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(Received 11 September 2005; published 23 November 2005)

We demonstrate explicit counterexamples to the correlated stability conjecture (CSC), which claims that the horizon of a black brane is unstable precisely if that horizon has a thermodynamic instability, meaning that its matrix of susceptibilities has a negative eigenvalue. These examples involve phase transitions near the horizon. Ways to restrict or revise the CSC are suggested. One of our examples shows that $\mathcal{N} = 1^*$ gauge theory has a second-order chiral symmetry breaking phase transition at a temperature well above the confinement scale.

DOI: [10.1103/PhysRevD.72.104019](https://doi.org/10.1103/PhysRevD.72.104019)

PACS numbers: 04.70.Bw, 04.50.+h, 11.10.Wx, 11.27.+d

I. INTRODUCTION

In [1,2] it was observed that the event horizons surrounding black strings and p -branes are often unstable toward linear perturbations with some sinusoidal dependence on a spatial direction parallel to the brane. According to general theorems, entropy must increase as such an instability develops. Indeed, entropy increase was used in [1,2] to motivate the existence of the instability, but the entropy argument in that case was that a final state consisting of separated black holes must have larger entropy than an initial state of a nonextremal uniform string. It has subsequently become a difficult question how and whether one can actually evolve in finite asymptotic time from one state to the other. This question was first raised in [3], and recent reviews [4,5] provide a summary of what is presently known as well as a guide to the (already extensive) literature.

In [6,7] it was suggested that the existence of a perturbative Gregory-Laflamme (GL) instability for a horizon which is infinite and translationally invariant in some spatial direction should be associated with a local thermodynamic instability. For uncharged black strings or branes, this simply means that a GL instability should occur precisely when the specific heat is negative. Analytic arguments in support of this link were advanced in [8]. These arguments are convincing in the case of pure gravity and for a limited class of charged black branes: see [9] for a recent extension. A more complicated case is when the black brane carries some charges or angular momenta which are capable of being spatially redistributed: see [10] for a discussion of how Reall's argument might generalize. An example is a black string in five or more dimensions which carries electric charge under a $U(1)$ gauge field A_μ . In such a case, local thermodynamic stability is a criterion that one applies to the Hessian matrix of second derivatives of the entropy with respect to the mass and all the conserved charges. If this matrix has a positive eigenvalue, then there is a local thermodynamic

instability. It should then be possible to locally redistribute mass density and/or conserved charge density in such a way as to increase entropy without changing the total value of the conserved charges. It makes sense therefore to conjecture that a GL instability occurs precisely when there is a local thermodynamic instability. This is the correlated stability conjecture (CSC) of [6,7]. The conjecture includes as a hypothesis that the horizon is infinite and translationally invariant in some spatial direction because only then should we entirely trust thermodynamic arguments.

Following [8], there has been debate over whether the CSC should be generally valid. It seems impossible for there to be a violation in the direction of having a GL instability in the presence of stable thermodynamics, because then the horizon area would decrease as one goes forward in time. Violations in the other direction do not spoil any broad properties of general relativity, but (to our knowledge) none have come to light as yet.

In this paper, we find violations of the CSC in the direction where the arguments of the previous paragraph indicate they are least likely—but entropy does not decrease. At the linearized level, the perturbatively unstable mode involves only a scalar field. This scalar acts as an order parameter of a phase transition that occurs near the black hole horizon. The violation of the CSC is associated with trying to keep the black hole in the disordered phase below the critical temperature at which spontaneous ordering is entropically favored. Entropy is expected to increase as the instability departs from the linear regime, and a definite endpoint of the evolution can be guessed: it is simply the ordered phase of the black hole horizon, possibly with some domain walls or other defects within the brane world volume.

The instability we find is a normalizable disturbance of the horizon which grows exponentially in time. Because it involves only a scalar field, it may seem rather different from the gravitational modes investigated in [1]. But there

are other examples [6,7] where there is a horizon instability involving only matter fields. (It can even be shown in these cases that one can interpolate between an instability involving only the metric and an instability involving only matter fields.) The main difference between the standard Gregory-Laflamme instability and the instability we study here is the dispersion relation below the critical wave number where the frequency ω becomes imaginary. In [1,2], the gravitational instabilities were found to have a characteristic dispersion relation $|\Im\omega| \approx \frac{k_c}{10} \sin \frac{\pi k}{k_c}$. This analytic expression is only a rough fit to the numerical results of [1,2]. The feature to note is that $\Im\omega \rightarrow 0$ as $k \rightarrow 0$. We may understand this heuristically as a consequence of conservation of energy: the $k=0$ mode cannot be excited at all because it would change the total mass of the black string. For the instabilities we will describe, $|\Im\omega|$ is a monotonically decreasing function of k up to some k_c where it vanishes. The instability is not associated with a conserved quantity, so there is no reason for $\Im\omega$ to vanish as $k \rightarrow 0$.

The instability we will describe is somewhat reminiscent of the gyrating strings proposal of [11]. Discussions of this proposed violation of the CSC have focused on nonuniqueness of black brane solutions carrying specified conserved quantum numbers. This is in the spirit of a near-horizon phase transition [12]. What makes a violation of the CSC possible is that knowing the conserved charges of a black brane is not enough to uniquely determine the classical solution. Local thermodynamic stability has to do only with the conserved quantities, but a dynamical horizon instability is a property of the entire solution.

Although the CSC as stated in [6,7] seems to us now to be violated by the examples of the present paper, it may be possible to save it by adding an extra hypothesis, namely, that the uniform brane solution is unique once all conserved quantities have been specified.

The organization of the rest of this paper is as follows. In Sec. II we consider a class of examples based on magnetically-charged branes with an unusual $\chi^2 F^2$ coupling of a scalar field χ to a gauge field strength F . In Sec. III, we explore an example in AdS₅ with two scalars, one of which describes a renormalization group (RG) flow, while the other describes the breaking of a spontaneous symmetry. In Sec. IV, we show that the $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ SYM exhibits a finite-temperature phase transition, and that a metastable branch with no gaugino condensate has a horizon instability in the same class as the examples discussed in Sec. III.

II. MAGNETICALLY-CHARGED BRANES

Consider the action

$$S = \int d^D x \sqrt{g} \left[R - \frac{1}{2} (\partial\chi)^2 - f(\chi) F_{D-p-2}^2 - V(\chi) \right], \quad (1)$$

where $V(0) = V'(0) = 0$, $f(0) = 1$, and $f'(0) = 0$. There is a magnetically-charged p -brane solution in which $\chi = 0$ identically:

$$\begin{aligned} ds^2 &= H^{-2/(p+1)} (-h dt^2 + d\vec{x}^2) + H^{2/(D-p-3)} \\ &\quad \times \left(\frac{dr^2}{h} + r^2 d\Omega_{D-p-2}^2 \right), \\ H &= 1 + \frac{r_0^{D-p-3} \sinh^2 \alpha}{r^{D-p-3}}, \\ h &= 1 - \frac{r_0^{D-p-3}}{r^{D-p-3}}, \\ F_{D-p-2} &= \sqrt{\frac{2(D-2)(D-p-3)}{p+1}} \\ &\quad \times r_0^{D-p-3} \cosh \alpha \sinh \alpha \text{vol}_{S^{D-p-2}} \end{aligned} \quad (2)$$

where $\text{vol}_{S^{D-p-2}}$ is the volume form on the unit sphere S^{D-p-2} . Special cases of the solutions (2) include the magnetically-charged Reissner-Nordstrom black hole ($D=4$, $p=0$), the M2-brane ($D=11$, $p=2$), and the M5-brane ($D=11$, $p=5$). We will assume $0 \leq p \leq D-4$ to avoid certain pathologies, like spacetimes which are not asymptotically flat.

To check for the sign of specific heat, it is easiest to first compute the temperature of the black brane solution in terms of α and r_0 :

$$\begin{aligned} T &= \frac{(D-p-3)}{4\pi r_0} (\cosh \alpha)^{2\mu} \\ &= \frac{(D-p-3)}{\pi r_0 2^{\mu+2}} \left(1 + \sqrt{1 + \frac{Q^2}{r_0^{2(D-p-3)}}} \right)^\mu, \end{aligned} \quad (3)$$

where $\mu = -(D-2)/(D-p-3)(p+1)$ and in the last equality we have eliminated the parameter α for Q which is proportional to the conserved charge $\int F_{D-p-2}$. It is straightforward to check that the specific heat, which is inversely proportional to and has the same sign as dT/dr_0 , is positive for these solutions for the charge to mass ratio $Q/r_0^{(D-p-3)}$ larger than some $O(1)$ lower bound. There is no conserved charge that is capable of being spatially redistributed, so positive specific heat means that there is local thermodynamic stability. In such a situation, the CSC predicts no dynamical instability of the horizon.

Consider however the equation of motion for χ , linearized around (2):

$$\begin{aligned}
 & [\square - f''(0)F_{D-p-2}^2 - V''(0)]\chi = 0, \\
 & \left[H^{2/(p+1)} \left(\frac{\omega^2}{h} - \vec{k}^2 \right) + \frac{1}{r^{D-p-2} H^{2/(D-p-3)}} \partial_r (r^{D-p-2} h \partial_r) - m_{\text{eff}}^2 \right] \chi_{\omega, \vec{k}} = 0, \\
 & m_{\text{eff}}^2 = V''(0) + f''(0) \frac{2(D-2)(D-p-3)}{p+1} r_0^{2(D-p-3)} \frac{H^{-2(D-p-2)/(D-p-3)}}{r^{2(D-p-2)}} \cosh^2 \alpha \sinh^2 \alpha,
 \end{aligned} \tag{4}$$

where in the second line we have specialized to a separated s -wave ansatz: $\chi = e^{-i\omega t + i\vec{k} \cdot \vec{x}} \chi_{\omega, \vec{k}}(r)$. If $V''(0) < 0$, then empty space is unstable toward development of a VEV for χ . This instability is also visible in the stress tensor as a violation of the dominant energy condition. Let us therefore assume $V''(0) \geq 0$. If we choose $f''(0)$ sufficiently negative, then (4) admits normalizable solutions with imaginary ω . This is a GL instability. Provided $f(\chi) > 0$ for all χ , the dominant energy condition is not violated. Thus our main conclusion: violations of the CSC can be arranged by choosing suitable couplings of a scalar field.

To verify the claims in the previous paragraph about the dominant energy condition, let us examine the stress tensor explicitly:

$$\begin{aligned}
 T_{\mu\nu} &= T_{\mu\nu}^{(\chi)} + T_{\mu\nu}^{(F)}, \\
 T_{\mu\nu}^{(\chi)} &= \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} (\partial\chi)^2 - g_{\mu\nu} V(\chi), \\
 T_{\mu\nu}^{(F)} &= f(\chi) \left[\frac{1}{(D-p-3)!} F_{\mu\mu_2 \dots \mu_{D-p-2}} F_{\nu}^{\mu_2 \dots \mu_{D-p-2}} \right. \\
 &\quad \left. - \frac{1}{2(D-p-2)!} g_{\mu\nu} F_{\mu_1 \dots \mu_{D-p-2}}^2 \right].
 \end{aligned} \tag{5}$$

The dominant energy condition is that when ξ^μ is timelike,

$T_{\mu\nu} \xi^\mu \xi^\nu \geq 0$ (heuristically: energy density is positive) and $T_{\nu}^{\mu} \xi^\nu$ is timelike or null (heuristically: the flow of energy is timelike or null). It is well known that $T_{\mu\nu}^{(\chi)}$ obeys dominant energy provided $V(\chi) \geq 0$. Assume also that $f(\chi) \geq 0$. Then $T_{\mu\nu}^{(F)}$ is a non-negative multiple of its form in the special case where $f(\chi) = 1$. This special case is a free massless theory with a positive definite Hamiltonian density, and it is intuitively clear that the flow of energy is timelike or null for any such theory. A formal demonstration that $T_{\nu}^{(F)\mu} \xi^\nu$ is timelike or null for timelike ξ^μ is slightly technical and will not be presented here.

What drives the existence of a normalizable eigenfunction $\chi_{\omega, \vec{k}}$ with negative ω^2 is that m_{eff}^2 becomes large and negative near the horizon when $f''(0)$ is sufficiently negative, holding all other quantities (including \vec{k}^2) fixed. Heuristically χ becomes tachyonic near the horizon, so it should condense. To verify this more rigorously, we may transform (4) into Schrödinger form. We assume $r_0 > 0$. The extremal case is somewhat different on account of the nature of horizon boundary conditions, but the same qualitative claim about condensation of χ in the presence of sufficiently negative $f''(0)$ should persist at extremality.

The radial equation in (4) can be cast in the form

$$\left[\omega^2 + \frac{J_1 h}{r^{D-p-2}} \partial_r (r^{D-p-2} h \partial_r) - h [J_2 V''(0) + \vec{k}^2 + K_1 f''(0)] \right] \chi_{\omega, \vec{k}} = 0, \tag{6}$$

where the functions J_i are explicitly given by:

$$\begin{aligned}
 J_1 &= H^{-2(D-2)/(D-p-3)(p+1)}, & J_2 &= H^{-2/(p+1)}, \\
 K_1 &= \frac{2(D-2)(D-p-3)}{p+1} r_0^{2(D-p-3)} \frac{H^{2(p^2+4p-d(p+2)+5)/(D-p-3)(p+1)}}{r^{2(D-p-2)}} \cosh^2 \alpha \sinh^2 \alpha.
 \end{aligned} \tag{7}$$

The functions J_i , considered on the range $r_0 \leq r < \infty$, are analytic, bounded above and below by positive numbers, with a limit of 1 as $r \rightarrow \infty$. The function K_1 , considered on the same range, is analytic, everywhere positive, bounded above, and with a finite nonzero limit at the horizon. Evidently, either class of functions may be added, multiplied, and raised to real powers without going outside that class of functions. Define

$$\begin{aligned}
 u &= \int^r \frac{d\tilde{r}}{P}, & P &= h\sqrt{J_1}, & \tilde{\chi} &= \frac{\chi_{\omega, \vec{k}}}{F}, \\
 F &= r^{-(D-p-2)/2} \sqrt{J_1},
 \end{aligned} \tag{8}$$

where the lower limit on the integration defining u is set so that $u - r \rightarrow 0$ as $r \rightarrow \infty$. Note that $u \rightarrow -\infty$ at the horizon, because h has a simple zero there as a function of r . The radial equation becomes

$$[-\partial_u^2 + V(u)]\tilde{\chi} = \omega^2 \tilde{\chi},$$

$$V = h[J_2 V''(0) + \vec{k}^2 + K_1 f''(0)] + (\partial_u \log F)^2 - \partial_u^2 \log F. \quad (9)$$

The extra terms $(\partial_u \log F)^2 - \partial_u^2 \log F$ introduced into V by the transformation of variables are uniformly bounded. They tend to 0 exponentially fast as $u \rightarrow -\infty$, and so does $V(u)$ itself. And they tend to 0 as $u \rightarrow \infty$ as well, but V has a positive nonzero limit there, namely $V''(0) + \vec{k}^2$. Moreover, $V(u)$ is analytic, and each of its derivatives is uniformly bounded over the real line.

Most importantly, by making $f''(0)$ sufficiently negative, $V(u)$ can be made as negative as one pleases over any specified finite range of u , without altering its smoothness or asymptotic properties at $\pm\infty$. It is then a standard fact of quantum mechanics that an L^2 -normalizable bound state appears in the spectrum for sufficiently negative $f''(0)$. Indeed, any number of such bound states, each corresponding to a different negative value of ω^2 , can be introduced into the spectrum by sending $f''(0)$ more and more negative. The eigenfunction for a negative ω^2 bound state decays exponentially as $u \rightarrow \pm\infty$. The existence of such a bound state to the Schrödinger problem (9) implies the existence of an exponentially growing perturbation of χ which is finite outside the horizon, regular at the horizon, and exponentially small at infinity: a GL instability.

Using (9) one may also argue in general that if all parameters except \vec{k}^2 are held fixed, and \vec{k}^2 is increased sufficiently, there is no GL instability. It is generally understood that when a GL instability exists for small \vec{k}^2 , it persists up to a critical value k_c^2 , and at k_c^2 there is a static, normalizable perturbation of the horizon. Demonstrating this from (9) involves a minor technical complication: for $\omega = 0$, there is not an L^2 -normalizable eigenfunction $\tilde{\chi}(u)$, but instead an eigenfunction which approaches a constant as $u \rightarrow -\infty$, is uniformly bounded, and decays exponentially as $u \rightarrow \infty$. The corresponding $\chi_{0,\vec{k}}$ is finite at the horizon (so clearly regular there), finite outside the horizon, and exponentially small at infinity—hence normalizable.¹

¹We have not given a precise definition of normalizability of a perturbation in the scalar field χ , but several could be used interchangeably in the current context. Most physically, the norm could be taken as the energy density in the perturbation integrated over a slice of constant t outside the horizon. Alternatively, $\chi_{\omega,\vec{k}}(r)$, considered as a function on a slice of constant t outside the horizon, could be required to be in all L^p spaces for $p > 1$. The “minor technical complication” amounts to showing that a function $\tilde{\chi}(u)$ which is not L^2 -normalizable nevertheless translates into a perturbation $\chi_{0,\vec{k}}(r)$ which is normalizable in either of the senses just described. An additional feature generally required of a perturbation in order for it to be a physically meaningful classical instability is that there should be no outgoing energy flux at the horizon. This is true of solutions with negative ω^2 because $\chi_{\omega,k}(r) \rightarrow 0$ at the horizon, and it is true of the $\omega = 0$ solution because $\dot{\chi} = 0$.

Unlike other examples of GL instabilities, in this case it is fairly clear that there are possible static endpoints of the evolution with uniformly controlled curvatures. In particular, there are static, spatially uniform solutions with χ nonzero but vanishing asymptotically as $r \rightarrow \infty$. This nonzero profile of χ is an example of hair for the black brane, because it is not determined by any conserved quantity. Static hairy solutions were studied in quantitative detail for the case $p = 0$, $D = 4$ in [12], and the qualitative features should be the same in other cases. In particular, it is not necessary for $V(\chi)$ or $f(\chi)$ to have extrema at nonzero χ in order for static hairy solutions to exist. If $V''(0) > 0$, then for given magnetic charge, hairy solutions exist only for r_0 sufficiently small. The hair develops smoothly as r_0 crosses a critical value r_c : slightly below this critical value, $\chi(r)$ in the hairy solution is uniformly small outside the horizon. The hairy solutions, when they exist, have greater entropy than the solutions (2). It is in this respect that the current example of a GL instability is especially reminiscent of gyrating branes [11]: the driving intuition in that case was that for a certain range of parameters, there was a more entropically favorable way to carry certain quantum numbers than the standard stationary black string solution.

If there is a \mathbf{Z}_2 symmetry of the classical action under $\chi \rightarrow -\chi$, then it should also be possible in principle to construct static domain wall solutions where a black brane has nonzero χ which is negative on one side of the domain wall (say, as $x^1 \rightarrow -\infty$) and positive on the other (say, as $x^1 \rightarrow \infty$). In fact, a variety of more or less intricate solitons can be contrived, depending on the topological structure of the space of solutions with spatially uniform horizons and a given set of conserved charges.

It is clear that the GL instability and hairy black brane solutions under discussion relate to second-order phase transitions on the world volume of the brane. The dictionary between the gravitational description and the world-volume theory is clearest when the spacetime is asymptotically anti-de Sitter rather than asymptotically flat. We therefore turn to this case in the next section.

III. AN EXAMPLE IN AdS₅

The horizon instability discussed in the previous section is driven by the $\chi^2 F^2$ coupling. This sort of coupling can arise in low-energy effective actions of compactified string theory, but it is not particularly familiar to us from other contexts. It is therefore interesting to look for violations of the CSC in systems that are well studied for other reasons. A class of examples in AdS₅ was suggested in [12]. The action is

$$S = \int d^5x \sqrt{g} \left[R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} (\partial\chi)^2 - V(\phi, \chi) \right],$$

$$V(\phi, \chi) = -\frac{12}{L^2} + \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} m_\chi^2 \chi^2 + g \phi^2 \chi^2,$$

$$g < 0. \quad (10)$$

This action is still “made up,” but in Sec. IV we will see that an interesting two-dimensional slice of the scalar manifold of $d = 5$, $\mathcal{N} = 8$ gauged supergravity gives an action of essentially this form.

We are interested in black brane solutions in the Poincaré slice of AdS_5 . The metric is

$$ds^2 = e^{2A(r)}[-h(r)dt^2 + d\vec{x}^2] + \frac{dr^2}{h(r)}. \quad (11)$$

It is assumed that ϕ has a nonzero profile corresponding to a deformation of the CFT Lagrangian. χ , on the other hand, may be zero or nonzero, but its asymptotics near the boundary of AdS_5 are required to indicate a VEV of the dual operator in the CFT rather than a deformation by it. It was demonstrated in [12] for a particular choice of m_ϕ, m_χ , and g that there is a continuous transition from a “disordered phase” where $\chi = 0$ (so that $\langle \mathcal{O}_\chi \rangle = 0$) to an “ordered” phase where $\chi \neq 0$ (so that $\langle \mathcal{O}_\chi \rangle \neq 0$). If we denote the value of ϕ at the horizon by ϕ_0 , then the transition happens for a particular value ϕ_c of ϕ_0 : for $\phi_0 < \phi_c$ one is in the disordered phase, and for $\phi_0 > \phi_c$ one is in the ordered phase.

The action (10) has a $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry, associated with $\phi \rightarrow -\phi$ and $\chi \rightarrow -\chi$. The first \mathbf{Z}_2 is broken explicitly by the boundary conditions on ϕ near the boundary of AdS_5 . The second \mathbf{Z}_2 is preserved in the ordered phase and broken in the disordered phase.

The Lagrangian of the dual field theory is

$$\mathcal{L} = \mathcal{L}_{\text{CFT}} + \Lambda_\phi^{4-\Delta_\phi} \mathcal{O}_\phi, \quad (12)$$

where Λ_ϕ is an energy scale and $\Delta_\phi = 2 + \sqrt{4 + m_\phi^2 L^2}$ is the dimension of \mathcal{O}_ϕ (assumed to be less than 4). \mathcal{L}_{CFT} is the Lagrangian of the undeformed conformal field theory. In Sec. IV we will encounter a specific example where the CFT is $\mathcal{N} = 4$ super-Yang-Mills theory and \mathcal{O}_ϕ is a fermion mass term—up to certain subtleties to be mentioned below. The Hawking temperature of the black hole horizon translates into a finite temperature for the field theory. It may be assumed that

$$\begin{aligned} A(r) &\rightarrow \frac{r}{l}, \\ h(r) &\rightarrow 1 \\ \phi(r) &\rightarrow X_1 e^{(\Delta_\phi - 4)r/L} + X_2 e^{-\Delta_\phi r/L}, \\ \chi(r) &\rightarrow Y_2 e^{-\Delta_\chi r/L} \end{aligned} \quad (13)$$

as $r \rightarrow \infty$. X_1 is proportional to $\Lambda_\phi^{4-\Delta_\phi}$, and it may be set equal to unity by making an appropriate dilation of \vec{x} and t .

²The asymptotics for $\phi(r)$ can be more complex, because the $e^{(\Delta_\phi - 4)r/L}$ behavior may be modified by subleading exponentials which are nevertheless larger than $e^{-\Delta_\phi r/L}$. This will not be a concern for present purposes.

Black brane solutions of the form(11) with the asymptotics (13) may be parametrized by ϕ_0 , or alternatively in dual field theory language, by T/Λ_ϕ . The precise relation between these two functions can be established through numerics. In particular, the critical value ϕ_c is associated with a critical temperature T_c at which the phase transition occurs.

Our goals in the remainder of this section are

- (1) to demonstrate that there is a Gregory-Laflamme instability for black branes with $\phi_0 > \phi_c$;
- (2) to extract two critical exponents of the phase transition.

In addressing point 1 we will be satisfied to show that, for black brane backgrounds with $\phi_0 > \phi_c$ and $\chi = 0$, there is a normalizable, static mode in the linearized equation of motion for χ with a nonzero wave number k_c . To understand point 2 we will solve the linearized equation for χ and extract $\langle \mathcal{O}_\chi(x)\mathcal{O}(0) \rangle$. The full equation of motion for χ is

$$\square\chi = \frac{\partial V}{\partial \chi}, \quad (14)$$

and with a separated ansatz $\chi = e^{ik\cdot\vec{x}}\tilde{\chi}(r)$ one obtains

$$\begin{aligned} e^{-4A(r)} \left[-k^2 e^{2A(r)} + \frac{d}{dr} \left(h(r) e^{4A(r)} \frac{d}{dr} \right) \right] \tilde{\chi}(r) \\ = [m_\chi^2 L^2 + 2g\phi^2(r)] \tilde{\chi}(r). \end{aligned} \quad (15)$$

Near the horizon ($r = 0$), one of the solutions to (15) diverges logarithmically with r , while the finite solution asymptotes to a constant:

$$\tilde{\chi}(r) \rightarrow \tilde{\chi}_h \left(1 + \frac{k^2 + m_\chi^2 L^2 + 2g\phi_0^2}{h'(0)} r \right). \quad (16)$$

Far away from the black hole, $\tilde{\chi}(r)$ dies off as

$$\tilde{\chi}(r) \rightarrow Y_1(k) e^{(\Delta_\chi - 4)r/L} + Y_2(k) e^{-\Delta_\chi r/L}. \quad (17)$$

Because the equation for $\tilde{\chi}(r)$ is linear, $Y_1(k)$ and $Y_2(k)$ are not individually meaningful. But their ratio,

$$G(k) \equiv Y_2(k)/Y_1(k), \quad (18)$$

is well defined, and up to a k -independent constant, it is the scaling part of the correlator $\langle \mathcal{O}_\chi(\vec{k})\mathcal{O}_\chi(-\vec{k}) \rangle$ evaluated in the thermal state. (This correlator also may include analytic terms in k , corresponding to contact terms. These are excluded by construction in $G(k)$.)

Figs. 1 and 2 show $G(k)$ for

$$g = -10, \quad \Delta_\phi = 2.2, \quad \Delta_\chi = 2.4, \quad (19)$$

on either side of the phase transition. On the disordered side, the singularity at a finite wave number k_c signals the existence of a GL instability. As $k \rightarrow k_c$, $Y_1(k) \rightarrow 0$ while $Y_2(k)$ remains finite. So at $k = k_c$, the static mode $\chi = e^{ik\cdot\vec{x}}\tilde{\chi}(r)$ is normalizable.

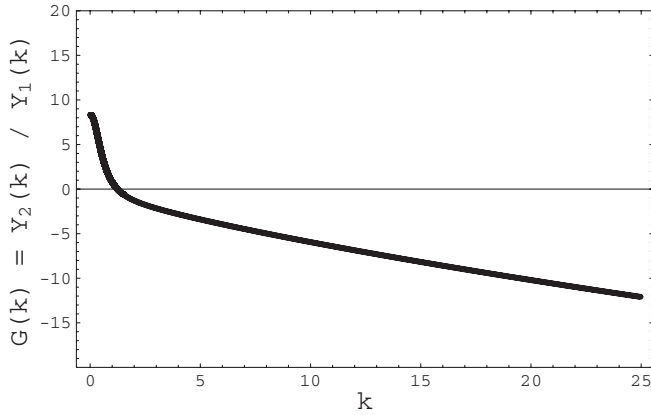


FIG. 1. Propagator for $\phi_0 = 0.95\phi_c$, where $\phi_c = 0.428$. The lack of a singularity indicates the absence of a normalizable mode.

As one approaches the phase transition, the static mode's wave number k_c should go to zero: this is the standard way in which a GL instability appears or disappears [13]. In Fig. 3 we show that indeed this happens for the choice $g = -10$, $\Delta_\phi = 2.2$, and $\Delta_\chi = 2.4$. It is convenient to plot the ratio k_c/k_* , where k_* is the wave number at which the scaling part of $G(k)$ vanishes. As $\phi \rightarrow \phi_c$ from above, k_* has a finite limit, while

$$\frac{k_c}{k_*} \sim \left(\frac{T_c - T}{T_c}\right)^\zeta \quad \zeta \approx 0.479. \quad (20)$$

Given the accuracy of the fits, this result for the exponent ζ is consistent with $\zeta = 1/2$, which was also found (approximately) for the GL instability of nonextremal D3-branes, M2-branes, and M5-branes in [13].

Another critical exponent can be defined as follows: let $\overline{\mathcal{O}}_\chi(x) = T \int_0^\beta dt_E \mathcal{O}_\chi(t_E, x)$. Then for $x \gg 1/T$, one expects a power-law behavior

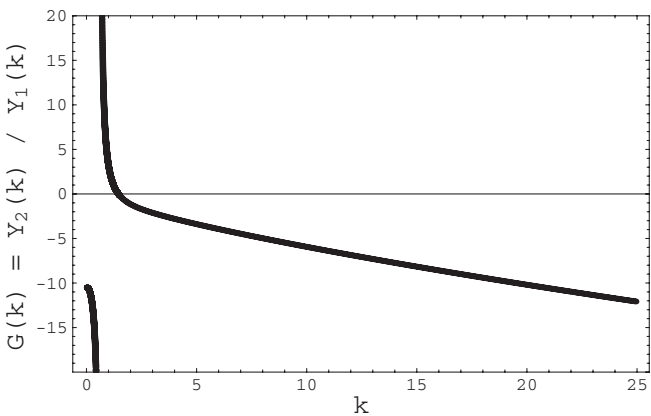


FIG. 2. Propagator for $\phi_0 = 1.05\phi_c$. The singularity indicates a normalizable, stationary mode, and the existence of a GL instability for $k < k_c$.

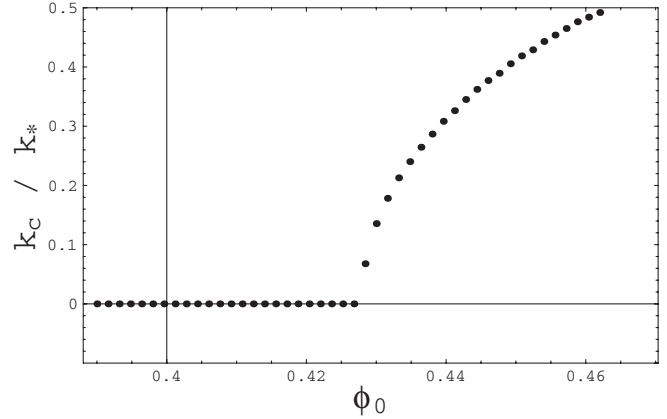


FIG. 3. Plot showing nonzero critical values of k_c for $\phi_0 > \phi_c$ (the ordered phase). k_* is the value of k at which $G(k) = 0$.

$$\langle \overline{\mathcal{O}}_\chi(x) \overline{\mathcal{O}}_\chi(0) \rangle \sim \frac{1}{x^{2\overline{\Delta}_\chi}}. \quad (21)$$

The expectation value in (21) is taken in the thermal state right at the critical point ($\phi_0 = \phi_c$), and the dimension $\overline{\Delta}_\chi$ is distinct from the dimension Δ_χ . Note that the correlator in (21) is the Fourier transform in only the three spatial directions \vec{x} of $\langle \mathcal{O}_\chi(\vec{k}) \mathcal{O}_\chi(-\vec{k}) \rangle$: we work with zero Matsubara frequency throughout. Carrying out the Fourier transform, one obtains

$$\langle \mathcal{O}_\chi(\vec{k}) \mathcal{O}_\chi(-\vec{k}) \rangle \sim k^{2\overline{\Delta}_\chi - 3}, \quad (22)$$

again for $T = T_c$.

In testing the power-law prediction (22) and obtaining the critical exponent, it is convenient for numerics to depart slightly from $T = T_c$ and examine a scaling region of k . In Fig. 4 we show an example with ϕ_0 slightly greater than ϕ_c . The scaling region is cut off in the infrared by k_c , below which there is a GL instability. And it is cut off in the ultraviolet by k_* , above which there is a gradual transition to a different power law, controlled by the dimension Δ_χ

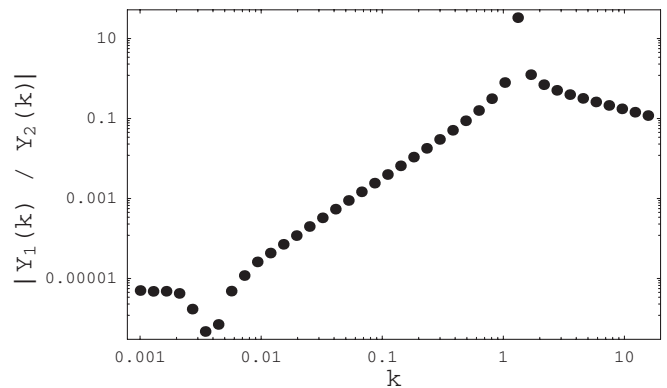


FIG. 4. Inverse propagator for $(\phi_0 - \phi_c)/\phi_0 \approx 10^{-6}$.

pertaining to the vacuum state. The numerical results for the slope are consistent with $\bar{\Delta}_\chi = 1/2$.

A notable feature of the present discussion is that we see scaling behavior for $x \gg 1/T$ without manifest conformal invariance in a similar limit. Conformal invariance in the supergravity background would be associated with recovering a four-dimensional anti-de Sitter space by making a Kaluza-Klein reduction in the Euclidean time direction—a procedure that amounts to restricting attention to zero Matsubara frequency. But such a reduction leads instead to a singular background where the scalar that controls the size of the circle in the Euclidean time direction diverges.

IV. A FINITE-TEMPERATURE PHASE TRANSITION IN $\mathcal{N} = 1^*$ SUPER-YANG-MILLS THEORY

So far, all of the examples of systems which violate the CSC have relied on Lagrangians without a string-theoretic origin. In this section we will explore an example drawn from type IIB string theory on $\text{AdS}_5 \times S^5$ —or, more precisely, from its consistent truncation to $d = 5$, $\mathcal{N} = 8$ gauged supergravity. The dual field theory is the so-called $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ super-Yang-Mills theory. The relevant deformation gives equal masses to three of the four adjoint fermion fields. The scalar ϕ dual to this dimension 3 deformation is nonzero because of the asymptotic boundary conditions at the boundary of AdS_5 . There is another $SU(3)$ -invariant, dimension 3 operator: it includes a bilinear of the fourth adjoint fermion, which is the $\mathcal{N} = 1$ partner of the gauge boson. The scalar χ dual to this operator has asymptotic boundary conditions that allow it to be zero or nonzero. If it is nonzero, the VEV of the dual operator \mathcal{O}_χ can be read off from it. This VEV breaks a chiral symmetry. The relevant part of the $d = 5$, $\mathcal{N} = 8$ supergravity Lagrangian is [14]

$$S = \int d^5x \sqrt{g} \left[R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} (\partial\chi)^2 - V(\phi, \chi) \right],$$

$$V(\phi, \chi) = -\frac{3}{2} \left(\cosh^2 \frac{\phi}{\sqrt{3}} + 4 \cosh \frac{\phi}{\sqrt{3}} \cosh \chi - \cosh^2 \chi + 4 \right). \quad (23)$$

In short, the situation is very much as in the examples of Sec. III. And the outcome is similar, too: if the horizon is sufficiently close to the boundary of AdS_5 —corresponding to a small value of ϕ_0 at the horizon—then χ and hence $\langle \mathcal{O}_\chi \rangle$ must vanish. But if $\phi_0 > \phi_c \approx 4.41$, the background with zero χ has a Gregory-Laflamme instability. There is a second-order transition at $\phi_0 = \phi_c$ to backgrounds with $\chi \neq 0$. The two-point function of \mathcal{O}_χ shows the same $1/k^2$ behavior in an infrared scaling region. These conclusions still hold if we replace the transcendental function $V(\phi, \chi)$ by a polynomial approximation to it of the form considered in Sec. III:

$$L^2 V = -12 - \frac{3}{2} (\phi^2 + \chi^2) - \frac{\phi^2 \chi^2}{2}. \quad (24)$$

In the language of Sec. III, $g = -1/2$. With the modified potential (24), the critical horizon value of ϕ_0 is $\phi_c \approx 2.96$.

In [14], holographic RG flows were considered of the form

$$ds^2 = e^{2A(r)} (-dt^2 + d\vec{x}^2) + dr^2, \quad \phi = \phi(r),$$

$$\chi = \chi(r), \quad \phi = \sqrt{3} \log \frac{1 + e^{-r}}{1 - e^{-r}}, \quad (25)$$

$$\chi = \log \frac{1 + e^{-3r+C}}{1 - e^{-3r+C}}.$$

These geometries are singular in the infrared for all values of C . It is hard to know which of them is physical. In [15], it was suggested that the ones with $C > 0$ are unphysical, that the $C = 0$ trajectory is probably physical, and that the

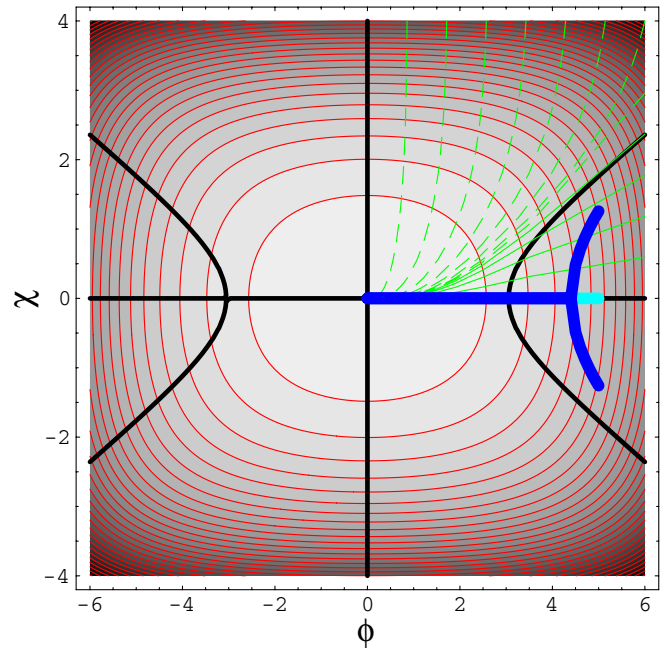


FIG. 5 (color online). The thick blue lines show the values (ϕ_0, χ_0) of the scalars at the horizons we were able to produce numerically. Dark blue indicates stable solutions. Light blue indicates solutions with a GL instability. The thin green trajectories are the holographic RG flows (25). They are found most simply as the gradient flows of a superpotential W , whose contours are shown in red. The aspect ratio of this figure is not 1:1, so it is not readily apparent that the green trajectories are orthogonal to the red contours. The $C = 0$ trajectory is asymptotic to the black curve, which is part of the locus where the gradient of W is parallel to the gradient of V . The $C < 0$ trajectories are shown in solid green, and the $C > 0$ trajectories are shown in dashed green.

$C < 0$ trajectories might also be physical. The reason to think this is that the $C > 0$ trajectories cannot be limits of backgrounds with regular horizons, while the $C = 0$ solution is in a special class that was conjectured in [15] to be precisely the limits of backgrounds with regular horizons. Our numerical results are consistent with these conjectures: for given $\phi_0 > \phi_c$, only one background with positive χ_0 was found, and the points (ϕ_0, χ_0) are not far from the $C = 0$ trajectory—see Fig. 5. Only a limited range of ϕ_0 was explored, so we cannot be sure that the $C = 0$ trajectory is the limit of black hole solutions. For larger ϕ_0 , numerical noise prevented reliable results.

It may be that the solutions with regular horizons that we have found are the only stable ones in the temperature range under consideration, even if we go beyond $d = 5$, $\mathcal{N} = 8$ gauged supergravity. Other solutions would correspond to turning on scalars in AdS_5 which have positive m^2 . Standard no-hair arguments suggest that this is impossible: only one of the two solutions to the linearized equations for such scalars is normalizable at infinity, and horizon boundary conditions generically prevent this normalizable mode from being turned on. Also, the total scalar potential must be more negative than $-12/L^2$ at the horizon [15], which suggests that large deformation by positive mass scalars is impossible.

In sum, the second-order chiral symmetry breaking phase transition exhibited in this section gives a tantalizing first glimpse of the finite-temperature behavior of $\mathcal{N} = 1^*$ super-Yang-Mills theory. Much more is known about its zero-temperature phases [16,17] and their ten-dimensional description in terms of five-branes [18]. The fact that the chiral symmetry breaking transition happens when there is a regular horizon with an entropy scaling as N^2 shows that this transition is well above the confinement scale. This is somewhat reminiscent of the model of [19] where unconfined quarks interact with a chiral condensate.

V. CONCLUSIONS

Although the correlated stability conjecture (CSC) of [6,7] has served as a useful guide to the Gregory-Laflamme (GL) instability in various settings, we have argued that it fails to correctly predict horizon instabilities that are unrelated to conserved quantities. Instead, these instabilities are associated (in the examples we have described) with second-order phase transitions in which the unstable horizon represents the disordered phase cooled below the critical temperature at which ordering should take place. The ordered phase is represented by a new (and presumably stable) uniform black brane solution.

The nature of these counter-examples leads us to conjecture that the CSC works provided that there is a unique background with a spatially uniform horizon and specified conserved charges. We know of no counter-examples to this restricted version of the conjecture.

A simple refinement of the CSC has been suggested to us by B. Kol:³ thermodynamic stability should be redefined by enlarging the Hessian matrix to include derivatives of the entropy with respect to quantities that characterize the asymptotics of scalar fields. (In the examples given, the asymptotics of χ are characterized by a single real parameter: Y_2 in the example of Sec. III, if one sets $X_1 = 1$: see (13).) Local thermodynamic stability then amounts to the absence of a positive eigenvalue for this enlarged Hessian matrix, and the revised version of the CSC is that GL instabilities occur precisely when local thermodynamic stability is lost, provided the brane has infinite volume and translation invariance in some spatial direction. It seems very likely that this refinement of the CSC survives all tests to date.

While we have not shown explicitly that there is a violation of the CSC (as originally phrased in [6,7]) in an asymptotically flat background of a well-defined string theory, there should be no difficulty in principle in extending the example of Sec. IV to an asymptotically flat background: one simply has to “reattach” flat space to the asymptotically $\text{AdS}_5 \times S^5$ throat region. The resulting background and its unstable perturbation should still possess a global $SU(3)$ symmetry, which may make it easier to find them explicitly.

For second-order phase transitions in AdS_5 , we found in an infrared scaling region, where spatial separations are much greater than $1/T$, that the dimension of the operator \mathcal{O}_χ (whose VEV is the order parameter) is $1/2$. Over the rather wide set of parameter choices that we checked, this result is independent of the dimension of \mathcal{O}_χ at the ultraviolet fixed point, as well as the dimension of \mathcal{O}_ϕ and the coupling constant g . We believe this result can be understood as a consequence of large N : when normalized to have an $O(1)$ two-point function, the higher point functions of \mathcal{O}_χ are suppressed by factors of N . So the state created by \mathcal{O}_χ in the three-dimensional effective theory describing physics in the infrared scaling region is, up to $1/N$ corrections, a free scalar, and the dimension of a free scalar in three dimensions is $1/2$.

ACKNOWLEDGEMENTS

The research of J.F. was supported in part by the NSF Graduate Research Fellowship Program, and by the Department of Energy under Grant No. DE-FG02-91ER40671. J.F. would like to thank H. Verlinde for a useful discussion and the organizers of TASI 2005 for their hospitality while this work was in progress. S.G. would like to thank I. Klebanov for a useful discussion and B. Kol for commenting on an early version of the paper. The research of S.G. was supported in part by the

³We thank B. Kol for his permission to explain this proposal here.

Department of Energy under Grant No. DE-FG02-91ER40671, and by the Sloan Foundation. I.M. was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231, in

part by the National Science Foundation under Grant No. PHY-00-98840, and by the Berkeley Center for Theoretical Physics. I.M. would also like to thank the Aspen Center for Physics for hospitality during the final stages of this work.

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