

Clash of positivities in topological density correlators

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We discuss the apparent conflict between reflection positivity and positivity of the topological susceptibility in two-dimensional nonlinear sigma models and in four-dimensional gauge theories. We pay special attention to the fact that this apparent conflict is already present on the lattice; its resolution puts some nontrivial restrictions on the short-distance behavior of the lattice correlator. It is found that these restrictions can be satisfied both in the case of asymptotic freedom and the dissident scenario of a critical point at finite coupling.

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I. INTRODUCTION

Topological density correlators have some positivity properties that may seem paradoxical at first sight. If we denote the topological density by $q(x)$ and (minus) its two-point function (in Euclidean space) by

$$F(x) \equiv -\langle q(0)q(x) \rangle, \quad (1)$$

reflection positivity (RP), i.e. positivity of the metric in Hilbert space demands that

$$F(x) \geq 0 \quad \text{for } x \neq 0, \quad (2)$$

as has been pointed out long ago (see [1,2]). Actually it is easy to see that F cannot vanish anywhere (unless it vanishes identically), i.e.

$$F(x) > 0 \quad \text{for } x \neq 0; \quad (3)$$

on account of the Lehmann-Källén spectral representation.

On the other hand the topological susceptibility

$$\chi_t \equiv \int dx \langle q(0)q(x) \rangle = -\sum_x F(x) \quad (4)$$

should be nonnegative on account of the positivity of the euclidean functional measure (at least if there is no non-zero θ angle), since it can be obtained as

$$\chi_t = \lim_{V \rightarrow \infty} \frac{1}{V} \langle Q_V^2 \rangle, \quad (5)$$

where Q_V is the topological charge in the finite volume V . As has been stressed repeatedly, these two properties can be reconciled only by requiring specific contact terms in $F(x)$, something that is of no physical relevance in axiomatic quantum field theory, because contact terms do not contribute to the analytic continuation from Euclidean to Minkowski space.

We want to approach this problem by considering the quantum field theory as a continuum limit of a lattice field theory in which both positivities are already satisfied at

nonzero lattice spacing. We are aware of the fact that (2) does not hold for all lattice versions of the models in question, but if we rely on the universality principle we should be allowed to restrict our attention to the lattice theories satisfying it. After all, RP (for gauge invariant fields) has to be true in the continuum limit, if the theory is to make physical sense. Similarly, there are nonlocal definitions of the topological density that do not satisfy (2), but again the violation should only be a lattice artefact.

II. TWO DIMENSIONS

We will discuss the case of the two-dimensional $O(3)$ nonlinear σ model in some detail and remark about the $\mathbb{C}P^{N-1}$ models and the massless and massive Schwinger models at the end.

The lattice $O(3)$ model is defined in terms of the standard lattice action

$$S = \sum_{\langle xy \rangle} s(x) \cdot s(y), \quad (6)$$

where $s(\cdot) \in S^2 \subset \mathbb{R}^3$ and the Gibbs density is proportional to $e^{-\beta S}$. We are working on the unit lattice \mathbb{Z}^2 in a regime $\beta < \beta_{crit}$ where the model shows exponential clustering with correlation length ξ . The dynamically defined lattice spacing a is proportional to the inverse correlation length

$$a = \frac{\ell_0}{\xi}, \quad (7)$$

where the constant ℓ_0 defines the standard of length.

Of course according to the standard wisdom $\beta_{crit} = \infty$, but Patrascioiu and Seiler have raised doubts about this over the years (for a recent summary see [3] and references given there) and the issue remains an open mathematical question [4].

The most natural definition of the topological density $q(x^*)$ on the lattice is associated with a plaquette or equivalently with a site x^* of the dual lattice; other definitions associate q with lattice sites. As examples we mention two choices that satisfy RP:

(a) “field theoretic definition” [5],

$$q_{fi}(x) = \frac{1}{32\pi} \sum_{\mu\nu} \sum_{ijk} \epsilon_{\mu\nu} \epsilon_{ijk} s_i(x) [s_j(x + \hat{\mu}) - s_j(x - \hat{\mu})] [s_k(x + \hat{\nu}) - s_k(x - \hat{\nu})] \quad (8)$$

(b) “geometric definition” [6]

$$q_{geom}(x^*) = \frac{1}{8\pi} \{A(s(1), s(2), s(3)) + A(s(1), s(3), s(4)) + A(s(1), s(2), s(4)) + A(s(2), s(3), s(4))\} \quad (9)$$

where the sites 1, 2, 3, 4 are the four corners of the plaquette dual to x^* and $A(., ., .)$ is the area of the spherical triangle spanned by the three points on the sphere appearing as arguments.

(9) arises from the expression found in [6] by symmetrization, so as to make it antisymmetric with respect to time reflections, a prerequisite for RP.

We study the two-point correlation function at a certain value of β , which we prefer to parameterize by $a(\beta) = \ell_0/\xi(\beta)$

$$F_a(x) = -a^{-4} \left\langle q(0) q\left(\frac{x}{a}\right) \right\rangle \quad (10)$$

where we inserted the prefactor a^{-4} in anticipation of the continuum limit

$$F_0(x) = \lim_{a \rightarrow 0} F_a(x), \quad (11)$$

which is not expected to require any divergent field strength renormalization.

Note that the whole lattice definition of the topological charge density (in particular, all contact terms of the two-point correlator arising from this definition) must be taken into account to analyze the interplay of the behavior of the correlator at $x = 0$ and at $x \neq 0$ necessary to fulfill positivity requirements. For instance, additive renormalizations suggested to define a “physical” topological susceptibility in the continuum limit should not be introduced here. We do not want to make any claims concerning the existence of the continuum limit of the topological susceptibility, which is a difficult issue in the case of the $O(3)$ model (see for instance [7,8]). The two-point correlator of the topological charge density could be well defined in this limit even if χ_t is not.

Since $q(x)$ is a dimension 2 operator, naively one would expect that the short-distance behavior of its two-point correlation function is

$$F_0(x) = O\left(\frac{1}{|x|^4}\right). \quad (12)$$

The two positivities satisfied by F_a are

$$F_a(x) > 0 \quad \text{for } x \neq 0 \quad (13)$$

and

$$\chi_t^a = -\sum_x a^2 F_a(x) \geq 0. \quad (14)$$

These two inequalities imply

$$\langle q(0)^2 \rangle \geq -\sum_{x \neq 0} \left\langle q(0) q\left(\frac{x}{a}\right) \right\rangle = \sum_{x \neq 0} a^2 F_a(x) \quad (15)$$

and if we rewrite (15) as

$$\chi_t^a = \langle q(0)^2 \rangle a^{-2} - \sum_{x \neq 0} a^2 F_a(x) \geq 0, \quad (16)$$

we see that the topological susceptibility is the remainder of the incomplete cancellation of the two sides of (15).

Replacing heuristically the right hand side of Eq. (15) by its continuum limit one is tempted to write

$$\int_{|x| \geq ad} F_0(x) d^2x \leq a^{-2} \langle q(0)^2 \rangle \quad (17)$$

with some constant d of order 1. Using the fact that according to tree level perturbation theory (which is uncontested) there is a constant c such that for β greater than some β_0

$$\langle q(0)^2 \rangle \leq \frac{c}{\beta^2} \quad (18)$$

we then would conclude that

$$\int_{|x| \geq ad} F_0(x) d^2x \leq \frac{c'}{\beta^2 a^2}. \quad (19)$$

We will later give a more precise derivation of a slightly weaker inequality than Eq. (19), that depends, however, on a certain assumption about the approach to the continuum.

Note that in this equation a should be considered as a function of β . It has to remain valid as $a \rightarrow 0$, i.e. $\beta \rightarrow \beta_{crit}$. So Eq. (19) expresses a remarkable link between the short-distance behavior of the topological correlator and the value of the critical coupling β_{crit} . If, as commonly believed, $\beta_{crit} = \infty$, it implies that the short-distance singularity of $F_0(x)$ has to be softer than $1/|x|^4$. As will be discussed, this is in fact consistent with RG improved perturbation theory. But Eq. (19) can obviously also easily be satisfied in the dissident scenario of a finite value of β_{crit} ; in this case the “classical” behavior (12) is allowed.

Another remarkable feature in the conventional scenario is this: according to asymptotic scaling the topological susceptibility should be exponentially small in β , but the first term on the right hand side of (16) is $O(1/\beta^2)$. That means that also the second term has to be of that order and the cancellation between the two terms has to be almost complete. It has of course been known for a long time that for instance the geometric definition does not satisfy

asymptotic scaling [6] numerically; it is an open question if it is satisfied for any definition that also obeys RP in the continuum limit. But maybe one should not worry about this point too much, since asymptotic scaling has also not been verified for the correlation length; the only interesting open question is the existence of a nontrivial continuum limit of χ_t^a , which is, however, not our concern here.

Let us now turn to the derivation of (19). It is certainly to be expected that the two-point function $F_a(x)$ converges to the continuum limit $F_0(x)$ pointwise. But one cannot expect that the approach is uniform in x ; it is to be expected that the convergence is slower the shorter the distance x is. We make the following assumption about the approach of F_a to the continuum: there are constants $d > 0$ (independent of a) and $a_0 > 0$ such that

$$\left| \frac{F_a(x)}{F_0(x)} - 1 \right| \leq \frac{1}{2} \quad \text{for } a \leq a_0 \quad \text{and} \quad \ell_0 \geq |x| \geq ad. \quad (20)$$

This assumption limits the amount of nonuniformity permitted in the approach to the continuum; it holds for correlators of free fields and can be checked in perturbation theory. In principle it can also be tested numerically. We omitted large distances because we are considering the massive continuum limit and the correlation function will decay exponentially in $\ell_0|x|$.

To use this assumption we reinterpret the lattice function $F_a(x)$ as a piecewise constant function in the continuum and the sum $\sum_{|x| \geq ad} F_a(x)$ as an integral. We get, using the triangle inequality

$$\begin{aligned} \sum_{\ell_0 \geq |x| \geq ad} F_a(x) &\geq \int_{\ell_0 \geq |x| \geq ad} F_0(x) d^2x - \int_{\ell_0 \geq |x| \geq ad} |F_0(x) \\ &\quad - F_a(x)| d^2x \\ &\geq \frac{1}{2} \int_{\ell_0 \geq |x| \geq ad} F_0(x) d^2x. \end{aligned} \quad (21)$$

Inserting this in (15) we get

$$\int_{\ell_0 \geq |x| \geq ad} F_0(x) d^2x \leq 2a^{-2} \langle q(0)^2 \rangle, \quad (22)$$

which is the announced replacement for (19).

Next we discuss inequality (22) in the conventional scenario. According to RG improved tree level perturbation theory we have (cf. [9,10])

$$F_0(x) = g^2(x) \frac{1}{|x|^4} + O(g^3(x)) \quad \text{for } x \rightarrow 0. \quad (23)$$

Inserting the leading order perturbative running coupling

$$g^2(x) \sim \frac{\text{const.}}{(\ln \mu |x|)^2} \quad (24)$$

we get

$$F_0(x) \sim \frac{\text{const.}}{|x|^4 (\ln \mu |x|)^2}, \quad (25)$$

i.e. the short-distance behavior is indeed softer than the naive one. It is now not hard to see that with this behavior one gets

$$\int_{\ell_0 \geq |x| \geq ad} F_0(x) d^2x = O\left(\frac{a^{-2}}{(\ln(\mu ad))^2}\right). \quad (26)$$

This is consistent with (22) if one assumes asymptotic scaling, because then to leading order $\beta^2 = O((\ln a)^2)$.

The above discussion carries over without any essential changes to the two-dimensional $\mathbb{C}\mathbb{P}^{N-1}$ models; in fact it is even simpler due to the fact that there is a very natural definition of the topological density as the field strength of the auxiliary Abelian gauge field in these models.

In the (massive or massless) Schwinger model the situation is slightly different: the value of β_{crt} is finite; in the massless version there is perfect cancellation between the two terms in (22), whereas in the massive Schwinger model the cancellation is incomplete. The Schwinger model is also atypical in that the topological density is really a dimension 0 field—this is due to the fact that there is a dimensional parameter (the electric charge) in this model.

III. FOUR DIMENSIONS

The discussion in four dimensions, in particular, QCD, parallels the one in two dimensions, so we will limit ourselves to pointing out the necessary modifications of the previous discussion.

Again there are different lattice definitions of the topological density to be considered. Among them the so-called field theoretic definition [11] satisfies RP in a straightforward manner. There are also geometric definitions satisfying RP [12,13]. The physically most relevant definitions, however, are based on the relation between chirality and topology; only these lead to a solution of the $U(1)$ problem of QCD via credible derivations of the Witten-Veneziano formula [2,14–19], and are generally nonlocal, making RP very nonobvious. In this context it is gratifying that recently the topological two-point function based on the overlap Dirac operator has been measured and found indeed to satisfy RP, at least for lattice distances greater than 2 [20].

The topological density, being given by $\frac{g^2}{32} \pi^2 F_{\mu\nu} \tilde{F}_{\mu\nu}$ in the continuum, is now a dimension 4 operator and hence its two-point correlator on the lattice should be defined as

$$-F_a(x) = a^{-8} \left\langle q(0) q\left(\frac{x}{a}\right) \right\rangle, \quad (27)$$

where x may be a site of the original or the dual lattice. The short-distance behavior of the continuum limit $F_0(x)$ is now naively

$$F_0(x) = O\left(\frac{1}{|x|^8}\right) \quad (28)$$

and the topological susceptibility is the difference of two almost cancelling positive terms:

$$\chi_t^a = \langle q(0)^2 \rangle a^{-4} - \sum_{x \neq 0} a^4 F_a(x) \quad (29)$$

as in two dimensions. Again the contact term satisfies

$$\langle q(0)^2 \rangle \leq \frac{c}{\beta^2} \quad (30)$$

just as in two dimensions.

The approach to the continuum should satisfy the same uniformity as in two dimensions (see Eq. (20)). By the same reasoning as above we obtain then

$$\int_{\ell_0 \geq |x| \geq ad} F_0(x) d^4x \leq 2a^{-4} \langle q(0)^2 \rangle \quad (31)$$

and again we find that this can be satisfied either by assuming the softened short-distance behavior

$$F_0(x) \sim \frac{1}{|x|^8 (\ln(\mu|x|))^2} \quad (32)$$

or, of course, by the existence of a critical point at finite β .

IV. CONCLUSIONS

The two positivities of the topological two-point function are superficially in conflict with each other. To reconcile them, one needs first of all specific contact terms. It is a remarkable fact that we obtain restrictions on nonuniversal “unphysical” quantities from these considerations.

In addition we found out that:

- (a) either the short-distance behavior of $F_0(x)$ is softened logarithmically compared to the naive tree level behavior, in a way consistent with RG improved tree level perturbation theory,
- (b) or there is a critical point at a finite value of β . In another paper [21] we report on a direct lattice perturbation calculation for the 2D $O(3)$ model, which verifies consistency with the RG improved tree level expression Eq. (23).

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