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Spacetime realization of κ-Poincaré algebra

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We study a Hamiltonian realization of the phase space of κ-Poincaré algebra that yields a definition of velocity consistent with the deformed Lorentz symmetry. We are also able to determine the laws of transformation of spacetime coordinates and to define an invariant spacetime metric, discussing some possible experimental consequences.

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Doubly special relativity (DSR) is a class of models which aim to give an effective description of quantum gravity effects on particle kinematics at energies near the Planck scale, $\kappa \sim 10^{19}$ GeV, by postulating a nonlinear (deformed) action of the Lorentz group on momentum space [1]. In spite of the recent advances in the understanding of this proposal, some problems are still open, as for example what should be the realization of the theory in position space. Related to this is the problem of defining the velocity and the dynamics of a particle in a way consistent with the deformed Lorentz transformations.

In a recent paper [2], we have proposed a method for introducing a Hamiltonian structure for DSR models in such a way that the velocity of a particle, defined classically in terms of the proper-time derivatives of the coordinates as $\mathbf{v} = \dot{x}_i/\dot{x}_0$, coincides with the definition proposed in Ref. [3], based on the observation that the velocity can be viewed as the parameter of the (deformed) boosts. The formalism of Ref. [2] also induces in a natural way a realization of the deformed Lorentz symmetry in position space.

Our prescription worked well for the model of Ref. [4], but led to inconsistencies in the case of the κ-Poincaré model of Ref. [5]. In particular, the expression for the velocity derived in [2] did not transform in the correct way and consequently it was not possible to define an invariant line element. In [6] it was noticed that this problem can be solved in general by imposing further constraints on the symplectic structure.

In this note we wish to show how the results of [6] can be applied also to the case of the κ-Poincaré model. We refer to [2,6] for further motivations and technical details. For simplicity, we work in $1 + 1$ dimensions. We denote the position and the momentum of a particle as q_a and p_a, $a = 0, 1$.

The κ-Poincaré model [5] is defined by the following nonlinear transformation law for the momentum of a particle under a boost of parameter $\xi = \tanh v$:

$$p'_0 = p_0 + \kappa \log\Gamma,$$
$$p'_1 = \frac{p_1 \cosh\xi + \frac{\kappa}{2}(1 - e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2}) \sinh\xi}{\Gamma}, \tag{1}$$

where

$$\Gamma = \frac{1}{2}\left(1 + e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2}\right) + \frac{1}{2}\left(1 - e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2}\right)\cosh\xi + \frac{p_1}{\kappa} \sinh\xi. \tag{2}$$

In infinitesimal form the transformation law (1) reads

$$\delta p_0 = p_1, \qquad \delta p_1 = \frac{\kappa}{2}\left(1 - e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2}\right). \tag{3}$$

The Hamiltonian for a free particle can be identified with the Casimir invariant

$$H = \frac{m}{2} = \frac{e^{2p_0/\kappa}}{2m}\left[\frac{\kappa^2}{4}\left(1 - e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2}\right)^2 - p_1^2\right]. \tag{4}$$

The physical mass M of the particle, i.e. its energy at rest, is related to m by $m = \kappa \sinh(M/\kappa)$.

The Hamiltonian is not uniquely defined. For example, in Ref. [7] it was chosen as

$$\tilde{H} = \frac{\kappa}{2} e^{p_0\kappa}\left(1 + e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2}\right), \tag{5}$$

which is related to ours by $H = (\tilde{H}^2 - \kappa^2)/2m$.

It is well-known that in DSR models the momentum p_a can be related by a nonlinear transformation to an unphysical momentum π_a that transforms linearly under deformed Lorentz transformations [8]. In our case,

$$\pi_0 = \frac{\kappa}{2} e^{p_0/\kappa}\left(1 - e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2}\right), \qquad \pi_1 = e^{p_0/\kappa} p_1. \tag{6}$$

According to Ref. [3], the definition of the velocity $\mathbf{v}$ compatible with its role of parameter of the Lorentz trans-

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formations is then

$$\mathbf{v} = \frac{\pi_1}{\pi_0} = \frac{2p_1/\kappa}{1 - e^{-2p_0/\kappa} + p_1^2/\kappa^2}. \tag{7}$$

If one postulates the standard κ-Poincaré symplectic structure [2,7]

$$\omega_{00} = 1, \qquad \omega_{01} = -\frac{p_1}{\kappa}, \qquad \omega_{10} = 0, \qquad \omega_{11} = -1, \tag{8}$$

with $\omega_{ab} \equiv \{q_a, p_b\}$, this expression for the velocity can be obtained from the basic definition $\mathbf{v} = \dot{q}_1/\dot{q}_0$, where $\dot{q}_a \equiv dq_a/d\tau$ are the derivatives of the coordinates with respect to the time parameter τ that follow from the Hamilton equations.

However, one may choose different Poisson brackets leading to the same expression for the velocity. In particular, in [6] it was shown that in order for the velocity to transform in the correct way under deformed boosts, one must impose some further conditions on the ω_{ab}. When these conditions hold, it is also possible to define a (momentum-dependent) metric invariant under deformed Lorentz transformations. In the present case, the conditions of [6] are not satisfied by (8), but rather by

$$\omega_{00} = \frac{1}{2}\left(1 + e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2}\right), \qquad \omega_{01} = -\frac{p_1}{\kappa} e^{-2p_0/\kappa}, \qquad \omega_{10} = \frac{p_1}{\kappa}, \qquad \omega_{11} = -e^{-2p_0/\kappa}. \tag{9}$$

Given the symplectic structure (9), the Jacobi identities imply that the coordinates obey nontrivial Poisson brackets

$$\{q_0, q_1\} = 2\frac{p_1 q_0}{\kappa^2} - \left(1 + \frac{p_1^2}{\kappa^2}\right)\frac{q_1}{\kappa}, \tag{10}$$

and transform as

$$\delta q_0 = q_1 - \frac{p_1}{\kappa} q_0, \qquad \delta q_1 = q_0 - \frac{p_1}{\kappa} q_1. \tag{11}$$

Moreover, the Hamilton equations arising from (4) and (9) read

$$\frac{dq_0}{d\tau} = \frac{\kappa e^{2p_0/\kappa}}{8m}\left(1 + e^{-2p_0} - \frac{p_1^2}{\kappa^2}\right)^2\left(1 - e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2}\right), \qquad \frac{dq_1}{d\tau} = \frac{e^{2p_0/\kappa}}{4m}\left(1 + e^{-2p_0} - \frac{p_1^2}{\kappa^2}\right)^2 p_1, \tag{12}$$

from which one can recover the velocity (7). From (12), making use of (4), one can also derive an expression for the differential of the time parameter $d\tau$ in terms of the dq_i:

$$d\tau^2 = \frac{16 e^{-2p_0/\kappa}}{(1 + e^{-2p_0} - p_1^2/\kappa^2)^4}(dq_0^2 - dq_1^2). \tag{13}$$

It is easy to check that $d\tau^2$ is invariant under the infinitesimal deformed Lorentz transformations (3) and (11), and can therefore be interpreted as the line element of the geometry.[1] Contrary to other known cases [2,6], it depends on both components of p_a, and not on the energy only. However, comparing with (5), one may write (13) in the simpler form

$$d\tau^2 = \left(1 + \frac{m^2}{\kappa^2}\right)^{-2} e^{2p_0/\kappa}(dq_0^2 - dq_1^2). \tag{14}$$

It should be noticed that in our interpretation the invariant parameter τ must not be identified with the physical proper time, but rather considered as an auxiliary variable, analogous to the external evolution parameter, or fifth coordinate, of Ref. [9].

The transformations (11) that, combined with (1), leave (13) invariant can also be written in finite form as

$$q_0' = \frac{q_0 \cosh\xi + q_1 \sinh\xi}{\Gamma}, \qquad q_1' = \frac{q_1 \cosh\xi + q_0 \sinh\xi}{\Gamma}. \tag{15}$$

Hence the coordinate transformations take the form of a product of standard Lorentz transformations with a function of the momentum.

The transformations (15) imply a modification of the relativistic formula for time dilation. For example, reasoning as in special relativity, it is easy to see that the relation between the coordinate time T measured in the laboratory and the coordinate time T_0 measured in the rest frame of a particle is given by

$$T = \frac{\gamma}{\Gamma_0} T_0, \tag{16}$$

where $\gamma = \cosh\xi = (1 - v^2)^{-1/2}$ and

$$\Gamma_0 \equiv \Gamma(p_1 = 0) = \frac{1}{2}(1 + e^{-2M/\kappa}) + \frac{\gamma}{2}(1 - e^{-2M/\kappa}). \tag{17}$$

Hence T becomes a function both of the velocity and the momentum, or equivalently the mass, of the particle, giving rise to corrections of order p_0/κ to the relativistic formula for the measured lifetime of high-velocity particles, which in principle may be susceptible of experimental verification. In fact, one has in first approximation in M/κ, $T \sim \gamma(1 - \frac{M}{\kappa}(\gamma - 1))T_0$. If $\gamma \gg 1$, with $\gamma M/\kappa \ll 1$, this reduces to $T \sim \gamma(1 - \frac{p_0}{\kappa})T_0$.[2] In this approximation, the size of the corrections is given by the ratio of the energy of

[1] In view of the previous argument, this seems to be the most natural choice for the line element. If one simply requires invariance under deformed Lorentz transformations, one may multiply $d\tau^2$ by any function of m. An especially interesting choice is for example $d\tau^2 = e^{2(p_0 - M)/\kappa}$, which reduces to the proper time in the rest frame of the particle (see below).

[2] We adopt this approximation in order to simplify the discussion, but one can of course always use the exact formula.

the particle and the Planck energy. If one takes for κ the standard value of the Planck energy, 10^{19} GeV, this would be an extremely small correction, not detectable experimentally even for particles of energy of order 10 GeV. However, in the context of some higher-dimensional theories, the effective four-dimensional Planck energy $\bar{\kappa}$ could be lowered up to 10^3 GeV [10], and in this case corrections might be observable.

To our knowledge, the relativistic formula for time delay has been checked for pions with $\gamma \sim 2.4$, with a confidence of 0.4% [11]. This fixes a lower limit for $\bar{\kappa}$ to 100 GeV. Improving the experimental limits could give evidence for a modification of the time delay formula, only if $\bar{\kappa}$ is not much greater than this value.

We conclude by remarking that the line element (13) may also take the role of the metric in a formulation of a κ-Poincaré extension of general relativity on the lines of the gravity rainbow formalism of Ref. [12]. In the present case, the metric would depend not only on the energy, but also on the space component of the momentum (or equivalently on the mass) of the particle. It must be also pointed out that in the present framework the speed of light is independent of the energy, contrary to the case of Ref. [12].

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