation operator  $\delta$  is replaced by the Lie derivative  $\mathcal{L}_N$  ( $\equiv di_N + i_N d$  on the components of form fields):

$$di_N \mathcal{L} \equiv \mathcal{L}_N \mathcal{L} \equiv d(\mathcal{L}_N \varphi \wedge p) + \mathcal{L}_N \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \mathcal{L}_N p. \quad (4)$$

This simply means that  $\mathcal{L}$  is a 4-form which depends on position only through the fields  $\varphi, p$ . (For this to be the case the set of fields in  $\mathcal{L}$  necessarily includes dynamic geometric variables, which means gravity.)

From (4) it directly follows that the 3-form

$$\mathcal{H}(N) := \mathcal{L}_N \varphi \wedge p - i_N \mathcal{L} \quad (5)$$

satisfies the identity

$$-d\mathcal{H}(N) \equiv \mathcal{L}_N \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \mathcal{L}_N p, \quad (6)$$

showing that it is a conserved “current” *on shell* (meaning: when the field equations are satisfied). Substituting (1) into (5) leads to the explicit expression  $\mathcal{H}(N) \equiv d(i_N \varphi \wedge p) + si_N \varphi \wedge dp + sd\varphi \wedge i_N p + i_N \Lambda$ , from which one can see that this conserved *Noether translation current* can be written as a 3-form linear in the displacement vector plus a total differential:

$$\mathcal{H}(N) := N^\mu \mathcal{H}_\mu + d\mathcal{B}(N). \quad (7)$$

Compare the differential of this expression,  $d\mathcal{H}(N) \equiv dN^\mu \wedge \mathcal{H}_\mu + N^\mu d\mathcal{H}_\mu$ , with (6); equating the  $dN^\mu$  coefficient on both sides reveals that

$$\mathcal{H}_\mu \equiv -i_\mu \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + s \frac{\delta \mathcal{L}}{\delta p} \wedge i_\mu p. \quad (8)$$

This identity is a necessary consequence of *local* diffeomorphism invariance (i.e., symmetry for nonconstant  $N^\mu$ ). From this relation one can see that  $\mathcal{H}_\mu$  vanishes on shell; hence the value of the conserved quantity associated with a local displacement  $N$  and a spatial region  $\Sigma$  is determined by a 2-surface integral over the boundary of the region:

$$E(N, \Sigma) := \int_\Sigma \mathcal{H}(N) = \oint_{\partial \Sigma} \mathcal{B}(N). \quad (9)$$

The value is *quasilocal*. For *any* choice of  $N$  this expression defines a conserved quasilocal quantity.

What do these values mean? In general we do not have a clear physical interpretation. However, at least if the geometry on the boundary is not so far from flat space, for a suitable timelike (spacelike) quasitranslation displacement on the boundary the expression defines a quasilocal energy (momentum), and for a suitable quasirotation(boost) it defines a quasilocal angular momentum (center of mass). (Here we do not explore the important question of how to specifically make these quasidisplacement choices for a general region in order to obtain good physical quasilocal values.)

#### IV. THE HAMILTONIAN FORMULATION

From the first order field Eqs. (3), by contraction with a “time evolution vector field”  $N$  and using  $i_N d\varphi = \mathcal{L}_N \varphi - di_N \varphi$ ,  $i_N dp = \mathcal{L}_N p - di_N p$ , we get a pair of Hamiltonian-like evolution equations for the “time derivatives”:  $\mathcal{L}_N \varphi$ ,  $\mathcal{L}_N p$ . A key identity involving these time derivatives is revealed by comparing two relations; on the one hand take the projection of  $\delta \mathcal{L}$  (2) along  $N$ :

$$\begin{aligned} i_N \delta \mathcal{L} &\equiv i_N d(\delta \varphi \wedge p) + i_N \left[ \delta \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \delta p \right] \\ &\equiv \mathcal{L}_N(\delta \varphi \wedge p) - di_N(\delta \varphi \wedge p) \\ &\quad + i_N \left[ \delta \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \delta p \right]; \end{aligned} \quad (10)$$

on the other hand take the projection of the Lagrangian 4-form  $i_N \mathcal{L}$ , which from (5) is just  $\mathcal{L}_N \varphi \wedge p - \mathcal{H}(N)$ , and vary:

$$\begin{aligned} \delta i_N \mathcal{L} &\equiv \delta(\mathcal{L}_N \varphi \wedge p) - \delta \mathcal{H}(N) \\ &\equiv \delta(\mathcal{L}_N \varphi) \wedge p + \mathcal{L}_N \varphi \wedge \delta p - \delta \mathcal{H}(N) \\ &\equiv \mathcal{L}_N \delta \varphi \wedge p + \mathcal{L}_N \varphi \wedge \delta p - \delta \mathcal{H}(N) \\ &\equiv \mathcal{L}_N(\delta \varphi \wedge p) - \delta \varphi \wedge \mathcal{L}_N p + \mathcal{L}_N \varphi \wedge \delta p \\ &\quad - \delta \mathcal{H}(N). \end{aligned} \quad (11)$$

Since  $N$  is not varied the two relations are identical:  $\delta i_N \mathcal{L} \equiv i_N \delta \mathcal{L}$ ; consequently,

$$\begin{aligned} \delta \mathcal{H}(N) &\equiv -\delta \varphi \wedge \mathcal{L}_N p + \mathcal{L}_N \varphi \wedge \delta p + di_N(\delta \varphi \wedge p) \\ &\quad - i_N \left[ \delta \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \delta p \right]. \end{aligned} \quad (12)$$

The last term vanishes “on shell.” This relation identifies the Noether translational current  $\mathcal{H}(N)$  as the *Hamiltonian 3-form* (i.e., density), as the following considerations show. The integral of  $\mathcal{H}(N)$  over a three-dimensional region,

$$H(N, \Sigma) := \int_\Sigma \mathcal{H}(N), \quad (13)$$

is the Hamiltonian which displaces this region along  $N$ , since the integral of its variation:

$$\delta H(N, \Sigma) = \int_\Sigma \delta \mathcal{H}(N), \quad (14)$$

yields, from (12), (on shell) the Hamilton equations:

$$\mathcal{L}_N \varphi = \frac{\delta \mathcal{H}(N)}{\delta p}, \quad \mathcal{L}_N p = -\frac{\delta \mathcal{H}(N)}{\delta \varphi}, \quad (15)$$

if the boundary term in the variation of the Hamiltonian *vanishes*. In this case that means when  $\delta \varphi$  vanishes on  $\partial \Sigma$ . Technically the variational derivatives of the Hamiltonian  $H(N, \Sigma)$  are only defined for variations satisfying this boundary condition. In other words this Hamiltonian is functionally differentiable on the phase space of fields satisfying this particular boundary condition.

#### V. REFINED BOUNDARY TERMS

In some important cases the fields of physical interest do not satisfy the aforementioned boundary condition naturally inherited from the Lagrangian (the main example is the spacetime metric for an asymptotically flat region). A suitably modified formulation is needed to deal with this situation. One alternative is to modify the Lagrangian 4-form itself by a total differential. As discussed in some detail in [9], this would modify the boundary condition on the whole three-dimensional boundary of the spacetime region, thus inducing the same type of modification on the spatial boundary at large spatial distances as on the initial time hypersurface. However we actually want the freedom to adjust the boundary condition on the two-dimensional boundary of the spacelike region  $\partial \Sigma$  independently of the type of initial conditions imposed within  $\Sigma$ . For this purpose we focus on the Hamiltonian. We note that the Hamiltonian (7) has two distinct parts; each plays a distinct role. The proper density  $N^\mu \mathcal{H}_\mu$ , although it has vanishing value on shell, generates the equations of motion, whereas the boundary term  $\mathcal{B}(N)$  determines both the *quasilocal value* (9) and the *boundary condition*. Now the boundary term can be adjusted—without changing the Hamilton equations or the conservation property (6)—indeed we can replace the 2-form  $\mathcal{B}(N) = i_N \varphi \wedge p$  inherited

from the Lagrangian by any other. Such an adjustment is in one respect just a special case of the conserved Noether current ambiguity [i.e. for any 2-form  $\chi$ ,  $J$  and  $J' := J + d\chi$  are both conserved currents (3-forms) if  $dJ = 0$ , even though they define different conserved values]. However here, in this Hamiltonian case, any such adjustment modifies—in *parallel*—not only the value of the quasilocal quantities but also the spatial boundary conditions. Thus the boundary term ambiguity is under physical control: each distinct choice of Hamiltonian boundary quasilocal expression is associated with a physically distinct boundary condition [8,9,21,22]. We thus find that the Hamiltonian density always takes the general form

$$\mathcal{H}_{\text{B.C.}}(N) = N^\mu \mathcal{H}_\mu + d\mathcal{B}_{\text{B.C.}}(N). \quad (16)$$

The subscript ‘‘B.C.’’ here refers to the particular choice of built in boundary condition.

In order to accommodate suitable boundary conditions we found that, in general, one must introduce (at the minimum actually only on the boundary, but to simplify the discussion here we presume it to be on any desired neighborhood of the boundary) certain reference values  $\bar{p}$ ,  $\bar{\varphi}$ , which represent the ground state of the field—the ‘‘vacuum’’ (or background field) values (this is necessary, in particular, for fields whose natural ground state is non-vanishing, e.g. the spacetime metric). We take our boundary terms to be linear in  $\Delta\varphi := \varphi - \bar{\varphi}$  and  $\Delta p := p - \bar{p}$ , so that they (and thus all the quasilocal quantities) vanish if the fields take on the ground state (i.e. reference) values. We presume that the reference values (like  $N$ ) are not varied:  $\delta\bar{\varphi} = 0$  and  $\delta\bar{p} = 0$ , consequently  $\delta\Delta\varphi = \delta\varphi$ ,  $\delta\Delta p = \delta p$ . Note that we presume the reference values (as well as  $N$ ) to be defined on the dynamic spacetime in the region of interest, independently of the dynamic fields; they are regarded as being fixed prior to (and thus independently of) the choice of  $\Sigma$  and  $S = \partial\Sigma$ .

From our investigations [7,8] we found two ‘‘covariant’’ boundary term alternates to  $\mathcal{B}(N) := i_N\varphi \wedge p$ , namely,

$$\mathcal{B}_{\text{Dirichlet}}(N) := i_N\varphi \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N\bar{p}, \quad (17)$$

$$\mathcal{B}_{\text{Neumann}}(N) := i_N\bar{\varphi} \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N p. \quad (18)$$

A short calculation [of the form  $\delta\mathcal{H}_{\text{B.C.}} = \delta(\mathcal{H} - d\mathcal{B}) + \delta d\mathcal{B}_{\text{B.C.}}$ ] shows that the variation of the associated Hamiltonian 3-forms (16) have (on shell) the respective forms

$$\begin{aligned} \delta\mathcal{H}_{\text{Dirichlet}}(N) &\equiv -\delta\varphi \wedge \mathcal{L}_N p + \mathcal{L}_N\varphi \wedge \delta p \\ &\quad + di_N(\delta\varphi \wedge \Delta p), \end{aligned} \quad (19)$$

$$\begin{aligned} \delta\mathcal{H}_{\text{Neumann}}(N) &\equiv -\delta\varphi \wedge \mathcal{L}_N p + \mathcal{L}_N\varphi \wedge \delta p \\ &\quad - di_N(\Delta\varphi \wedge \delta p), \end{aligned} \quad (20)$$

corresponding to holding fixed certain (after integration

certain projected) covariant sets of components, respectively, the value of the field and the value of its canonically conjugate momentum; whence our labels.

Moreover, we found two more physical interesting choices [9]:

$$\mathcal{B}_{\text{dynamic}}(N) := i_N\bar{\varphi} \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N\bar{p}, \quad (21)$$

$$\mathcal{B}_{\text{constrained}}(N) := i_N\varphi \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N p. \quad (22)$$

The variation of the associated Hamiltonian 3-forms have (on shell) the indicated forms

$$\begin{aligned} \delta\mathcal{H}_{\text{dynamic}}(N) &\equiv -\delta\varphi \wedge \mathcal{L}_N p + \mathcal{L}_N\varphi \wedge \delta p \\ &\quad + d(\varsigma\delta\varphi \wedge i_N\Delta p - i_N\Delta\varphi \wedge \delta p), \end{aligned} \quad (23)$$

$$\begin{aligned} \delta\mathcal{H}_{\text{constrained}}(N) &\equiv -\delta\varphi \wedge \mathcal{L}_N p + \mathcal{L}_N\varphi \wedge \delta p \\ &\quad + d(\delta i_N\varphi \wedge \Delta p - \varsigma\Delta\varphi \wedge \delta i_N p), \end{aligned} \quad (24)$$

corresponding to holding fixed certain (after integration certain projected) components of, respectively, the ‘‘dynamic parts’’ (the spatial pullback) of  $\varphi$ ,  $p$  and the ‘‘constrained parts’’ (the ‘‘time’’ projections)  $i_N\varphi$ ,  $i_N p$ .

In each of these cases the boundary term in the Hamiltonian variation has a certain *symplectic* structure which pairs certain *control* and *response* quantities [23]. Within each of our two sets of expressions the pairs are simply related by an interchange of ‘‘control’’ and ‘‘response,’’ formally  $\delta \rightarrow \Delta$ ,  $\Delta \rightarrow -\delta$ .

Note that one expression stands out: for fields which allow trivial reference values,  $\bar{\varphi} = 0 = \bar{p}$ , the boundary term  $\mathcal{B}_{\text{dynamic}}(N)$  vanishes. These fields, with this choice of boundary condition, make no explicit contribution to the quasilocal boundary term. Thus there is a certain preferred boundary expression—and thus a *preferred boundary condition*—for this large class of fields, a class which includes all the necessary physical fields—aside from dynamic geometry gravity.

An instructive discussion of boundary conditions associated with variational principles appears in Lanczos [24], Sec. II.15. The variational principle always gives us the correct number of boundary conditions. However it should be remarked that it cannot *guarantee* that the boundary conditions are the proper ones for the existence and uniqueness of solutions; that would depend on the particular quality and type of the equations, which is influenced especially by the metric signature (our formalism ‘‘formally’’ does not care) and the details of the Lagrangian ‘‘potential’’  $\Lambda(\varphi, p)$ . Our general formalism cannot take such particulars into account.

Our philosophy is that normally one should control on the boundary the indicated variations. Lanczos also discusses an interesting alternative: one could instead take ‘‘free’’ variations on the boundary. To have the boundary

term in the variation vanish with free variations of the control variables one must then require that the associated response vanishes; this yields what is referred to as the *natural boundary condition*. For our expressions it would simply mean that the response fields would take, on the boundary, the prespecified “reference” values. Hence, as far as imposing boundary conditions is concerned, one can achieve the same result by these two different approaches: (i) control the variable to the desired boundary value, or (ii) take free variations with the appropriate reference fields chosen to have the desired boundary values. Note, however, that there are differences in the resultant quasilocal values. In particular a free variation for  $\mathcal{H}_{\text{Dirichlet}}$  requires from (19) that  $p = \bar{p}$ . Consequently the value of the Hamiltonian is determined by a reduced form of the boundary expression (17):  $-\varsigma \Delta \varphi \wedge i_N p$ . Similarly, free variation for  $\mathcal{H}_{\text{Neumann}}$  leads to  $\varphi = \bar{\varphi}$  and the reduced boundary term  $i_N \varphi \wedge \Delta p$ . On the other hand, for  $\mathcal{H}_{\text{dynamic}}$ , free variations lead to the vanishing of  $i_N \Delta \varphi$  and  $i_N \Delta p$ , which yields no formal reduction of  $\mathcal{B}_{\text{dynamic}}$ , while free variation for  $\mathcal{H}_{\text{constrained}}$  leads to  $\Delta p = 0 = \Delta \varphi$ , consequently  $\mathcal{B}_{\text{constrained}}$  vanishes. Hence we find a curious fact: for free variations there is a boundary expression which is distinguished by having a vanishing value; it is the expression which is complementary to the one which vanishes for trivial reference values.

Lanczos has discussed, via a detailed example, how one could do more complicated things, where certain variables are “controlled” and others are varied “freely.” In all cases the boundary variation principle yields the correct number of boundary conditions. For our expressions there are many possible combinations of free and controlled variations. Using the “natural boundary conditions” associated with “free variation” is an interesting option which merits further exploration. Based on our present understanding, however, for most applications we favor purely controlled variations.

## VI. THE PHASE SPACE AND ASYMPTOTICS

For finite regions (which is our primary interest) these boundary terms in the variation of the Hamiltonian tell us exactly what needs to be held fixed (not constant but rather controlled, i.e. the function on the boundary is predetermined by some outside agent). For asymptotically flat regions, however, it is not sufficient to just say we want the field to vanish at infinity. Rather one should take into account the asymptotic falloff rates. Asymptotically we want to allow the variations and responses to be like the differences between generic solutions. The various boundary terms we have constructed will all enable the Hamiltonian to be well defined on the phase space of fields with asymptotic behavior for all typical physical fields. Detailed investigations have been done, in particular, for Einstein’s gravity theory, general relativity beginning with the pioneering work of Regge and Teitelboim [1]. This work was

later improved by Beig and Ó Murchadha [3] and more recently further refined by Szabados [4]. Here we give a simplified summary of certain relevant conclusions of these works.

It turns out that one should impose different falloff rates for the terms with different parity. For the fields it is sufficient to take the respective asymptotic falloffs and parities to be

$$\Delta \varphi \approx O_1^+ + O_2^-, \quad \Delta p \approx O_2^- + O_3^+, \quad (25)$$

where  $r^M O_M^\pm$  can be finite as  $r \rightarrow \infty$  and  $\pm$  indicates the parity. Here, for all types of tensors and forms, we define the parity to mean the parity of the components in an asymptotically Cartesian reference frame  $dx^\nu$ . Now the Cartesian components of the 2-surface area element have odd parity, consequently even parity 2-forms automatically have vanishing 2-surface integral.

For asymptotically flat spaces the displacement should asymptotically be a Killing vector, i.e. an infinitesimal Poincaré displacement. More precisely one can allow

$$N^\mu \approx (\alpha_0 + \alpha_1^+)^\mu + (\lambda_0 + \lambda_1^+)^\mu{}_\nu x^\nu, \quad (26)$$

where  $\alpha_0^\mu$  is a constant translation parameter and  $\lambda_0^{\mu\nu} = \lambda_0^{[\mu\nu]}$  is a constant asymptotic infinitesimal Lorentz boost/rotation parameter; the *even* parity part of the perturbations in these parameters can have the indicated  $1/r$  falloff, but any *odd* parity perturbation to  $\lambda$  should fall off faster. Note that the  $\lambda_1^+$  correction, because of its coordinate coefficient, means that it can be regarded as an odd parity perturbation of  $\alpha$ . Thus we can, without any change in the conserved total values or vanishing of the boundary term in the Hamiltonian variation, allow *supertranslations*—i.e., asymptotically nonconstant ( $\equiv$  angular dependent) terms of finite magnitude—as long as they are odd parity. Even parity supertranslations, on the other hand, would *change* the value of the quasilocal quantities and also would yield in general a *nonvanishing boundary term* in the Hamiltonian variation. If one really wanted them one could allow even parity supertranslations, but only at the expense of requiring boundary conditions on the fields more strict than (25).

With the asymptotics (25) and (26), it is straightforward to verify, as we just stated, that all four of our Hamiltonians are differentiable on the specified phase space—since all the boundary terms in the variations of the Hamiltonians vanish asymptotically; moreover the quasilocal expressions all give *the same* (since they differ by terms of the form  $N \Delta \varphi \Delta p$  which vanish asymptotically) *finite* constant values (independent of the perturbation  $\alpha_1^+$  and supertranslation  $\lambda_1^+ x$ ) for the energy momentum and angular momentum/center of mass—even for fields, like the metric or frame, with  $\bar{\varphi} \approx 1$ .

Note that the indicated asymptotic behavior is sufficient but hardly necessary. In practice only the specific projected component combinations that actually show up in the

boundary integrand need be so restricted (moreover one need only require this behavior up to a closed form). However it is not possible to give a general *covariant* formula for such details; for such refinements one must examine how each component for a particular field in a specific theory actually occurs in the expression.

We also note that one could even turn things around and take the finiteness of the quasilocal expressions and the vanishing of the boundary term in the variation of the Hamiltonian as defining a norm that would determine the largest possible phase space. We do not pursue this idea here.

At spatial infinity the aforementioned asymptotics are physically reasonable. Let us now consider what can be expected if the boundary of our 2-surface  $\partial\Sigma$  approaches null infinity. One can imagine it following the characteristic propagation surfaces of outgoing radiation. Long-range radiation fields (e.g. electromagnetism) have slower falloffs, like  $\Delta p \approx d\varphi \approx O_1$ . Then it may seem that we will have a serious problem, since the boundary terms in the variation of our various Hamiltonians will not vanish, so the Hamiltonian is no longer functionally differentiable. This seeming calamity is actually providential—as was recognized long ago (see e.g. the remark on page 160 in [25]) but not investigated until more recently—it is directing us to additional physics contained within the formalism, namely, energy flux expressions.

## VII. FLUX EXPRESSIONS

Given any vector field  $M = d/d\lambda$  one could calculate (on shell) the associated change in any quasilocal quantity directly from the boundary expression:

$$\frac{d}{d\lambda} E_{\text{B.C.}}(N, \Sigma) = \int_{\Sigma} \mathcal{L}_M \mathcal{H}_{\text{B.C.}}(N) = \oint_{\partial\Sigma} \mathcal{L}_M \mathcal{B}_{\text{B.C.}}(N). \quad (27)$$

In this fashion one could calculate from such a “flux” expression, for example, the change in the quasilocal linear momentum under a rotation or the time rate of change of the angular momentum. In particular this approach can be specialized to calculate the “time rate of change” of the “energy” associated with the “time displacement”  $N = d/dt$  itself

$$\frac{d}{dt} E_{\text{B.C.}}(N, \Sigma) = \int_{\Sigma} \mathcal{L}_N \mathcal{H}_{\text{B.C.}}(N) = \oint_{\partial\Sigma} \mathcal{L}_N \mathcal{B}_{\text{B.C.}}(N). \quad (28)$$

The 2-surface integral then defines an “energy flux.” (Note: we use the labels time and energy here for convenience in our description; however, the actual meaning is the change along  $N$  in the conserved quantity associated with  $N$ ; e.g. we could consider a rotation:  $N = \partial/\partial\phi$ .) Of course there is another natural way to calculate the “time rate of change” of any conserved quantity, namely, one simply rearranges the conserved current formula, such as

$\partial_\mu j^\mu = 0$  to  $\partial_i j^0 = -\partial_k j^k$ , and then integrates over a spatial region; the spatial divergence via the divergence theorem yields a boundary 2-surface flux integral. Such expressions are often referred to as “balance equations.” Expressed in terms of a closed three-form they follow just from integrating  $\mathcal{L}_N j \equiv di_N j$ . In particular for energy this leads (on shell) to  $\mathcal{L}_N \mathcal{H}_{\text{B.C.}}(N) \equiv di_N \mathcal{H}_{\text{B.C.}}(N) \equiv di_N d\mathcal{B}_{\text{B.C.}}(N) \equiv d\mathcal{L}_N \mathcal{B}_{\text{B.C.}}(N)$ . Thus this alternative approach also leads to (28).

For the flux of energy (and *only* for the flux of *energy*, i.e. the change of  $E(N)$  along  $N$ , not for the time rate of change of any other quantity like linear or angular momentum, etc.) there is another way of calculating—the analog of the classical mechanics calculation (for conservative Hamiltonian systems) of

$$\delta H = \dot{q}^k \delta p_k - \dot{p}_k \delta q^k \quad \longrightarrow \quad \dot{E} := \dot{H} \equiv 0 \quad (29)$$

under the replacement  $\delta \rightarrow d/dt$ , where the remarkable cancellation is a consequence of the particular (symplectic) form of the Hamilton equations. Specializing the relations (19), (20), (24), and (25) with  $\delta \rightarrow \mathcal{L}_N$  (here we are assuming that  $\mathcal{L}_N \bar{\varphi}$ ,  $\mathcal{L}_N \bar{p}$  vanish; this could be a strong restriction on the reference or on  $N$ ; examples appear in later sections), we note that the three-form parts vanish identically (again this is associated with the symplectic form of the Hamilton equations), hence the respective boundary flux expressions are the integrals of

$$\mathcal{L}_N \mathcal{H}_{\text{Dirichlet}}(N) = di_N(\mathcal{L}_N \varphi \wedge \Delta p), \quad (30)$$

$$\mathcal{L}_N \mathcal{H}_{\text{Neumann}}(N) = di_N(-\Delta \varphi \wedge \mathcal{L}_N p), \quad (31)$$

$$\mathcal{L}_N \mathcal{H}_{\text{dynamic}}(N) = d(\mathfrak{s} \mathcal{L}_N \varphi \wedge i_N \Delta p - i_N \Delta \varphi \wedge \mathcal{L}_N p), \quad (32)$$

$$\mathcal{L}_N \mathcal{H}_{\text{constraint}}(N) = d(i_N \mathcal{L}_N \varphi \wedge \Delta p - \mathfrak{s} \Delta \varphi \wedge i_N \mathcal{L}_N p). \quad (33)$$

These expressions hold even if  $\mathcal{H}_\mu$  does not vanish (i.e. when we have global but not local translation symmetry). Note that they are significantly different in appearance from those obtained using (28) along with (17), (18), (21), and (22):

$$\mathcal{L}_N \mathcal{H}_{\text{Dirichlet}}(N) = d\mathcal{L}_N(i_N \varphi \wedge \Delta p - \mathfrak{s} \Delta \varphi \wedge i_N \bar{p}), \quad (34)$$

$$\mathcal{L}_N \mathcal{H}_{\text{Neumann}}(N) = d\mathcal{L}_N(i_N \bar{\varphi} \wedge \Delta p - \mathfrak{s} \Delta \varphi \wedge i_N p), \quad (35)$$

$$\mathcal{L}_N \mathcal{H}_{\text{dynamic}}(N) = d\mathcal{L}_N(i_N \bar{\varphi} \wedge \Delta p - \mathfrak{s} \Delta \varphi \wedge i_N \bar{p}), \quad (36)$$

$$\mathcal{L}_N \mathcal{H}_{\text{constraint}}(N) = d\mathcal{L}_N(i_N \varphi \wedge \Delta p - \mathfrak{s} \Delta \varphi \wedge i_N p), \quad (37)$$

which hold only as long as  $\mathcal{H}_\mu$  vanishes. Via a straightforward but nontrivial calculation using the field Eqs. (15) it can, of course, be explicitly verified that the two respective forms are actually equivalent—when  $\mathcal{H}_\mu$  vanishes.

### VIII. APPLICATION TO ELECTROMAGNETISM

To illustrate these ideas first in a familiar setting let us examine (vacuum) electromagnetism. One could consider this as a source field for gravity, but it is more instructive, and still sufficient for our needs here, to more simply just consider vacuum electromagnetism in Minkowski space. In that case the formalism developed above, with the important exception of the “on shell” vanishing of  $\mathcal{H}_\mu$ , is still applicable. The first order Lagrangian four-form for the (source free) U(1) gauge field one-form  $A$  is

$$\mathcal{L}_{\text{EM}} = dA \wedge H - \frac{1}{2\lambda_0} {}^*H \wedge H, \quad (38)$$

yielding the pair of first order equations

$$dH = 0, \quad dA - \frac{1}{\lambda_0} {}^*H = 0. \quad (39)$$

These are just the vacuum Maxwell equations with  ${}^*H = \lambda_0 F := \lambda_0 dA$ ; hence  $H = -\lambda_0 {}^*F$ , and  $d{}^*F = 0$ . (Here  $\lambda_0^{-1}$  is the vacuum impedance.) With  $N = \partial_t$  and the decomposition  $A = (-\phi, A_k)$  we find that  $i_N F = i_N dA = \mathcal{L}_N A - di_N A$  corresponds to  $F_{0k} = \dot{A}_k + \partial_k \phi = -E_k$ . The magnetic field strength is  $F_{ij} := \partial_i A_j - \partial_j A_i := \epsilon_{ijk} B^k$ . Hence  $H_{0i} = -\lambda_0 {}^*F_{0i} = -\lambda_0 B_i$ ,  $H_{ij} = -\lambda_0 {}^*F_{ij} = -\lambda_0 \epsilon_{ijk} E^k$ . The natural reference choice is  $\bar{A} = 0 = \bar{H}$ .

The Hamiltonian three-form is

$$\begin{aligned} \mathcal{H}_{\text{B.C.}}^{\text{EM}}(N) &= -i_N A dH - dA \wedge i_N H + i_N \left( \frac{1}{2\lambda_0} {}^*H \wedge H \right) \\ &+ d\mathcal{B}_{\text{B.C.}} \end{aligned} \quad (40)$$

In conventional tensor index notation the volume density part is

$$\begin{aligned} \mathcal{H}^{\text{EM}} &= \lambda_0 [-\phi \partial_k E^k + \frac{1}{2}(\partial_i A_j - \partial_j A_i) F^{ij} + \frac{1}{2} E^k E_k \\ &- \frac{1}{4} F^{ij} F_{ij}]. \end{aligned} \quad (41)$$

After eliminating the 2nd class constraint,  $F_{ij} = \partial_i A_j - \partial_j A_i$ , it corresponds to the familiar  $\lambda_0 [\frac{1}{2}(E^2 + B^2) - \phi \partial_k E^k]$ ; the scalar potential acts as a Lagrange multiplier to enforce the Gauss constraint  $\partial_k E^k = 0$ .

For the four considered boundary choices we have

$$\mathcal{B}_{\text{Dir}}^{\text{EM}} = i_N A H = \lambda_0 \phi E^k dS_k, \quad (42)$$

$$\mathcal{B}_{\text{Neu}}^{\text{EM}} = A \wedge i_N H = -\lambda_0 A_i B_j \epsilon^{ijk} dS_k, \quad (43)$$

$$\mathcal{B}_{\text{dyn}}^{\text{EM}} = 0, \quad (44)$$

$$\mathcal{B}_{\text{con}}^{\text{EM}} = i_N A H + A \wedge i_N H = \lambda_0 (\phi E^k - A_i B_j \epsilon^{ijk}) dS_k. \quad (45)$$

The Hamiltonian variations have the respective forms

$$\begin{aligned} \delta \mathcal{H}_{\text{Dir}}^{\text{EM}}(N) &= \text{field equation terms} \\ &+ d(\delta i_N A H - \delta A \wedge i_N H), \end{aligned} \quad (46)$$

$$\begin{aligned} \delta \mathcal{H}_{\text{Neu}}^{\text{EM}}(N) &= \text{field equation terms} \\ &+ d(-i_N A \delta H + A \wedge \delta i_N H), \end{aligned} \quad (47)$$

$$\begin{aligned} \delta \mathcal{H}_{\text{dyn}}^{\text{EM}}(N) &= \text{field equation terms} \\ &+ d(-i_N A \delta H - \delta A \wedge i_N H), \end{aligned} \quad (48)$$

$$\begin{aligned} \delta \mathcal{H}_{\text{con}}^{\text{EM}}(N) &= \text{field equation terms} \\ &+ d(\delta i_N A H + A \wedge \delta i_N H). \end{aligned} \quad (49)$$

Here our interest is not in the field equations but in the total differential term which, upon integration, becomes a boundary term indicating the boundary condition. Briefly: the choice  $\mathcal{H}_{\text{Dir}}^{\text{EM}}$  corresponds to fixing the scalar potential and the components of the vector potential parallel to the 2-surface (the gauge-independent part of the latter fixes the normal component  $B_\perp$  of the magnetic field).  $\mathcal{H}_{\text{Neu}}^{\text{EM}}$  enables the fixing of certain projected components of  $F$ , namely,  $E_\perp$  and a part of  $B_\parallel$ . The choice  $\mathcal{H}_{\text{dyn}}^{\text{EM}}$  is used for fixing  $E_\perp$  and  $B_\perp$ , while  $\mathcal{H}_{\text{con}}^{\text{EM}}$  is used for fixing the scalar potential and a part of  $B_\parallel$ .

The physical meaning of such boundary conditions are well known especially in the electrostatics case. Fixing  $E_\perp$  on the boundary corresponds to fixing the surface charge density. Here is an instructive physical application: use a battery to first charge up a capacitor which contains a dielectric which can be inserted/removed, then disconnect the battery and measure the work needed to remove/insert the dielectric (in the process the potential will vary but the charge is fixed—no current or power will flow into/out of the system—the system is *adiabatically insulated*). Alternately leave the battery connected and measure the work needed to displace the dielectric—then the potential is fixed although the charge will now vary, so in this latter case current and hence power flows into or out of the system. The respective boundary terms in the variation of the Hamiltonian are  $-\phi \delta E^k dS_k$  and  $\delta \phi E^k dS_k$ . The point we wish to emphasize is that both boundary condition choices are physically meaningful; they correspond to real situations actually encountered in practice. Nevertheless there is a preferred choice.

One expression stands out: the  $\mathcal{H}_{\text{dyn}}^{\text{EM}}$  choice is the only one in which the value of the Hamiltonian is *gauge invariant*. Moreover, in addition to the neat property of enjoying a vanishing boundary term, it has an extra virtue of considerable physical importance: namely, the Hamiltonian density is *nonnegative*. Consequently the associated energy has a lower bound and the system has a natural vacuum or ground state: zero energy for vanishing fields. The value of the Hamiltonian  $H_{\text{dyn}}^{\text{EM}}$  can be interpreted as the *internal*

energy, whereas the other expressions include some additional energy on the boundary of the system associated with maintaining the boundary condition.

The respective energy flux expressions from (30)–(33) are

$$\begin{aligned}\mathcal{L}_N \mathcal{H}_{\text{Dir}}^{\text{EM}} &= d(i_N di_N A \wedge H - di_N A \wedge i_N H - i_N F \wedge i_N H) \\ &= \lambda_0 d[(\partial_t A_0 B_j - \frac{1}{2} \partial_t A_0 \epsilon_{ijk} E^k - E_i B_j) dx^i \wedge dx^j],\end{aligned}\quad (50)$$

$$\begin{aligned}\mathcal{L}_N \mathcal{H}_{\text{Neu}}^{\text{EM}} &= d(A \wedge i_N di_N H - i_N A \wedge di_N H) \\ &= \lambda_0 d[(A_0 \partial_i B_j - A_i \partial_t B_j) dx^i \wedge dx^j],\end{aligned}\quad (51)$$

$$\mathcal{L}_N \mathcal{H}_{\text{dyn}}^{\text{EM}} = -d(i_N F \wedge i_N H) = \lambda_0 d(-E_i B_j dx^i \wedge dx^j),\quad (52)$$

$$\begin{aligned}\mathcal{L}_N \mathcal{H}_{\text{con}}^{\text{EM}} &= d(i_N di_N A \wedge H + A \wedge i_N di_N H) \\ &= \lambda_0 d[-(\frac{1}{2} \partial_t A_0 \epsilon_{ijk} E^k + A_i \partial_t B_j) dx^i \wedge dx^j].\end{aligned}\quad (53)$$

(Note we cannot calculate the correct energy flux in this case by expressions like (28) and (34)–(37) unless we include an additional nonvanishing contribution from  $\mathcal{H}_{\mu}$ .) All of these Hamiltonians and their associated energies are really describing the same physical laws (the differences in the right-hand sides just correspond to differences in the left-hand sides); however, all except  $\mathcal{H}_{\text{dyn}}^{\text{EM}}$  are boundary condition choices which are gauge dependent descriptions. Clearly the  $\mathcal{H}_{\text{dyn}}^{\text{EM}}$  choice, associated with fixing the normal components  $E_{\perp}$  and  $B_{\perp}$  on the boundary, is preferred; it is the one suitable for most physical applications. It gives the usual energy density and Poynting flux.

## IX. APPLICATION TO EINSTEIN'S GRAVITY THEORY

Einstein's gravity theory, general relativity, can be formulated in several ways. For our purposes the most convenient is to take the *orthonormal coframe*  $\vartheta^{\mu} = \vartheta_k^{\mu} dx^k$  and the *connection one-form* coefficients  $\omega^{\alpha}_{\beta} = \Gamma^{\alpha}_{\beta k} dx^k$  as our geometric potentials. Moreover we take the connection to be *a priori* metric compatible:  $Dg_{\alpha\beta} := dg_{\alpha\beta} - \omega^{\gamma}_{\alpha} g_{\gamma\beta} - \omega^{\gamma}_{\beta} g_{\alpha\gamma} \equiv 0$ . Restricted to orthonormal frames where the metric coefficients are constant, this condition reduces to the algebraic constraint  $\omega^{\alpha\beta} \equiv \omega^{[\alpha\beta]}$ .

We consider the vacuum (source free) case for simplicity (the inclusion of sources is straightforward). GR can be obtained from the first order Lagrangian four-form

$$\begin{aligned}\mathcal{L}_{\text{GR}} &= \Omega^{\alpha\beta} \wedge \rho_{\alpha\beta} + D\vartheta^{\mu} \wedge \tau_{\mu} \\ &\quad - V^{\alpha\beta} \wedge \left( \rho_{\alpha\beta} - \frac{1}{2\kappa} \eta_{\alpha\beta} \right),\end{aligned}\quad (54)$$

where  $\Omega^{\alpha\beta} := d\omega^{\alpha}_{\beta} + \omega^{\alpha}_{\gamma} \wedge \omega^{\gamma}_{\beta}$  is the *curvature* two-form,  $D\vartheta^{\mu} := d\vartheta^{\mu} + \omega^{\mu}_{\nu} \wedge \vartheta^{\nu}$  is the *torsion* two-form, and we have made use of the convenient dual form basis  $\eta^{\alpha\beta\cdots} := *(\vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \cdots)$ . The two-forms  $\Omega^{\alpha\beta}$ ,  $V^{\alpha\beta}$ , and  $\rho_{\alpha\beta}$  are antisymmetric. We take  $\kappa := 8\pi G/c^4 = 8\pi$ .

In this Lagrangian the frame and connection conjugate momenta  $\tau_{\mu}$ ,  $\rho_{\alpha\beta}$  and the auxiliary field  $V^{\alpha\beta}$  all function like Lagrange multiplier fields; their variation imposes, respectively, the torsion vanishing condition  $D\vartheta^{\mu} = 0$ , the multiplier value  $V^{\alpha\beta} = \Omega^{\alpha\beta}$ , and the conjugate momentum value  $\rho_{\alpha\beta} = (2\kappa)^{-1} \eta_{\alpha\beta}$ . The connection one-form variation gives (in vacuum)

$$D\rho_{\alpha\beta} + \vartheta_{[\beta} \wedge \tau_{\alpha]} = 0. \quad (55)$$

Since  $D\rho_{\alpha\beta} \propto D\eta_{\alpha\beta} = D\vartheta^{\mu} \wedge \eta_{\alpha\beta\mu} = 0$  we get  $\vartheta_{[\beta} \wedge \tau_{\alpha]} = 0$ , from which it follows that  $\tau_{\mu} = 0$  (in vacuum). The frame variation gives

$$D\tau_{\mu} + \frac{1}{2\kappa} V^{\alpha\beta} \wedge \eta_{\alpha\beta\mu} = 0. \quad (56)$$

Substituting the already determined values for  $\tau_{\mu}$  and  $V^{\alpha\beta}$  yields  $\Omega^{\alpha\beta} \wedge \eta_{\alpha\beta\mu} = 0$ , the vanishing of the Einstein tensor three-form, i.e. the vacuum Einstein equation.

For most physical fields we can get by with a trivial reference. However in the case of GR we certainly need the refinements of both adjusting the Hamiltonian boundary term “by hand” — as was first clearly argued by Regge and Teitelboim [1] — and introducing a reference. With just the boundary term in the Hamiltonian *naturally* inherited from the Lagrangian,  $\mathcal{B}(N) = i_N \omega^{\alpha\beta} \wedge \rho_{\alpha\beta}$ , the boundary term in the variation of the Hamiltonian has the form  $i_N(\delta\omega^{\alpha\beta} \wedge \eta_{\alpha\beta})$ , which does not vanish for asymptotically flat falloffs. A simple reason for introducing the reference values, at least for the connection, is to render covariant the manifestly noncovariant “natural” Hamiltonian boundary term. (By the way, the need for reference values for GR was not apparent in the ADM [2] and Regge and Teitelboim formulations; there it was hidden in the choice of asymptotically Cartesian coordinates. The explicit need for a reference metric in the asymptotic Hamiltonian boundary term was, to our knowledge, first clearly apparent in the work of Beig and Ó Murchadha [3].)

In the vacuum case (or more generally as long as our boundary is in the vacuum region) the frame conjugate momentum field  $\tau_{\mu} = 0$ , so only the connection and its conjugate momentum make explicit contributions to our gravitational quasilocal expressions. (When sources are included we can always choose the boundary conditions so that the source fields have vanishing quasilocal boundary term, however via the gravitational field equations the sources, of course, indirectly influence the values of the gravitational field variables on the boundary and thereby do contribute to the quasilocal values.)

Now one of our dynamic gravitational variables is not a tensor field, so there are certain terms in our quasilocal expressions which include the noncovariant factors  $i_N \omega^{\alpha\beta}$  or  $i_N \bar{\omega}^{\alpha\beta}$ . As discussed in more detail in [8,9], the physical contribution due to these terms is obtained by replacing them by  $D^{[\beta} N^{\alpha]}$  or  $\bar{D}^{[\beta} N^{\alpha]}$ . Taking these considerations into account, along with the identification  $\rho_{\alpha\beta} = (2\kappa)^{-1} \eta_{\alpha\beta}$ , we obtain our quasilocal GR Hamiltonian boundary term expressions for gravitating systems (in vacuum regions):

$$\mathcal{B}_\vartheta^{\text{GR}}(N) := \frac{1}{2\kappa} (\Delta \omega^{\alpha\beta} \wedge i_N \eta_{\alpha\beta} + \bar{D}^{[\beta} N^{\alpha]} \Delta \eta_{\alpha\beta}), \quad (57)$$

$$\mathcal{B}_\omega^{\text{GR}}(N) := \frac{1}{2\kappa} (\Delta \omega^{\alpha\beta} \wedge i_N \bar{\eta}_{\alpha\beta} + D^{[\beta} N^{\alpha]} \Delta \eta_{\alpha\beta}), \quad (58)$$

$$\mathcal{B}_{\text{dyn}}^{\text{GR}}(N) := \frac{1}{2\kappa} (\Delta \omega^{\alpha\beta} \wedge i_N \bar{\eta}_{\alpha\beta} + \bar{D}^{[\beta} N^{\alpha]} \Delta \eta_{\alpha\beta}), \quad (59)$$

$$\mathcal{B}_{\text{con}}^{\text{GR}}(N) := \frac{1}{2\kappa} (\Delta \omega^{\alpha\beta} \wedge i_N \eta_{\alpha\beta} + D^{[\beta} N^{\alpha]} \Delta \eta_{\alpha\beta}). \quad (60)$$

As we already mentioned these quasilocal expressions will certainly yield finite values when integrated over an asymptotic 2-sphere for asymptotic Killing displacements of the form (26), at least when the variables satisfy the asymptotic parity and falloff conditions

$$\Delta \vartheta \approx O_1^+ + O_2^-, \quad \Delta \omega \approx O_2^- + O_3^+. \quad (61)$$

(At null infinity some parts of the connection actually have slower falloff yet; as we shall see, the quasilocal integrals are all still finite.)

The variations of the associated Hamiltonians have the respective forms

$$\begin{aligned} \delta \mathcal{H}_\vartheta^{\text{GR}}(N) = & \text{field equation terms} \\ & + \frac{1}{2\kappa} d(-i_N \Delta \omega^{\alpha\beta} \delta \eta_{\alpha\beta} \\ & + \Delta \omega^{\alpha\beta} \wedge \delta i_N \eta_{\alpha\beta}), \end{aligned} \quad (62)$$

$$\begin{aligned} \delta \mathcal{H}_\omega^{\text{GR}}(N) = & \text{field equation terms} + \frac{1}{2\kappa} d(\delta i_N \omega^{\alpha\beta} \Delta \eta_{\alpha\beta} \\ & - \delta \omega^{\alpha\beta} \wedge i_N \Delta \eta_{\alpha\beta}), \end{aligned} \quad (63)$$

$$\begin{aligned} \delta \mathcal{H}_{\text{dyn}}^{\text{GR}}(N) = & \text{field equation terms} \\ & + \frac{1}{2\kappa} d(-i_N \Delta \omega^{\alpha\beta} \delta \eta_{\alpha\beta} \\ & - \delta \omega^{\alpha\beta} \wedge i_N \Delta \eta_{\alpha\beta}), \end{aligned} \quad (64)$$

$$\begin{aligned} \delta \mathcal{H}_{\text{con}}^{\text{GR}}(N) = & \text{field equation terms} \\ & + \frac{1}{2\kappa} d(\delta i_N \omega^{\alpha\beta} \Delta \eta_{\alpha\beta} + \Delta \omega^{\alpha\beta} \wedge \delta i_N \eta_{\alpha\beta}). \end{aligned} \quad (65)$$

Here our interest is not in the field equations but in the total differential term which, upon integration, becomes a boundary term indicating the respective boundary conditions. Briefly:  $\mathcal{H}_\vartheta^{\text{GR}}$  requires fixing (after integration: certain projected components of) the orthonormal coframe, while  $\mathcal{H}_\omega^{\text{GR}}$  requires fixing (certain projected components of) the connection. Whereas  $\mathcal{H}_{\text{dyn}}^{\text{GR}}$  is associated with fixing the spatial projections of the frame and connection, and  $\mathcal{H}_{\text{con}}^{\text{GR}}$  is associated with fixing the time components of the frame and connection. The boundary terms in these Hamiltonian variations vanish asymptotically (spatially) with the aforementioned falloff and parity conditions. Asymptotically at spatial infinity the quasilocal quantities obtained from all four of our expressions are compatible with the analysis of Beig and Ó Murchadha [3], or more precisely with the refinement thereof of Szabados [4].

The associated energy flux expressions, calculated according to the respective prescriptions (30)–(33), (presuming that  $\mathcal{L}_N \bar{\vartheta} = 0 = \mathcal{L}_N \bar{\omega}$ , i.e.  $N$  is a Killing field of the reference geometry) take the form

$$\begin{aligned} \mathcal{L}_N \mathcal{H}_\vartheta^{\text{GR}}(N) = & \frac{1}{2\kappa} d(-i_N \Delta \omega^{\alpha\beta} \mathcal{L}_N \eta_{\alpha\beta} \\ & + \Delta \omega^{\alpha\beta} \wedge \mathcal{L}_N i_N \eta_{\alpha\beta}), \end{aligned} \quad (66)$$

$$\begin{aligned} \mathcal{L}_N \mathcal{H}_\omega^{\text{GR}}(N) = & \frac{1}{2\kappa} d(\mathcal{L}_N i_N \omega^{\alpha\beta} \Delta \eta_{\alpha\beta} \\ & - \mathcal{L}_N \omega^{\alpha\beta} \wedge i_N \Delta \eta_{\alpha\beta}), \end{aligned} \quad (67)$$

$$\begin{aligned} \mathcal{L}_N \mathcal{H}_{\text{dyn}}^{\text{GR}}(N) = & \frac{1}{2\kappa} d(-i_N \Delta \omega^{\alpha\beta} \mathcal{L}_N \eta_{\alpha\beta} \\ & - \mathcal{L}_N \omega^{\alpha\beta} \wedge i_N \Delta \eta_{\alpha\beta}), \end{aligned} \quad (68)$$

$$\mathcal{L}_N \mathcal{H}_{\text{con}}^{\text{GR}}(N) = \frac{1}{2\kappa} d(\mathcal{L}_N i_N \omega^{\alpha\beta} \Delta \eta_{\alpha\beta} + \Delta \omega^{\alpha\beta} \wedge \mathcal{L}_N i_N \eta_{\alpha\beta}). \quad (69)$$

Only one,  $\mathcal{L}_N \mathcal{H}_\vartheta^{\text{GR}}$  (66), is free from noncovariant factors.

Of course one should check the actual values given by these expressions. We have reexpressed (57) and (66) in the asymptotically null regime using the NP spin coefficient formalism and obtained good results [26]. Moreover we are presently working on adapting our expressions to conformal infinity; the results of that investigation will be reported in due course. Meanwhile, similar to the calculations in [27], we have tested all of these energy and energy flux expressions at null infinity on the full Bondi-Sachs [12] form of the metric using REDUCE and EXCALC. In particular, for the value of the expression  $\mathcal{B}_\vartheta^{\text{GR}}$  (which had been calculated earlier [27]) we obtain results essentially identical to those reported in [16] obtained using SHEEP to evaluate the Freud holonomic expression (which is the expression Trautman used in his original work [10]). This is not at all surprising, since this particular orthonor-

mal frame expression, for any frame satisfying the asymptotic ‘‘no rotation’’ gauge condition  $\vartheta_{[ak]} \approx O_2$ , is asymptotically equivalent to both the Freud superpotential [8,9] and the expression of Katz and coworkers [14]; the latter also gives good results at null infinity.

Our three other boundary expressions give similar but not identical results. Both for brevity and to more clearly show the main similarities and differences, we here present only the essential details for the simpler, axisymmetric Bondi metric [11]. We take the co-frame to be of the form

$$\begin{aligned} \vartheta^t &= e^{\beta+\phi} du + e^{\beta-\phi} dr, & \vartheta^r &= e^{\beta-\phi} dr, \\ \vartheta^\theta &= r e^\gamma (d\theta - r^{-2} U du), & \vartheta^\varphi &= r e^{-\gamma} \sin\theta d\varphi, \end{aligned} \quad (70)$$

where  $e^{2\phi} = 1 - 2m(u, \theta)/r + O_2$ ,  $\gamma = c(u, \theta)/r + O_2$ ,  $\beta = -c^2/4r^2 + O_4$ , and  $U = -\partial_\theta c - 2c \cot\theta + O_1$ . For the reference we take  $\phi = \gamma = \beta = U = 0$ . For  $N = \partial_u$  we find for the asymptotically contributing part of the energy expressions

$$\mathcal{B}_\vartheta^{\text{GR}} = \mathcal{B}_{\text{con}}^{\text{GR}} = \frac{1}{2\kappa} [4m \sin\theta + 2\partial_\theta(U \sin\theta)] d\theta \wedge d\varphi, \quad (71)$$

$$\begin{aligned} \mathcal{B}_\omega^{\text{GR}} &= \mathcal{B}_{\text{dyn}}^{\text{GR}} \\ &= \frac{1}{2\kappa} [4(m + cc_u) \sin\theta + 2\partial_\theta(U \sin\theta)] d\theta \wedge d\varphi. \end{aligned} \quad (72)$$

Upon integration over the sphere the terms with  $\partial_\theta(U \sin\theta)$  make a vanishing contribution due to the regularity conditions on  $c$ ,  $\partial_\theta c$  at the poles (note: similar terms appear in the 2-surface integrands in [16,27]). Then the two expressions in (71) give the standard Bondi mass aspect, while the other two do not.

The asymptotically contributing part of the associated quasilocal flux values were found to be

$$\mathcal{E}_{\partial_u} \mathcal{H}_\vartheta^{\text{GR}} = \mathcal{E}_{\partial_u} \mathcal{H}_{\text{con}}^{\text{GR}} = \frac{1}{2\kappa} d[-4c_u^2 \sin\theta d\theta \wedge d\varphi], \quad (73)$$

$$\mathcal{E}_{\partial_u} \mathcal{H}_\omega^{\text{GR}} = \mathcal{E}_{\partial_u} \mathcal{H}_{\text{dyn}}^{\text{GR}} = \frac{1}{2\kappa} d[4cc_{uu} \sin\theta d\theta \wedge d\varphi]. \quad (74)$$

Thus two expressions (73) directly give the standard Bondi flux loss. The other two expressions are actually describing exactly the same physics albeit for a different energy expression, as can be easily seen from comparing the time derivative of the expression (71) equated to (73) with the time derivative of the expression (72) equated to (74). Note that one can equally well compute an incoming flux from past null infinity by assuming a dependence on the advanced time  $v \simeq t + r$  in place of the retarded time  $u \simeq t - r$ .

From these calculations we can see that the functions  $\beta$  and  $U$  do not play a major role in this limit. This observation justifies the following simpler calculation, which still includes enough of the essential qualities while showing that our expressions capture both the incoming and outgoing quasilocal flux. Let us evaluate the energy and energy flux expressions for the following simplified asymptotic form of the orthonormal co-frame:

$$\begin{aligned} \vartheta^t &= e^\phi dt, & \vartheta^r &= e^{-\phi} dr, & \vartheta^\theta &= e^\gamma r d\theta, \\ \vartheta^\varphi &= e^{-\gamma} r \sin\theta d\varphi, \end{aligned} \quad (75)$$

where  $\phi = \phi(t, r) = O(1/r)$ ,  $\gamma = \gamma(t, r) = O(1/r)$ . For the reference we take  $\phi = 0 = \gamma$ . The connection one-form components are

$$\begin{aligned} \omega^{tr} &= \phi' e^{2\phi} dt - \dot{\phi} e^{-2\phi} dr, \\ \omega^{t\theta} &= r \dot{\gamma} e^{-\phi+\gamma} d\theta, \\ \omega^{t\varphi} &= -r \dot{\gamma} e^{-\phi-\gamma} \sin\theta d\varphi, \\ \omega^{r\theta} &= -(1 + r\gamma') e^{-\phi+\gamma} d\theta, \\ \omega^{r\varphi} &= -(1 - r\gamma') e^{-\phi-\gamma} \sin\theta d\varphi, \\ \omega^{\theta\varphi} &= -e^{-2\gamma} \cos\theta d\varphi. \end{aligned} \quad (76)$$

For our expressions with  $N = \partial_t$  the  $DN$  terms make no asymptotic contribution. The key factors are

$$\begin{aligned} \Delta \omega^{r\theta} &= [1 - (1 + r\gamma') e^{\phi+\gamma}] d\theta, \\ \Delta \omega^{r\varphi} &= [1 - (1 - r\gamma') e^{\phi-\gamma}] \sin\theta d\varphi. \end{aligned} \quad (77)$$

From these we find the asymptotically contributing part of the quasilocal energy boundary expressions:

$$\mathcal{B}_\vartheta^{\text{GR}} = \mathcal{B}_{\text{con}}^{\text{GR}} = \frac{1}{2\kappa} 4m \sin\theta d\theta \wedge d\varphi, \quad (78)$$

$$\begin{aligned} \mathcal{B}_\omega^{\text{GR}} &= \mathcal{B}_{\text{dyn}}^{\text{GR}} = \frac{1}{2\kappa} 4[m - r^2 \gamma \gamma'] \sin\theta d\theta \wedge d\varphi \\ &= \frac{1}{2\kappa} 4[m - r^2 \gamma (\gamma_v - \gamma_u)] \sin\theta d\theta \wedge d\varphi, \end{aligned} \quad (79)$$

where, as usual,  $m := (r/2)(1 - e^{2\phi})$  and  $u := t - r$ ,  $v := t + r$ . Upon integration over the 2-sphere  $r = \text{const}$  we find that two boundary conditions, corresponding to the boundary terms  $\mathcal{B}_\theta^{\text{GR}}$  and  $\mathcal{B}_{\text{con}}^{\text{GR}}$ , give the Bondi mass.

The associated quasilocal flux values are found to be

$$\begin{aligned} \mathcal{E}_N \mathcal{H}_\vartheta^{\text{GR}} &= \mathcal{E}_N \mathcal{H}_{\text{con}}^{\text{GR}} = \frac{1}{2\kappa} d[4r^2 \dot{\gamma} \gamma' \sin\theta d\theta \wedge d\varphi] \\ &= \frac{1}{2\kappa} d[4r^2 (\gamma_v^2 - \gamma_u^2) \sin\theta d\theta \wedge d\varphi], \end{aligned} \quad (80)$$

$$\begin{aligned}\mathcal{L}_N \mathcal{H}_\omega^{\text{GR}} &= \mathcal{L}_N \mathcal{H}_{\text{dyn}}^{\text{GR}} = \frac{1}{2\kappa} d[-4r^2 \gamma \dot{\gamma}' \sin\theta d\theta \wedge d\varphi] \\ &= \frac{1}{2\kappa} d[-4r^2 \gamma (\partial_v^2 \gamma - \partial_u^2 \gamma) \sin\theta d\theta \wedge d\varphi].\end{aligned}\quad (81)$$

Integrating (78)–(81) over a large 2-sphere at constant  $r$ , we find that all control modes are consistent with a Bondi news type energy flux loss and gain from both outgoing and incoming radiation:

$$\dot{m} = r^2 \dot{\gamma} \gamma' = r^2 (\gamma_v^2 - \gamma_u^2). \quad (82)$$

However only the Hamiltonians  $\mathcal{H}_\theta^{\text{GR}}$  and  $\mathcal{H}_{\text{con}}^{\text{GR}}$  give this relation *directly*. For most applications  $\mathcal{H}_\theta^{\text{GR}}$  would be the preferred choice, because the associated flux relation (66) is free of noncovariant factors.

## X. DISCUSSION

The Hamiltonian for a gravitating system in a finite or infinite region necessarily includes a boundary term. We have inquired into the significance and best form of this boundary term. We found that it determines not only the quasilocal values but also, via the *boundary variation principle*, the boundary conditions necessary for a well-defined Hamiltonian. We noted that it is always possible (and necessary, at least for gravity) to include in it non-dynamic reference values for the dynamic variables. Using our covariant Hamiltonian formalism and an identity associated with the variation of the Hamiltonian, we have identified, for each dynamic field, four special quasilocal energy-momentum boundary term expressions; each corresponds to a physically distinct and geometrically clear boundary condition. We showed how a fundamental Hamiltonian variation identity naturally forces us, for radiating asymptotics, to relax the well-defined Hamiltonian requirement and thereby obtained the associated quasilocal energy flux expressions. When the formalism is applied to electromagnetism one of the four is distinguished by gauge invariance (it gives the familiar energy density and Poynting flux). When the formalism is applied to Einstein's general relativity, two different boundary condition choices correspond to quasilocal expressions which asymptotically give the ADM energy, the Trautman-Bondi energy and, moreover, an associated energy flux (both outgoing and incoming), but once again there is a distinguished expression: in this case it is the one which is covariant.

Here we make a few further remarks. First we mention that our ideas regarding variational principles, symplectic structure, and the role of boundary terms have been much influenced by several sources, especially Tulczyjew and Kijowski [23,28] and his co-workers. Although we considered radiation using the Hamiltonian, there are many aspects associated with the Hamiltonian in the radiating

regime that we did not discuss; these issues are nicely treated in [17]. The uniqueness of the Trautman-Bondi mass is established in [16].

We, like many others, took a Hamiltonian approach. While some of our expressions are similar to those that appear in other approaches, e.g. the Noether charge approach [29], our formalism includes certain features that are unusual. Features which distinguish our formalism include:

- (i) We have made extensive use of differential forms—because of their convenience for integration over domains with boundaries, the spacetime decomposition, as well as the representation of geometric and gauge fields.
- (ii) We started from a first order Lagrangian—mainly because it facilitates the passage from the Lagrangian to the Hamiltonian but also because of its linearity re coupling via connections.
- (iii) We used the (co)frame and connection as independent variables.
- (iv) The Hamiltonian formalism we have developed is 4-covariant.

Our aim has been to find 4D covariant two-form and three-form expressions which can be integrated over any suitable desired regions; essentially we want to find a good Hamiltonian density for the region of interest. In this approach the particular 3-surface  $\Sigma$  and its boundary  $\partial\Sigma$  are quite incidental. Accordingly, our reference values and displacement are independent of these surfaces. Note also that we do not decompose our expressions into intrinsic and extrinsic parts with respect to such surfaces, as was done both traditionally and in many modern works, e.g. [30,31] wherein some of the themes considered here have been treated with the aid of such decompositions.

Aside from some important clarifications of the formalism, the main new ingredients here involve our variational identity, the relaxation of the boundary variational principle, and the flux expressions associated with our four quasilocal expressions, along with the fact that in each application one expression is distinguished (by being *gauge invariant* for the relevant type of gauge covariance— $U(1)$  or Lorentz in our electromagnetic and gravitational examples, respectively). Although in our discussion we referred to *energy* and *energy flux*, what we really considered is the value of the Hamiltonian for a prescribed displacement  $N$  and the associated flux expression that gives the rate of change of this value along  $N$ . Our formal results may have other useful applications besides energy.

Localization of energy-momentum has been an outstanding problem from the very beginning of GR. Traditional methods, both Noether spacetime translation symmetry and decomposition of the field equations, have only led to a variety of reference frame dependent expressions (generally referred to as pseudotensors) for the energy-momentum density. Thus in addition to the ambi-

guity of which expression should be preferred, there was also the ambiguity of the choice of reference frame. The boundary variation principle along with the Hamiltonian formalism tamed these inherent ambiguities, giving them a clear physical and geometric interpretation: the choice of expression is related to the choice of boundary conditions and the choice of reference frame is associated with the choice of ground state [9,21,22]. Within the covariant Hamiltonian formalism using this principle we had identified certain special “covariant-symplectic” Hamiltonian boundary term quasilocal expressions [7–9]. Here, based on a fundamental Hamiltonian identity, we have identified the associated *energy flux* expressions. Moreover we found that, for both source fields and gravity, among the four, *one* particular quasilocal Hamiltonian boundary term and its associated flux expression was distinguished.

An additional virtue enjoyed by the respective distinguished boundary terms (44) and (57) is that the Hamiltonian for this quasilocal choice has *positive energy*, not only for electromagnetism (as we discussed above), but also for gravity. More precisely (for acceptable choices of the variables) the gravitational Hamiltonian with the distinguished boundary term (57) is nonnegative, at least when  $\Sigma$  is maximal or nearly maximal. Here we just make mention of two lines of argument that lead to this

conclusion. One can obtain this result by adapting to finite regions the global positivity proof [32] using the special orthonormal frame gauge conditions [33]; one can instead use an adaptation of the Shi and Tam proof [34], which guarantees the positivity (for mean convex 2-surfaces) of the Brown-York [35] quasilocal expression (the latter, as shown in [8], agrees with (57) for certain choices). We are presently working on a detailed account of these two arguments. Of course it would be nice to have a stronger and more general energy-momentum positivity proof for our distinguished covariant-symplectic quasilocal GR Hamiltonian boundary term.

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