Precise bounds on the Higgs boson mass

P. Kielanowski*

Departamento de Física, Centro de Investigación y Estudios Avanzados del IPN, Mexico

S. R. Juárez W.† and H. G. Solís-Rodríguez‡

Departamento de Física, Escuela Superior de Física y Matemáticas, IPN, Mexico

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We study the renormalization group evolution of the Higgs quartic coupling λ_H. The one loop equation for λ_H is nonlinear and it is of the Riccati type which we analytically and numerically solve in the energy range $[m_t, E_{GU}]$ where m_t is the mass of the top quark and $E_{GU} = 10^{14}$ GeV. We find that depending on the value of $\lambda_H(m_t)$ the solution for $\lambda_H(E)$ may have a singularity or a zero and become negative in the former energy range so the ultraviolet cutoff of the standard model should be below or equal to the energy where the zero or singularity of λ_H occurs. We then numerically solve the two loop renormalization group equation for λ_H and compare it with the one loop solution. We find that the two loop running of λ_H is very sensitive to the evolution of the top quark Yukawa coupling Y_t. This implies a strong dependence on the top quark mass m_t and suggests that the choice of m_t as the renormalization point, that we use, reduces theoretical errors. We find that in the approximation of one loop for $0.397 \leq \lambda_H(m_t) \leq 0.618$ the standard model is valid in the whole range $[m_t, E_{GU}]$ while for two loops the bound is $0.368 \leq \lambda_H(m_t) \leq 0.621$. From the properties of λ_H we then study the predictions for the Higgs mass. We use the effective potential to derive the relation between the Higgs mass and λ_H and obtain that this relation is not very sensitive to the particular choice of the effective potential but for the large Higgs masses the two loop corrections are significant. We determine that the standard model is valid in the whole range $[m_t, E_{GU}]$ for the Higgs masses $153.5 \leq M_H \leq 191.1$ for one loop case and $148.5 \leq M_H \leq 193.1$ for two loops. The pattern of the behavior of $\lambda_H(E)$ for different values of $\lambda_H(m_t)$ indicates the existence of a phase transition in the standard model for $\lambda_H(m_t) = 0.5$ which corresponds to the value of the Higgs mass $M_H = m_t$.

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I. INTRODUCTION

The standard model (SM) provides a very precise description of all the present elementary particle data [1]. On the other hand it has relatively many free parameters (~ 19) what is rather unsatisfactory from the fundamental point of view. The idea of grand unification (GU) [2] is to look for additional symmetries in the SM at very high energies. The most notable sign of the presence of GU is the (approximate) convergence of the three gauge couplings to one common value at the energies 10^{14}–10^{15} GeV. This allows to substitute the gauge group $SU(3) \times SU(2) \times U(1)$ of the SM by only one group and to reduce the number of gauge couplings to one.

The main tool of the GU models are the renormalization group equations (RGE) [3] that relate various observables (like couplings or masses) at different energies and also allow the study of their asymptotic behavior.

In the perturbative quantum field theory the RGE are differential equations for the observables which are obtained from the condition that the S-matrix elements do not depend on the renormalization scheme or renormalization point. The right hand side of the RGE is an infinite series expanded according to the number of loops. Most of the numerical and analytical results for the RGE [4] are to the order of one loop only possibly with a partial inclusion of two loops, while the RGE for most of the observables have been given for the SM and its extensions up to two loops [4–6].

The right hand side of the RGE is constructed from the following terms:

$$g_l^2, \quad y_u y_u^\dagger, \quad y_d y_d^\dagger, \quad y_l y_l^\dagger, \quad y_\nu y_\nu^\dagger, \quad \lambda_H, \tag{1}$$

where the g_l's are the gauge couplings, y_u, y_d, y_l, y_ν, are the Yukawa couplings of the up and down quarks, charged leptons and neutrinos, respectively, and λ_H is the Higgs quartic coupling. The RGE form a set of nonlinear coupled differential equations and even at the one loop order there exist only approximate or numerical solutions [4,7].

The one loop RGE for the best measured observables g_l's, quark and lepton Yukawa couplings and the Cabibbo-Kobayashi-Maskawa (CKM) matrix are independent of the Higgs quartic coupling. This allows one to derive the running of those observables at the lowest order without the knowledge of the λ_H. On the other hand, at the two loops level, the quartic coupling λ_H appears in the RGE for many observables like the quark masses or the CKM matrix and it has an important influence on their behavior and therefore cannot be neglected.

The one loop equation for λ_H is also nonlinear and has been used to obtain the limits on the Higgs mass from the

*Electronic address: kiel@physics.utexas.edu
†Electronic address: rebeca@esfm.ipn.mx
‡Electronic address: hgabriel@esfm.ipn.mx

triviality condition of the $\lambda\phi^4$ theory and the existence of the Landau pole. This equation has been also considered in Refs. [8–10] to study the dependence of the Higgs mass and the UV cutoff on the energy and it was solved for the simplified case when the gauge couplings and the top quark Yukawa coupling are constant.

In this paper we study the one and two loop equation for λ_H without any simplifying assumptions in the energy range, starting at the top quark mass m_t. We find that the one loop equation is of the Riccati type and we solve this equation explicitly. We find that for the values of λ_H at the top quark mass, $\lambda_H(m_t) \geq 0.618$, the function $\lambda_H(E)$ has a Landau singularity. For the values of $\lambda_H(m_t) \leq 0.397$ there is no Landau pole below the energies E_{GU} and the solution $\lambda_H(E)$ passes through zero and then becomes negative. This means that for the latter values of $\lambda_H(m_t)$ the theory becomes unstable and the UV cutoff should appear below the energy value corresponding to the zero of $\lambda_H(E)$. As is well known the coupling λ_H is related to the Higgs mass, so our results are also presented in terms of the Higgs mass.

To estimate the precision of the one loop RGEs we numerically analyze the two loop equations for all the observables. We find that the inclusion of two loops has the most significant influence on the top Yukawa coupling Y_t and on the Higgs quartic coupling λ_H.

We present the range of validity of the standard model as a function of the Higgs quartic coupling λ_H and the physical Higgs mass M_H. We find that the standard model is valid up to the energy of the grand unification for the physical Higgs mass in the range $148.5\text{ GeV} < M_H < 193.1\text{ GeV}$. The discovery of the Higgs mass in this range would be a strong argument in favor of the idea of grand unification at the energy $\sim 10^{14}$ GeV.

Finally we analyze the pattern of evolution of the Higgs quartic coupling for the different initial values $\lambda_H(m_t)$. We find that there are two patterns of evolution: one for $\lambda_H(m_t) < 0.5$ and the other for $\lambda_H(m_t) > 0.5$. These results demonstrate that there is a sharp transition in the behavior of the standard model at $\lambda_H(m_t) = 0.5$ which corresponds to the Higgs mass $M_H = m_t$. This may be an indication of the phase transition in the standard model.

II. ONE AND TWO LOOP RENORMALIZATION GROUP EQUATIONS

The two loop RGE are the following

$$\frac{dg_l}{dt} = \frac{1}{(4\pi)^2} b_l g_l^3 - \frac{1}{(4\pi)^4} G_l g_l^3, \qquad l = 1, 2, 3, \tag{2a}$$

$$\frac{dy_{u,d,e,\nu}}{dt} = \left[\frac{1}{(4\pi)^2}\beta^{(1)}_{u,d,e,\nu} + \frac{1}{(4\pi)^4}\beta^{(2)}_{u,d,e,\nu}\right] y_{u,d,e,\nu}, \tag{2b}$$

$$\frac{d\lambda_H}{dt} = \left[\frac{1}{(4\pi)^2}\beta^{(1)}_{\lambda_H} + \frac{1}{(4\pi)^4}\beta^{(2)}_{\lambda_H}\right], \tag{2c}$$

$$\frac{dm^2}{dt} = \left[\frac{1}{(4\pi)^2}\beta^{(1)}_{m^2} + \frac{1}{(4\pi)^4}\beta^{(2)}_{m^2}\right], \tag{2d}$$

where $t = \ln(E/m_t)$, E is the energy, m_t is the top quark mass and the Higgs potential is $m^2\phi^\dagger\phi + (\lambda_H/2)(\phi^\dagger\phi)^2$. In Eqs. (2) b_l's are constant and G_l, $\beta^{(1)}_{u,d,e,\nu}$, $\beta^{(2)}_{u,d,e,\nu}$, $\beta^{(1)}_{\lambda_H}$, $\beta^{(2)}_{\lambda_H}$, $\beta^{(1)}_{m^2}$, $\beta^{(2)}_{m^2}$ are functions of the standard model couplings and the squares of the Yukawa couplings $H^{(1)}_{u,d,e,\nu} = y_{u,d,e,\nu}y^\dagger_{u,d,e,\nu}$, (for the definition of these functions and constants see [4] or [7]).

In the previous papers [7], we have discussed a consistent approximation scheme for the solution of the RGE that was based on the expansion of the solutions in terms of the powers of λ, where $\lambda \simeq 0.22$ is the absolute value of the $|V_{us}|$ element of the CKM matrix.

In such an approximation the lowest order RGE have the following form [5,6,11]

$$\frac{dg_l}{dt} = \frac{1}{(4\pi)^2} b_l g_l^3, \tag{3a}$$

$$\frac{dy_u}{dt} = \frac{1}{(4\pi)^2}\{\alpha_1^u(t) + \alpha_2^u y_u y_u^\dagger + \alpha_3^u \mathrm{Tr}(y_u y_u^\dagger)\} y_u, \tag{3b}$$

$$\frac{dy_d}{dt} = \frac{1}{(4\pi)^2}\{\alpha_1^d(t) + \alpha_2^d y_u y_u^\dagger + \alpha_3^d \mathrm{Tr}(y_u y_u^\dagger)\} y_d, \tag{3c}$$

$$\frac{d\lambda_H}{dt} = \frac{12}{(4\pi)^2}\left\{\lambda_H^2 + \left[\mathrm{Tr}(y_u y_u^\dagger) - \frac{3}{4}\left(\frac{1}{5}g_1^2 + g_2^2\right)\right]\lambda_H + \frac{3}{16}\left(\frac{3}{25}g_1^4 + \frac{2}{5}g_1^2 g_2^2 + g_2^4\right) - \mathrm{Tr}(y_u y_u^\dagger)^2\right\}, \tag{3d}$$

$$\frac{d\ln m^2}{dt} = \frac{1}{(4\pi)^2}\left\{6\lambda_H + 6Tr(y_u y_u^\dagger) - \frac{9}{2}\left(\frac{1}{5}g_1^2 + g_2^2\right)\right\}. \tag{3e}$$

The constants b_i and α's in Eqs. (3) are equal

$$(b_1, b_2, b_3) = \left(\frac{41}{10}, -\frac{19}{6}, -7\right),$$

$$\alpha_1^u(t) = -\left(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right), \qquad \alpha_2^u = \frac{3}{2}b,$$

$$\alpha_3^u = 3, \qquad \alpha_1^d(t) = -\left(\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right),$$

$$\alpha_2^d = \frac{3}{2}c, \qquad \alpha_3^d = 3a, \qquad (a, b, c) = (1, 1, -1).$$

Equations (3a)–(3c) can be *explicitly* solved and the most important results and properties of these solutions are [7]

(1) g_l's, y_u and y_d are all regular functions of energy in the range $[m_t, E_{GU}]$.

(2) The running of the gauge couplings $g_l(t)$ is ($t_0 = \ln(E/m_t)|_{E=m_t} = 0$)

$$(g_l(t))^2 = \frac{(g_l(t_0))^2}{1 - \frac{2}{(4\pi)^2}(g_l(t_0))^2 b_l(t - t_0)}. \tag{4}$$

(3) The running of the up quark Higgs couplings $y_u(t)$ has the following property:

$$\mathrm{Tr}\,(y_u y_u^\dagger) = Y_t^2(t) = \frac{Y_t^2(t_0) r(t)}{1 - \frac{2(\alpha_2^u + \alpha_3^u)}{(4\pi)^2} Y_t^2(t_0) \int_{t_0}^{t} r(\tau) d\tau} \tag{5}$$

where Y_t is the largest eigenvalue of the up quark Higgs coupling matrix y_u and $r(t) = \exp((2/(4\pi)^2) \times \int_{t_0}^{t} \alpha_1^u(\tau) d\tau) = \prod_{k=1}^{3} [g_k^2(t_0)/g_k^2(t)]^{c_k/b_k}$, $c_k = (17/20, 9/4, 8)$. Using Eqs. (4) and (5) as input into Eq. (3d) we obtain the uncoupled differential equation for the quartic coupling constant λ_H.

Equation (3d) for λ_H has been considered earlier by various authors [8–10] but in all these papers the effects of the running of the gauge couplings and of Y_t^2 have not been considered. The importance of λ_H for the evolution of other observables comes from the fact that λ_H appears at the two loops order in the RGE for y_u and y_d and at the one loop order for m_H.

III. ONE LOOP EQUATION FOR λ_H

The one loop equation for λ_H given in Eq. (3d) is rewritten in the form

$$\frac{d\lambda_H}{dt} = f_0(t) + f_1(t)\lambda_H + f_2(t)\lambda_H^2, \tag{6}$$

where the definition of the functions $f_i(t)$ can be deduced from Eq. (3d). This equation is of the Riccati type [12]. The behavior of the gauge coupling g_l's is given in Eq. (4) and the explicit energy dependence of $\mathrm{Tr}(y_u y_u^\dagger)$ is given in Eq. (5). As discussed before the g_l's and $\mathrm{Tr}(y_u y_u^\dagger)$, as functions of energy, have no singularities in the range $[m_t, E_{GU}]$. On the other hand the solutions of the Riccati's equations can become singular even if the coefficients of the equation are smooth and regular functions.

The solution of Eq. (6) is obtained by substituting the λ_H by the following expression containing the auxiliary function $W(t)$:

$$\lambda_H(t) = -\frac{1}{f_2(t)} \frac{W'(t)}{W(t)} \tag{7}$$

which fulfills the linear second order differential equation

$$W'' - \left(\frac{f_2'(t)}{f_2(t)} + f_1(t)\right) W' + f_0(t) f_2(t) W = 0. \tag{8}$$

Any solution of Eq. (8) generates the solutions of Eq. (6). Equation (8) is of the Frobenius type [13] and the solution $W(t)$ is a *regular* function of energy t in the region where the coefficients of Eq. (8) are *regular*. One can look for the solutions of this equation in terms of an infinite series. We look for the two solutions of this equation with the following properties:

$$\begin{aligned} W_1(t)|_{t_0} &= 1, & W_1'(t)|_{t_0} &= 0, \\ W_2(t)|_{t_0} &= 0, & W_2'(t)|_{t_0} &= 1. \end{aligned} \tag{9}$$

The solution of (6) for λ_H in terms of the functions $W_1(t)$ and $W_2(t)$ has the following form [note that $f_2(t) = 12/(4\pi)^2$]

$$\lambda_H(t) = -\frac{(4\pi)^2}{12} \frac{W_1'(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2'(t)}{W_1(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2(t)}. \tag{10}$$

The most important property of the solution (10) is that the singularities (simple poles) of the solution $\lambda_H(t)$ are determined from the zeros of the denominator

$$W_1(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2(t) = 0 \tag{11}$$

and the zeros of $\lambda_H(t)$ are determined from the zeros of the numerator

$$W_1'(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2'(t) = 0. \tag{12}$$

Using Eqs. (11) and (12) one can precisely determine the position of the singularities and zeros and their dependence on the initial value of the Higgs quartic coupling $\lambda_H(t_0)$. The detailed discussion of the solutions is given in the next section.

IV. RUNNING OF λ_H

A. One loop case

In this section we will discuss the explicit solutions of Eqs. (3d) and (8). Let us start with Eq. (8). The form of the functions $-(f_2'(t)/f_2(t) + f_1(t))$ and $f_0(t)f_2(t)$ is too complicated to be able to solve Eq. (8) explicitly. To find the solution of this equation we use the fact that they are smooth functions of energy so we approximate these two functions in the energy range $[m_t, E_{GU}]$ by the ratio of two polynomials. These functions perfectly approximate both coefficients in Eq. (8) in the whole energy range and this allows to find the solution of Eq. (8) in terms of a power series of the variable t [14]. In Fig. 1 we show the dependence on the energy of the two solutions of Eq. (8) and their derivatives [15]. As expected they are smooth functions of t.

From Eq. (10) we find now the dependence of $\lambda_H(t)$ on the energy t and important properties of its behavior. It is the most interesting to investigate how $\lambda_H(t)$ depends on the initial values of $\lambda_H(t_0)$ and to find out the range of validity of the SM. As discussed earlier, for the SM to be valid $\lambda_H(t)$ *must be positive and cannot be singular*. Since $\lambda_H(t_0) > 0$, it means that the SM is valid for energies between m_t and the zero or singularity of $\lambda_H(t)$ which can be determined from Eqs. (11) and (12).

Let us first consider the singularity (a simple pole) of $\lambda_H(t)$. For this purpose we plot in Fig. 2 the ratio of the two solutions $(12/(4\pi)^2) W_2(t)/W_1(t)$ from which we can determine the value of t for which the pole occurs depending on the value of $\lambda_H(t_0)$. If we impose the condition that $\lambda_H(t)$ is regular in the whole range of the energies

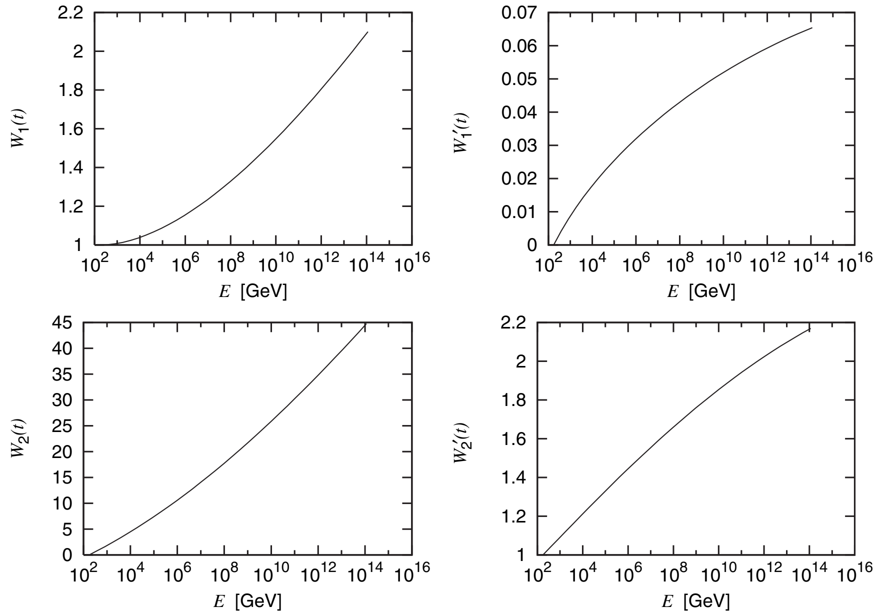

FIG. 1. The solutions of Eq. (8) and their derivatives with initial conditions defined in Eq. (9).

$[m_t, E_{GU}]$ then the value of $1/\lambda_H(t_0)$ should lie above the curve in Fig. 2 what gives the following condition:

$$\lambda_H(m_t) \leq 0.618. \tag{13}$$

For the SM to be valid the quartic coupling $\lambda_H(t)$ should not become negative. We use Eq. (12) to find the first zero of $\lambda_H(t)$. In Fig. 3 we have plotted $((4\pi)^2/12)W_1'(t)/W_2'(t)$ which determines at which energy in t occurs the first zero of $\lambda_H(t)$ depending on the value of $\lambda_H(t_0)$. Now from the condition that $\lambda_H(t)$ should not have zeros in the whole range of the energies $[m_t, E_{GU}]$ we obtain

$$\lambda_H(m_t) \geq 0.397. \tag{14}$$

We thus see that the consistency of the SM in the range of the energies up to the grand unification energy E_{GU} permits a very narrow band on the admissible values of the $\lambda_H(m_t)$.

$$0.397 \leq \lambda_H(m_t) \leq 0.618. \tag{15}$$

B. Two loop corrections and improvements

The results obtained so far were based on the one loop equations. To analyze possible improvements we have numerically analyzed the two loop RGE equations and investigated the influence of the two loop corrections on the behavior of the observables. The situation is the following:

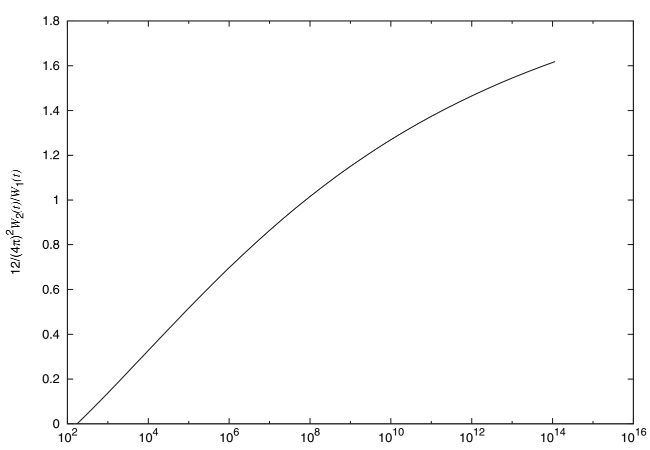

FIG. 2. The ratio of the solutions $(12/(4\pi)^2)W_2(t)/W_1(t)$ of Eq. (8). This ratio determines the value of t at which $1/\lambda_H(t)$ vanishes, i.e. $\lambda_H(t)$ has a pole.

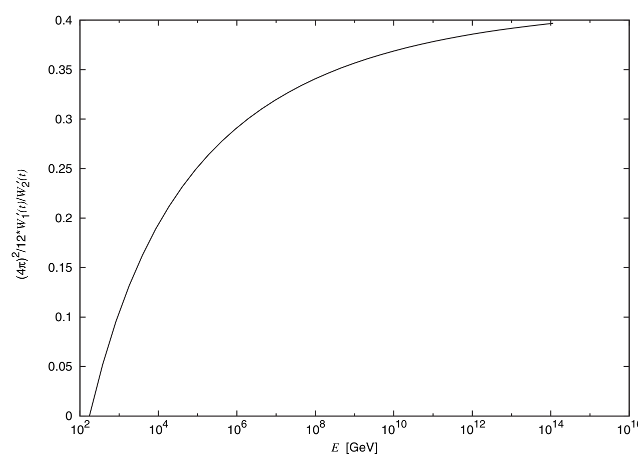

FIG. 3. The ratio of the derivatives $((4\pi)^2/12)W_1'(t)/W_2'(t)$ of the solutions of Eq. (8). This ratio determines the position t at which $\lambda_H(t)$ has a zero.

(1) The difference for the one and the two loop solutions for the gauge couplings g_l's is small (less than 1% in the whole range $[m_t, E_{GU}]$). The two loop corrections have more significant influence on the square of the Yukawa coupling Y_t^2 and the relative difference between the one and two loop solution is of the order of 10% at the energy E_{GU}. The comparison of the one and two loop running for these observables is given in Fig. 4.

(2) The next important point is the condition imposed on λ_H that follows from the consistency of the standard model. Since the RGEs are known only perturbatively so the values of the running masses and the coupling constants have to be such that the perturbation series is convergent. In the region where λ_H increases and has a pole the following criteria have been used [9] for the determination of the point Λ_p where the standard model becomes invalid [the functions $\beta^{(1)}_{\lambda_H}$ and $\beta^{(2)}_{\lambda_H}$ are defined in Eq. (2c)]:

$$\Lambda_p \text{ is the point where } \lambda(\Lambda_p) \text{ has a pole (Landau pole).} \tag{16a}$$

$$\frac{1}{(4\pi)^2}\frac{\beta^{(2)}_{\lambda_H}}{\beta^{(1)}_{\lambda_H}}\bigg|_{\Lambda_p} \approx 0.25 \quad \text{(condition 1)}. \tag{16b}$$

$$\frac{1}{(4\pi)^2}\frac{\beta^{(2)}_{\lambda_H}}{\beta^{(1)}_{\lambda_H}}\bigg|_{\Lambda_p} \approx 0.5 \quad \text{(condition 2)}. \tag{16c}$$

The condition Eq. (16a) can only be used for the one loop case. An important question is how the conditions in Eqs. (16) are related between them. We have verified that the value of Λ_p obtained from the one loop condition Eq. (16a) lies below and is close to the value of Λ_p obtained from the two loop condition Eq. (16c). It means that the physical meaning of conditions given in Eqs. (16a) and (16c) is very close and any condition of the two can be equivalently used. We will thus use the condition in Eq. (16a) in the one loop case and Eq. (16c) in the two loop case.

From the condition that the energy has to be positive definite we obtain that the SM is valid where λ_H is positive. For the one loop case the value of Λ_0 above which the standard model is not valid is obtained from the equation

$$\lambda_H(\Lambda_0) = 0. \tag{17}$$

For the two loop the condition Eq. (17) is only approximate and it becomes [16]

$$\tilde{\lambda}_H(\Lambda_0) = 0 \tag{18}$$

where $\tilde{\lambda}_H(t)$ has corrections from the effective potential $V_{\rm eff}$ (see Sec. V)

$$\tilde{\lambda}_H = \lambda_H - \frac{1}{16\pi^2}\bigg\{6Y_t^4\bigg[\ln\frac{Y_t^2}{2} - 1\bigg] - \frac{3}{4}g_2^4\bigg[\ln\frac{g_2^2}{4} - \frac{1}{3}\bigg] - \frac{3}{8}\bigg(\frac{3}{5}g_1^2 + g_2^2\bigg)^2\bigg[\ln\frac{3g_1^2 + 5g_2^2}{20} - \frac{1}{3}\bigg]\bigg\}. \tag{19}$$

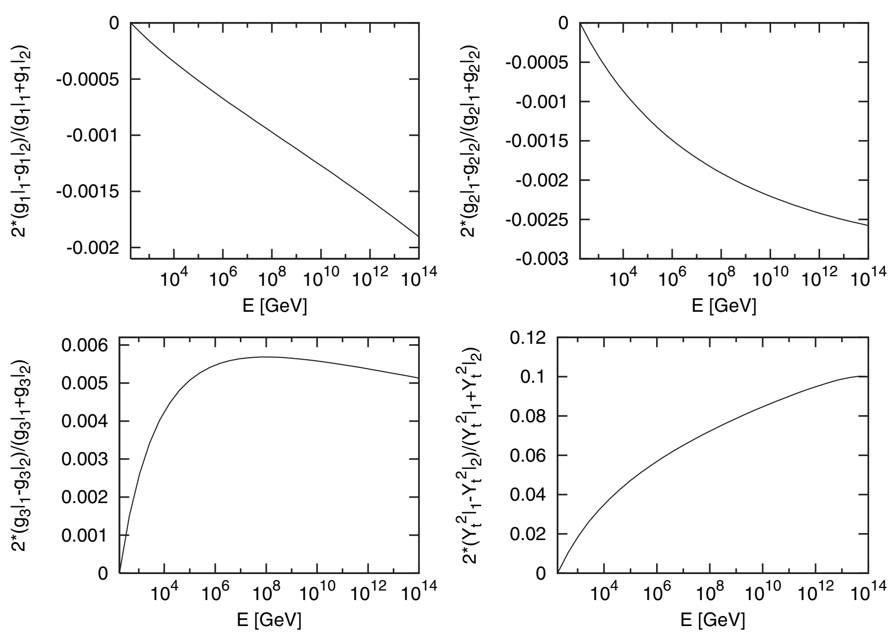

FIG. 4. The relative difference of the one and two loop evolution of the gauge couplings g_1, g_1, g_3 and the square of the t-quark Yukawa coupling constant Y_t^2. The relative difference of an observable F ($F = g_1, g_2, g_3, Y_t^2$) is given by the formula $2 * (F|_1 - F|_2)/(F|_1 + F|_2)$, where $F|_i$ is the value of the observable F calculated up to i loops.

(3) The one and the two loop solutions for λ_H differ significantly. The most striking difference is that in the two loop solutions for λ_H there is no singularity (pole) [17]. Instead of the pole there is a big jump and then the solution levels out at the value $\lambda_H \sim 24$ which is above the validity of the perturbative calculations. For low values of $\lambda_H(m_t)$ the two loop $\lambda_H(t)$ falls off slower than the one loop solution. We have studied the origin of the different behavior of the one and the two loop λ_H and have found that the two loop term in the equation for λ_H has very little influence on λ_H except that it wipes out the singularity. The position of the pole (or the jump) is determined by the one loop term of the equation. We have also found that the solution for λ_H is very sensitive to the behavior of the running of the top quark Yukawa coupling. A small difference in the running of $Y_t^2(t)$ causes important changes in the running of the *one* and *two* loop $\lambda_H(t)$. The function $\lambda_H(t)$ is not sensitive to small variations of the gauge couplings g_l's. Summarizing, we can conclude that to obtain the precise results for $\lambda_H(t)$ one has to have a very precise (two loops) knowledge of Y_t^2. The position of the pole (or jump) in $\lambda_H(t)$ on the other hand is not very sensitive to the two loop corrections in Y_t^2. The strong dependence of the quartic coupling $\lambda_H(t)$ on the top quark Yukawa coupling $Y_t^2(t)$ is an important reason to use the mass of the top quark as the renormalization point. The comparison of the one and two loop renormalization group evolution for various values of $\lambda_H(m_t)$ is given in Fig. 5.

As remarked earlier the two loop solution $\lambda_H(t)$ does not have a pole and we cannot apply the one loop criterion [position of the pole, Eq. (16a)] to determine the validity of the standard model. Instead we use the argument about the validity of the perturbation expansion [condition 1, Eq. (16b) and condition 2, Eq. (16c)]. From these conditions and the requirement $\lambda_H \geq 0$ we find that the SM is valid in the whole range of energies $[m_t, E_{GU}]$ for the following values of $\lambda_H(m_t)$

$$0.368 \leq \lambda_H(m_t) \leq 0.603 \quad \text{(condition 1)}, \tag{20a}$$

$$0.368 \leq \lambda_H(m_t) \leq 0.621 \quad \text{(condition 2)}. \tag{20b}$$

Comparing Eqs. (15) and (20) we see that the two loop correction to the upper limit for $\lambda_H(m_t)$ is of the order of 1%. The two loop deviation of the lower limit is larger but one has to note that the physical meaning of the lower limit (obtained from the condition $\lambda_H \geq 0$) for two loops is rather obscure. We will discuss the two loop improved condition and the limits for the Higgs mass in the next section.

The results on the range of the validity of the SM depending on the number of loops and the value of $\lambda_H(m_t)$ are given in Fig. 6. On this figure it is shown how the conditions following from the triviality and stability depend on energy. The two lower curves follow from the conditions $\lambda_H(t) = 0$ and $\tilde{\lambda}_H(t) = 0$. We see that these two curves are parallel and close one to the other. The influence of the two loop corrections is thus small and weakly dependent on energy. The three upper curves follow from the triviality condition of the standard model. The position of the Landau pole and the two conditions $\lambda_H(\Lambda_p) = 6$ and

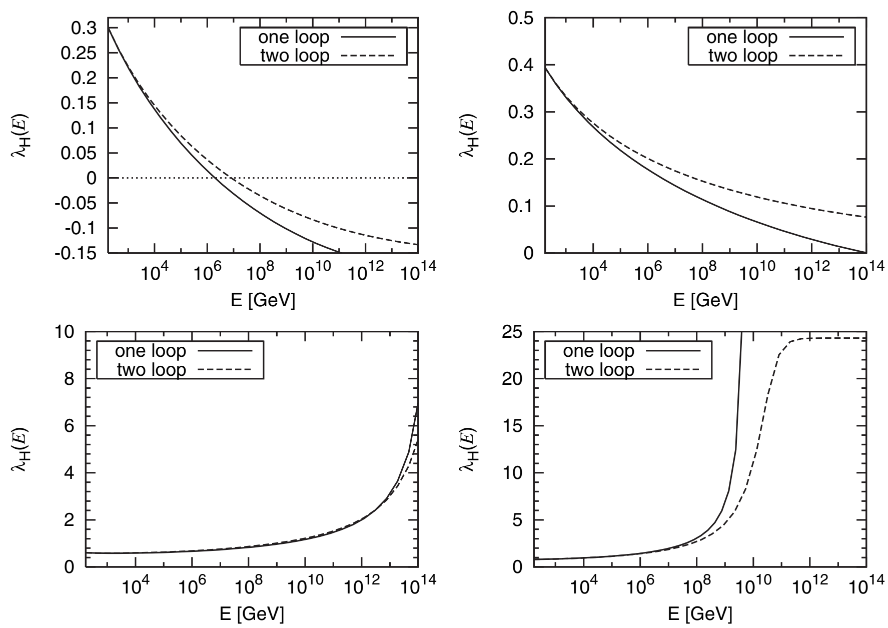

FIG. 5. The one and two loop running of λ_H for different initial values of $\lambda_H(m_t)$. The pattern of the evolution is different for small ($\lambda_H(m_t) \leq 0.4$) and large ($\lambda_H(m_t) \geq 0.6$) values of $\lambda_H(m_t)$.

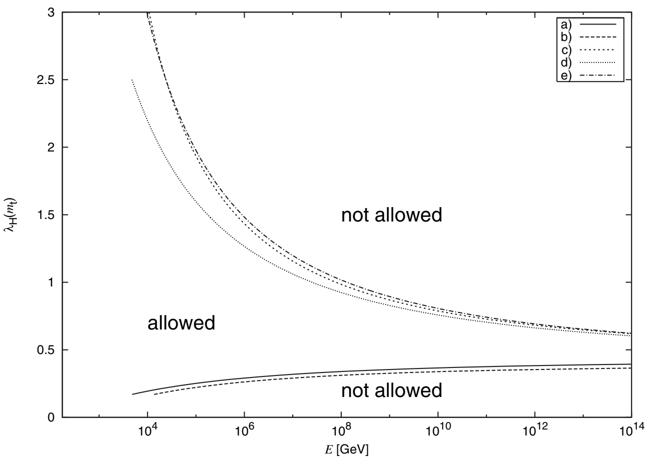

FIG. 6. The plot of the energy E_{lim} at which the SM breaks down as the function of $\lambda_H(m_t)$. The standard model is valid for energies to the left of the value indicated by $\lambda_H(m_t)$. The limits are derived from the conditions: (a) $\lambda_H(t) = 0$, (b) $\tilde{\lambda}_H(t) = 0$, (c) one loop pole position of $\lambda_H(t)$, (d) $\lambda_H(t) = 6$ and (e) $\lambda_H(t) = 12$. It is remarkable that the one loop condition (c) and the two loop condition (e) give very close results.

$\lambda_H(\Lambda_p) = 12$. It is surprising that the curves for the one loop Landau pole and the two loop condition $\lambda_H(\Lambda_p) = 12$ almost coincide and the range of the validity of the SM for both cases is almost identical. We will thus consider the condition $\lambda_H(\Lambda_p) = 6$ to be too strong and will not include it in further considerations.

V. RUNNING OF THE HIGGS MASS

The most important physical conclusions that follow from the limits on the quartic coupling constant λ_H are those for the physical Higgs mass. At the tree level, the relation between the Higgs mass m_H and λ_H is

$$m_H^2 = \lambda_H v^2 \tag{21}$$

where v is the vacuum expectation value of the Higgs field, $v(M_Z) = (\sqrt{2}G_\mu)^{-1/2} \approx 246.2$ GeV. Equation (21) is obtained from the tree level Higgs potential [Ref. [5], Eq. (13)]

$$V_0 = \frac{1}{2}m^2\phi^2 + \frac{1}{8}\lambda_H\phi^4. \tag{22}$$

The mass m_H in Eq. (21) is not the physical mass (pole mass). The proper mechanism to obtain the physical Higgs mass is to use the effective potential [18] and to add the contribution of the self energy of the Higgs field. The effective potential can be written in the following form:

$$V_{\text{eff}} = V_0 + V_1 + \cdots, \tag{23}$$

where V_0 is the tree level potential and V_1 is the one loop correction. The method of the effective potential has the property that the L loop improved effective potential together with the $(L+1)$ RGEs resums all the L to leading logarithms [19]. We will thus use the tree level potential for the one loop RGEs and the one loop corrected potential for the two loop RGEs.

The physical Higgs mass M_H has two contributions: the first one coming from the effective potential and the second one is the renormalized self energy of the Higgs field (Ref. [20], Appendix B)

$$M_H^2 = \frac{\partial^2 V_{\text{eff}}}{\partial\phi^2(t)}\Big|_{\min} + \text{Re}(\Pi(p^2 = M_H^2) - \Pi(p^2 = 0)). \tag{24}$$

The Higgs mass defined in Eq. (24) does not depend on the renormalization point t if V_{eff} is taken to all orders. If we truncate V_{eff} then M_H^2 defined by Eq. (24) becomes dependent on t and the issue of the proper choice of the renormalization point becomes important. In Ref. [9] it has been shown that for the Higgs mass $M_H \leq 1.7m_t$ the errors are the smallest for the renormalization point at the mass of the top quark m_t (i.e. $t = 0$), while for $M_H > m_t$ the optimal is the renormalization point at M_H, $[t = \ln(M_H/m_t)]$. Guided by this discussion we will use the following prescription

$$M_H^2 = M_{H,\text{perturbative}}^2(t_p), \qquad t_p = \begin{cases} 0 & \text{for } M_H \leq m_t \\ \ln\frac{M_H}{m_t} & \text{for } M_H > m_t. \end{cases} \tag{25}$$

The effective potential method determines the physical Higgs mass in terms of the quartic Higgs coupling $\lambda_H(m_t)$, gauge coupling constants $g_l(m_t)$, Yukawa coupling of the top quark mass and the physical masses of the gauge bosons M_W and M_Z and the top quark mass m_t. Since the values of all these parameters except $\lambda_H(m_t)$ are known so we will present the results of this section as the relation between the physical Higgs mass M_H and $\lambda_H(m_t)$. Let us first discuss the effective potential for different cases.

A. Tree level effective potential

We will use the effective potential defined in Eq. (22) from which we obtain the running Higgs mass

$$m_H^2(t) = \lambda_H(t)Z^2(t)v^2, \tag{26}$$

where $Z(t)$ is the renormalization factor of the Higgs field and the functions $Z(t)$ and $\lambda_H(t)$ are determined up to one loop. $Z(t)$ fulfills the RGE [Ref. [5], Eq. (14)] and [Ref. [21], Eq. (10)]

$$\frac{d\ln Z}{dt} = -\gamma_\phi(g_l^2, Y_t^2) = \frac{3}{(4\pi)^2}\left(\frac{3}{20}g_1^2 + \frac{3}{4}g_2^2 - Y_t^2\right) \tag{27}$$

which has the solution

$$Z(t) = h_m^{-3}(t)\left(\frac{g_1(t)}{g_1(t_0)}\right)^{9/20b_1}\left(\frac{g_2(t)}{g_2(t_0)}\right)^{9/4b_2} \quad (28)$$

and $h_m(t)$ is equal [7]

$$h_m = \exp\left(\frac{1}{(4\pi)^2}\int_{t_0}^{t}\mathrm{Tr}(y_u y_u^\dagger)dt\right). \quad (29)$$

From Eqs. (10) and (26) we obtain the following result for the running Higgs mass:

$$m_{H,1}^2(t) = -\frac{(4\pi)^2}{12}\frac{W_1'(t) - \frac{12}{(4\pi)^2}\lambda_H(t_0)W_2'(t)}{W_1(t) - \frac{12}{(4\pi)^2}\lambda_H(t_0)W_2(t)}h_m^{-6}(t) \times\left(\frac{g_1(t)}{g_1(t_0)}\right)^{9/10b_1}\left(\frac{g_2(t)}{g_2(t_0)}\right)^{9/2b_2}v^2. \quad (30)$$

The physical Higgs mass is thus equal (note that in the case of one loop we omit the self-energy contribution to the Higgs mass)

$$M_H^2 = m_{H,1}^2(t_p). \quad (31)$$

B. Two loop case

1. Effective potential with one loop corrections (Ref. [21])

The effective potential with one loop corrections used in Ref. [21] has the following form:

$$V_{\mathrm{eff}} = \frac{1}{2}m^2\phi^2 + \frac{1}{8}\lambda_H\phi^4 + \frac{1}{64\pi^2}\left\{6m_W^4(t)\left[\ln\frac{m_W^2(t)}{\mu^2(t)} - \frac{5}{6}\right] + 3m_Z^4(t)\left[\ln\frac{m_Z^2(t)}{\mu^2(t)} - \frac{5}{6}\right] - 12m_t^4(t)\left[\ln\frac{m_t^2(t)}{\mu^2(t)} - \frac{3}{2}\right]\right\} \quad (32)$$

where the functions $m_W(t)$, $m_Z(t)$, $m_t(t)$, $\mu(t)$ and $\phi(t)$ are defined as follows:

$$m_W^2(t) = \tfrac{1}{4}g_2^2(t)\phi^2(t), \qquad m_Z^2(t) = \tfrac{1}{4}(\tfrac{3}{5}g_1^2(t) + g_2^2(t))\phi^2(t),$$
$$m_t^2(t) = \tfrac{1}{2}Y_t^2(t)\phi^2(t), \qquad \mu(t) = m_t e^t, \qquad \phi(t) = Z(t)v. \quad (33)$$

From Eq. (32) we obtain the formula for the running Higgs mass

$$m_{H,2}^2(t) = \lambda_H(t)Z^2(t)v^2 + \frac{3}{64\pi^2}Z^2(t)v^2\left\{g_2^4(t)\left(\ln\frac{m_W^2(t)}{\mu^2(t)} + \frac{2}{3}\right) + \frac{1}{2}\left(\frac{3}{5}g_1^2(t) + g_2^2(t)\right)^2\left(\ln\frac{m_Z^2(t)}{\mu^2(t)} + \frac{2}{3}\right) - 8Y_t^4(t)\ln\frac{m_t^2(t)}{\mu^2(t)}\right\}. \quad (34)$$

The physical Higgs mass is obtained from the equation

$$M_H^2 = m_{H,2}^2(t_p) + \mathrm{Re}(\Pi(p^2 = M_H^2) - \Pi(p^2 = 0)). \quad (35)$$

2. Effective potential from analytic calculations (Ref. [22])

Recently the *analytic* calculation of the full two loop corrections to the pole masses in the standard model have been performed [22]. It has been shown that the one loop effective potential is not modified from the tree level form

$$V_{\mathrm{eff}} = \frac{1}{2}m^2\phi^2 + \frac{1}{8}\lambda_H\phi^4. \quad (36)$$

The same form was also postulated in Ref. [5]. The running Higgs mass is thus equal

$$m_{H,\mathrm{analytical}}^2(t) = \lambda_H(t)Z^2(t)v^2. \quad (37)$$

The form of Eqs. (26) and (37) for the Higgs running mass is the same but in case of Eq. (26) the functions $Z(t)$ and $\lambda_H(t)$ run according to one loop equations and in Eq. (37) they run according to the two loop equation. Also in the physical Higgs mass we include the self energy of the Higgs field

$$M_H^2 = m_{H,\mathrm{analytical}}^2(t_p) + \mathrm{Re}(\Pi(p^2 = M_H^2) - \Pi(p^2 = 0)). \quad (38)$$

C. Physical Higgs mass as the function of $\lambda_H(m_t)$

The main result of this section is to show what is the physical Higgs mass as a function of the coupling constant $\lambda_H(m_t)$ for different choices of the effective potential. This function permits to express the constraints Eqs. (15) and (20) for the coupling constant $\lambda_H(m_t)$ in terms of the

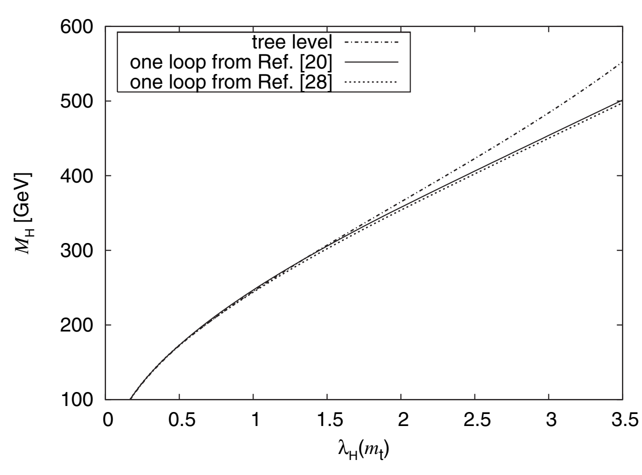

FIG. 7. The dependence of the physical Higgs mass M_H on the value of the Higgs quartic coupling constant $\lambda_H(m_t)$. The tree level and one loop Higgs masses differ significantly at large Higgs masses. On the other hand the two choices of the one loop effective potential yield equivalent results. It suggests that the theoretical error that comes out from the uncertainty of the effective potential is small. Tree level (dot-dashed line); one loop from Ref. [20] (solid line); one loop from Ref. [28] (dotted line).

physical Higgs mass. We have considered here three possible effective potentials: tree level and the two versions of the effective potential with the one loop corrections. The results for these three cases, Eqs. (31), (35), and (38) are shown in Fig. 7. It is remarkable that the two choices of the one loop effective potential yield very similar results in the whole range of the values of $\lambda_H(m_t)$, e.g. the difference between the predictions of the two versions of the one loop effective potential at the Higgs mass $M_H \sim 500$ GeV is 3.7 GeV which is less that 1%. We thus conclude that the theoretical error of the prediction of the Higgs mass is 1%.

VI. CONCLUSIONS

The most important predictions are presented in Figs. 7 and 8. Figure 7 represents the relation between the physical Higgs mass and the value of the Higgs quartic coupling $\lambda_H(m_t)$. One can see that the one and two loop relations differ significantly for large Higgs masses. The choice of the one loop effective potential for the two loop running has very little influence. For the Higgs masses $M_H \leq 250$ GeV the two loop corrections are negligible.

Figure 8 contains the upper value for the UV cutoff as a function of the Higgs mass. To obtain this figure we used the Higgs boson matching scale t_p defined in Eq. (25) and equal to $t_p = \max\{0, t_H\}$, where $t_H = \ln(M_H/m_t)$. The lower part of Fig. 8 consists of the two curves that are obtained from the conditions $\lambda_H(t) = 0$ and $\tilde{\lambda}_H(t) = 0$. For the Higgs masses that allow this condition there is no Landau pole up to the GU energy E_{GU} and the values of λ_H are small and the function $\lambda_H(t)$ is monotonically decreasing (see Fig. 9) so the two loop and higher corrections are small and the perturbation series does not diverge. The upper part of Fig. 8 consists also of two curves that are almost identical. It is remarkable that the one loop condition (curve c) and the two loop condition (curve d) are so close. This similarity strongly supports the idea that the one loop Landau pole position is a very good approximation (probably beyond one loop) of a point where the standard model breaks down.

From Fig. 8 we see that for the Higgs masses $M_H \lesssim 150$ GeV the UV cutoff is growing as a function of the Higgs mass. The energy of grand unification E_{GU} is reached at $M_H \approx 150$ GeV and then there is a narrow window of the Higgs masses

$$\begin{aligned} 153.5 \leq M_H \leq 191.1 \text{ GeV} \quad \text{for one loop,} \\ 148.5 \leq M_H \leq 193.1 \text{ GeV} \quad \text{for two loops,} \end{aligned} \tag{39}$$

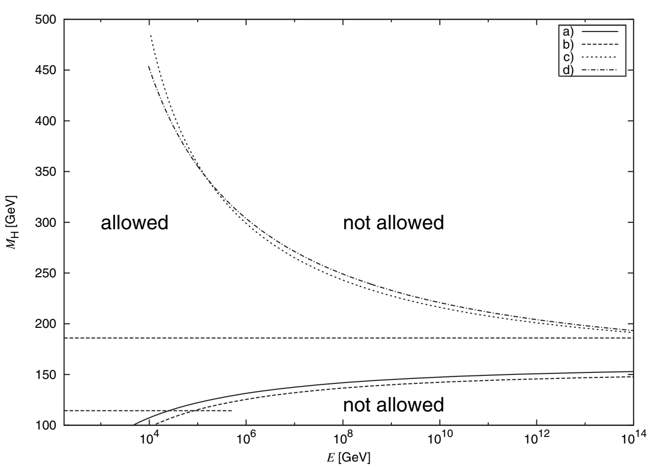

FIG. 8. The plot of the energy E_{lim} at which the SM breaks down as the function of the Higgs mass. This figure is obtained from Fig. 6 by transforming $\lambda_H(m_t)$ into the physical Higgs mass M_H. The standard model is valid for energies to the left of the value indicated by M_H. The energy limits are derived from the conditions: (a) $\lambda_H(t) = 0$, (b) $\tilde{\lambda}_H(t) = 0$, (c) one loop pole position of $\lambda_H(t)$, (d) two loop $\lambda_H(t) = 12$. The two dashed horizontal lines correspond to the experimental limits of the Higgs mass given in Eqs. (40) and (41). From the crossing points of these lines one determines the range of the validity of the SM for each of the Higgs masses.

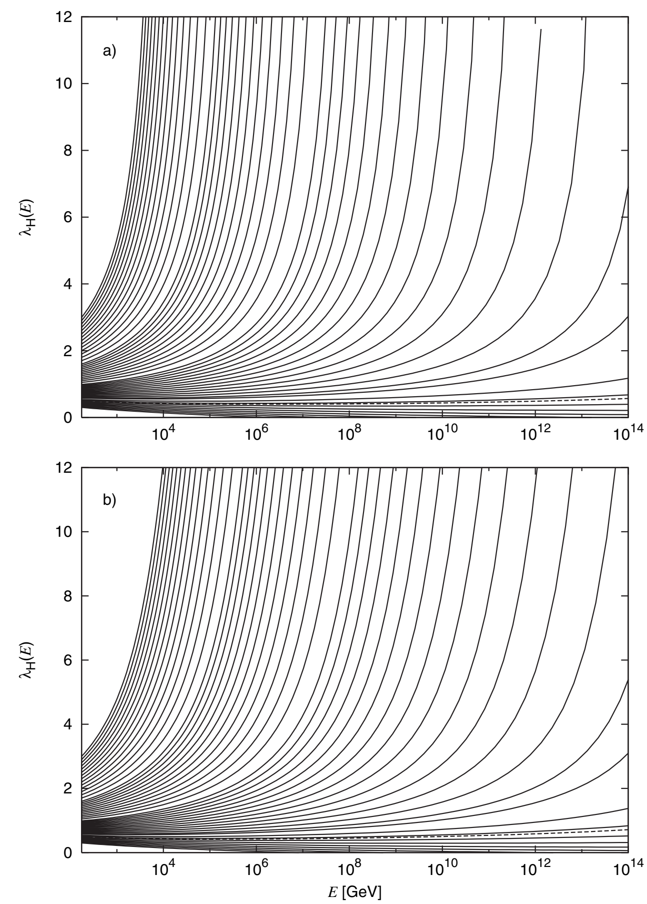

FIG. 9. Running of $\lambda_H(t)$ for different values of $\lambda_H(m_t)$: (a) was obtained from the one loop RGE and (b) from the two loop equations. On both parts the dashed lines correspond the running of the of $\lambda_H(t)$ with the initial value of $\lambda_H(m_t) = 0.5$ which corresponds to the Higgs mass $M_H = m_t$. From both parts one can see that the behavior of $\lambda_H(t)$ is different for $M_H < m_t$ and $M_H > m_t$. This indicates that for $M_H = m_t$ there may be a phase transition in the standard model.

for which the UV cutoff exceeds the E_{GU} scale. For the Higgs masses $M_H \gtrsim 178$ there appears the Landau pole and the UV cutoff is decreasing as a function of the Higgs mass.

In Fig. 9 we show the evolution of the coupling $\lambda_H(t)$ given in Eq. (10). One can see that the behavior of $\lambda_H(t)$ is in agreement with the earlier discussion and for values of $\lambda_H(t_0) \lesssim 0.37$ the function $\lambda_H(t)$ has a zero and for $\lambda_H(t_0) \gtrsim 0.61$ it has a pole.

From the evolution of the $\lambda_H(t)$ one can see the appearance of two patterns of the high energy behavior of the standard model. One, for $\lambda_H(m_t) < 0.5$ and the other for $\lambda_H(m_t) > 0.5$. Such a change of the pattern can be an indication of a phase transition. The point $\lambda_H(m_t) = 0.5$ corresponds to $M_H = m_t$.

The limits on the Higgs mass have been previously discussed theoretically and analyzed experimentally. The most recent limits on the Higgs mass from LEP's Electroweak Working Group [23], following from the high Q^2 precision electroweak measurements, are

$$M_H = 114 \begin{matrix} +69 \\ -45 \end{matrix} \text{ GeV} \quad (\text{at } 68\% \text{ C.L.})$$
$$\text{and} \quad M_H < 186 \text{ GeV} \quad (\text{at } 95\% \text{ C.L.}). \tag{40}$$

The direct Higgs boson search [24] gives the lower experimental limit

$$M_H > 114.4 \text{ GeV} \quad (\text{at } 95\%\text{C.L.}). \tag{41}$$

The experimental limits thus indicate that the Higgs mass might be in a range compatible with the ultraviolet cutoff of the order 10^4–10^{14} GeV.

If the mass of the Higgs boson is below 150 GeV the lower curve from Fig. 8 should be used for the determination of the cutoff. A similar curve has been presented in Ref. [9] and it contains a band including the theoretical error. Instead, we present two curves, one that follows from the condition $\lambda_H = 0$ and the other from $\tilde{\lambda}_H = 0$. Both of these curves for small values of the cutoff lie above the band from Ref. [9] (see also Refs. [16,18,21]). For the condition $\lambda_H = 0$ our results are compatible with those of Ref. [9] for the cutoff above the 10^8 GeV and for the condition $\tilde{\lambda}_H = 0$ they are compatible for the cutoff above 10^5 GeV. This discrepancy means that we predict a lower cutoff than Ref. [9]. We have tried to trace this discrepancy by the variation of the input parameters of our analysis: the strong coupling constant $\alpha_s(M_Z)$ and the top quark mass m_t but this could not explain the difference. A possible source of the discrepancy may be that we use other renormalization point (we use the top quark mass m_t while the Z boson mass m_Z is used in Ref. [9]) and the predictions are rather sensitive to the details of the evolution of the top quark Yukawa coupling.

Theoretically the Higgs mass limits have been obtained from the analysis of the renormalization group equations and from the lattice calculations. The RGE's most recent analysis [9] gives results that are compatible with ours. It should be stressed that our analysis of the one loop equations is based on the analytical results and gives explicit formulae for the position of the Landau pole and zero of the quartic coupling $\lambda_H(t)$.

The nonperturbative lattice limits for the Higgs mass are obtained from the triviality condition [25]. These results are complementary to the RGE analysis and yield the following limit [26]

$$M_H < 620 \text{ GeV}. \tag{42}$$

This result may slightly depend on the regularization scheme and also recently there was some discussion [27] about the correct treatment of the cutoff in the renormalization of the theory. This, however, does not seem to influence the result.

To conclude let us stress that the key point of the paper is the treatment of the one loop RGE for λ_H and the linearization of the problem by the substitution in Eq. (7). This linearization permitted a very precise analysis of the positions of the Landau pole and the point where λ_H vanishes. Moreover it also gave an intimate relation between the positions of these two points: Eq. (12) is the derivative of Eq. (11). Such a relation between these two important quantities is a new result. Additionally it should be also stressed that the *analytical* results for the running of λ_H up to one loop is a very good starting point for a precise analysis of the two loop effects. We also demonstrate that the position of the one loop pole of $\lambda_H(t)$ coincides with the two loop condition $\lambda_H(t) = 12$, from Ref. [9], where the perturbation series should break. In our opinion the condition based on the position of the pole is more transparent and less arbitrary.

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