Formalism for testing theories of gravity using lensing by compact objects: Static, spherically symmetric case

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(Received 7 October 2005; published 4 November 2005)

We are developing a general, unified, and rigorous analytical framework for using gravitational lensing by compact objects to test different theories of gravity beyond the weak-deflection limit. In this paper we present the formalism for computing corrections to lensing observables for static, spherically symmetric gravity theories in which the corrections to the weak-deflection limit can be expanded as a Taylor series in one parameter, namely, the gravitational radius of the lens object. We take care to derive coordinate-independent expressions and compute quantities that are directly observable. We compute series expansions for the observables that are accurate to second order in the ratio $\varepsilon = \vartheta_\bullet/\vartheta_E$ of the angle subtended by the lens's gravitational radius to the weak-deflection Einstein radius, which scales with mass as $\varepsilon \propto M_\bullet^{1/2}$. The positions, magnifications, and time delays of the individual images have corrections at both first and second order in ε, as does the differential time delay between the two images. Interestingly, we find that the first-order corrections to the total magnification and centroid position vanish in all gravity theories that agree with general relativity in the weak-deflection limit, but they can remain nonzero in modified theories that disagree with general relativity in the weak-deflection limit. For the Reissner-Nordström metric and a related metric from heterotic string theory, our formalism reveals an intriguing connection between lensing observables and the condition for having a naked singularity, which could provide an observational method for testing the existence of such objects. We apply our formalism to the galactic black hole and predict that the corrections to the image positions are at the level of 10 μarc s (microarcseconds), while the correction to the time delay is a few hundredths of a second. These corrections would be measurable today if a pulsar were found to be lensed by the galactic black hole, and they should be readily detectable with planned missions like MAXIM.

DOI: 10.1103/PhysRevD.72.104006 PACS numbers: 04.70.Bw, 04.80.Cc

I. INTRODUCTION

The gravitational deflection of light provided one of the first observational tests of general relativity. Now it is routinely observed in a broad array of astrophysical contexts ranging from stars through galaxies and clusters of galaxies up to the large-scale structure of the universe. (See [1,2] for thorough discussions of gravitational lensing, and [3] for a recent review of astrophysical applications.) All of the effects seen so far occur in the weak-deflection, quasi-Newtonian limit of general relativity. Thus, while gravitational lensing has proven valuable for measuring masses in astrophysics, it has not yet been able to test fundamental theories of gravity in physics.

There has been significant theoretical effort over several decades to understand lensing in the strong-deflection regime (e.g., [2,4–12]), in particular, for situations in which (i) the lens is compact, static, and spherically symmetric, (ii) the metric is asymptotically flat far from the lens, and (iii) the source and observer lie in the asymptotically flat regime. (See [10] for an approach that does not require asymptotic flatness.) The resulting theory has yielded a remarkable prediction: There should be an infinite series of images very close to and on either side of the black hole's photon sphere, corresponding to light rays that loop around the black hole once, twice, etc. before traveling to the observer (e.g., [4–8,11]). The possibility that these "relativistic images" could be used to test strong-deflection gravity has created considerable excitement, inspiring extensive studies of their properties in various familiar metrics (e.g., [4–8,11,12]) as well as those arising from string theory and braneworld gravity [13–15], together with assessments of prospects for detecting the images [16,17].

Unfortunately, the relativistic images are exceedingly faint (a flux correction of order 10^{-14} for the galactic black hole [8,16]), which makes it important to consider whether there are any other observable effects that can be used to test theories of gravity. The primary and secondary lensed images—which travel from the source to the observer without looping around the lens—do not usually pass close enough to the lens's photon sphere to experience extreme strong-deflection lensing, but they may nevertheless be affected by various orders of post-post-Newtonian (PPN) correction terms. Since these images are much easier to detect, it is valuable to compute the corrections to their observable properties. The first-order corrections

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have been studied for the Schwarzschild metric (which is standard point-mass lensing [1,2]), the Reissner-Nordström metric [12], and metrics with general PPN terms [18,19] and mass currents [20]. Certain aspects of higher-order PPN corrections have been studied [21]. The lowest-order (i.e., weak-deflection) theory has been studied for metrics from string theory and braneworld gravity [13,15,22]. Weyl gravity has been investigated extensively [23] and so will not be treated here. Unfortunately, much of the existing work seems to comprise a diverse collection of results, some of them rediscovered several times, that makes it difficult to draw general conclusions. Also, most of the previous studies have focused on the light bending angle rather than on quantities that can be observed directly in extra-solar lensing scenarios.

Our goal is to develop a general framework for computing corrections to a core set of observable properties of the primary and secondary lensed images in a general geometric theory of gravity. In the process, we shall unify the previous lensing analyses into a common framework, and extend them to a higher order of approximation. At the same time, we shall demonstrate how to handle general gravity theories using PPN terms up to third order. One crucial part of our formalism is that we take care to work with *observable* quantities and avoid coordinate dependence. For instance, most previous studies of corrections to the weak-deflection bending angle have expressed the results in terms of the light ray's radial distance of closest approach to the lens. However, such a radial distance is a coordinate-dependent quantity. Ambiguities created by different choices of coordinates can be alleviated by working instead with the impact parameter of the light ray [24]. The impact parameter is an invariant of the light ray (a constant of motion), and is given geometrically by the perpendicular distance, relative to initial observers at infinity, from the center of the lens to the asymptotic tangent line to the light ray trajectory at the observer. It is the quantity to use when defining the observable angular position of a lensed image and will play a key role in our formalism.

There are several reasons to compute corrections to a higher order than has been done before. One is that certain lensing observables have first-order corrections that vanish in general relativity (see, e.g., [19]). Our higher-order formalism will allow us to understand the fortuitous cancellations that cause those terms to vanish, and to find the lowest-order nonvanishing corrections. By being very general, we will also determine whether the cancellations are generic or restricted to specific families of gravity theories; we shall show, for example, that the cancellations also occur in all theories of gravity that agree with general relativity in the weak-deflection limit. A second reason for working to high order is that carrying the expansion far enough will, in principle, allow us to bridge the gap that now exists between weak- and strong-deflection analyses of lensing by black holes and compact objects.

Our generality will also allow us to find an unexpected connection between lensing observables and certain kinds of naked singularities. The Reissner-Nordström metric is usually taken to describe a charged black hole in general relativity, but if the charge parameter exceeds a threshold value, then the metric describes a naked singularity instead. We shall discover that the corrections to certain lensing observables can have a negative value only if the condition for a naked singularity is satisfied. This could provide a direct observational test for these exotic singularities, which are conventionally ruled out by the still-unproved cosmic censorship conjecture [25].

This is the first in a series of papers intended to develop the complete lensing framework. Here we begin with a thorough analysis of lensing by a static, spherically symmetric compact body, using a formalism that can handle all gravity theories in which the metric can be expressed as a series expansion in the single parameter $\mathsf{m}_\bullet$ (the gravitational radius of the compact body). In separate papers we will generalize to metrics with two parameters and metrics that describe a rotating compact body. In addition to presenting the formalism, we will discuss some possible astrophysical applications. It is our hope that these studies will be relevant to the next generation of black hole imagers, such as MAXIM [26].

II. BASIC ASSUMPTIONS OF THE FORMALISM

Since we seek to lay out a general framework, we should begin by stating the assumptions very clearly. Consider a compact body of mass $M_\bullet$, perhaps a black hole or neutron star, that is described by a geometric theory of gravity. This means that the body's gravitational field is determined by a spacetime metric appropriate to the theory of gravity in question. (See Sec. III A for the form of the metric.) Possibilities include general relativity, whose spacetime metric obeys the Einstein equation, and modified gravity theories, so named because their metrics are not governed by Einstein's equation.

We study how the body acts as a gravitational lens by considering light rays that travel directly from the source to the observer without looping around the compact body. Figure 1 gives a schematic diagram of the lensing situation and defines standard quantities: $\mathcal{B}$ is the angular position of the unlensed source; ϑ is the angular position of an image; $\hat{\alpha}$ is the bending angle of the light ray; and d_L, d_S, and d_{LS} are the observer-lens, observer-source, and lens-source angular diameter distances, respectively. Also important is the impact parameter b, which is an invariant of the light ray (a constant of motion) and is given geometrically by the perpendicular distance, relative to inertial observers at infinity, from the center of the compact body to the asymptotic tangent line to the light ray trajectory at the observer. As noted in Sec. I, when defining observables it is crucial to

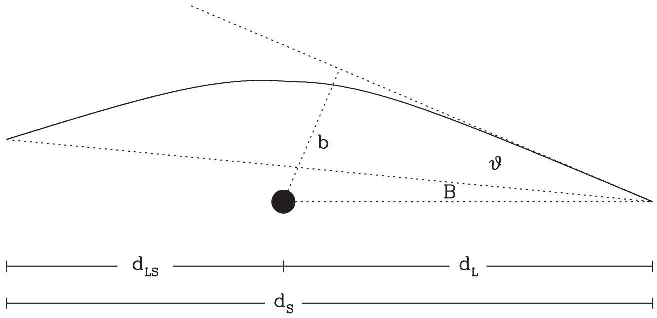

FIG. 1. Schematic diagram of the lensing geometry.

use the invariant impact parameter rather than the coordinate-dependent distance of closest approach (also see [24]). The angular image position is defined in terms of the impact parameter as $\vartheta = \sin^{-1}(b/d_L)$.

From the figure, elementary trigonometry establishes the relationship (see [8])

$$\tan\mathcal{B} = \tan\vartheta - D(\tan\vartheta + \tan(\hat{\alpha} - \vartheta)), \tag{1}$$

where $D = d_{LS}/d_S$. This equation agrees very well with the full relativistic formalism for light propagation [9]. In this paper, we take it as the general form of the gravitational lens equation.

Henceforth, the angles describing image positions are assumed to be positive, which then forces the source's angular position to take on a positive or negative value depending on the image's location. Explicitly, the source is assumed to be fixed at an angle $\mathcal{B}$, which is taken to be positive when studying an image on the same side of the compact object as the source, and negative when studying an image on the opposite side. (Note the different conventions for the signs of ϑ and $\mathcal{B}$.)

We make the following assumptions:

A1 The gravitational lens is compact, static, and spherically symmetric, with an asymptotically flat spacetime geometry sufficiently far from the lens. The spacetime is vacuum outside the lens and flat in the absence of the lens.

A2 The observer and source lie in the asymptotically flat regime of the spacetime.

A3 The light ray's distance of closest approach r_0 and impact parameter b both lie well outside the gravitational radius $\mathsf{m}_\bullet = GM_\bullet/c^2$, namely, $\mathsf{m}_\bullet/r_0 \ll 1$ and $\mathsf{m}_\bullet/b \ll 1$. The bending angle can then be expressed as a series expansion as follows in the single quantity $\mathsf{m}_\bullet/b$:

$$\hat{\alpha}(b) = A_1\left(\frac{\mathsf{m}_\bullet}{b}\right) + A_2\left(\frac{\mathsf{m}_\bullet}{b}\right)^2 + A_3\left(\frac{\mathsf{m}_\bullet}{b}\right)^3 + \mathcal{O}\left(\frac{\mathsf{m}_\bullet}{b}\right)^4. \tag{2}$$

The coefficients A_i are independent of $\mathsf{m}_\bullet/b$, but may include other fixed parameters of the spacetime. Since b and $\mathsf{m}_\bullet$ are invariants of the light ray, Eq. (2) is independent of coordinates. Note that the subscript of A_i conveniently indicates that the component is affiliated with a term of order i in $\mathsf{m}_\bullet/b$.

Assumption A1 is a natural first step to take for the study of lensing by neutron stars and black holes. The spherical symmetry conveniently allows us to restrict attention to light rays moving in a plane. As we shall see in Sec. VI, assumption A2 certainly holds in interesting astrophysical settings.

As for assumption A3, the form of the general bending angle $\hat{\alpha}$ in Eq. (2) is a series expansion expressing corrections to the standard weak-deflection bending angle in general relativity,

$$\hat{\alpha}_{\mathrm{wf}}(\vartheta) = 4\left(\frac{\mathsf{m}_\bullet}{b}\right) \approx 4\left(\frac{\vartheta_\bullet}{\vartheta}\right), \tag{3}$$

where $\vartheta_\bullet = \tan^{-1}(\mathsf{m}_\bullet/d_L)$ is the angle subtended by the gravitational radius of the compact object, and "≈" simply indicates that in the last expression we have used the standard small-angle approximation. (This approximation is not valid when we work beyond linear order, as discussed in Sec. IV A.) By writing the coefficient of the linear term as A_1, we allow for the possibility that gravity theories can differ slightly from general relativity in the quasi-Newtonian regime. In practice, though, observational analyses imply that A_1 is quite close to 4: for example, recent compilations of 20 years' worth of data imply that $A_1 = 3.999\,66 \pm 0.000\,90$ [27]. There are as yet no good observational constraints on the higher-order coefficients A_2 and A_3 in (2), so they are to be determined from the gravity theory in question (as discussed below).

III. LIGHT BENDING ANGLE IN VARIOUS GRAVITY THEORIES

A key part of any lensing framework is the light bending angle. We state the general form of the metric we consider, and derive the exact, general expression for the bending angle. We obtain the desired series expansion of the bending angle for the Schwarzschild metric, working to higher order than has been done previously, and taking care to express the result in terms of the invariant impact parameter. We then repeat the analysis for a general PPN metric, and illustrate how various gravity theories can be studied with this approach.

A. Form of the metric and bending angle

From assumption A1, the spacetime geometry is postulated to be static, spatially spherically symmetric, and asymptotically Minkowski:

$$ds^2 = -\bar{A}(\bar{r})dt^2 + \bar{B}(\bar{r})d\bar{r}^2 + \bar{C}(\bar{r})d\Omega^2, \tag{4}$$

where $d\Omega^2$ is the standard unit sphere metric, and $\bar{A}(\bar{r}) \to 1$, $\bar{B}(\bar{r}) \to 1$, $\bar{C}(\bar{r}) \to \bar{r}^2$ as $\bar{r} \to \infty$. Again by A1, the metric is Minkowski in the absence of the lens and we make the natural mathematical assumptions (for regularity in coordinate changes and constancy of causal structure) that $\bar{A}$, $\bar{B}$, $\bar{C}$, $d\bar{A}/d\bar{r}$, and $d\bar{C}/d\bar{r}$ are all positive in the region outside the lens through which the light rays of interest propagate. Note that we do not require that the metric approaches the general relativity weak-deflection limit for $\bar{r}$ sufficiently large. Since $d\bar{C}/d\bar{r} > 0$, the function $\bar{C}(\bar{r})$ is invertible and allows a new radial coordinate $r = \sqrt{\bar{C}(\bar{r})}$. This transforms the metric as follows:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2. \tag{5}$$

Because of the spherical symmetry, the geodesics of (5) lie in a plane, which we can take to be the equatorial plane. It then suffices to work with a metric of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\varphi^2, \tag{6}$$

where φ is the azimuthal angle. The Lagrangian f determining the null geodesics of the metric (6) is given by

$$f(r, v_t, v_r, v_\varphi) = \tfrac{1}{2}[-A(r)v_t^2 + B(r)v_r^2 + r^2 v_\varphi^2], \tag{7}$$

where (v_t, v_r, v_φ) is a velocity vector with components along the ∂_t, ∂_r, and ∂_φ coordinate directions, respectively. Note that this Lagrangian is independent of t and φ. The stationary curves are governed by the Euler-Lagrange equations, which yield the geodesic equations and provide two constants E and L of their motion:

$$\frac{\partial f}{\partial \dot{t}} = E, \qquad \frac{\partial f}{\partial \dot{\varphi}} = L, \tag{8}$$

which give

$$\dot{t}(s) = \frac{E}{A(r(s))}, \qquad \dot{\varphi}(s) = \frac{L}{r^2(s)}. \tag{9}$$

Here, a dot denotes differentiation with respect to an affine parameter s along the geodesic. In addition, restricting the Lagrangian to the position-velocity path of the null geodesic (in the tangent bundle over spacetime) gives a third constant of the motion:

$$\begin{aligned} 0 &= f(r(s), \dot{t}(s), \dot{r}(s), \dot{\varphi}(s)) \\ &= -A(r(s))\dot{t}^2(s) + B(r(s))\dot{r}^2(s) + r^2(s)\dot{\varphi}^2(s). \end{aligned} \tag{10}$$

This yields

$$\dot{r} = \pm |L| \sqrt{\frac{1/b^2 - A/r^2}{AB}}, \tag{11}$$

where $b = |L/E|$ and is called the *impact parameter*. If the light ray's distance r_0 of closest approach to the lens occurs at affine parameter $s = s_0$ such that $\dot{r}(s_0) = 0$, then b and r_0 are simply related by

$$\frac{1}{b^2} = \frac{A(r_0)}{r_0^2}. \tag{12}$$

Equations (9) and (11) then yield

$$\frac{d\varphi}{dr} = \pm \frac{1}{r^2} \sqrt{\frac{AB}{1/b^2 - A/r^2}} \tag{13}$$

(since $C > 0$ and $d\varphi/dr = \dot{\varphi}/\dot{r} = \pm|\dot{\varphi}/\dot{r}|$). The plus (minus) sign corresponds to the portion of the light's trajectory where r increases (decreases) as a function of φ. Similarly, Eqs. (9) and (11) also give

$$\frac{dt}{dr} = \pm \frac{1}{bA} \sqrt{\frac{AB}{1/b^2 - A/r^2}}, \tag{14}$$

where the plus (minus) sign corresponds to the portion of the orbit where r increases (decreases) as a function of t.

Consider a light ray that originates in the asymptotically flat region of spacetime and is deflected by the compact body before arriving at an observer in the flat region. Equation (13) then yields the following expression for the bending angle (e.g., [28]):

$$\begin{aligned} \hat{\alpha}(r_0) &= 2 \int_{r_0}^{\infty} \left| \frac{d\varphi}{dr} \right| dr - \pi \\ &= 2 \int_{r_0}^{\infty} \frac{1}{r^2} \sqrt{\frac{AB}{1/b^2 - A/r^2}} dr - \pi. \end{aligned} \tag{15}$$

B. General relativity: Schwarzschild metric

1. Schwarzschild metric

For a spherical, electrically neutral compact body, general relativity yields a unique metric that is asymptotically flat and obeys the vacuum Einstein equation: the Schwarzschild metric. It is characterized by the single parameter $\mathsf{m}_\bullet$ and can be written in the form of (5) with

$$A(r) = 1 - \frac{2\mathsf{m}_\bullet}{r}, \qquad B(r) = \left(1 - \frac{2\mathsf{m}_\bullet}{r}\right)^{-1}. \tag{16}$$

We need to relate the distance of closest approach r_0 to the impact parameter b. Using Eq. (12), we can write b in terms of r_0:

$$\frac{r_0}{b} = \sqrt{1 - 2\frac{\mathsf{m}_\bullet}{r_0}}. \tag{17}$$

Inverting this to find r_0 in terms of b yields (cf. [29])

$$\frac{r_0}{b} = \frac{2}{\sqrt{3}} \cos\left[\frac{1}{3}\cos^{-1}\left(-\frac{3^{3/2}\mathsf{m}_\bullet}{b}\right)\right]. \tag{18}$$

Since $\mathsf{m}_\bullet/b \ll 1$ by A3, we can Taylor expand the right-hand side in powers of $\mathsf{m}_\bullet/b$ to obtain

$$\begin{aligned} r_0 = b\bigg[1 &- \frac{\mathsf{m}_\bullet}{b} - \frac{3}{2}\left(\frac{\mathsf{m}_\bullet}{b}\right)^2 - 4\left(\frac{\mathsf{m}_\bullet}{b}\right)^3 - \frac{105}{8}\left(\frac{\mathsf{m}_\bullet}{b}\right)^4 \\ &- 48\left(\frac{\mathsf{m}_\bullet}{b}\right)^5 - \frac{3003}{16}\left(\frac{\mathsf{m}_\bullet}{b}\right)^6 + \mathcal{O}\left(\frac{\mathsf{m}_\bullet}{b}\right)^7 \bigg]. \end{aligned} \tag{19}$$

2. Schwarzschild bending angle

From Eqs. (15) and (16), the exact form of the light bending angle can be written as

$$\hat{\alpha}(r_0) = 2\int_{r_0}^{\infty} \frac{dw}{w^2\sqrt{1/b^2 - 1/w^2 + 2\mathsf{m}_\bullet/w^3}} - \pi. \quad (20)$$

We change variables to $x = r_0/w$, and also substitute for the impact parameter b using (17). This yields

$$\hat{\alpha}(r_0) = 2\int_0^1 \frac{dx}{\sqrt{1 - 2h - x^2 + 2hx^3}} - \pi, \quad (21)$$

where $h = \mathsf{m}_\bullet/r_0$. (Note that for computational purposes, a numerically stable expression can be obtained by further changing variables to $x = \cos\eta$.) For pedagogical purposes, rewrite the integral as

$$\int_0^1 \frac{dx}{\sqrt{1 - 2h - x^2 + 2hx^3}} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - 2h(1 - x^3)/(1 - x^2)}}. \quad (22)$$

By assumption A3 we have $h < 1/3$, which means physically that the light ray is outside the photon sphere at $3\mathsf{m}_\bullet$. The rational function $(1 - x^3)/(1 - x^2)$ is monotonically increasing on $[0, 1]$ with a maximum value of $3/2$ at $x = 1$. This means that $0 \leq 2h(1 - x^3)/(1 - x^2) < 1$, so we can Taylor expand $[1 - 2h(1 - x^3)/(1 - x^2)]^{-1/2}$ in a geometric series. Carrying out the integration in Eq. (22) term by term then gives

$$\begin{aligned}\hat{\alpha}(h) = {}& 4h + \left(-4 + \frac{15}{4}\pi\right)h^2 + \left(\frac{122}{3} - \frac{15}{2}\pi\right)h^3 \\ &+ \left(-130 + \frac{3465}{64}\pi\right)h^4 + \left(\frac{7783}{10} - \frac{3465}{16}\pi\right)h^5 \\ &+ \left(-\frac{21\,397}{6} + \frac{310\,695}{256}\pi\right)h^6 + \mathcal{O}(h)^7. \end{aligned} \quad (23)$$

To obtain the bending angle in terms of the invariant impact parameter b, we use Eq. (19) to write r_0 in terms of b. This yields

$$\begin{aligned}\hat{\alpha}(b) = {}& A_1\left(\frac{\mathsf{m}_\bullet}{b}\right) + A_2\left(\frac{\mathsf{m}_\bullet}{b}\right)^2 + A_3\left(\frac{\mathsf{m}_\bullet}{b}\right)^3 + A_4\left(\frac{\mathsf{m}_\bullet}{b}\right)^4 \\ &+ A_5\left(\frac{\mathsf{m}_\bullet}{b}\right)^5 + A_6\left(\frac{\mathsf{m}_\bullet}{b}\right)^6 + \mathcal{O}\left(\frac{\mathsf{m}_\bullet}{b}\right)^7, \end{aligned} \quad (24)$$

where

$$A_1 = 4, \qquad A_2 = \frac{15\pi}{4}, \qquad A_3 = \frac{128}{3}, \qquad A_4 = \frac{3465\pi}{64}, \qquad A_5 = \frac{3584}{5}, \qquad A_6 = \frac{255\,255\pi}{256}. \quad (25)$$

It follows that Eq. (24) has the form required by assumption A3. We shall not actually use terms beyond $\mathcal{O}(\mathsf{m}_\bullet/b)^3$, but we have included some higher-order terms because they have not appeared in the literature before.

C. PPN approach

1. PPN metric to third order

The post-post-Newtonian formalism is a convenient way to handle the wide range of gravity theories in which the weak-deflection limit can be expressed as a series expansion in the single variable $\mathsf{m}_\bullet$. The formalism was extended to third order in [21]. However, we reformulate that treatment to have a better fit with our approach, and present new results.

Express the coefficients of the standard metric (5) in a PPN series to third order as follows:

$$A(r) = 1 + 2a_1\left(\frac{\phi}{c^2}\right) + 2a_2\left(\frac{\phi}{c^2}\right)^2 + 2a_3\left(\frac{\phi}{c^2}\right)^3 + \cdots, \quad (26)$$

$$B(r) = 1 - 2b_1\left(\frac{\phi}{c^2}\right) + 4b_2\left(\frac{\phi}{c^2}\right)^2 - 8b_3\left(\frac{\phi}{c^2}\right)^3 + \cdots, \quad (27)$$

where ϕ is the three-dimensional Newtonian potential with

$$\frac{\phi}{c^2} = -\frac{\mathsf{m}_\bullet}{r}. \quad (28)$$

If the metric is in isotropic form, namely,

$$ds^2 = -A'(r')dt^2 + B'(r')[dr'^2 + r'^2 d\Omega^2], \quad (29)$$

then the PPN convention is to write

$$A'(r') = 1 + 2\alpha'\frac{\phi'}{c^2} + 2\beta'\left(\frac{\phi'}{c^2}\right)^2 + \frac{3}{2}\xi'\left(\frac{\phi'}{c^2}\right)^3 + \cdots, \quad (30)$$

$$B'(r') = 1 - 2\gamma'\frac{\phi'}{c^2} + \frac{3}{2}\delta'\left(\frac{\phi'}{c^2}\right)^2 - \frac{1}{2}\eta'\left(\frac{\phi'}{c^2}\right)^3 + \cdots, \quad (31)$$

where $\phi'/c^2 = -\mathsf{m}_\bullet/r'^2$, and $(\alpha', \beta', \gamma', \delta', \xi', \eta')$ denote the Eddington-Robertson parameters (with primes added to avoid confusion with standard lensing quantities). The parameters are chosen so that the Schwarzschild metric has

$$\alpha' = \beta' = \gamma' = \delta' = \xi' = \eta' = 1. \quad (32)$$

We can relate the parameters in the standard and isotropic forms of the metric by comparing the time, radial, and angular parts of the two metrics, yielding the relations

$$A(r) = A'(r'), \qquad r^2 = B'(r')r'^2, \qquad B(r)dr^2 = B'(r')dr'^2. \quad (33)$$

The second and third relations yield

$$\ln r' = \int \frac{\sqrt{B(r)}}{r} dr + \text{const}, \tag{34}$$

where the constant is chosen so that $r'/r \to 1$ as $r \to \infty$. Plugging r' into (33) and identifying terms on the left- and right-hand sides that have the same order in $\mathsf{m}_\bullet/r$, we find the following correspondence between the standard and isotropic coefficients:

$$a_1 = \alpha', \tag{35}$$

$$b_1 = \gamma', \tag{36}$$

$$a_2 = \beta' - \alpha'\gamma', \tag{37}$$

$$b_2 = \frac{3\delta' + \gamma'^2}{4}, \tag{38}$$

$$a_3 = \frac{3\xi' + 3\alpha'\delta' - 8\beta'\gamma' + 2\alpha'\gamma'^2}{4}, \tag{39}$$

$$b_3 = \frac{3\eta' + 15\delta'\gamma' - 2\gamma'^3}{16}. \tag{40}$$

The correspondences (39) and (40) have not been worked out before in the literature. For reference, the Schwarzschild metric has

$$a_1 = b_1 = b_2 = b_3 = 1, \qquad a_2 = a_3 = 0. \tag{41}$$

We need to relate the distance of closest approach r_0 to the invariant impact parameter b. Using Eq. (12), we can write b in terms of r_0:

$$b = r_0\left[1 + a_1\frac{\mathsf{m}_\bullet}{r_0} + \frac{3a_1^2 - 2a_2}{2}\left(\frac{\mathsf{m}_\bullet}{r_0}\right)^2 + \frac{5a_1^3 - 6a_1a_2 + 2a_3}{2}\left(\frac{\mathsf{m}_\bullet}{r_0}\right)^3 + \mathcal{O}\left(\frac{\mathsf{m}_\bullet}{r_0}\right)^4\right]. \tag{42}$$

To invert this relation and find r_0 in terms of b, we postulate a relation of the form

$$r_0 = b\left[1 + c_1\frac{\mathsf{m}_\bullet}{b} + c_2\left(\frac{\mathsf{m}_\bullet}{b}\right)^2 + c_3\left(\frac{\mathsf{m}_\bullet}{b}\right)^3 + \mathcal{O}\left(\frac{\mathsf{m}_\bullet}{b}\right)^4\right], \tag{43}$$

plug this into (42), and solve for the constants c_i by requiring that the coefficient of each power of $\mathsf{m}_\bullet$ vanishes. This yields

$$r_0 = b\left[1 - a_1\frac{\mathsf{m}_\bullet}{b} + \frac{2a_2 - 3a_1^2}{2}\left(\frac{\mathsf{m}_\bullet}{b}\right)^2 - (a_3 - 4a_1a_2 + 4a_1^3)\left(\frac{\mathsf{m}_\bullet}{b}\right)^3 + \mathcal{O}\left(\frac{\mathsf{m}_\bullet}{b}\right)^4\right]. \tag{44}$$

The result (44) is also new.

2. PPN bending angle

To compute the light bending angle, we take the exact expression (15), plug in the PPN metric functions (26) and (47), and change integration variables to $x = r_0/w$. We also substitute for the impact parameter b using Eq. (42). Finally, we introduce $h = \mathsf{m}_\bullet/r_0$, so that the series expansion becomes a Taylor series in h.

The first step yields

$$\begin{aligned}\hat{\alpha}(h) = 2\int_0^1 \frac{dx}{\sqrt{x^2-1}}\Bigg\{&1 + h\left(b_1x + \frac{a_1}{1+x}\right) + h^2\left[-a_2 - \frac{1}{2}(b_1^2 - 4b_2)x^2 + \frac{3a_1^2}{2(1+x)^2} + (2a_1^2 + a_1b_1)\frac{x}{1+x}\right] \\ &+ \frac{h^3}{2}\Bigg[a_1(8a_1^2 + 4a_1b_1 - b_1^2 + 4b_2)(x-1) + 4a_1a_2(1-2x) + 2x(a_3 - a_2b_1) + (b_1^3 - 4b_1b_2 + 8b_3)x^3 \\ &+ \frac{5a_1^3}{(1+x)^3} - \frac{3a_1^2(4a_1 + b_1)}{(1+x)^2} + \frac{2a_3}{1+x} + \frac{a_1(20a_1^2 - 10a_2 + 7a_1b_1 - b_1^2 + 4b_2}{1+x}\Bigg] + \mathcal{O}(h)^4\Bigg\} - \pi.\end{aligned} \tag{45}$$

Carrying out the integration term by term then yields

$$\begin{aligned}\hat{\alpha}(h) = {}& 2(a_1 + b_1)h + h^2\left[-2a_1(a_1 + b_1) + \left(2a_1^2 - a_2 + a_1b_1 - \frac{b_1^2}{4} + b_2\right)\pi\right] \\ &+ h^3\Bigg[\frac{67}{3}a_1^3 - 18a_1a_2 + 4a_3 + 9a_1^2b_1 - 2b_1(a_2 + a_1b_1) + 8a_1b_2 + \frac{2}{3}(b_1^3 - 4b_1b_2 + 8b_3) \\ &- a_1\left(4a_1^2 - 2a_2 + 2a_1b_1 - \frac{b_1^2}{2} + 2b_2\right)\pi\Bigg] + \mathcal{O}(h)^4,\end{aligned} \tag{46}$$

which agrees with [21]. However, the expression (46) is coordinate dependent since it is written in terms of the distance of closest approach. To obtain an invariant expression (so we can discuss observable quantities), we use (44) to write r_0 in terms of b and obtain

$$\hat{\alpha}(b) = A_1\left(\frac{\mathsf{m}_\bullet}{b}\right) + A_2\left(\frac{\mathsf{m}_\bullet}{b}\right)^2 + A_3\left(\frac{\mathsf{m}_\bullet}{b}\right)^3 + \mathcal{O}\left(\frac{\mathsf{m}_\bullet}{b}\right)^4, \tag{47}$$

where

$$A_1 = 2(a_1 + b_1), \tag{48}$$

$$A_2 = \left(2a_1^2 - a_2 + a_1 b_1 - \frac{b_1^2}{4} + b_2\right)\pi, \tag{49}$$

$$A_3 = \tfrac{2}{3}[35a_1^3 + 15a_1^2 b_1 - 3a_1(10a_2 + b_1^2 - 4b_2) + 6a_3 + b_1^3 - 6a_2 b_1 - 4b_1 b_2 + 8b_3]. \tag{50}$$

This invariant bending angle expression has not appeared in the literature before.

D. Sample gravity theories

In general relativity, the Reissner-Nordström metric describes a black hole with physical charge Q. Although the charge is generally expected to become neutralized, in certain circumstances it could persist for 1000 to 10 000 years [30], perhaps allowing detection of such black holes. The Reissner-Nordström metric has the form of (5) with the metric functions

$$A(r) = B(r)^{-1} = 1 - \frac{2\mathsf{m}_\bullet}{r} + \frac{q^2 \mathsf{m}_\bullet^2}{r^2}, \tag{51}$$

where $q = \sqrt{G}Q/(c^2 \mathsf{m}_\bullet)$ is a dimensionless parameter. If $q^2 > 1$, the metric actually describes a naked singularity rather than a charged black hole (e.g., [11,12]). It is natural to view (51) as a series expansion in $\mathsf{m}_\bullet/r$. We can then identify the PPN coefficients:

$$\begin{aligned} a_1 &= 1, \qquad a_2 = \frac{q^2}{2}, \qquad a_3 = 0, \\ b_1 &= 1, \qquad b_2 = 1 - \frac{q^2}{4}, \qquad b_3 = 1 - \frac{q^2}{2}. \end{aligned} \tag{52}$$

Using Eqs. (48)–(50), we can determine the coefficients in the expansion of the bending angle:

$$A_1 = 4, \qquad A_2 = (5 - q^2)\frac{3\pi}{4}, \qquad A_3 = \frac{128}{3} - 16q^2. \tag{53}$$

Notice that if $q \to 0$ we recover the results for the Schwarschild metric. Also notice that the condition for a naked singularity corresponds to $A_2 < 3\pi$. Turning this around, we may say that for the Reissner-Nordström metric of general relativity, the coefficient A_2 can be negative only if there is a naked singularity.

In heterotic string theory, modifications of the Einstein equation lead to a different charged black hole solution, which is often called the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) black hole [31]. In [13], the metric's lensing properties were computed to lowest order in both the weak- and strong-deflection regimes, although only with a coordinate-dependent approach. We can include the weak-deflection limit in our invariant formalism, and thereby obtain new results. For a black hole with gravitational radius $\mathsf{m}_\bullet$ and charge Q, the GMGHS metric is often written in the form

$$ds^2 = -\left(1 - \frac{2\mathsf{m}_\bullet}{\bar{r}}\right)dt^2 + \left(1 - \frac{2\mathsf{m}_\bullet}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2\left(1 - \frac{q^2\mathsf{m}_\bullet}{\bar{r}}\right)d\Omega^2, \tag{54}$$

where again $q = \sqrt{G}Q/(c^2 \mathsf{m}_\bullet)$. As in the Reissner-Nordström case, this metric describes a naked singularity if the charge parameter exceeds some threshold, in this case $q^2 > 2$. We convert to standard coordinates by setting

$$r^2 = \bar{r}^2\left(1 - \frac{q^2\mathsf{m}_\bullet}{\bar{r}}\right). \tag{55}$$

This puts the metric into the form of (5) with the metric functions

$$A(r) = 1 - \frac{4\mathsf{m}_\bullet}{q^2\mathsf{m}_\bullet + \sqrt{q^4\mathsf{m}_\bullet^2 + 4r^2}}, \tag{56}$$

$$B(r) = \left(1 - \frac{4\mathsf{m}_\bullet}{q^2\mathsf{m}_\bullet + \sqrt{q^4\mathsf{m}_\bullet^2 + 4r^2}}\right)^{-1} \frac{4r^2}{\mathsf{m}_\bullet^2 q^4 + 4r^2}. \tag{57}$$

Expanding the metric functions as a Taylor series in $\mathsf{m}_\bullet/r$, we can identify the PPN parameters

$$\begin{aligned} a_1 &= 1, \qquad a_2 = \frac{q^2}{2}, \qquad a_3 = \frac{q^4}{8}, \\ b_1 &= 1, \qquad b_2 = 1 - \frac{q^2}{4} - \frac{q^4}{16}, \qquad b_3 = 1 - \frac{q^2}{2} - \frac{q^4}{32}. \end{aligned} \tag{58}$$

The coefficients in the expansion of the bending angle are then

$$\begin{aligned} A_1 &= 4, \qquad A_2 = (60 - 12q^2 - q^4)\frac{\pi}{16}, \\ A_3 &= \frac{128}{3} - 16q^2. \end{aligned} \tag{59}$$

In this case, the condition for a naked singularity corresponds to $A_2 < 2\pi$. Or, we may again say that for the GMGHS metric A_2 can be negative only if there is a naked singularity.

IV. LENSING FRAMEWORK IN VARIOUS GRAVITY THEORIES

We can now move beyond the bending angle to compute corrections to the observable properties of the primary and secondary lensed images. In this section we focus on the positions and magnifications of the images; time delays are deferred to Sec. V.

A. Lens equation

We start with the general lens equation,

$$\tan\mathcal{B} = \tan\vartheta - D(\tan\vartheta + \tan(\hat{\alpha} - \vartheta)), \tag{60}$$

and seek an appropriate series expansion. First, we change variables to match the scalings commonly used in the astrophysical lensing literature. A natural scale is the weak-deflection angular Einstein ring radius,

$$\vartheta_E = \sqrt{\frac{4GM_\bullet d_{LS}}{c^2 d_L d_S}}. \tag{61}$$

We then define

$$\beta = \frac{\mathcal{B}}{\vartheta_E}, \qquad \theta = \frac{\vartheta}{\vartheta_E}, \qquad \varepsilon = \frac{\vartheta_\bullet}{\vartheta_E} = \frac{\vartheta_E}{4D}. \tag{62}$$

In other words, β and θ are the scaled angular positions of the source and image, respectively. The quantity ε represents the angle subtended by the gravitational radius normalized by the angular Einstein radius, and it becomes our new expansion parameter.

The second step is to postulate that the solution of the lens equation can be written as a series expansion of the form

$$\theta = \theta_0 + \theta_1\varepsilon + \theta_2\varepsilon^2 + \mathcal{O}(\varepsilon)^3, \tag{63}$$

where θ_0 is expected to be the image position in the weak-deflection limit, and the coefficients θ_1 and θ_2 of the correction terms remain to be determined. (This is a standard perturbation theory analysis, e.g., [32].)

After making these substitutions, we first find that we can write the bending angle as

$$\hat{\alpha} = \frac{A_1}{\theta_0}\varepsilon + \frac{A_2 - A_1\theta_1}{\theta_0^2}\varepsilon^2 + \frac{1}{\theta_0^3}\bigg[A_3 - 2A_2\theta_1 + A_1\bigg(\frac{8}{3}D^2\theta_0^4 + \theta_1^2 - \theta_0\theta_2\bigg)\bigg]\varepsilon^3 + \mathcal{O}(\varepsilon)^4. \tag{64}$$

Note that since we are expanding beyond linear order, it is important to use the exact geometric relations between physical and angular radii: $\vartheta = \sin^{-1}(b/d_L)$ and $\vartheta_\bullet = \tan^{-1}(\mathsf{m}_\bullet/d_L)$. The standard small-angle approximations ($\vartheta \approx b/d_L$ and $\vartheta_\bullet \approx \mathsf{m}_\bullet/d_L$) are valid only at linear order.

Now making the substitutions in the lens equation and Taylor expanding in ε, we find

$$\begin{aligned} 0 = {} & D\bigg[-4\beta + 4\theta_0 - \frac{A_1}{\theta_0}\bigg]\varepsilon + \frac{D}{\theta_0^2}[-A_2 + (A_1 + 4\theta_0^2)\theta_1]\varepsilon^2 \\ & + \frac{D}{3\theta_0^3}\{-A_1^3 - 3A_3 + 12A_1^2D\theta_0^2 - A_1(56D^2\theta_0^4 + 3\theta_1^2 - 3\theta_0\theta_2) + 2[32D^2\theta_0^3(\theta_0^3 - \beta^3) \\ & + 3A_2\theta_1 + 6\theta_0^3\theta_2]\}\varepsilon^3 + \mathcal{O}(\varepsilon)^4. \end{aligned} \tag{65}$$

This is the desired series expansion of the lens equation.

B. Image positions

We now solve Eq. (65) term by term to find the coefficients θ_i in the series expansion for the image position. The idea is to fix the source position β and find the values of θ_i that make each term in (65) vanish. The first-order term is just the standard weak-deflection lens equation,

$$0 = -\beta + \theta_0 - \frac{1}{\theta_0}, \tag{66}$$

which yields the weak-deflection image position

$$\theta_0 = \tfrac{1}{2}(\beta + \sqrt{\beta^2 + 4}). \tag{67}$$

We neglect the negative solution because we have explicitly specified that angles describing image positions are positive. We find the positive-parity image θ_0^+, which lies on the same side of the lens as the source, by using a positive angular source position, $\beta > 0$. We then find the negative-parity image θ_0^-, which lies on the opposite side of the lens from the source, by using $\beta < 0$. In other words, we can rewrite (67) as

$$\theta_0^{\pm} = \tfrac{1}{2}(\sqrt{4 + \beta^2} \pm |\beta|). \tag{68}$$

One curious feature of the positive- and negative-parity image positions in the weak-deflection limit is that

$$\theta_0^+\theta_0^- = 1. \tag{69}$$

Note that (68) makes it clear that (69) does not depend on the sign of β.

Next, we choose θ_1 to make the second-order term in (65) vanish. This yields

$$\theta_1 = \frac{A_2}{A_1 + 4\theta_0^2}. \tag{70}$$

For the third-order term, we still have θ_0 given by (67) and θ_1 given by (70), and we must choose θ_2 to make the coefficient of ε^3 in (65) vanish. This yields

$$\theta_2 = \frac{1}{3\theta_0(A_1+4\theta_0^2)^3}\{A_1(A_1^4-3A_2^2+3A_1A_3-64A_1D^2)-4[6A_2^2-2A_1(A_1^3+3A_3)+3A_1^4D-16A_1D^2(3A_1-8)]\theta_0^2 + 8[2A_1^3(1-6D)+6A_3-128D^2+A_1D^2(192+A_1(-24+7A_1))]\theta_0^4+64D[48D-24A_1D+A_1^2(-3+7D)]\theta_0^6 + 128D^2(-24+7A_1)\theta_0^8\}. \tag{71}$$

Note that we have used the relation $\beta = \theta_0 - 1/\theta_0$ to replace β on the right-hand side. It is possible to rewrite the right-hand side in terms of β instead of θ_0, but at this point we believe that is less useful.

For gravity theories with $A_1 = 4$ (including general relativity) the correction terms simplify slightly to

$$\theta_1 = \frac{A_2}{4(1+\theta_0^2)}, \tag{72}$$

$$\theta_2 = \frac{1}{48\theta_0(1+\theta_0^2)^3}[-3A_2^2(1+2\theta_0^2)+4(1+\theta_0^2)^2(3A_3 + 64-64D^2-192D\theta_0^2+192D^2\theta_0^2+32D^2\theta_0^4)]. \tag{73}$$

It is worth pointing out that the image position can be written to first order in ε as

$$\theta = \theta_0 + \frac{A_2}{A_1+4\theta_0^2}\varepsilon + \mathcal{O}(\varepsilon)^2. \tag{74}$$

The sign of the first-order correction to the weak-deflection image position is given by the sign of A_2 (for both the positive- and negative-parity images). We have already seen that the sign of A_2 is connected with the possible presence of naked singularities: in the Reissner-Nordström and GMGHS metrics, A_2 can be negative only if there is a naked singularity (see Sec. III D). Thus, the correction to the image positions can provide a possible observational test for naked singularities.

C. Magnifications

The signed magnification μ of a lensed image at angular position ϑ is

$$\mu(\vartheta) = \left[\frac{\sin\mathcal{B}(\vartheta)}{\sin\vartheta}\frac{d\mathcal{B}(\vartheta)}{d\vartheta}\right]^{-1}. \tag{75}$$

We change to our scaled variables as in Eqs. (62) and (63), make a Taylor expansion in ε, and then substitute for θ_1 and θ_2 using (70) and (71). In this way we obtain a series expansion for the magnification:

$$\mu = \mu_0 + \mu_1\varepsilon + \mu_2\varepsilon^2 + \mathcal{O}(\varepsilon)^3, \tag{76}$$

where

$$\mu_0 = \frac{16\theta_0^4}{16\theta_0^4 - A_1^2}, \tag{77}$$

$$\mu_1 = -\frac{16A_2\theta_0^3}{(A_1+4\theta_0^2)^3}, \tag{78}$$

$$\mu_2 = \frac{8\theta_0^2}{3(A_1-4\theta_0^2)^2(A_1+4\theta_0^2)^5}\{A_1^4D^2(512-9A_1^3)+4A_1^3\theta_0^2[-384A_1D^2+A_1^3(4+12D-9D^2)+4(3A_3+256D^2)] + 64A_1\theta_0^4[A_1^4(1+3D)-9A_2^2+3A_1A_3+128A_1D^2-192A_1^2D^2+24A_1^3D^2]-256\theta_0^6[-9A_2^2+3A_1A_3+96A_1^2D^2 - 48A_1^3D^2+A_1^4(1+3D+2D^2)]-256\theta_0^8[12A_3-96A_1^2D^2+A_1^3(4+12D+7D^2)]+1024A_1^2D^2\theta_0^{10}\}. \tag{79}$$

Recall that the sign of the signed magnification indicates the parity of the image: $\mu > 0$ for θ^+, the positive-parity primary image; while $\mu < 0$ for θ^-, the negative-parity secondary image.

For a gravity theory with $A_1 = 4$, these expressions reduce to

$$\mu_0 = \frac{\theta_0^4}{\theta_0^4 - 1}, \tag{80}$$

$$\mu_1 = -\frac{A_2\theta_0^3}{4(1+\theta_0^2)^3}, \tag{81}$$

$$\mu_2 = -\frac{\theta_0^2}{24(\theta_0^2-1)(\theta_0^2+1)^5}\{768D\theta_0^2(\theta_0^2+1)^2 - 64D^2(\theta_0^2+1)^2(1+16\theta_0^2+\theta_0^4) + \theta_0^2[256+(512-9A_2^2)\theta_0^2+256\theta_0^4 + 12A_3(\theta_0^2+1)^2]\}. \tag{82}$$

It is worth pointing out that if $A_1 = 4$ the first-order changes in the magnifications of the positive- and negative-parity images have the following relation:

$$\mu_1^+ = -\frac{A_2(\theta_0^+)^3}{4[1+(\theta_0^+)^2]^3} = -\frac{A_2(\theta_0^-)^3}{4[(\theta_0^-)^2+1]^3} = \mu_1^-. \tag{83}$$

In the second equality we used (69) to write $\theta_0^+ = 1/\theta_0^-$. In gravity theories with $A_1 = 4$ and $A_2 > 0$, the μ_1 perturbation is negative, so it makes μ^+ less positive (fainter) and μ^- more negative (brighter) by exactly the same amount. (The opposite occurs for $A_1 = 4$ and $A_2 < 0$.) In other words, the magnifications of the positive- and negative-parity images are shifted by the same amount but in the opposite sense. This has important implications for the total magnification and centroid (see below).

Notice that the sign of the first-order magnification correction depends on the sign of A_2, so this provides another possible observational test for naked singularities.

D. Total magnification and centroid

If the two images are too close together to be resolved (as in microlensing), the main observables are the total magnification and the magnification-weighted centroid position. Using our results above, we find the total magnification $\mu_{\text{tot}} = |\mu^+| + |\mu^-|$ to be

$$\mu_{\text{tot}} = \frac{16A_1^2(\theta_0^8 - 1)}{(16\theta_0^4 - A_1^2)(A_1^2\theta_0^4 - 16)} - \frac{16(A_1 - 4)A_2\theta_0^3}{(A_1 + 4\theta_0^2)^3(4 + A_1\theta_0^2)^3}\{[16 + A_1(4 + A_1)] \times (\theta_0^6 - 1) + 12A_1\theta_0^2(\theta_0^2 - 1)\}\varepsilon + \mathcal{O}(\varepsilon)^2. \qquad (84)$$

The second-order term can be worked out from (79) if desired, but is too complicated to write here. Notice that the sign of A_2 again determines the sign of the first-order correction.

It is important to see the factor of $(A_1 - 4)$ multiplying the first-order term in (84). It means that the first-order correction to the total magnification vanishes with full generality in any gravity theory with $A_1 = 4$. In such theories, the first-order changes in the magnifications of the positive- and negative-parity images exactly cancel [see Eq. (83)]. A correction then enters at second order, which is given by

$$\mu_{\text{tot}} = \frac{2 + \beta^2}{\beta\sqrt{4 + \beta^2}} + \frac{1}{12\beta(4 + \beta^2)^{5/2}}\{9A_2^2 - 12A_3(4 + \beta^2) - 64(4 + \beta^2)[4 + 12D - (18 + \beta^2)D^2]\}\varepsilon^2 + \mathcal{O}(\varepsilon)^3. \qquad (85)$$

Here we have found it convenient to express the result in terms of the source angular position β.

The magnification-weighted centroid position is defined by

$$\Theta_{\text{cent}} = \frac{\theta^+|\mu^+| - \theta^-|\mu^-|}{|\mu^+| + |\mu^-|} = \frac{\theta^+\mu^+ + \theta^-\mu^-}{\mu_+ - \mu_-}. \qquad (86)$$

(The sign in the numerator may be understood by recalling that we use positive angles, $\theta_\pm > 0$.) Our perturbation analysis then yields

$$\Theta_{\text{cent}} = \frac{A_1^2(\theta_0^8 - \theta_0^6 + \theta_0^4 - \theta_0^2 + 1) - 16\theta_0^4}{A_1^2\theta_0(\theta_0^6 - \theta_0^4 + \theta_0^2 - 1)} + \frac{A_2(A_1 - 4)\theta_0^2}{A_1^4(\theta_0^4 - 1)(\theta_0^4 + 1)^2(A_1 + 4\theta_0^2)(4 + A_1\theta_0^2)} \times [A_1^2(16 + 4A_1 + A_1^2)(1 + \theta_0^{12}) - 4A_1(32 - A_1^2)\theta_0^2(1 + \theta_0^8) + (256 - 64A_1 - 32A_1^2 + 8A_1^3 + A_1^4)\theta_0^4(1 + \theta_0^4) + (256 + 32A_1^2 + 24A_1^3 + A_1^4)\theta_0^6]\varepsilon + \mathcal{O}(\varepsilon)^2. \qquad (87)$$

As before the second-order term can be worked out but is too complicated to write here. Notice again that the sign of A_2 yet again determines the sign of the first-order correction.

We again see a factor of $(A_1 - 4)$ multiplying the first-order term. Thus, the first-order correction to the centroid vanishes exactly in any gravity theory with $A_1 = 4$. In such theories, a correction appears at second order and is given in terms of β by

$$\Theta_{\text{cent}} = \frac{\beta(3 + \beta^2)}{2 + \beta^2} - \frac{\beta}{24(4 + \beta^2)(2 + \beta^2)^2} \times \{9A_2^2 - 12A_3(4 + \beta^2) - 128(4 + \beta^2)(2 - D^2) - 64(4 + \beta^2)[(9 + \beta^2)D - 6]D\beta^2\}\varepsilon^2 + \mathcal{O}(\varepsilon)^3. \qquad (88)$$

Notice that in this case ($A_1 = 4$), the sign of A_2 does not affect the corrections, but the sign of A_3 does.

V. TIME DELAYS IN VARIOUS GRAVITY THEORIES

Having studied the positions and brightnesses of the primary and secondary images, we are now ready to compute their time delays. The analysis parallels some of what has gone before, but is different enough to warrant a separate treatment.

A. General formalism

Let R_{src} and R_{obs} be the radial coordinates of the source and observer, respectively. From geometry relative to the flat metric of the distant observer, who is assumed to be at rest in the natural coordinates of the metric (6), we can work out (see Fig. 1)

$$R_{\text{obs}} = d_L, \qquad R_{\text{src}} = (d_{LS}^2 + d_S^2 \tan^2\mathcal{B})^{1/2}. \qquad (89)$$

The radial distances are very nearly the same as angular diameter distances since the source and observer are in the asymptotically flat region of the spacetime. In other words, the distortions in distances near the black hole are assumed to have little impact on the total flat metric distance from the compact body to the observer or source. We focus on spacetimes that would be flat in the absence of the lens (see assumption A1), and in that case the light ray would travel along a linear path from the source to the observer with length $d_S/\cos\mathcal{B}$.

The time delay is the difference between the light travel time for the actual ray, and the travel time for the ray the light would have taken had the lens been absent. This can be written as

$$c\tau = T(R_{\rm src}) + T(R_{\rm obs}) - \frac{d_S}{\cos\mathcal{B}}, \tag{90}$$

with

$$T(R) = \int_{r_0}^{R} \left| \frac{dt}{dr} \right| dr = \frac{1}{b}\int_{r_0}^{R} \frac{1}{A(w)}\sqrt{\frac{A(w)B(w)}{1/b^2 - A(w)/w^2}} dw, \tag{91}$$

where we have used Eq. (14) for dt/dr. Unlike Eq. (15), this integral cannot extend to infinity because the travel time would diverge.

Note that there are no redshift factors in these equations because we are assuming a spacetime that is static and asymptotically flat. This assumption is not overly restrictive because expected applications involve noncosmological lens systems (see Sec. VI).

B. General relativity: Schwarzschild metric

To compute $T(R)$ for the Schwarzschild metric, we take the metric functions from Eq. (16), change integration variables to $x = r_0/w$, and use (17) to relate b to r_0, so that we find

$$T(R) = r_0 \int_{r_0/R}^{1} x^{-2}(1-2hx)^{-1}\left(1 - x^2\frac{1-2hx}{1-2h}\right)^{-1/2} dx, \tag{92}$$

where $h = \mathsf{m}_\bullet/r_0$. Expanding the integrand as a Taylor series in h and integrating term by term yields

$$\begin{aligned} T(R) &= \sqrt{R^2 - r_0^2} + hr_0\left[\frac{\sqrt{1-\xi^2}}{1+\xi} + 2\ln\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right)\right] \\ &\quad + h^2 r_0\left[\frac{15}{2}\left(\frac{\pi}{2} - \sin^{-1}\xi\right) - \frac{(4+5\xi)\sqrt{1-\xi^2}}{2(1+\xi)^2}\right] \\ &\quad + h^3 r_0\left[-\frac{15}{2}\left(\frac{\pi}{2} - \sin^{-1}\xi\right) \right. \\ &\quad \left. + \frac{(60 + 157\xi + 133\xi^2 + 35\xi^3)\sqrt{1-\xi^2}}{2(1+\xi)^3}\right] \\ &\quad + \mathcal{O}(h)^4, \end{aligned} \tag{93}$$

where $\xi = r_0/R$.

This expression is currently written in terms of the distance of closest approach r_0, but we want to rewrite it in terms of the invariant impact parameter b. We can use (19) to make the translation, but we must then consider the nature of the series expansion. The expansion in $h = \mathsf{m}_\bullet/r_0$ naturally becomes an expansion in $\mathsf{m}_\bullet/b$. It may be less obvious, but we also want to expand in b/R, which is small because both the source and the observer lie far from the lens. To do the joint expansion, we need to consider the amplitudes of $\mathsf{m}_\bullet/b$ and b/R. In terms of angular variables, we have two fairly simple cases,

$$\frac{\mathsf{m}_\bullet}{b} \sim \frac{\vartheta_\bullet}{\vartheta_E} \sim \varepsilon, \tag{94}$$

$$\frac{b}{R_{\rm obs}} = \frac{b}{d_L} \sim \vartheta_E \sim D\varepsilon, \tag{95}$$

and one that is slightly more involved,

$$\begin{aligned} \frac{b}{R_{\rm src}} &= \frac{b}{\sqrt{d_{LS}^2 + d_S^2\tan^2\mathcal{B}}} = \frac{1}{\sqrt{D^2 + \tan^2\mathcal{B}}}\frac{b}{d_S} \\ &= \frac{1-D}{\sqrt{D^2+\tan^2\mathcal{B}}}\frac{b}{d_L} \sim \frac{D(1-D)}{\sqrt{D^2+\tan^2\mathcal{B}}}\varepsilon. \end{aligned} \tag{96}$$

The bottom line is that $\mathsf{m}_\bullet/b$, $b/R_{\rm obs}$, and $b/R_{\rm src}$ are all similar in amplitude, up to factors that depend on D. We will do the rigorous series expansion of the time delay momentarily, but for now it is instructive to expand (93) in both $\mathsf{m}_\bullet/b$ and b/R, taking care to collect terms of a given order in any combination of the two quantities. Working to third order, we find

$$\begin{aligned} \frac{T(R)}{R} &= 1 - \left\{\frac{1}{2}\frac{b}{R}\left[\frac{b}{R} - 2\frac{\mathsf{m}_\bullet}{b} + 4\frac{\mathsf{m}_\bullet}{b}\ln\left(\frac{1}{2}\frac{b}{R}\right)\right]\right\} \\ &\quad + \left\{\frac{15\pi}{4}\frac{b}{R}\left(\frac{\mathsf{m}_\bullet}{b}\right)^2\right\} + \cdots. \end{aligned} \tag{97}$$

There is no term that is linear in $\mathsf{m}_\bullet/b$ or b/R. The first term in braces is of second order, while the second is of third order. We have neglected terms of fourth order and higher.

That was meant to be pedagogical. To properly compute the series expansion for the full time delay (90), we first compute $T(R_{\rm src})$ and $T(R_{\rm obs})$ by plugging (89) into (93). We replace r_0 with b using (19). We change to angular variables using $b = d_L\sin\vartheta$, and then reintroduce the scaled angular variables θ and β defined in (62). Finally, we take a formal Taylor series in our expansion parameter ε. Putting the pieces together, we find

$$\begin{aligned} c\tau = 8\frac{d_L d_{LS}}{d_S}\left\{\left[1 + \beta^2 - \theta_0^2 - \ln\left(\frac{d_L\theta_0^2\vartheta_E^2}{4d_{LS}}\right)\right]\varepsilon^2 \right. \\ \left. + \frac{15\pi - 8(1+\theta_0^2)\theta_1}{4\theta_0}\varepsilon^3 + \mathcal{O}(\varepsilon)^4\right\}. \end{aligned} \tag{98}$$

The interpretation of this result becomes more clear when we recognize that a characteristic scale for the time delay is

$$\tau_E \equiv \frac{d_L d_S}{c d_{LS}} \vartheta_E^2 = 4\frac{\mathsf{m}_\bullet}{c}. \tag{99}$$

Although it is not obvious from the definition, the second equality shows that τ_E is independent of the distances. Using this definition, we can write (98) as

$$\frac{\tau}{\tau_E} = \frac{1}{2}\left[1 + \beta^2 - \theta_0^2 - \ln\left(\frac{d_L \theta_0^2 \vartheta_E^2}{4 d_{LS}}\right)\right] + \frac{15\pi - 8(1+\theta_0^2)\theta_1}{8\theta_0}\varepsilon + \mathcal{O}(\varepsilon)^2. \tag{100}$$

Now rewrite θ_1 (the first-order correction to the image position) using (70), with $A_1 = 4$ and $A_2 = 15\pi/4$ for the Schwarzschild metric, to obtain

$$\frac{\tau}{\tau_E} = \frac{1}{2}\left[1 + \beta^2 - \theta_0^2 - \ln\left(\frac{d_L \theta_0^2 \vartheta_E^2}{4 d_{LS}}\right)\right] + \frac{15\pi}{16\theta_0}\varepsilon + \mathcal{O}(\varepsilon)^2. \tag{101}$$

Though not yet apparent, the zeroth-order term in (101) reduces to the familiar lensing time delay in the weak-deflection limit of general relativity. Rearranging, we can write the zeroth-order term as

$$\tau = \frac{d_L d_S}{c d_{LS}}\left[\frac{(\vartheta_0 - \mathcal{B})^2}{2} - \vartheta_E^2 \ln\vartheta_0\right] + C, \tag{102}$$

where we have used the identity $\beta = \theta_0 - \theta_0^{-1}$ from (66), and we have also used the definitions $\theta_0 = \vartheta_0/\vartheta_E$ and $\beta = \mathcal{B}/\vartheta_E$ from (62). The "constant" term C in this expression is independent of ϑ_0 and $\mathcal{B}$; it depends on ϑ_E and the distances as $C = \tau_E[1 + \ln(d_L/4d_{LS})]/2$.

Finally, to make contact with conventional calculations we examine the differential time delay $\Delta\tau = \tau_- - \tau_+$ between the positive-parity primary image and the negative-parity secondary image. This can be written as

$$\Delta\tau = \Delta\tau_0 + \varepsilon\Delta\tau_1 + \mathcal{O}(\varepsilon)^2, \tag{103}$$

where

$$\Delta\tau_0 = \tau_E\left[\frac{(\theta_0^-)^{-2} - (\theta_0^+)^{-2}}{2} - \ln\left(\frac{\theta_0^-}{\theta_0^+}\right)\right], \tag{104}$$

$$\Delta\tau_1 = \tau_E \frac{15\pi}{16}\frac{(\theta_0^+ - \theta_0^-)}{\theta_0^+\theta_0^-}. \tag{105}$$

It is possible to derive the second-order correction terms in Eqs. (101) and (103), but they are more complicated. Since we have already found the first correction terms for both absolute and differential time delays, it seems unnecessary to write down the higher-order terms.

C. Time delay via the PPN metric

To compute $T(R)$ for the PPN approach, we take the metric functions from Eqs. (26) and (27), use (42) to relate b to r_0, change integration variables to $x = r_0/w$, and finally carry out the integration to find

$$T(R) = \sqrt{R^2 - r_0^2} + h r_0\left[a_1\frac{\sqrt{1-\xi^2}}{1+\xi} + (a_1+b_1)\ln\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right)\right] + h^2 r_0\left[\left(4a_1^2 - 2a_2 + 2a_1b_1 - \frac{1}{2}b_1^2 + 2b_2\right)\left(\frac{\pi}{2} - \sin^{-1}\xi\right) - a_1\left(a_1 + b_1 + \frac{3}{2}a_1\xi + b_1\xi\right)\frac{\sqrt{1-\xi^2}}{2(1+\xi)^2}\right] + h^3 r_0\left[F(\xi)\frac{\sqrt{1-\xi^2}}{2(1+\xi)^2} - a_1\left(4a_1^2 - 2a_2 + 2a_1b_1 - \frac{1}{2}b_1^2 + 2b_2\right)\left(\frac{\pi}{2} - \sin^{-1}\xi\right)\right] + \mathcal{O}(h)^4, \tag{106}$$

where $\xi = r_0/R$. Here $F(\xi)$ is a cubic polynomial in ξ, which is long and not particularly enlightening so we have not written it out (but we do write complete expressions in what follows).

As in the Schwarzschild case, it is instructive to convert from the distance of closest approach r_0 to the impact parameter b [now using Eq. (44)], and then expand in both $\mathsf{m}_\bullet/b$ and b/R to find

$$\frac{T(R)}{R} = 1 - \left\{\frac{1}{2}\frac{b}{R}\left[\frac{b}{R} - 2a_1\frac{\mathsf{m}_\bullet}{b} + 2(a_1+b_1)\frac{\mathsf{m}_\bullet}{b}\ln\left(\frac{1}{2}\frac{b}{R}\right)\right]\right\} + \left\{(8a_1^2 - 4a_2 + 4a_1b_1 - b_1^2 + 4b_2)\frac{\pi}{4}\frac{b}{R}\left(\frac{\mathsf{m}_\bullet}{b}\right)^2\right\} + \cdots. \tag{107}$$

However, again we need to be more careful to obtain a rigorous series expansion of the time delay. Repeating the analysis discussed around Eqs. (98)–(100), we obtain

$$\frac{\tau}{\tau_E} = \frac{1}{2}\left[a_1 + \beta^2 - \theta_0^2 - \frac{a_1+b_1}{2}\ln\left(\frac{d_L\theta_0^2\vartheta_E^2}{4d_{LS}}\right)\right] + \frac{1}{8\theta_0}[(8a_1^2 - 4a_2 + 4a_1b_1 - b_1^2 + 4b_2)\pi - 4(a_1 + b_1 + 2\theta_0^2)\theta_1]\varepsilon + \mathcal{O}(\varepsilon)^2. \tag{108}$$

We can rewrite θ_1 (the first-order correction to the image position) using (70), with A_1 and A_2 given by (48) and (49), to obtain

$$\frac{\tau}{\tau_E} = \frac{1}{2}\left[a_1 + \beta^2 - \theta_0^2 - \frac{a_1 + b_1}{2}\ln\left(\frac{d_L\theta_0^2\vartheta_E^2}{4d_{LS}}\right)\right] + \frac{\pi}{16\theta_0} \times (8a_1^2 - 4a_2 + 4a_1b_1 - b_1^2 + 4b_2)\varepsilon + \mathcal{O}(\varepsilon)^2. \tag{109}$$

Finally, the differential time delay between the positive- and negative-parity images is

$$\Delta\tau = \Delta\tau_0 + \varepsilon\Delta\tau_1 + \mathcal{O}(\varepsilon)^2, \tag{110}$$

where

$$\Delta\tau_0 = \tau_E\left[\frac{(\theta_0^-)^{-2} - (\theta_0^+)^{-2}}{2} - \frac{a_1 + b_1}{2}\ln\left(\frac{\theta_0^-}{\theta_0^+}\right)\right], \tag{111}$$

$$\Delta\tau_1 = \tau_E\frac{\pi}{16}(8a_1^2 - 4a_2 + 4a_1b_1 - b_1^2 + 4b_2)\frac{(\theta_0^+ - \theta_0^-)}{\theta_0^+\theta_0^-}. \tag{112}$$

VI. APPLICATION TO THE GALACTIC BLACK HOLE

To illustrate our results, we consider gravitational lensing by the supermassive black hole at the center of our Galaxy. The black hole has a mass of $M_\bullet = (3.6 \pm 0.2) \times 10^6 M_\odot$ [33] and is located at a distance of $d_L = 7.9 \pm 0.4$ kpc [34] from Earth. (For illustration purposes, we adopt the nominal values and neglect the small uncertainties.) Its gravitational radius is therefore $\mathsf{m}_\bullet = 1.1 \times 10^{12}$ cm $= 3.4 \times 10^{-7}$ pc, which corresponds to an angle of $\theta_\bullet = 9.0$ μarc s (microarcseconds).

Suppose the source lies a distance d_{LS} beyond the black hole. Typical distances will be $d_{LS} \sim 1$–100 pc, so we have $d_L \approx d_S \gg d_{LS} \gg \mathsf{m}_\bullet$ which means that both the source and the observer lie in the asymptotically flat regime of the spacetime, confirming assumption A2. The angular Einstein radius is $\vartheta_E = 0.068(d_{LS}/10\text{ pc})^{1/2}$ arc s, which is much larger than $\theta_\bullet$. Since the primary and secondary images both lie near the Einstein radius, each has a light ray with an impact parameter and distance of closest approach that are much larger than the gravitational radius, confirming that part of assumption A3. Put another way, our dimensionless expansion parameter is

$$\varepsilon = \frac{\theta_\bullet}{\vartheta_E} = 1.3 \times 10^{-4}\left(\frac{d_{LS}}{10\text{ pc}}\right)^{-1/2}, \tag{113}$$

which is small enough to justify working with series expansions in ε. The natural lensing time scale is $\tau_E = 71$ s [independent of the distances; see Eq. (99)].

We obviously want to compute the corrections to lensing observables for the Schwarzschild metric in general relativity. For comparison, we also consider three other cases: a charged black hole described by the Reissner-Nordström metric in general relativity, with charge parameter $q = 0.5$; and a naked singularity described by either the Reissner-Nordström metric of general relativity or the GMGHS metric of heterotic string theory (modified gravity), with "charge" parameter $q = 2.5$. (A charged black hole described by the GMGHS metric with $q = 0.5$ is very similar to the Reissner-Nordström case with $q = 0.5$, so we do not show it.) Note that all four cases are from two gravity theories in which the coefficient of the leading term in the expansion of the bending angle is $A_1 = 4$.

Figures 2 and 3 show the corrections to the individual image positions and magnifications, as a function of the scaled angular source position β, for the four sample cases. The scaled first-order correction term for the image position θ_1 is of order unity, so the full correction to the image position is of order $\varepsilon\vartheta_E \sim 10$ μas. The correction is small, but detectable with high-resolution radio interferometry; and it should be readily measurable with planned

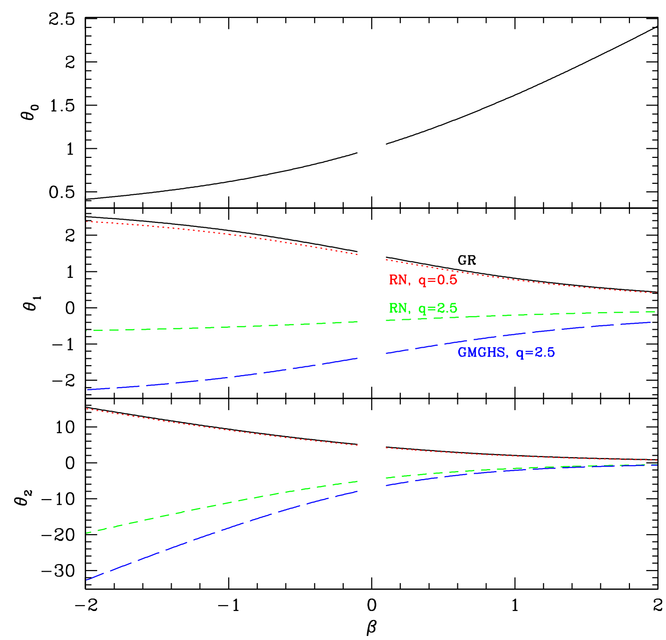

FIG. 2 (color online). Terms in the series expansion (63) for the angular image position, as a function of the angular source position. Recall that $\beta > 0$ corresponds to the positive-parity image while $\beta < 0$ corresponds to the negative-parity image. Sources very close to the origin $|\beta| < 0.1$ are not shown. (Top panel) The zeroth-order image position θ_0, which is the same for all gravity theories. (Middle panel) The first-order term θ_1; recall that the full correction term is $\varepsilon\theta_1$. Solid curve: Schwarzschild metric in general relativity. Dotted and short-dashed curves: Reissner-Nordström metric in general relativity, with charge parameter $q = 0.5$ and 2.5, respectively. Long-dashed curve: GMGHS metric in string theory (modified gravity), with charge parameter $q = 2.5$. (The GMGHS case with $q = 0.5$ is very similar to the Reissner-Nordström case with $q = 0.5$.) (Bottom panel) The second-order term θ_2. Again, the full correction term is $\varepsilon^2\theta_2$. All angular lengths are in units of the angular Einstein radius ϑ_E. For the galactic black hole, $\vartheta_E = 0.068(d_{LS}/10\text{ pc})^{1/2}$ arc s and the dimensionless expansion parameter is $\varepsilon = 1.3 \times 10^{-4} \times (d_{LS}/10\text{ pc})^{-1/2}$.

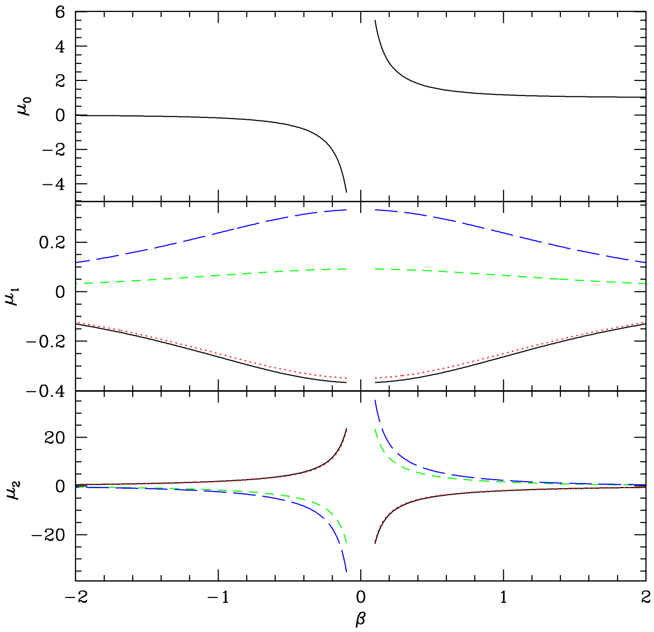

FIG. 3 (color online). Similar to Fig. 2, but showing the terms in the series expansion (76) of the individual image magnifications. Again recall that the full first- and second-order correction terms are $\varepsilon\mu_1$ and $\varepsilon^2\mu_2$, respectively.

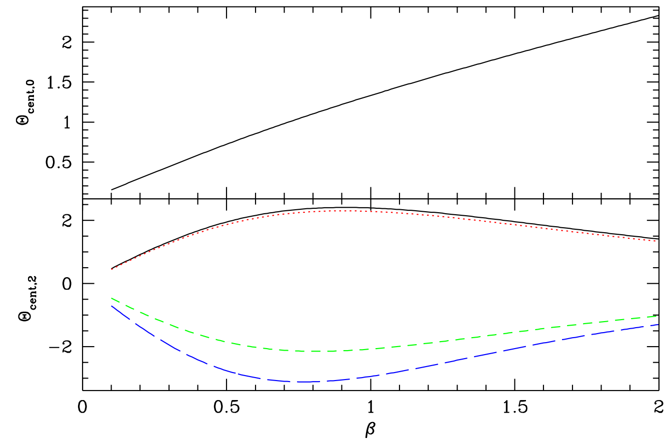

FIG. 5 (color online). Similar to Fig. 4, but showing the zeroth- and second-order terms in the series expansion (88) of the magnification-weighted centroid position, in units of ϑ_E. Again, for our sample gravity theories having $A_1 = 4$, the first-order correction term vanishes identically and is not shown.

microarcsecond-resolution missions such as MAXIM [26]. While the difference between neutral (Schwarzschild) and charged (Reissner-Nordström) black hole cases is fairly small, the naked singularity case stands out for having the opposite sign (see Sec. IV B). In other words, it will be challenging but not impossible to detect the corrections to the usual weak-deflection lensing. If the corrections can be detected, high precision will be needed to distinguish between different kinds of black holes. However, it will be easy to rule out certain kinds of naked singularities observationally.

Figures 4 and 5 show the corrections to the total magnification and magnification-weighted centroid position. The corrections to these observables are much smaller—the centroid correction is of order $\varepsilon^2\vartheta_E \sim 10^{-9}$ arc s—because the first-order terms vanish in gravity theories with $A_1 = 4$. Although interesting, these quantities will be very challenging to use in the near future for realistic observational tests of gravity theories.

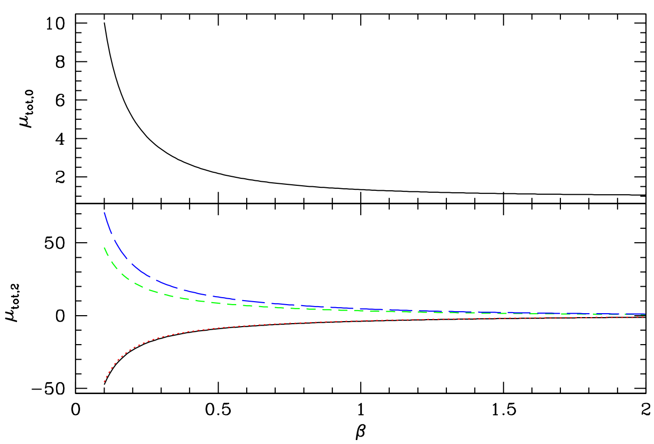

FIG. 4 (color online). Zeroth- and second-order terms in the series expansion (85) of the total magnification. Our examples involve gravity theories with $A_1 = 4$, so the first-order correction term vanishes identically and is not shown. For each $\beta > 0$, we find the positive-parity image using the source position β, and the negative-parity image using the source position $-\beta$, and then combine them to obtain the total magnification. The line types are the same as in Fig. 2.

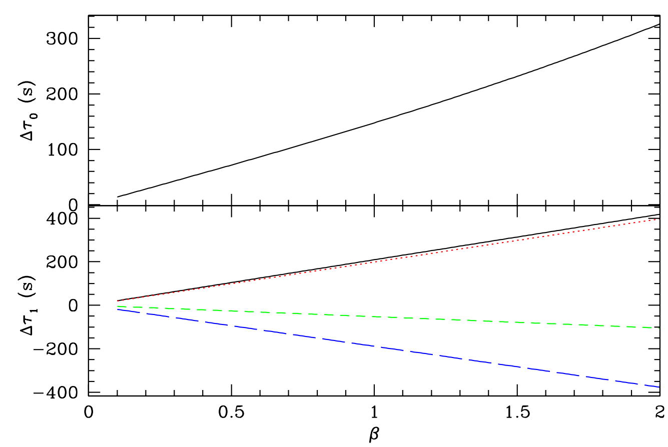

FIG. 6 (color online). Zeroth- and first-order terms in the series expansion (110) of the differential time delay between the two images. Note that we have defined $\Delta\tau_0$ and $\Delta\tau_1$ to have dimensions of time, so they are expressed in seconds. Recall that the full first-order correction is $\varepsilon\Delta\tau_1$, where the expansion parameter is $\varepsilon = 1.3 \times 10^{-4} \times (d_{LS}/10\ \mathrm{pc})^{-1/2}$ for the galactic black hole.

Finally, Fig. 6 shows the corrections to the differential time delay between the positive- and negative-parity images. Remarkably, the first-order correction $\varepsilon\Delta\tau_1$ is predicted to be as large as a few hundredths of a second—which would be relatively easy to measure if we could find a pulsar lensed by the galactic black hole. As with the image positions, distinguishing between charged and neutral black holes would require higher precision (not unfeasible in the case of a pulsar source), but ruling out certain kinds of naked singularities would be easy because they lead to corrections with the opposite sign. Although we have not explicitly computed the second-order corrections, we can estimate that they would be of order $\varepsilon^2\tau_E \sim 10^{-6}$ s.

VII. CONCLUSIONS

We have introduced a rigorous and general framework for computing corrections to standard weak-deflection lensing observables. In this paper we have presented a formalism for handling any static, spherically symmetric theory of gravity in which corrections to the weak-deflection regime can be expanded as a Taylor series in the gravitational radius of the compact body acting as a gravitational lens. Conceptually, three points distinguish our framework from previous studies. First, we take care to avoid coordinate dependence by expressing our results in terms of invariant quantities. Second, we go beyond the bending angle to study quantities that are directly observable for extra-solar lensing studies, including the positions, magnifications, and time delays of the lensed images. Third, our general approach allows us to unify the diverse results that have been presented before.

Besides the framework itself, our main results are series expansions for the lensed image positions and magnifications that are accurate to second order in $\varepsilon = \vartheta_{\bullet}/\vartheta_E$, or the ratio of the angle subtended by the lens's gravitational radius to the (weak-deflection) angular Einstein radius, as well as series expansions for the lensing time delays with corrections at first order in ε. The signs of the first-order corrections are determined by the sign of the coefficient A_2 in the series expansion of the light bending angle. If $A_2 > 0$, both the positive- and negative-parity images are shifted away from the lens, and the positive-parity image gets fainter while the negative-parity image gets brighter. If $A_2 < 0$, the opposite occurs.

The sign of A_2 appears to be connected with the possible existence of naked singularities. In the sample gravity theories we have considered—namely the Schwarzshild and Reissner-Nordström metrics in general relativity, and the GMGHS metric in heterotic string theory—A_2 can be negative only if naked singularities exist. This connection between hypothetical naked singularities and observable lensing quantities is exciting because it offers the possibility that (certain kinds of) naked singularities can be ruled out observationally rather than just by the still-unproved cosmic censorship conjecture. Note that we must use the qualifying phrase "certain kinds of" because not all naked singularities lead to $A_2 < 0$. In these less extreme cases, we must go beyond a gross feature like the sign of A_2 and consider the extent to which fine, quantitative constraints on A_2 from realistic data could rule out all or at least most interesting types of naked singularities.

It has been known that for the Schwarzschild metric the first-order corrections to the total magnification and magnification-weighted centroid vanish (see [19]). We have shown that this result is not generic: it depends on precise cancellations that can occur only for gravity theories in which the leading-order term in the series expansion of the bending angle has the value $A_1 = 4$. In practice, A_1 does equal 4 in the sample gravity theories we have considered, and it is constrained by observational data to be quite close to 4 (e.g.,[27]). Nevertheless, it is important to understand why these particular first-order corrections vanish, and to recognize that it is possible to devise gravity theories for which that is not the case.

We have applied our formalism to lensing by the galactic black hole. We predict the corrections to the image positions to be at the level of 10 μas, and the correction to the time delay between the images to be a few hundredths of a second. The position corrections would be measurable today with radio interferometry, if we could find a radio source that is lensed by the galactic black hole. The time delay correction would be measurable if the source were a pulsar. Even if such a convenient source cannot be found, the corrections should be measurable with planned missions such as MAXIM [26].

ACKNOWLEDGMENTS

This work was supported by NSF Grants No. DMS-0302812, No. AST-0434277, and No. AST-0433809. A. O. P. would like to acknowledge the hospitality of the American Institute of Mathematics for hosting a workshop on Kerr black holes, where part of this work was conducted. He enthusiastically thanks the participants, which included researchers from Germany, Italy, Russia, and the U.S.A., for informative presentations and discussions on the state of the art in the field of black hole lensing. Volker Perlick is also acknowledged for emphasizing the importance of using the terms "weak-deflection" and "strong-deflection" as a precise way to describe the various limits in black hole lensing.

[1] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer, Berlin, 1992).
[2] A. O. Petters, H. Levine, and J. Wambsganss, *Singularity Theory and Gravitational Lensing* (Birkhauser, Boston, 2001).
[3] C. S. Kochanek, P. Schneider, and J. Wambsganss, in "Gravitational Lensing: Strong, Weak, and Micro. Lecture Notes of the 33rd Saas-Fee Advanced Course," edited by G. Meylan, P. Jetzer, and P. North (Springer-Verlag, Berlin, to be published).
[4] C. Darwin, Proc. R. Soc. A **249**, 180 (1958); **263**, 39 (1958).
[5] R. D. Atkinson, Astron. J. **70**, 517 (1965).
[6] J.-P. Luminet, Astron. Astrophys. **75**, 228 (1979).
[7] H. Ohanian, Am. J. Phys. **55**, 428 (1987).
[8] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, Astron. Astrophys. **337**, 1 (1998); K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D **62**, 084003 (2000).
[9] S. Frittelli and E. T. Newman, Phys. Rev. D **59**, 124001 (1999); S. Frittelli, T. P. Kling, and E. T. Newman, Phys. Rev. D **61**, 064021 (2000).
[10] V. Perlick, Commun. Math. Phys. **220**, 403 (2001); Phys. Rev. D **69**, 064017 (2004).
[11] V. Bozza, S. Capozziello, G. Iovane, and G. Scarpetta, Gen. Relativ. Gravit. **33**, 1535 (2001); V. Bozza, Phys. Rev. D **66**, 103001 (2002); **67**, 103006 (2003); V. Bozza and L. Mancini, Gen. Relativ. Gravit. **36**, 435 (2004); V. Bozza, F. De Luca, G. Scarpetta, and M. Sereno, Phys. Rev. D **72**, 083003 (2005).
[12] E. F. Eiroa, G. E. Romera, and D. F. Torres, Phys. Rev. D **66**, 024010 (2002).
[13] A. Bhadra, Phys. Rev. D **67**, 103009 (2003).
[14] A. S. Majumdar and N. Mukherjee, astro-ph/0403405; E. F. Eiroa, Phys. Rev. D **71**, 083010 (2005); R. Whisker, Phys. Rev. D **71**, 064004 (2005).
[15] A. S. Majumdar and N. Mukherjee, Int. J. Mod. Phys. D **14**, 1095 (2005).
[16] A. O. Petters, Mon. Not. R. Astron. Soc. **338**, 457 (2003).
[17] V. Bozza and L. Mancini, Astrophys. J. **627**, 790 (2005).
[18] R. Epstein and I. Shapiro, Phys. Rev. D **22**, 2947 (1980); E. Fishbach and B. Freeman, Phys. Rev. D **22**, 2950 (1980); G. Richter and R. Matzner, Phys. Rev. D **26**, 1219 (1982); **26**, 2549 (1982).
[19] J. Ebina, T. Osuga, H. Asada, and M. Kasai, Prog. Theor. Phys. **104**, 1317 (2000); G. F. Lewis and X. R. Wang, Prog. Theor. Phys. **105**, 893 (2001).
[20] M. Sereno, Phys. Rev. D **69**, 023002 (2004).
[21] A. F. Sarmiento, Gen. Relativ. Gravit. **14**, 793 (1982).
[22] S. Kar and M. Sinha, Gen. Relativ. Gravit. **35**, 1775 (2003).
[23] S. Pireaux, Classical Quantum Gravity **21**, 1897 (2004); **21**, 4317 (2004).
[24] J.-M. Gérard and S. Pireaux, gr-qc/9907034; J. Bodener and C. Will, Am. J. Phys. **71**, 770 (2003).
[25] R. Penrose, Riv. Nuovo Cimento Numero speciale **1**, 252 (1969); R. Penrose, in *Sixth Texas Symposium on Relativistic Astrophysics*, edited by Dennis J. Hegyi, Annals of the New York Academy of Sciences Vol. 224 (The New York Academy of Sciences, New York, 1973), p. 125; J. Astrophys. Astron. **20**, 233 (1999).
[26] See http://maxim.gsfc.nasa.gov
[27] S. S. Shapiro, J. L. Davis, D. E. Lebach, and J. S. Gregory, Phys. Rev. Lett. **92**, 121101 (2004); G. Will, Living Rev. Relativity **4**, 4 (2001), www.livingreviews.org/Articles/volume4/2001-4will
[28] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[29] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[30] B. Punsly, Astrophys. J. **498**, 640 (1998).
[31] G. W. Gibbons, Nucl. Phys. **B207**, 337 (1982); G. W. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988); D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991); **45**, 3888 (1992).
[32] R. E. Bellman, *Perturbation Techniques in Mathematics, Physics, and Engineering* (Hold, Rinehart and Winston, New York, 1964).
[33] A. M. Ghez, S. Salim, S. D. Hornstein, A. Tanner, J. R. Lu, M. Morris, E. E. Becklin, and G. Duchêne, Astrophys. J. **620**, 744 (2005).
[34] F. Eisenhauer, R. Schödel, R. Genzel, T. Ott, M. Tecza, R. Abuter, A. Eckart, and T. Alexander, Astrophys. J. **597**, L121 (2003).