Model-independent properties of the *B*-meson distribution amplitude

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The operator product expansion is used to obtain model-independent predictions for the first two moments of the renormalized B-meson light-cone distribution amplitude $\phi_+^B(\omega, \mu)$, defined with a cutoff $\omega \leq \Lambda_{\rm UV}$. The leading hadronic power corrections are given in terms of the parameter $\bar{\Lambda} = m_B - m_b$. From the cutoff dependence of the zeroth moment an analytical expression for the asymptotic behavior of the distribution amplitude is derived, which exhibits a negative radiation tail for $\omega \gg \mu$. By solving the evolution equation for the distribution amplitude, an integral representation for $\phi_+^B(\omega, \mu)$ is obtained in terms an initial function $\phi_+^B(\omega, \mu_0)$ defined at a lower renormalization scale. A realistic model of the B-meson light-cone distribution amplitude is proposed, which satisfies the moment relations and has the correct asymptotic behavior. This model provides an estimate for the first inverse moment and the associated parameter λ_B.

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I. INTRODUCTION

Exclusive decays of B mesons such as $B \to \pi l \nu$ and $B \to \pi\pi, \pi K$ are important tools to search for physics beyond the Standard Model as well as to measure fundamental parameters in the flavor sector. In processes where large momentum is transferred to the soft spectator quark via hard gluon exchange, the B-meson light-cone distribution amplitude (LCDA) enters in the parameterization of hadronic matrix elements of bilocal current operators [1]. The past few years have seen a lot of progress in the theoretical framework for the analysis of exclusive B-meson decays, mainly based on QCD factorization theorems [2–5] and perturbative QCD methods [6–9]. However, in many cases the extraction of important physics from experimental data is still limited by theoretical uncertainties, often due to our ignorance of the functional form of the B-meson LCDA and other hadronic matrix elements. For example, using the soft-collinear effective theory [10–14], the large-recoil heavy-to-light form factors relevant to weak B decays have been studied at leading order in a $1/E$ expansion [15–17]. The analysis of spin-symmetry violating contributions to these form factors, in particular, relies on knowledge about the B-meson LCDA [18–21].

In spite of the importance of the B-meson LCDA, so far most studies of its properties have been limited to model-dependent analyses based on QCD sum rules [1,22,23]. In the present work, we employ the operator product expansion (OPE) to explore some model-independent properties of the LCDA. We calculate the first two moments of the distribution amplitude, derive its asymptotic behavior, and study its properties under renormalization-group evolution, thereby obtaining strong constraints on model building. Using the results of this analysis, we propose a realistic model of the B-meson LCDA and use it to estimate the important hadronic parameter λ_B [2], which enters in many analyses based on QCD factorization.

II. MOMENT ANALYSIS

The leading-twist, two-particle LCDA ϕ_+^B of the B-meson is defined in terms of the B-meson matrix element of a renormalized bilocal heavy-quark effective theory (HQET) operator relative to the matrix element of the corresponding local operator. The bilocal operator is made up of a soft spectator quark q_s and a heavy quark h at lightlike separation z, connected by a straight soft Wilson line $S_n(z, 0)$. Specifically, one defines [1]

$$\tilde{\phi}_+^B(\tau, \mu) = \frac{\langle 0|\bar{q}_s(z) S_n(z, 0) \not{n} \Gamma h(0)|\bar{B}(v)\rangle}{\langle 0|\bar{q}_s(0) \not{n} \Gamma h(0)|\bar{B}(v)\rangle}, \qquad (1)$$

where $\tau = v \cdot z - i\epsilon$. Our notation is such that z is proportional to a lightlike vector n, v is the B-meson velocity, and for convenience we choose $n \cdot v = 1$. The object Γ represents an arbitrary Dirac matrix chosen such that the operators have nonzero overlap with the B meson. The momentum-space LCDA is given by the Fourier transform

$$\phi_+^B(\omega, \mu) = \frac{1}{2\pi} \int d\tau e^{i\omega\tau} \tilde{\phi}_+^B(\tau, \mu). \qquad (2)$$

The analytic properties of the function $\tilde{\phi}_+^B(\tau, \mu)$ in the complex τ plane imply that $\phi_+^B(\omega, \mu) = 0$ if $\omega < 0$.

We start by defining regularized moments of the B-meson LCDA as (for integer $N \geq 0$)

$$M_N(\Lambda_{\rm UV}, \mu) = \int_0^{\Lambda_{\rm UV}} d\omega \omega^N \phi_+^B(\omega, \mu). \qquad (3)$$

A hard cutoff $\Lambda_{\rm UV}$ is imposed on the integral so as to avoid singularities from the region of large ω values, which are not regularized by renormalizing the bilocal operator in (1) [1]. The reason is that the position-space LCDA $\tilde{\phi}_B^+(\tau, \mu)$ and its derivatives are singular at $\tau = 0$. Only cut moments of the renormalized LCDA are UV finite. For a sufficiently large value of $\Lambda_{\rm UV}$ the moments $M_N(\Lambda_{\rm UV}, \mu)$ can be expanded in a series of B-meson matrix elements of local

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HQET operators. The basic idea is the same as that used in previous work on cut moments of the B-meson shape function entering the analysis of inclusive decays [24,25]. From the structure of the bilocal HQET operator in (1) and the Feynman rules of HQET it follows that the resulting local operators have Dirac structure

$$\bar{q}_s(\gamma\gamma\dots\gamma)\not{n}\Gamma h, \tag{4}$$

where the number of Dirac matrices inside the parenthesis is even if light quarks are treated as massless. By using the equations of motion $i\not{D}q_s = 0$ and $iv\cdot Dh = 0$, it is straightforward to find the corresponding operators of a given dimension D. For $D = 3$, the only possibility is the operator

$$Q_0 = \bar{q}_s\not{n}\Gamma h, \tag{5}$$

which appears in the denominator in (1). For $D = 4$, there are naively four subleading operators with one derivative, namely

$$\begin{aligned} Q_{1a} &= \bar{q}_s iv\cdot\overleftarrow{D}\not{n}\Gamma h, \qquad & Q_{1c} &= \bar{q}_s in\cdot D\not{n}\Gamma h, \\ Q_{1b} &= \bar{q}_s in\cdot\overleftarrow{D}\not{n}\Gamma h, \qquad & Q_{1d} &= \bar{q}_s i\not{v}\not{D}\not{n}\Gamma h. \end{aligned} \tag{6}$$

However, the Wilson coefficients of the operators Q_{1c} and Q_{1d} are zero, because the residual momentum k of the external heavy-quark field only appears as $v\cdot k$ in HQET diagrams. Hence, these operators can be ignored. For $D \geq 5$, the situation becomes more complicated, since operators containing the gluon field $G^{\mu\nu}$ need to be included. For our current analysis, we restrict the discussion to operators of dimension less than 5.

The resulting expansion of the moments to subleading power in $1/\Lambda_{\rm UV}$ takes the form

$$\begin{aligned} M_N(\Lambda_{\rm UV},\mu) = \Lambda_{\rm UV}^N\bigg\{ & K_0^{(N)}(\Lambda_{\rm UV},\mu) + \sum_{i=a,b}\frac{K_{1i}^{(N)}(\Lambda_{\rm UV},\mu)}{\Lambda_{\rm UV}} \\ & \times\frac{\langle 0|Q_{1i}|\bar{B}(v)\rangle}{\langle 0|Q_0|\bar{B}(v)\rangle} + \dots\bigg\}, \end{aligned} \tag{7}$$

where the ellipses denote terms of order $(\Lambda_{\rm QCD}/\Lambda_{\rm UV})^2$ and higher. The short-distance coefficients $K_n^{(N)}(\Lambda_{\rm UV},\mu)$ can be calculated using on-shell external quark states and employing partonic expressions for the LCDA and for the matrix elements of the local operators Q_n to evaluate both sides of the matching relation (7). The relevant one-loop diagrams are shown in Fig. 1. Wave-function renormalization contributions cancel in the matching and thus can be omitted. We assign incoming residual momentum k to the heavy quark and incoming momentum p to the light quark, subject to the on-shell conditions $v\cdot k = 0$ and $p^2 = 0$. The Feynman amplitude is expanded to linear order in p before loop integrations are performed. This ensures that loop corrections to the matrix elements of the local operators vanish in dimensional regularization, because all integrals are scaleless. We thus obtain

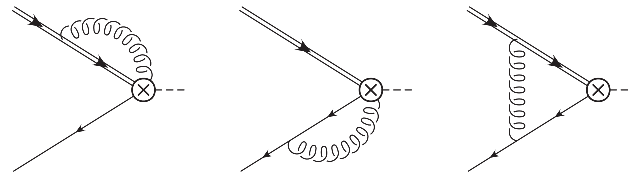

FIG. 1. One-loop diagrams contributing to the partonic matrix elements of bilocal and local operators in HQET. A crossed circle denotes an operator insertion. Double lines represent effective heavy-quark fields.

$$\begin{aligned} \langle 0|Q_{1a}|\bar{B}(v)\rangle_{\rm parton} &= v\cdot p\ \langle 0|Q_0|\bar{B}(v)\rangle_{\rm parton}, \\ \langle 0|Q_{1b}|\bar{B}(v)\rangle_{\rm parton} &= n\cdot p\ \langle 0|Q_0|\bar{B}(v)\rangle_{\rm parton}. \end{aligned} \tag{8}$$

The result for the one-loop matrix element of the bilocal HQET operator is nontrivial. According to (1), it provides us with a partonic expression for the LCDA. After $\overline{\rm MS}$ subtractions, we obtain at one-loop order

$$\begin{aligned} \phi_+^B(\omega,\mu)_{\rm parton} = {} & \delta(\omega)\left(1 - \frac{C_F\alpha_s}{4\pi}\frac{\pi^2}{12}\right) \\ & + \frac{C_F\alpha_s}{4\pi}\left[-4\left(\frac{\ln\frac{\omega}{\mu}}{\omega}\right)_*^{[\mu]} + 2\left(\frac{1}{\omega}\right)_*^{[\mu]}\right] \\ & + \delta'(\omega)\left\{-n\cdot p\left[1 - \frac{C_F\alpha_s}{4\pi}\left(1+\frac{\pi^2}{12}\right)\right]\right. \\ & \left. + v\cdot p\frac{C_F\alpha_s}{4\pi}\right\} + \frac{C_F\alpha_s}{4\pi}\left[-4n\cdot p\left(\frac{\ln\frac{\omega}{\mu}}{\omega^2}\right)_*^{[\mu]}\right. \\ & \left. + (5n\cdot p + 4v\cdot p)\left(\frac{1}{\omega^2}\right)_*^{[\mu]}\right], \end{aligned} \tag{9}$$

where $\alpha_s \equiv \alpha_s(\mu)$ throughout, unless indicated otherwise. We have retained terms of linear order in p, which will be sufficient to extract the matching coefficients $K_0^{(N)}$ and $K_{1i}^{(N)}$ in (7). The star distributions are generalized plus distributions defined as

$$\begin{aligned} \int_0^\Lambda d\omega F_\omega\left(\frac{1}{\omega}\right)_*^{[\mu]} &= \int_0^\Lambda d\omega\frac{F_\omega - F_0}{\omega} + F_0\ln\frac{\Lambda}{\mu}, \\ \int_0^\Lambda d\omega F_\omega\left(\frac{\ln\frac{\omega}{\mu}}{\omega}\right)_*^{[\mu]} &= \int_0^\Lambda d\omega\frac{F_\omega - F_0}{\omega}\ln\frac{\omega}{\mu} + \frac{F_0}{2}\ln^2\frac{\Lambda}{\mu}, \\ \int_0^\Lambda d\omega F_\omega\left(\frac{1}{\omega^2}\right)_*^{[\mu]} &= \int_0^\Lambda d\omega\frac{F_\omega - F_0 - \omega F_0'}{\omega^2} - \frac{F_0}{\Lambda} \\ & \quad + F_0'\ln\frac{\Lambda}{\mu}, \\ \int_0^\Lambda d\omega F_\omega\left(\frac{\ln\frac{\omega}{\mu}}{\omega^2}\right)_*^{[\mu]} &= \int_0^\Lambda d\omega\frac{F_\omega - F_0 - \omega F_0'}{\omega^2}\ln\frac{\omega}{\mu} \\ & \quad - \frac{F_0}{\Lambda}\left(\ln\frac{\Lambda}{\mu} + 1\right) + \frac{F_0'}{2}\ln^2\frac{\Lambda}{\mu}, \end{aligned} \tag{10}$$

where $F(\omega)$ is a smooth test function, and we use the shorthand notation $F_\omega \equiv F(\omega)$ and $F'_\omega \equiv F'(\omega)$.

Given the results (8) and (9), it is straightforward to derive expressions for both sides of the matching relation (7) in the parton model, and to extract the desired expressions for the Wilson coefficients. At one-loop order, we find

$$K_0^{(0)} = 1 + \frac{C_F\alpha_s}{4\pi}\left(-2\ln^2\frac{\Lambda_{\rm UV}}{\mu} + 2\ln\frac{\Lambda_{\rm UV}}{\mu} - \frac{\pi^2}{12}\right),$$
$$K_{1a}^{(0)} = \frac{C_F\alpha_s}{4\pi}(-4), K_{1b}^{(0)} = \frac{C_F\alpha_s}{4\pi}\left(4\ln\frac{\Lambda_{\rm UV}}{\mu} - 1\right) \tag{11}$$

for the zeroth moment, and

$$K_0^{(1)} = \frac{C_F\alpha_s}{4\pi}\left(-4\ln\frac{\Lambda_{\rm UV}}{\mu} + 6\right),$$
$$K_{1a}^{(1)} = \frac{C_F\alpha_s}{4\pi}\left(4\ln\frac{\Lambda_{\rm UV}}{\mu} - 1\right), \tag{12}$$
$$K_{1b}^{(1)} = 1 + \frac{C_F\alpha_s}{4\pi}\left(-2\ln^2\frac{\Lambda_{\rm UV}}{\mu} + 5\ln\frac{\Lambda_{\rm UV}}{\mu} - 1 - \frac{\pi^2}{12}\right)$$

for the first moment.

We have repeated the entire calculation outlined above in a different regularization scheme, where the dependence of the Feynman amplitudes on the component $n \cdot p$ of the light-quark momentum is kept exactly, whereas we linearize in the remaining components of p. In this scheme the loop corrections to the matrix elements of the local operators in (8) no longer vanish, and the result for the LCDA is far more complicated than that displayed in (9). Nevertheless, we obtain the same expressions for the Wilson coefficients $K_n^{(0)}$ and $K_n^{(1)}$ as given above. This is a highly nontrivial check, which gives us confidence in the correctness of our results.

The Wilson coefficients describe the short-distance physics associated with the large cutoff scale $\Lambda_{\rm UV}$, and hence it was legitimate to obtain them using a partonic calculation. Long-distance effects, on the other hand, reside in the hadronic matrix elements of the local operators Q_n, which cannot be calculated reliably using perturbation theory. However, these matrix elements are constrained by heavy-quark symmetry and can be parameterized in terms of universal form factors [26]. The results are particularly simple in the case of the operators Q_{1i}. Using relations derived in [27], we find that

$$\frac{\langle 0|Q_{1a}|\bar{B}(v)\rangle}{\langle 0|Q_0|\bar{B}(v)\rangle} = \bar{\Lambda}, \qquad \frac{\langle 0|Q_{1b}|\bar{B}(v)\rangle}{\langle 0|Q_0|\bar{B}(v)\rangle} = \frac{4\bar{\Lambda}}{3}, \tag{13}$$

where the quantity $\bar{\Lambda} = m_B - m_b$ is the only hadronic parameter needed at this order. The first-order power corrections to the moments M_N can now be expressed in terms of $\bar{\Lambda}$. At one-loop order, and to subleading order in the power expansion in $1/\Lambda_{\rm UV}$, the results are

$$M_0 = 1 + \frac{C_F\alpha_s}{4\pi}\left(-2\ln^2\frac{\Lambda_{\rm UV}}{\mu} + 2\ln\frac{\Lambda_{\rm UV}}{\mu} - \frac{\pi^2}{12}\right) + \frac{16\bar{\Lambda}}{3\Lambda_{\rm UV}}\frac{C_F\alpha_s}{4\pi}\left(\ln\frac{\Lambda_{\rm UV}}{\mu} - 1\right),$$
$$M_1 = \Lambda_{\rm UV}\frac{C_F\alpha_s}{4\pi}\left(-4\ln\frac{\Lambda_{\rm UV}}{\mu} + 6\right) + \frac{4\bar{\Lambda}}{3}\left[1 + \frac{C_F\alpha_s}{4\pi}\times\left(-2\ln^2\frac{\Lambda_{\rm UV}}{\mu} + 8\ln\frac{\Lambda_{\rm UV}}{\mu} - \frac{7}{4} - \frac{\pi^2}{12}\right)\right]. \tag{14}$$

These are our final expressions for the first two moments of the renormalized B-meson LCDA. As long as $\Lambda_{\rm UV} \gg \Lambda_{\rm QCD}$, they are model-independent predictions of QCD, valid up to higher-order terms in α_s and $1/\Lambda_{\rm UV}$. The fixed-order perturbative expressions derived here are applicable if the two scales $\Lambda_{\rm UV}$ and μ are of the same order, so that the logarithms in the matching coefficients are not parametrically large.

Taking the derivative of the zeroth moment M_0 in (14) with respect to the cutoff, we can obtain a model-independent description of the asymptotic behavior of the B-meson LCDA [24], i.e.

$$\phi_+^B(\omega, \mu) = \frac{dM_0(\Lambda_{\rm UV}, \mu)}{d\Lambda_{\rm UV}}\Bigg|_{\Lambda_{\rm UV}=\omega}. \tag{15}$$

At one-loop order, the result reads

$$\phi_+^B(\omega, \mu) = \frac{C_F\alpha_s}{\pi\omega}\left[\left(\frac{1}{2} - \ln\frac{\omega}{\mu}\right) + \frac{4\bar{\Lambda}}{3\omega}\left(2 - \ln\frac{\omega}{\mu}\right) + \dots\right]. \tag{16}$$

This relation holds for $\omega \gg \Lambda_{\rm QCD}$, up to power corrections of order $\Lambda_{\rm QCD}^2/\omega^3$. We observe that the radiation tail of the B-meson LCDA becomes negative at $\omega \approx \sqrt{e}\mu$ for a sufficiently large value of μ. This model-independent prediction for the asymptotic behavior of $\phi_+^B(\omega, \mu)$ agrees qualitatively with the findings of the QCD sum-rule analysis in [23].

III. ELIMINATION OF THE POLE MASS

Our calculations so far have been performed in the on-shell (pole) scheme, where $\bar{\Lambda} = m_B - m_b^{\rm pole}$ is defined in terms of the b-quark pole mass. However, it is well known that the pole mass suffers from infrared renormalon ambiguities [28,29]. Hence, it is desirable to eliminate the pole-scheme parameter $\bar{\Lambda}$ in favor of a new, short-distance parameter $\bar{\Lambda}_{\rm RS}$ defined in some renormalization scheme. For our purposes it is most convenient to employ a so-called ‘‘low-scale subtracted’’ heavy-quark mass defined with the help of a hard subtraction scale μ_f. Examples are the ‘‘kinetic mass’’ [30], the ‘‘potential-subtracted mass’’ [31], the ‘‘1S mass’’ [32], and the ‘‘shape-function mass’’ [24,33]. Using the last definition as an example, we would use the relation

$$\bar{\Lambda} = \bar{\Lambda}_{\rm SF}(\mu_f, \mu) + \mu_f \frac{C_F\alpha_s}{4\pi}\left(8\ln\frac{\mu_f}{\mu} - 4\right) + \dots \quad (17)$$

to eliminate the pole-scheme parameter $\bar{\Lambda}$ in the moment relations (14), identifying the subtraction scale μ_f with the cutoff $\Lambda_{\rm UV}$. As always, $\alpha_s \equiv \alpha_s(\mu)$.

Alternatively, the moment relations themselves can be used to define a new subtraction scheme. Guided by the tree-level relations $M_1 = 4\bar{\Lambda}/3$ and $M_0 = 1$, we are led to define a running parameter (the subscript "DA" stands for "distribution amplitude")

$$\bar{\Lambda}_{\rm DA}(\mu_f, \mu) \equiv \frac{3M_1(\mu_f, \mu)}{4M_0(\mu_f, \mu)} \quad (18)$$

to all orders in perturbation theory. From (14), it follows that

$$\bar{\Lambda} = \bar{\Lambda}_{\rm DA}(\mu_f, \mu)\left[1 - \frac{C_F\alpha_s}{4\pi}\left(6\ln\frac{\mu_f}{\mu} - \frac{7}{4}\right)\right] + \mu_f \frac{C_F\alpha_s}{4\pi}\left(3\ln\frac{\mu_f}{\mu} - \frac{9}{2}\right) + \dots . \quad (19)$$

By taking the ratio of M_1 and M_0 in (18) the double-logarithmic radiative corrections are eliminated. Like the other short-distance mass definitions mentioned above, the parameter $\bar{\Lambda}_{\rm DA}$ can be regarded as a "physical" quantity in the sense that it is free of renormalon ambiguities. Perturbative relations can be used to transform from our new scheme to any other mass-definition scheme. For example, from (17) and (19) it follows that at one-loop order the parameter $\bar{\Lambda}_{\rm DA}$ is related to the parameter $\bar{\Lambda}_{\rm SF}$ in the shape-function scheme through

$$\bar{\Lambda}_{\rm DA}(\mu_f, \mu) = \bar{\Lambda}_{\rm SF}(\mu_*, \mu_*)\left[1 + \frac{C_F\alpha_s}{4\pi}\left(6\ln\frac{\mu_f}{\mu} - \frac{7}{4}\right)\right] - \mu_f \frac{C_F\alpha_s}{4\pi}\left(3\ln\frac{\mu_f}{\mu} - \frac{9}{2} + \frac{4\mu_*}{\mu_f}\right). \quad (20)$$

A rather precise value for $\bar{\Lambda}_{\rm SF}$ has been extracted from moment analyses of various spectra in the inclusive decays $B \to X_s\gamma$ and $B \to X_u l\nu$, yielding $\bar{\Lambda}_{\rm SF}(\mu_*, \mu_*) = (0.65 \pm 0.06)$ GeV at $\mu_* = 1.5$ GeV (and at leading order in $1/m_b$) [33,34]. This value will be used as an input when we compute the running parameter $\bar{\Lambda}_{\rm DA}(\mu_f, \mu)$ from the above relation.

IV. RENORMALIZATION-GROUP EVOLUTION

In Sec. II we have derived model-independent predictions for moments of the B-meson LCDA and for its asymptotic behavior for large ω. The renormalization group can be used to obtain a model-independent description of how $\phi_+^B(\omega, \mu)$ changes under variation of the scale μ. The integro-differential evolution equation obeyed by the LCDA was derived in [35], where an analytic solution was presented in the form of a double integral. One finds that the distribution amplitude at a scale μ can be expressed in terms of that at a lower scale $\mu_0 < \mu$ by

$$\phi_+^B(\omega, \mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\varphi_0(t) f(\omega, \mu, \mu_0, it), \quad (21)$$

where

$$\varphi_0(t) = \int_0^\infty \frac{d\omega'}{\omega'}\phi_+^B(\omega', \mu_0)\left(\frac{\omega'}{\mu_0}\right)^{-it} \quad (22)$$

denotes the Fourier transform with respect to $\ln\omega$ of the function $\phi_+^B(\omega, \mu_0)$ at the initial scale μ_0. At leading order in perturbation theory, the kernel f takes the form

$$f(\omega, \mu, \mu_0, it) = e^{V(\mu,\mu_0)}\left(\frac{\omega}{\mu_0}\right)^{it+g} \times e^{-2\gamma_E g}\frac{\Gamma(1-it-g)\Gamma(1+it)}{\Gamma(1+it+g)\Gamma(1-it)}, \quad (23)$$

where

$$V(\mu, \mu_0) = -\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\left[\Gamma_{\rm cusp}(\alpha)\int_{\alpha_s(\mu_0)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')} + \gamma(\alpha)\right], \quad (24)$$

and

$$g \equiv g(\mu, \mu_0) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)} \approx \frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}. \quad (25)$$

In these expressions $\beta = d\alpha_s/d\ln\mu$ is the β-function, and $\Gamma_{\rm cusp} = C_F\alpha_s/\pi + \dots$, $\gamma = -C_F\alpha_s/2\pi + \dots$ are anomalous dimensions. The perturbative expansion of $V(\mu, \mu_0)$ at next-to-leading order can be found in [36].

Here we take a step further and simplify the solution obtained in [35] by performing the integration over t in (21) analytically. Substituting the expression for f from (23), we observe that the integrand has poles situated on the imaginary axis in the complex t plane. The poles on the negative imaginary axis are located at $t = -i(n-g)$ with $n \geq 1$ an integer (we assume $0 < g < 1$, which is satisfied for all reasonable values of scales), while those on the positive imaginary axis are located at $t = in$ with $n \geq 1$ an integer. Using the theorem of residues, we obtain

$$\phi_+^B(\omega, \mu) = e^{V(\mu,\mu_0)}e^{-2\gamma_E g}\frac{\Gamma(2-g)}{\Gamma(g)}\int_0^\infty \frac{d\omega'}{\omega'}\phi_+^B(\omega', \mu_0) \times \left(\frac{\omega_>}{\mu_0}\right)^g \frac{\omega_<}{\omega_>}\,{}_2F_1\left(1-g, 2-g; 2; \frac{\omega_<}{\omega_>}\right), \quad (26)$$

where $\omega_< = \min(\omega, \omega')$ and $\omega_> = \max(\omega, \omega')$. The hypergeometric function ${}_2F_1(a, b; c; z)$ has the series expansion

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad (27)$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$. In the limit $\mu \to \mu_0$ we have $V(\mu, \mu_0) \to 0$, $g \to 0$, and ${}_2F_1(1-g, 2-g; 2; x) \to (1-x)^{2g-1}$. Then the right-hand side in (26) reduces to the left-hand one.

Equation (26) provides the most compact expression possible for calculating the evolution of the LCDA under changes of the renormalization scale. It is tempting to conjecture that this is the exact solution to the evolution equation for the LCDA, valid to all orders in perturbation theory. An analogous statement is indeed true for the *B*-meson shape function [37]. In the present case, to prove this assertion one would need to show that the exact evolution equation for the LCDA is given by

$$\left[\frac{d}{d\ln\mu} + \Gamma_{\rm cusp}(\alpha_s)\ln\frac{\mu}{\omega} + \gamma(\alpha_s)\right]\phi_+^B(\omega,\mu) = \Gamma_{\rm cusp}(\alpha_s)\int_0^\infty d\omega'\,\frac{\omega}{\omega_>}\,\frac{\phi_+^B(\omega',\mu) - \phi_+^B(\omega,\mu)}{|\omega'-\omega|}, \quad (28)$$

where $\Gamma_{\rm cusp}$ is the universal cusp anomalous dimension of Wilson loops with lightlike segments [38,39], and γ is some other anomalous dimension. In [35], the above relation was confirmed at one-loop order.

V. PHENOMENOLOGICAL MODEL

The model-independent properties of the *B*-meson distribution amplitude derived in this work provide useful constraints on model building. In this section we suggest a realistic form for $\phi_+^B(\omega, \mu)$, which satisfies these constraints. For phenomenological purposes such a model is needed at a renormalization scale of order $\mu \sim \sqrt{m_b\Lambda_{\rm QCD}}$, as this is the characteristic "hard-collinear" scale for hard spectator scattering in exclusive *B* decays [14,36]. Our model consists of the two-component ansatz

$$\phi_+^B(\omega,\mu) = N\frac{\omega}{\omega_0^2}e^{-\omega/\omega_0} + \theta(\omega-\omega_t)\frac{C_F\alpha_s}{\pi\omega}\times\left[\left(\frac{1}{2} - \ln\frac{\omega}{\mu}\right) + \frac{4\bar\Lambda_{\rm DA}}{3\omega}\left(2 - \ln\frac{\omega}{\mu}\right)\right], \quad (29)$$

where $\bar\Lambda_{\rm DA} \equiv \bar\Lambda_{\rm DA}(\mu,\mu)$ is defined in our new scheme (19), and we set $\mu_f = \mu$ for simplicity. The first term on the right-hand side is based on the exponential form proposed in [1], while the second piece is a radiation tail added so as to ensure the correct asymptotic behavior as shown in (16). The tail is "glued" onto the exponential at a position ω_t chosen such that the resulting function is continuous. This yields

$$\omega_t = \sqrt{e}\mu\left(1 + \frac{2\bar\Lambda_{\rm DA}}{\sqrt{e}\mu} - \frac{14\bar\Lambda_{\rm DA}^2}{3e\mu^2} + \ldots\right). \quad (30)$$

The normalization constant N and the parameter ω_0 can be fixed by matching the expressions for the first two moments in (14) with the corresponding results obtained by substituting the model function (29) into (3), neglecting exponentially small terms $\sim e^{-\Lambda_{\rm UV}/\omega_0}$. All remaining terms involving the cutoff $\Lambda_{\rm UV}$ are reproduced by construction, so that the results for N and ω_0 are independent of the cutoff, as they must be. At first order in α_s, we obtain

$$N = 1 + \frac{C_F\alpha_s}{4\pi}\left[-2\ln^2\frac{\omega_t}{\mu} + 2\ln\frac{\omega_t}{\mu} - \frac{\pi^2}{12} + \frac{16\bar\Lambda_{\rm DA}}{3\omega_t}\left(\ln\frac{\omega_t}{\mu} - 1\right)\right] = 1 + \frac{C_F\alpha_s}{4\pi}\left(\frac{1}{2} - \frac{\pi^2}{12} - \frac{8\bar\Lambda_{\rm DA}}{3\sqrt{e}\mu} + \ldots\right), \quad (31)$$

and

$$\omega_0 = \frac{2\bar\Lambda_{\rm DA}}{3}\left\{1 + \frac{C_F\alpha_s}{4\pi}\left[6\ln\frac{\omega_t}{\mu} - \frac{16\bar\Lambda_{\rm DA}}{3\omega_t}\left(\ln\frac{\omega_t}{\mu} - 1\right)\right]\right\} - \frac{C_F\alpha_s}{4\pi}\left[\omega_t\left(2\ln\frac{\omega_t}{\mu} - 3\right) + 3\mu\right] = \frac{2\bar\Lambda_{\rm DA}}{3}\left(1 + 3\frac{C_F\alpha_s}{4\pi}\right) + (2\sqrt{e} - 3)\mu\frac{C_F\alpha_s}{4\pi} + \ldots. \quad (32)$$

The expanded expressions for ω_t, N, and ω_0 are given only for the purpose of illustration. The exact expressions will be used in our numerical analysis.

The model ansatz (29) has the attractive feature that it is to a good approximation invariant under renormalization-group evolution. Table I collects the parameters entering this function for different values of μ, obtained using the central value $\bar\Lambda_{\rm SF}(\mu_*, \mu_*) = 0.65$ GeV in (20). For $\mu = 1$ GeV and 2.5 GeV the corresponding functions are shown in Fig. 2. For comparison, we also show the result at $\mu = 2.5$ GeV obtained by applying the evolution formula (26) to the model function at $\mu = 1$ GeV. Both curves are very similar, indicating that the functional form (29) is approximately preserved under evolution.

We have mentioned earlier that a QCD sum-rule analysis of the *B*-meson LCDA at next-leading order in α_s performed by Braun *et al.* [23] has exhibited an asymptotic behavior similar to that of our perturbative QCD analysis. These authors have proposed the model form

TABLE I. Parameters of the model function (29) for different values of the renormalization scale

μ [GeV]	$\bar\Lambda_{\rm DA}$ [GeV]	ω_t [GeV]	N	ω_0 [GeV]
1.0	0.519	2.33	0.963	0.438
1.5	0.635	3.35	0.974	0.509
2.0	0.709	4.32	0.978	0.557
2.5	0.770	5.26	0.981	0.596

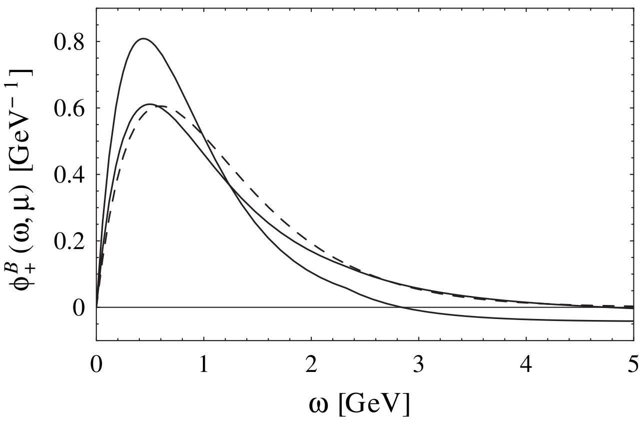

FIG. 2. Model ansatz for the B-meson LCDA at $\mu = 1$ GeV (narrow solid curve) and 2.5 GeV (wide solid curve). The dashed curve shows the result at 2.5 GeV obtained by evolving the distribution amplitude from 1 to 2.5 GeV.

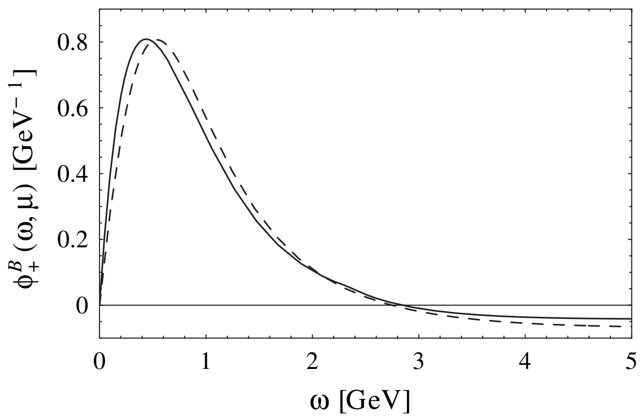

FIG. 3. Two different models for the B-meson LCDA at $\mu = 1$ GeV, constrained to have the same normalization and first moment. The solid curve corresponds to (29), the dashed one to (33).

$$\phi_+^B(\omega,\mu) = \frac{4\lambda_B^{-1}}{\pi}\frac{k}{k^2+1}\left[\frac{1}{k^2+1} - \frac{2(\sigma_B - 1)}{\pi^2}\ln k\right] \tag{33}$$

at $\mu = 1$ GeV, where $k = \omega/1$ GeV. The two parameters entering this functions are defined in terms of the integrals

$$\begin{aligned}\lambda_B^{-1} &= \int_0^\infty d\omega \frac{\phi_+^B(\omega,\mu)}{\omega},\\ \sigma_B\lambda_B^{-1} &= -\int_0^\infty d\omega \frac{\phi_+^B(\omega,\mu)}{\omega}\ln\frac{\omega}{\mu}.\end{aligned} \tag{34}$$

The parameter ranges obtained from the sum-rule analysis are $\lambda_B^{-1} = (2.15 \pm 0.50)$ GeV^{-1} and $\sigma_B = 1.4 \pm 0.4$ at $\mu = 1$ GeV. On the other hand, if we require that the function (33) obey the moment constraints (14) at a large value of the cutoff, say $\Lambda_{\rm UV} = 3$ GeV, then we find $\lambda_B^{-1} = (1.79 \pm 0.06)$ GeV^{-1} and $\sigma_B = 1.57 \pm 0.27$. These values are consistent with the findings of [23]. It is interesting that, once the moment constraints are imposed, the two models in (29) and (33) are nearly indistinguishable, in spite of the rather different functional forms (exponential vs powerlike fall-off). This fact is illustrated in Fig. 3.

We are now in a position to investigate how moments of the LCDA computed using the model functions (29) and (33) compare with the model-independent predictions (14) of the OPE, which are valid for $\Lambda_{\rm UV} \gg \Lambda_{\rm QCD}$. In Fig. 4, we show in black the model results for the moments M_0 and M_1 at $\mu = 1$ GeV as a function of the cutoff $\Lambda_{\rm UV}$. For comparison, the gray curves show the predictions of the OPE. We observe that our model curves quickly converge toward the OPE predictions for $\Lambda_{\rm UV} > 2.5$ GeV. For large cutoff values the agreement is perfect, since by construction our function has the correct asymptotic behavior. The model of Braun *et al.* agrees qualitatively with the OPE for large $\Lambda_{\rm UV}$, but exact agreement can only be enforced at a single value of the cutoff (3 GeV in our case). Note that for small values of $\Lambda_{\rm UV}$ there are significant deviations between the OPE predictions and the model results. This is

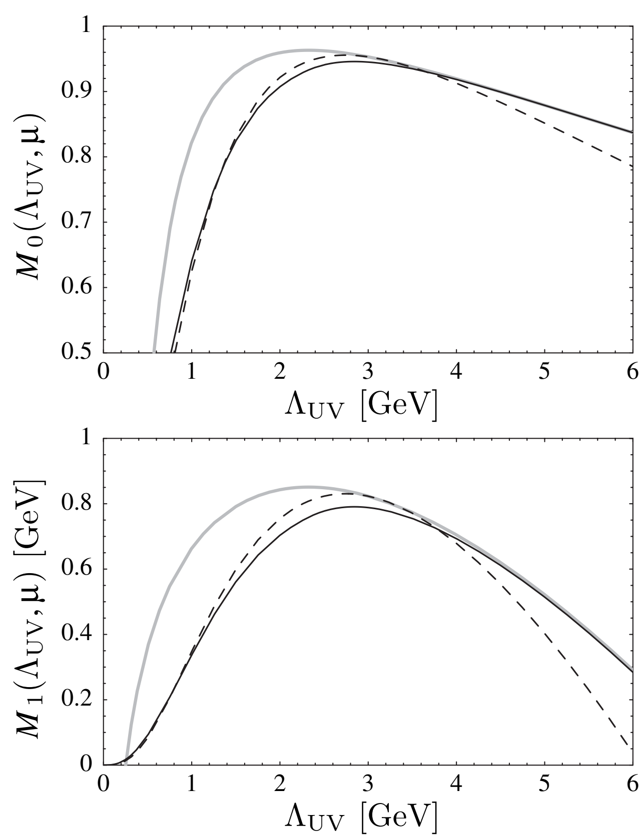

FIG. 4. Comparison of model results (black) and OPE predictions (gray) for the first two moments of the LCDA, evaluated at $\mu = 1$ GeV and for different values of the cutoff. The solid black curves are obtained in our model (29), the dashed ones in the model (33) of [23].

expected, given that the OPE is only valid for $\Lambda_{\rm UV} \gg \Lambda_{\rm QCD}$. For $\Lambda_{\rm UV} = 2$ GeV, for example, we expect unknown corrections of order $(\bar\Lambda/\Lambda_{\rm UV})^2 \sim 0.1$ to M_0, and of order $\bar\Lambda^2/\Lambda_{\rm UV} \sim 0.2$ GeV to M_1. This is consistent with the deviations seen in the figure.

VI. ESTIMATES FOR INVERSE MOMENTS

The "inverse moments" defined in (34) play an important role in the analysis of many exclusive *B*-meson decays. They control the strength of the leading-power spectator interactions in leptonic decays such as $B \to \gamma l \nu$, semileptonic decays such as $B \to \pi l \nu$, and hadronic decays such as $B \to \pi\pi$. The quantity σ_B enters these analyses as soon as one goes beyond the tree approximation. Given that we have constructed highly constrained models for the distribution amplitude which satisfy the QCD predictions for moments and have the correct asymptotic behavior, it is interesting to ask what estimates we can obtain for the parameters λ_B and σ_B.

In Table II we collect the results for the two inverse moments obtained using the model ansatz (29). The error bars reflect the variation of the results with the input parameter $\bar\Lambda_{\rm SF} = (0.65 \pm 0.06)$ GeV. In addition, there are other theoretical uncertainties related to the neglect of higher-order terms in the OPE and, more importantly, to nonperturbative hadronic uncertainties in the precise shape of the LCDA for small values of ω. For instance, comparing the results in the table with those obtained using the model (33) at $\mu = 1$ GeV, we observe shifts in λ_B^{-1} and σ_B by 0.3 GeV^{-1} and 0.04, respectively. We believe that the true theoretical uncertainties are about twice as large as the errors shown in the table. A graphical representation of the results is shown in Fig. 5, where the light gray bands are an estimate of the total theoretical uncertainty.

Our findings are in good agreement with the QCD sum-rule estimates at next-to-leading order in α_s obtained by Braun *et al.* [23], indicated by the data points in the figure. We may also compare with earlier estimates of λ_B^{-1} derived from lowest-order QCD sum rules, where the scale dependence is not controlled. Grozin *et al.* [1] found $\lambda_B^{-1} = 3/(2\bar\Lambda) \approx 2.2$ GeV^{-1} (for a typical value $m_b \approx 4.6$ GeV), while Ball *et al.* [22] obtained $\lambda_B^{-1} \approx 1.7$ GeV. Both are consistent with our findings.

TABLE II. Inverse moments λ_B^{-1} and σ_B calculated using the model function (29)

μ [GeV]	λ_B^{-1} [GeV^{-1}]	σ_B
1.0	2.09 ± 0.24	1.61 ± 0.09
1.5	1.86 ± 0.17	1.79 ± 0.08
2.0	1.72 ± 0.14	1.95 ± 0.07
2.5	1.62 ± 0.12	2.09 ± 0.07

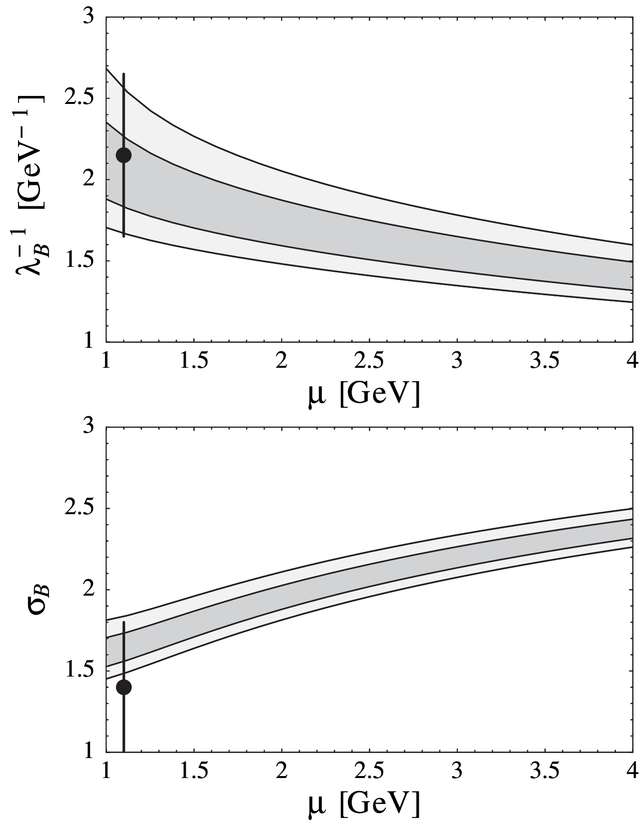

FIG. 5. Model estimates of the inverse moments λ_B^{-1} and σ_B for different values of the renormalization scale. The dark bands reflect the uncertainty in the value of $\bar\Lambda$, whereas the light bands represent an estimate of the total theoretical error. The data points show the results obtained from the QCD sum-rule analysis of [23].

VII. CONCLUSIONS

Using rigorous methods based on the operator product expansion, we have studied some model-independent properties of the *B*-meson light-cone distribution amplitude $\phi_+^B(\omega, \mu)$. We have derived explicit expressions for the first two moments of the distribution amplitude as a function of the renormalization scale μ and a hard Wilsonian cutoff $\Lambda_{\rm UV}$ applied to integrals over ω. The ratio M_1/M_0 of the first two moments can be used to define a physical subtraction scheme for the parameter $\bar\Lambda = m_B - m_b$ of heavy-quark effective theory. This links the only nonperturbative hadronic parameter entering the moment predictions at next-to-leading power in $1/\Lambda_{\rm UV}$ in a calculable way to the *b*-quark mass. From the cutoff dependence of the moment M_0 we have derived an analytic expression for the asymptotic behavior of the distribution amplitude for large $\omega \gg \Lambda_{\rm QCD}$, valid at first order in α_s and at next-to-leading order in $1/\omega$. Finally, we have presented a new, compact evolution formula that expresses the distribution amplitude at some scale μ in terms the function $\phi_+^B(\omega, \mu_0)$ at a lower scale μ_0.

Based on our analysis we have proposed a realistic model of the B-meson distribution amplitude, which is consistent with the moment relations. With the help of this function we have obtained estimates for the inverse-moment parameters λ_B and σ_B, which play an important role in many phenomenological applications of the QCD factorization approach to exclusive B decays. We find $\lambda_B^{-1} = (2.1 \pm 0.5)\ \mathrm{GeV}^{-1}$ and $\sigma_B = 1.6 \pm 0.2$ at $\mu = 1$ GeV with conservative errors.

We hope that our analysis will not only supply a guideline for understanding the B-meson distribution amplitude without relying on a specific model, but also open a new strategy for further, more detailed studies of $\phi_+^B(\omega, \mu)$ using a systematic short-distance approach. Ultimately, this may help to reduce the theoretical uncertainties in predictions for exclusive B-meson decays.

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