Topological charge in 1 + 1 dimensional lattice ϕ^4 theory

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We investigate the topological charge in 1 + 1 dimensional ϕ^4 theory on a lattice with antiperiodic boundary condition (APBC) in the spatial direction. We propose a simple order parameter for the lattice theory with APBC and we demonstrate its effectiveness. Our study suggests that kink condensation is a possible mechanism for the order-disorder phase transition in the 1 + 1 dimensional ϕ^4 theory. With renormalizations performed on the lattice with periodic boundary condition (PBC), the topological charge in the renormalized theory is given as the ratio of the order parameters in the lattices with APBC and PBC. We present a comparison of topological charges in the bare and the renormalized theory and demonstrate invariance of the charge of the renormalized theory in the broken symmetry phase.

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I. INTRODUCTION

Topology has an important role in nonperturbative quantum field theory on a lattice. For early work see, for example, [1]. It is of utmost importance in the dual superconductivity picture of quark confinement in QCD [2]. In field theories with nontrivial topology, different topological sectors are characterized by a conserved topological charge which depends on the boundary behavior of the theory. This conserved quantity is well-defined in classical continuum field theory.

In 1 + 1 dimensional classical ϕ^4 field theory, there exists a kink solution with a conserved topological charge [3]. In the quantum version of the above theory using lattice regularization, past studies on kinks focussed on the calculation of the kink mass [4,5]. Recently these kinks have also been studied in discrete light cone quantization [6].

In this paper we investigate the topological charge in the context of the ordered (broken symmetry) phase of 1 + 1 dimensional lattice ϕ^4 field theory. The short range quantum fluctuations lead to the renormalization of the mass and the coupling constant in the topologically trivial sector of the theory. On the other hand, long range fluctuations lead to a phase transition in the strong coupling region which leads to a breaking of the conservation of topological current and hence the vanishing of the topological charge in the disordered (symmetric) phase. We show that the topological charge in the renormalized theory remains invariant in the ordered phase. To the best of our knowledge, topological charge and its invariance in this quantum theory has not been investigated before.

Given that the ϕ^4 theory and the Ising model are in the same universality class, one can also define a topological charge in the Ising model when considered as a Euclidean quantum field theory in 1 + 1 dimensions [7]. However, our aim in this work is to look at the renormalizations of the parameters of the ϕ^4 theory so that a sensible definition of topological charge, invariant in the ordered phase of the theory, emerges.

Condensation of the topological objects has been associated with the mechanism of phase transitions in various statistical and quantum field theories [7–9] (also see works cited in [7]). Our study indicates that condensation of kink-antikinks is a possible mechanism for the order-disorder phase transition in 1 + 1 dimensional ϕ^4 theory.

The plan of this paper is as follows. In Sec. II we present our notation and definition of the topological charge in the bare and renormalized lattice theory. Sections III, IV, and V are devoted to the numerical work with details of various issues that we encounter in the calculation. Finally, Sec. VI contains discussion of our results and conclusion.

II. TOPOLOGICAL CHARGE AND RENORMALIZATION

We start from the Lagrangian density in Minkowski space (in usual notation)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2.1}$$

which leads to the Lagrangian density in Euclidean space

$$\mathcal{L}_E = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{2.2}$$

Note that in one space and one time dimensions, the field ϕ is dimensionless and the quartic coupling λ has dimension of mass^2.

The Euclidean action is

$$S_E = \int d^2x\mathcal{L}_E. \tag{2.3}$$

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Next we put the system on a lattice of spacing a with

$$\int d^2x = a^2 \sum_x. \tag{2.4}$$

Because of the periodicity of the lattice sites in a toroidal lattice the surface terms will cancel among themselves (irrespective of the boundary conditions on fields) enabling us to write

$$(\partial_\mu \phi)^2 = -\phi \partial_\mu^2 \phi \tag{2.5}$$

and on the lattice

$$\partial_\mu^2 \phi = \frac{1}{a^2}[\phi_{x+\mu} + \phi_{x-\mu} - 2\phi_x]. \tag{2.6}$$

Introducing dimensionless lattice parameters m_0^2 and λ_0 by $m_0^2 = m^2 a^2$ and $\lambda_0 = \lambda a^2$ we arrive at the lattice action in two Euclidean dimensions

$$S = -\sum_x \sum_\mu \phi_x \phi_{x+\mu} + \left(2 + \frac{m_0^2}{2}\right)\sum_x \phi_x^2 + \frac{\lambda_0}{4!}\sum_x \phi_x^4. \tag{2.7}$$

With antiperiodic boundary condition (APBC), we need to identify an order parameter to characterize different phases of the theory since $\langle \phi \rangle$ is zero in both symmetric and broken phases. In previous works on kinks in two dimensional lattice field theory [4,5], vanishing of the kink mass in the symmetric phase was used to identify the critical coupling. Since the calculation of the kink mass is rather involved, it is preferable to have a simpler choice.

For a L^2 lattice, we propose $\overline{\phi}_{\rm diff} = \frac{1}{2}[\overline{\phi}_{L-1} - \overline{\phi}_0]$ as the order parameter where $\overline{\phi}_x \equiv \langle \text{kink g.s}|\phi_x|\text{kink g.s}\rangle$ with $|\text{kink g.s}\rangle$ denoting the kink ground state. $\overline{\phi}_x$ is computed by taking the average of ϕ_x over configurations with APBC in the spatial direction (which effectively performs importance sampling around the kink configurations in a Monte Carlo simulation).

In the classical theory, the topological charge is given by

$$Q_{\rm classical} = \sqrt{\frac{\lambda}{6m^2}}\phi_{\rm diff} = \sqrt{\frac{\lambda}{3m_B^2}}\phi_{\rm diff} \tag{2.8}$$

where $\phi_{\rm diff} = \frac{1}{2}[\phi(\infty) - \phi(-\infty)]$ and we have used the fact that the mass of the elementary boson is $m_B = \sqrt{2}m$ in the broken phase. In the classical theory, the relation between $\phi_{\rm diff}$, m_B and λ is given by

$$\phi_{\rm diff} = \sqrt{\frac{3m_B^2}{\lambda}} \tag{2.9}$$

which guarantees that the topological charge is $+1$ (kink sector) and -1 (antikink sector) in the broken symmetric phase and zero in the symmetric phase.

In the lattice theory with APBC in the "spatial" direction, the bare topological charge is defined by

$$Q_0 = \sqrt{\frac{\lambda_0}{3m_{B_0}^2}}\overline{\phi}_{\rm diff}. \tag{2.10}$$

Because of quantum fluctuations, the classical relation analogous to Eq. (2.9) between $\overline{\phi}_{\rm diff}$, m_{B_0} and λ_0 is not obeyed and as a consequence, Q_0 does not remain $+1$ or -1 in the ordered phase.

We propose to define the topological charge in the renormalized theory as

$$Q_R = \sqrt{\frac{\lambda_R}{3m_R^2}}\overline{\phi}_{R_{\rm diff}} \tag{2.11}$$

where $\phi_R = (1/\sqrt{Z})\phi$, Z being the field renormalization constant. The conventional definition of the renormalized coupling λ_R in ϕ^4 theory is in terms of the renormalized four-point vertex function. The calculation of the four-point vertex function on the lattice is computationally demanding. Fortunately, in the broken phase, we can choose a definition [10] of λ_R in terms of the renormalized mass m_R and the renormalized vacuum expectation value $\langle \phi_R \rangle$, determined with periodic boundary condition (PBC):

$$\lambda_R = 3\frac{m_R^2}{\langle \phi_R \rangle^2}. \tag{2.12}$$

This definition of λ_R involves only the computation of one-point function for $\langle \phi \rangle$ and the two point function for m_R and Z. The details of these computations are provided in Ref. [11]. Using this definition of λ_R in Eq. (2.11), we get

$$Q_R = \frac{\overline{\phi}_{R_{\rm diff}}}{\langle \phi_R \rangle} = \frac{\overline{\phi}_{\rm diff}}{\langle \phi \rangle}. \tag{2.13}$$

The second equality of Eq. (2.13) assumes Z to be the same for $\phi_{\rm diff}$ and ϕ and therefore involves two unrenormalized quantities, $\overline{\phi}_{\rm diff}$ calculated with APBC and $\langle \phi \rangle$ calculated with PBC. If $\overline{\phi}_{\rm diff}$ and $\langle \phi \rangle$ are of the same magnitude, the renormalized topological charge is ± 1 in the ordered phase except for the statistical and systematic errors that occur in the numerical computations of the quantities in the numerator and the denominator of Eq. (2.13).

III. CALCULATION WITH PBC: $\langle \phi \rangle$

First we discuss the calculation of $\langle \phi \rangle$ with PBC. As is well known, spontaneous symmetry breaking cannot occur at finite volume due to the quantum mechanical tunneling phenomena. This is due to the fact that at finite volume we have finite degrees of freedom. Tunneling probability is, however, inversely proportional to the exponential of the volume and hence is exponentially suppressed at large

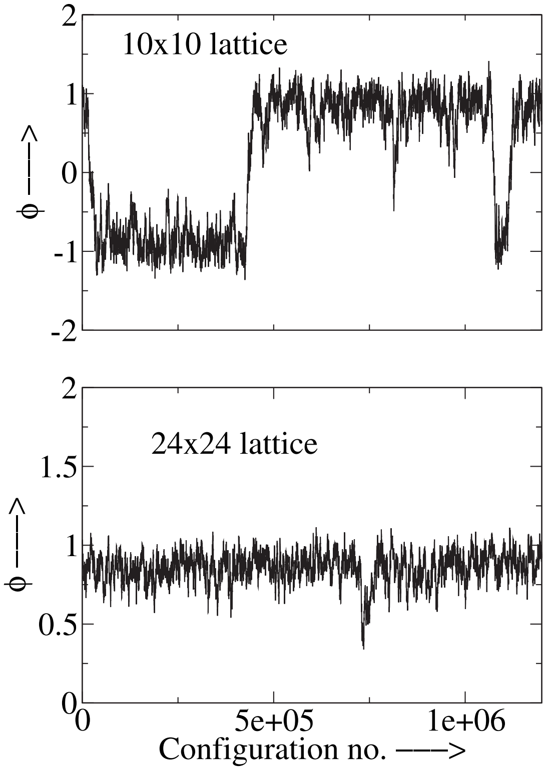

FIG. 1. Comparison of tunneling in ϕ^4 theory for couplings $m_0^2 = -0.5$, $\lambda_0 = 1.8$.

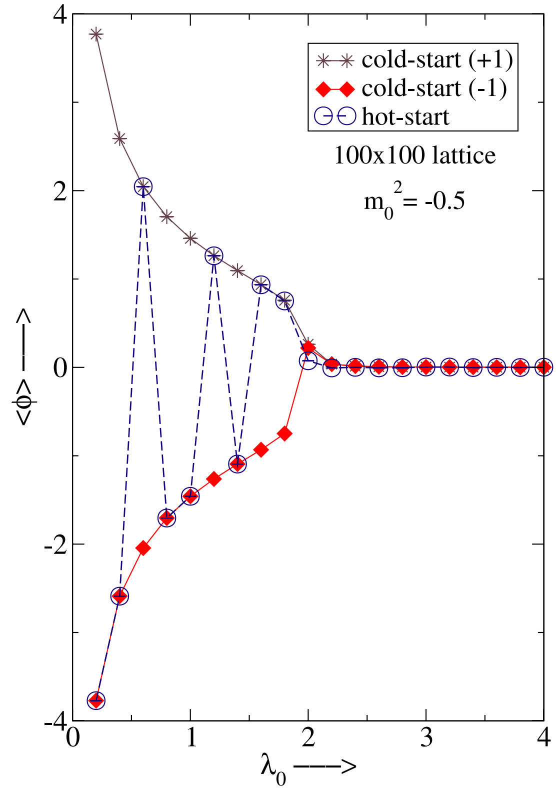

FIG. 2 (color online). Sensitivity to initial configuration (Metropolis algorithm).

volume [10,12]. Thus in practice, the issue is how large a volume we need for the suppression to occur.

For ϕ^4 theory, tunneling is present for 10^2 lattice but is practically absent for 24^2 lattice. This is demonstrated in Fig. 1 where ϕ averaged over all sites for a given configuration is plotted against the configuration number. These calculations were done using single hit Metropolis algorithm.

We find $\langle \phi \rangle$ to be sensitive to the various choices of the initial configuration (see Fig. 2) in the simulation using the Metropolis algorithm. For cold starts [$\phi = +1$ (or -1) for all sites] the selection of the vacuum is controlled by the choice of the sign of ϕ irrespective of the couplings. On the other hand for hot starts ($\phi = \pm 1$ sprinkled randomly over the entire lattice), the vacuum is picked randomly from the two degenerate vacua.

A more serious problem with the Metropolis algorithm is critical slowing down which makes computation near the critical point prohibitively expensive. In the Monte Carlo simulation, to reduce critical slowing down, we have thus resorted to the standard Metropolis update *combined* with a cluster algorithm [13] to update the embedded Ising variables in ϕ^4 theory, following the method of Brower and Tamayo [14]. We use Wolff's single cluster variant of the algorithm. This method has been used previously in determining the phase diagram of two dimensional lattice ϕ^4 theory [15].

In the cluster algorithm however, since the sign of the field of all the members of the cluster are flipped in every updation cycle, the algorithm actually enforces tunneling and the configuration average of ϕ is always zero. Thus to get the appropriate nonzero value for the condensate we measure $\langle |\phi| \rangle$ where $\phi = \frac{1}{\mathrm{Volume}} \sum_{\mathrm{sites}} \phi(x)$. To understand the mod let us consider a local order parameter $\langle \phi(x) \rangle$. Since the configurations will be selected at random dominantly from the neighborhood of either vacua in the broken phase, $\langle \phi(x) \rangle$ will vanish when averaged over configurations thus wiping out the signature of a broken phase. If one uses $\langle |\phi(x)| \rangle$ as the order parameter then in the broken phase it correctly projects itself onto one of the vacua yielding the appropriate nonzero value. The use of this mod, unfortunately, destroys the signal in the symmetric phase completely by wiping out the significant fluctuations in sign. However if we choose to use $\langle | \frac{1}{\mathrm{Volume}} \sum_{\mathrm{sites}} \phi(x) | \rangle$, it correctly captures the broken phase as well as the symmetric phase. While the sign fluctuation over configurations are still masked, the fluctuations over sites survive producing $\langle |\phi| \rangle = 0$ correctly in the symmetric phase.

The volume dependence of $\langle |\phi| \rangle$ is presented in Fig. 3 where we also compare with the classical value of the condensate. As can be seen, for small volumes, the signal for phase transition is very weak to detect, but 50^2 lattice is big enough to observe the transition.

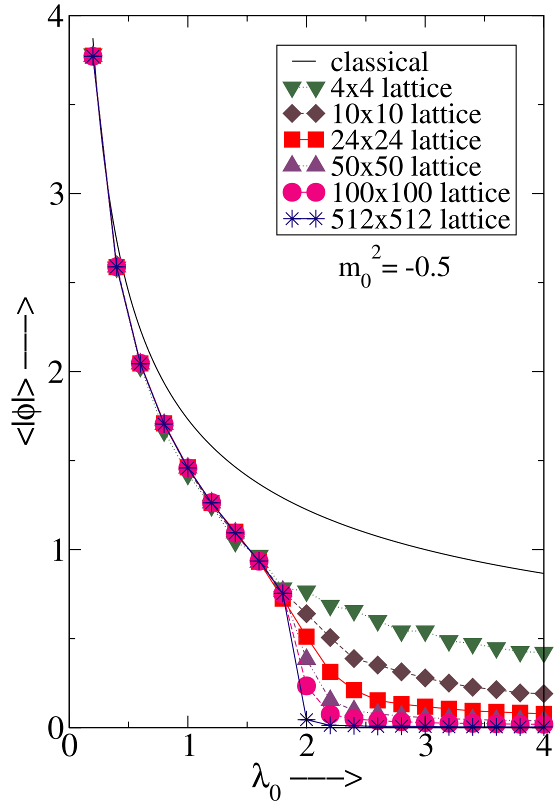

FIG. 3 (color online). $\langle|\phi|\rangle$ versus λ_0 for various lattice volumes. For comparison, classical result is also given.

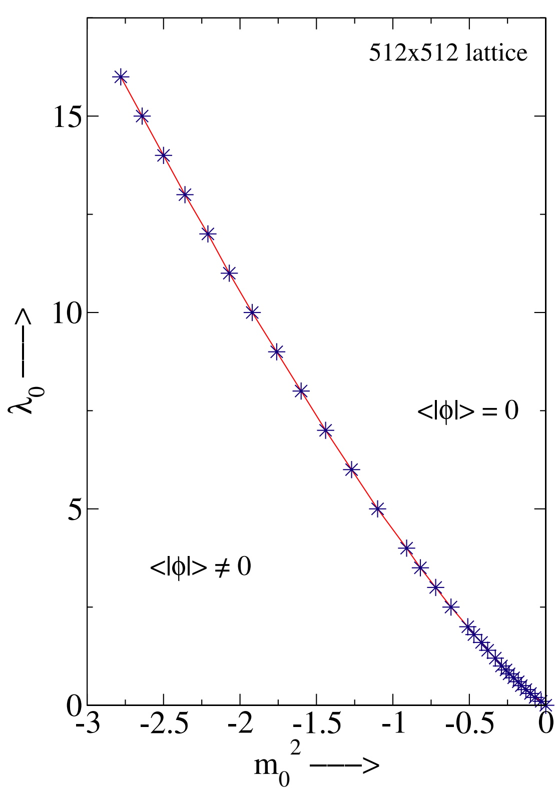

FIG. 4 (color online). Phase diagram.

In Figs. 2 and 3 and all the figures to follow the standard error bars, if not visible, are smaller than the symbols unless otherwise stated.

The phase diagram we obtained for a 512^2 lattice using PBC is presented in Fig. 4. This agrees with the phase diagram obtained in [4,5,15]. In [15] the authors extrapolate their results to infinite volume. We observe that our 512^2 lattice results are as good as the infinite volume result in Ref. [15].

Most of our calculations are performed at $m_0^2 = -0.5$. The corresponding critical coupling λ_0^c is around 1.95.

IV. CALCULATION WITH APBC: KINK CONFIGURATIONS AND $\overline{\phi}_{\mathrm{diff}}$

Cluster algorithms are known to fail with antiperiodic boundary conditions in ϕ^4 theory [16]. For the calculation of $\overline{\phi}_{\mathrm{diff}}$, because of APBC, we cannot use the cluster algorithm and we resort to the standard Metropolis algorithm. In addition to the problems associated with critical slowing down, we also face problems associated with computing the profile of an extended object, the topological kink in this case and also in fixing its location for the measurement of topological charge.

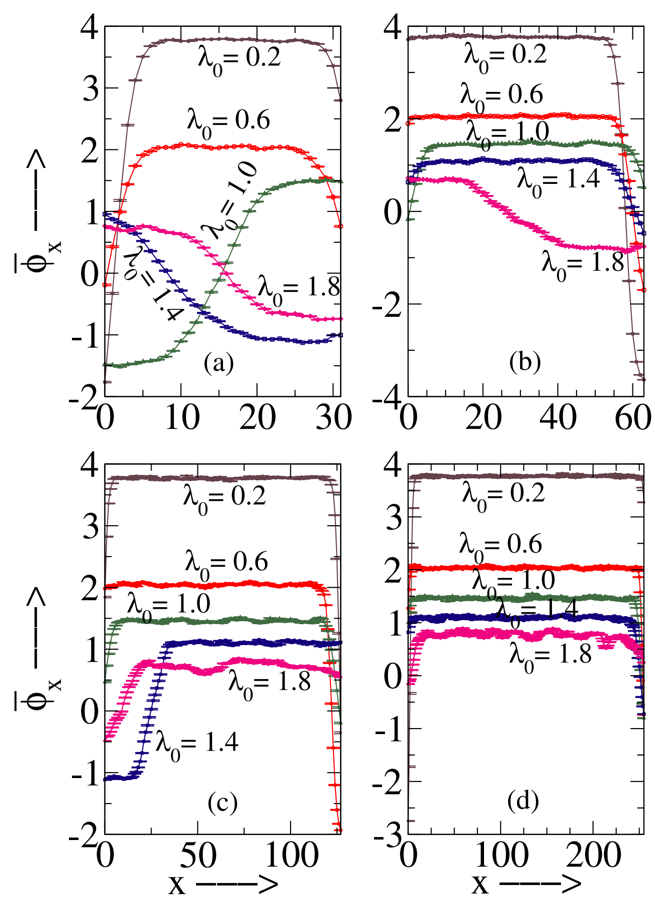

FIG. 5 (color online). $\overline{\phi}_x$ for cold-start(+1) at $m_0^2 = -0.5$ for lattices (a) 32^2, (b) 64^2, (c) 128^2 and (d) 256^2.

First we note that the location of the kink cannot be controlled with a cold start as we show in Fig. 5. It is perhaps not easy at first sight, to recognize that most of the configurations seen in Fig. 5 are actually kink configurations. To bring it out we refer to Fig. 7 in which we demonstrate schematically, how a kink centered near the boundary of a toroidal lattice with APBC on the fields should look like. Note that the $(L + i)$th site is identified with the ith site on a toroidal lattice but the sign of the field is flipped on account of APBC on the fields giving it a plateaulike appearance. Most of the configurations in Fig. 5 resemble this plateau.

All such profiles in Fig. 5 are therefore basically kinks or antikinks located near the edge. We think that the formation of kinks near the edge are favored by the algorithm because it is clearly much easier to generate the plateaulike configurations from a cold start ($\phi = +1$) compared to a kink which involves a flipping of the sign of fields over larger region. Near the critical region of course, we expect the formation of kink like configurations to be much easier and it is consistent with our observation ($\lambda_0 = 1.8$ in Fig. 5). Let us mention at this point that because of translational invariance, the position of a kink is of no significance as long as it does not tend to the spatial infinities.

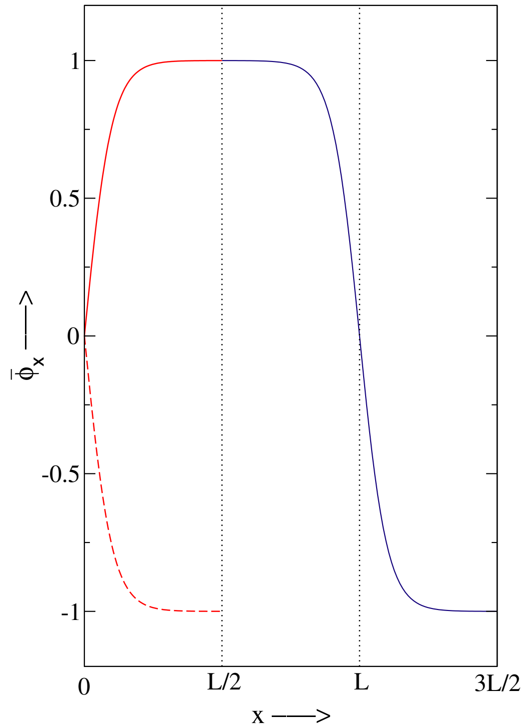

FIG. 7 (color online). The part of the kink between L to $3L/2$ gets translated to 0 to $L/2$ (short dash) due to periodicity of toroidal lattice and finally gets inverted (solid line between 0 to $L/2$) due to APBC on the field.

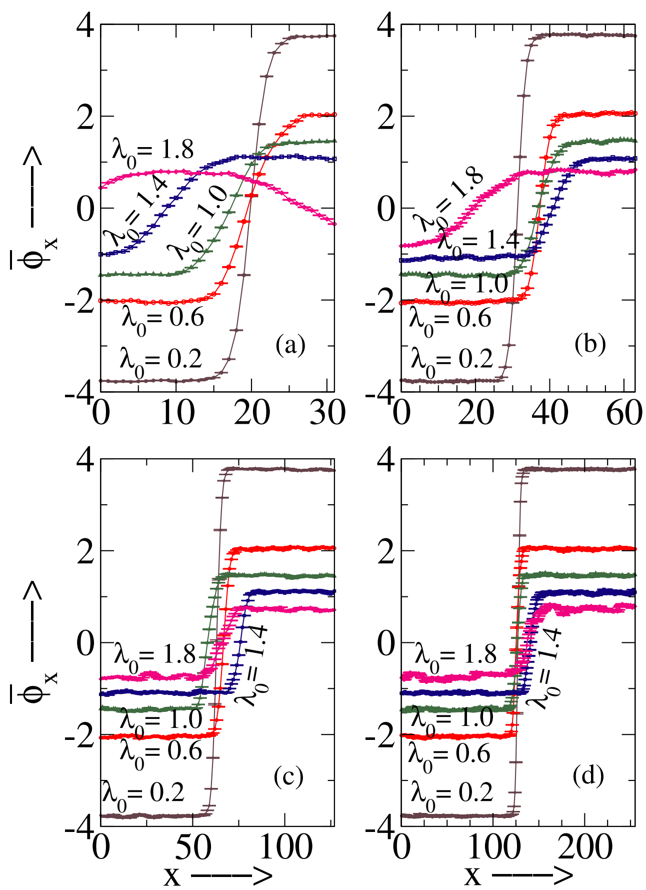

FIG. 6 (color online). $\overline{\phi}_x$ for kink start at $m_0^2 = -0.5$ for lattices (a) 32^2, (b) 64^2, (c) 128^2 and (d) 256^2.

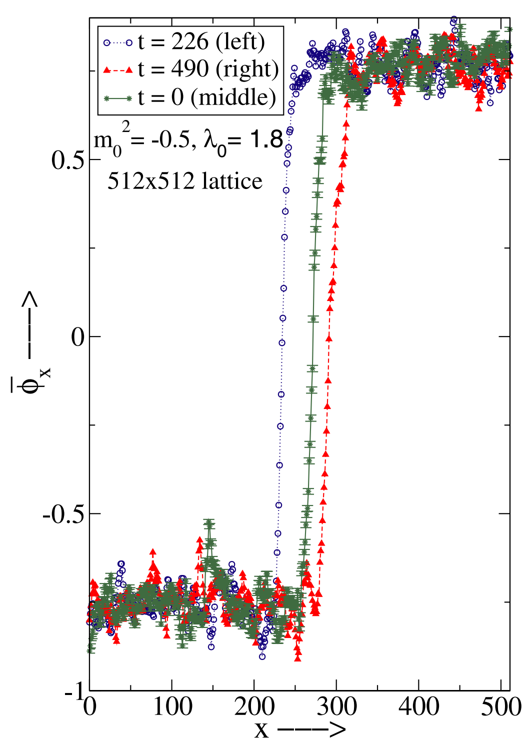

FIG. 8 (color online). $\overline{\phi}_x$ for kink start for three different time slices showing motion of the kink.

To make our results more transparent we intended to work with kink configurations that are centered near the middle of the lattice. The definition of the topological charge that we use [Eq. (2.11)] also presupposes that our kink is actually located near the center.

To obtain such kink configurations we generated configurations by using the kink start ($\phi_x = -1$ for $0 \le x < \frac{L}{2}$ and $\phi_x = +1$ for $\frac{L}{2} \le x < L$) which nicely generates the kink configurations near the center (Fig. 6).

In passing we remark that the data in both Fig. 5 and 6 are seen to deteriorate with smaller volumes owing to finite size effects which are as usual more serious near the critical point (large coupling in this case). We have also observed that there is a small but noticeable increase in the *size* of the kinks (spatial extent over which ϕ_x changes) with an increase of λ_0 which affects the sharpness of the kinks in a small volume near the critical region.

Reasonably good kink configurations for the measurement of topological charge (with stable flat regions) are obtained with the kink start with 128^2 and 256^2 lattices.

We have observed kink motion in the vicinity of the critical region. This motion of the kink is clearly visible in Fig. 8 where we present $\overline{\phi}_x$ for a 512^2 lattice for three different time slices. Probably because the kink mass is small enough at the couplings, we see this motion. This is consistent with the vanishing of the kink mass at the critical point. Incidentally, the equality in the magnitude of $\overline{\phi}_{\rm diff}$ among different time slices demonstrates the expected conservation of topological charge. One would have ideally liked to increase the statistics by averaging over the time slices at a given x, but the movement of kink makes this not viable in general. However, taking into account the fact that the location of the kink is of no consequence in the measurement of topological charge, we have actually forced the formation of kinks at the middle (by fixing the value of the field at the midpoint in every time slice, to zero) and then taken the average over time slices.

V. KINK CONDENSATION AND TOPOLOGICAL CHARGE

Let us now come to the discussion of the topological charge. For the use of Eq. (2.13) for the topological charge, it is necessary that we get the same phase diagram and critical behavior for the numerator and the denominator of this expression. The simulation results with PBC and APBC shown in Fig. 9 for 48^2 lattice indeed confirm this within our numerical accuracy. Here we would also like to remark that Fig. 9 shows that the disappearance of the kink configuration (characterized by $|\overline{\phi}_{\rm diff}| = 0$) and the onset of the phase transition from broken to symmetric phase (characterized by $\langle|\phi|\rangle = 0$) coincide. It therefore suggests "kink condensation" as a possible mechanism for the order-disorder phase transition in $1 + 1$ dimensional ϕ^4 theory. Such a connection is known to exist in 2-

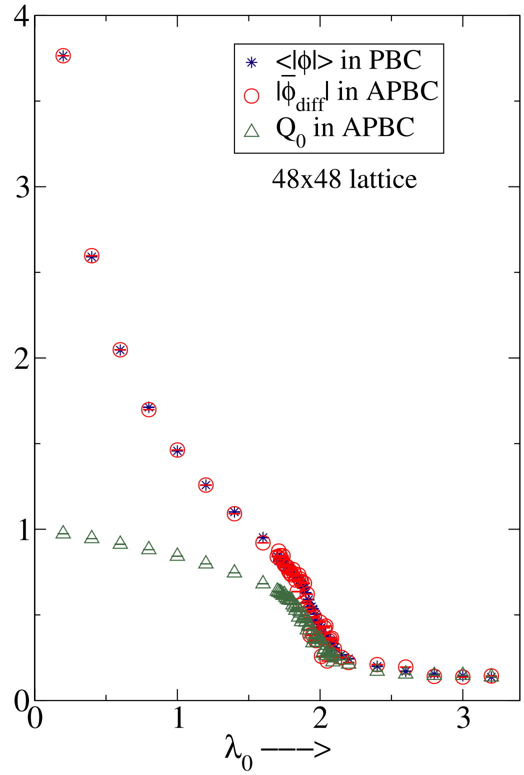

FIG. 9 (color online). $\langle|\phi|\rangle$ for PBC and $|\overline{\phi}_{\rm diff}|$ for APBC versus λ_0 at $m_0^2 = -0.5$. Q_0 for APBC also plotted.

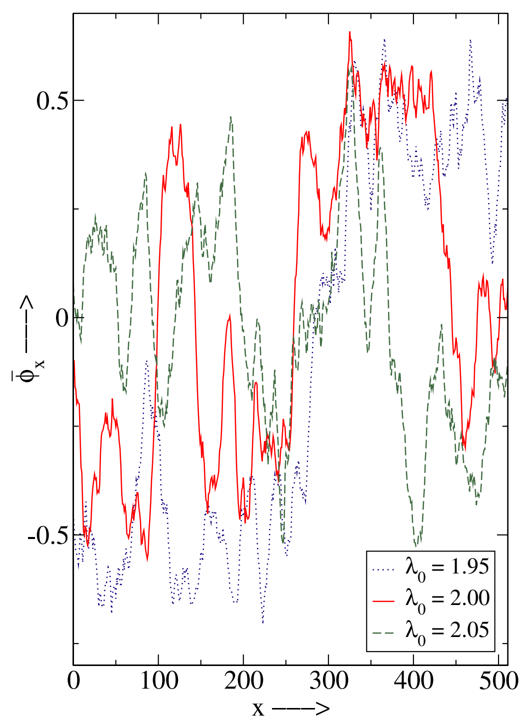

FIG. 10 (color online). Kink condensation shown for 512^2 lattice at $m_0^2 = -0.5$.

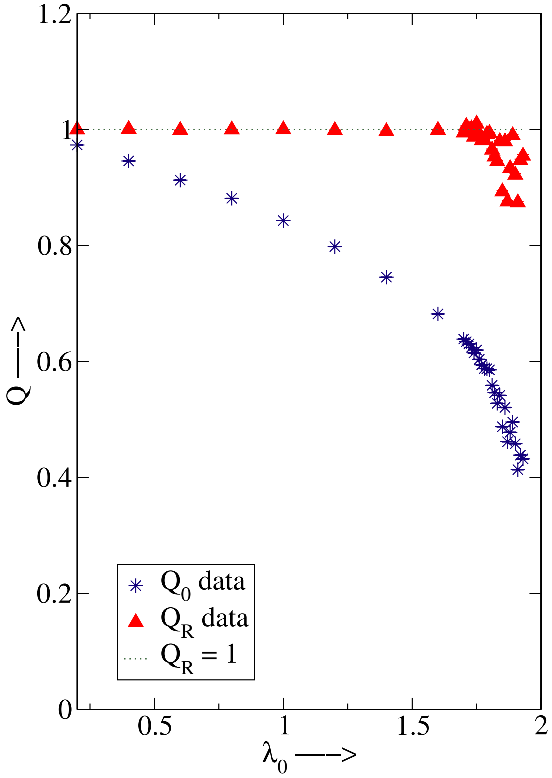

FIG. 11 (color online). Topological charge calculated with 48^2 lattice at $m_0^2 = -0.5$.

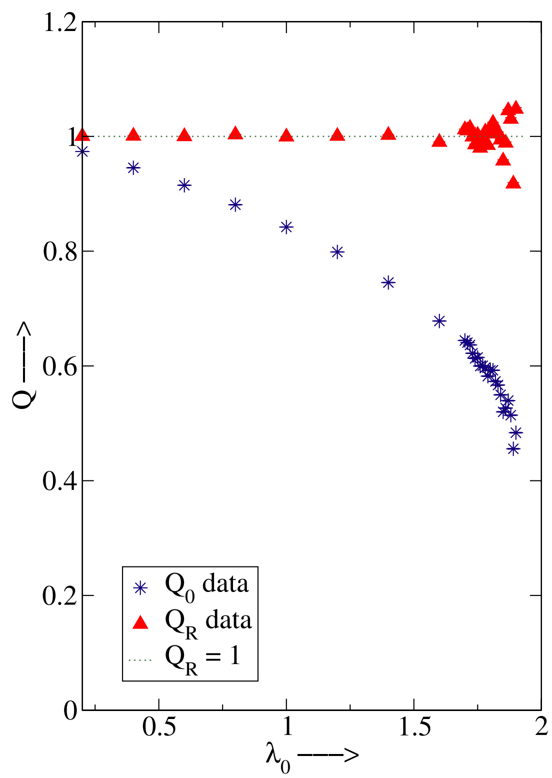

FIG. 12 (color online). Topological charge calculated with 512^2 lattice at $m_0^2 = -0.5$.

dimensional Ising model [7–9] which is believed to be in the same universality class and is not totally unexpected in ϕ^4 theory since it has an *embedded* Ising variable. Condensation of topological excitation has generally been associated with the mechanism of phase transition in many statistical and quantum field theories. Kink condensation is clearly visible in Fig. 10 where we present $\bar{\phi}_x$ for three couplings very close to the critical region. At $\lambda_0 = 1.95$, kink configuration is just barely visible, i.e., the boundary value of $\bar{\phi}_x$ is still nonzero and $\bar{\phi}_x$ passes through zero only once. As the coupling increases, the boundary value of $\bar{\phi}_x$ reduces to zero signalling disappearance of the single kink. However, $\bar{\phi}_x$ passes through zero many times showing closely packed kink-antikink configurations of varying amplitudes.

Finally we present the result for the topological charge in Figs. 11 and 12. The behavior of the topological charge for a range of λ_0 is depicted in Figs. 11 and 12 for 48^2 and 512^2 lattices, respectively. The figures clearly demonstrate the need for renormalization: It restricts Q_R to $+1$ for the whole range of λ_0 investigated in conformity with our expectations. Near the critical region the fluctuation of Q_R around unity is due to different systematic and statistical errors associated with different numerical algorithms used for the calculation of the numerator and the denominator of Eq. (2.13) especially because the numerator is evaluated with Metropolis algorithm which is known to suffer from critical slowing down. We also see by comparing Figs. 11 and 12 that there is noticeable improvement of data with volume near the critical region.

VI. SUMMARY AND CONCLUSIONS

In this work we have investigated the topological charge in $1 + 1$ dimensional lattice ϕ^4 field theory. We have shown that with APBC in the spatial direction, lowest energy configuration is a kink or an antikink. In order to characterize the different phases of the theory with APBC we have proposed a simple order parameter $\overline{\phi}_{\rm diff}$ and we have demonstrated its effectiveness.

In the process of computing the topological charge we have observed that as the system moves on from ordered to disordered phase, the single kink (or antikink) gives way to the occurrence of a multitude of kink-antikink pairs suggesting kink condensation as a possible mechanism for this phase transition.

A major issue in extracting the topological charge is the renormalization. By a particular choice of the renormalized coupling (in the topologically trivial sector), we are able to express the topological charge in the renormalized theory as the ratio of renormalized order parameters in the lattice theories with APBC and PBC. Provided the field renormalization constants are the same in the two cases, the expression for the topological charge in the renormalized theory becomes the ratio of unrenormalized order parameters. Making use of this, we have computed the topological charge in the renormalized theory and demonstrated that it indeed maintains the value +1 in the broken phase.

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