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Nonbirefringence conditions for spacetime

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Within the axiomatic premetric approach to classical electrodynamics, we derive under which covariant conditions the quartic Fresnel surface represents a unique light cone without birefringence in vacuum.

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The classical electrodynamics theory has been reformulated recently by Hehl and Obukhov in an axiomatic premetric form; for a comprehensive account, see [1] and the references given therein. In this approach, the electromagnetic field equations

$$dF = 0, \qquad dH = J \tag{1}$$

are accepted as consequences of the flux and charge conservation laws, respectively. Here $F = (1/2)F_{ij}dx^i \wedge dx^j$ is the electromagnetic strength 2-form, while $H = (1/2)H_{ij}dx^i \wedge dx^j$ is the electromagnetic excitation 2-form. So far, the electromagnetic field (F, H) is undetermined. It has 12 independent components restricted only by 8 independent equations (1). The remaining conditions are specified by the spacetime relation linking the excitation to the field strength. In the simplest case, this relation is assumed to be local and linear,

$$H_{ij} = \tfrac{1}{4}\kappa_{ij}{}^{kl}F_{kl}, \qquad H_{ij} = \tfrac{1}{4}\epsilon_{ijmn}\chi^{mnkl}F_{kl}. \tag{2}$$

Here the tensor density ϵ_{ijmn} is $+1$, -1, or 0 depending on whether i, j, m, n is an even, an odd, or no permutation of $0, 1, 2, 3$. Recall that the physical space is considered as a bare manifold without metric or connection. All information on its geometry is encoded into the constitutive tensor $\kappa_{ij}{}^{kl}$ or, equivalently, in χ^{ijkl}. By definition, this tensor density inherits the symmetries of the electromagnetic field (F, H)

$$\chi^{ijkl} = \chi^{[ij]kl} = \chi^{ij[kl]}. \tag{3}$$

Consequently, the fourth-order constitutive tensor χ^{ijkl} has only 36 independent components. Wave propagation in the premetric electrodynamics was studied by the method of geometric optics [1,2]. An output of this approach is a generalized Fresnel equation

$$\mathcal{G}^{ijkl}q_iq_jq_kq_l = 0 \tag{4}$$

for a wave-covector q_i. The coefficients of this equation form the fourth-order Tamm-Rubilar tensor density [2] of weight $+1$, which is completely symmetric and cubic in the constitutive tensor

$$\mathcal{G}^{ijkl} = \frac{1}{4!}\epsilon_{mnpq}\epsilon_{rstu}\chi^{mnr(i}\chi^{j|ps|k}\chi^{l)qtu}. \tag{5}$$

When the premetric scheme is applied on a manifold with a prescribed metric tensor g_{ij}, the standard Maxwell electrodynamics is reinstated with a special (Maxwell-Lorentz) constitutive tensor

$$^{(\mathrm{Max})}\chi^{ijkl} = \lambda_0\sqrt{-g}(g^{ik}g^{jl} - g^{il}g^{jk}). \tag{6}$$

Substitution of these expressions in the Fresnel equation (4) yields

$$(g_{ij}q^iq^j)^2 = 0. \tag{7}$$

Consequently, for a prescribed metric, $ds^2 = g_{ij}dx^idx^j$, the Fresnel equation yields the proper light cone, which turns out to be a double light cone.

The idea of the premetric approach to the electrodynamics is that the metric tensor has to be reinstated from the pure electromagnetic data, in particular, from the constitutive tensor χ^{ijkl}. An examination of the general Fresnel equation (4) indicates that, for a general constitutive tensor, it does not yield a unique double light cone. The birefringence effect of distinct light cones is known from crystal optics. Consequently, the generalized Fresnel equation of premetric electrodynamics predicts the possibility of birefringence in vacuum. Such an effect was predicted in the Lorentz-violating electrodynamics [3,4]. When a nonzero torsion of spacetime is coupled nonminimally to the electromagnetic field, birefringence is a generic effect [5–7]. Moreover, classical electrodynamics modified by an axion field, which yields a violation of Lorentz symmetry, can, if one goes beyond the geometrical optics limit, induce birefringence of the vacuum [8,9].

In this context, it is desirable to have the exact conditions on the Tamm-Rubilar tensor and consequently on the constitutive tensor which forbid the effect of birefringence. In Refs. [10,11], the absence of birefringence is attributed by a requirement for Eq. (4) to have two solutions of multiplicity 2. In this analysis, the exact Ferrari solution of the quartic equation was used and a *necessary* condition between the coefficients was derived. We are starting from this point with an aim to derive a complete set of *necessary and sufficient* conditions which guarantee the uniqueness of the light cone. Since birefringence is a proper physical effect which is independent of a choice of a coordinate

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system, the nonbirefringence conditions have to be formulated in a covariant form.

Let us decompose the wave-covector in the time and spatial $(a = 1, 2, 3)$ components $q_i = (q_0, q_a)$. Correspondingly, the Fresnel equation (4) represents a quartic surface

$$\mathcal{G}^{ijkl} q_i q_j q_k q_l = M_0 q_0^4 + M_1 q_0^3 + M_2 q_0^2 + M_3 q_0 + M_4 = 0, \tag{8}$$

where

$$M_0 = \mathcal{G}^{0000}, \qquad M_1 = 4\mathcal{G}^{000a} q_a,$$
$$M_2 = 6\mathcal{G}^{00ab} q_a q_b, \qquad M_3 = 4\mathcal{G}^{0abc} q_a q_b q_c, \tag{9}$$
$$M_4 = \mathcal{G}^{abcd} q_a q_b q_c q_d.$$

A generic quartic polynomial with real coefficients can always be decomposed into a product of two quadratic polynomials with real coefficients. Let us write this decomposition as

$$\mathcal{G}^{ijkl} q_i q_j q_k q_l = M_0 (q_0^2 + a q_0 + b)(q_0^2 + c q_0 + d). \tag{10}$$

Since the Fresnel surface is defined only up to a scalar factor, we can chose

$$M_0 > 0 \tag{11}$$

without loss of generality. Because of the Fresnel equation, these quadratic factors can vanish independently. Since every factor represents a relation between the components of an arbitrary wave-covector, it determines a metric on the manifold. The quadratic factors in (10) can be of two types—positive definite or indefinite. Consequently, we consider four distinct possibilities:

(i) Both factors are indefinite and coincide. Consequently, the induced metric is Lorentzian and unique.

(ii) Both factors are indefinite and distinct. There are two distinct Lorentzian metrics; i.e., there is birefringence in wave propagation.

(iii) One factor is positive definite while the second is indefinite. There are two metrics—one Lorentzian and one Euclidean.

(iv) Both factors are positive definite; i.e., two metrics are Euclidean.

Our aim is to extract the first possibility, which corresponds to a unique (double) light cone. In cases (ii) and (iii), the sign of the left-hand side of (4) is indefinite. Consequently, we can remove these two cases by the requirement:

Condition 1.—The Fresnel quartic form has to be positive definite, i.e., for an arbitrary covector $q = q_i dx^i$

$$\mathcal{G}^{ijkl} q_i q_j q_k q_l \geq 0. \tag{12}$$

Thus, we come to a condition

$$M_0 q_0^4 + M_1 q_0^3 + M_2 q_0^2 + M_3 q_0 + M_4 = M_0 (q_0^2 + a q_0 + b)^2. \tag{13}$$

Equating the coefficients of the same powers of q_0 on both sides of (13), we have

$$\frac{M_1}{M_0} = 2a, \qquad \frac{M_2}{M_0} = a^2 + 2b, \tag{14}$$

$$\frac{M_3}{M_0} = 2ab, \qquad \frac{M_4}{M_0} = b^2. \tag{15}$$

From (14),

$$a = \frac{M_1}{2M_0}, \qquad b = \frac{4M_0 M_2 - M_1^2}{8M_0^2}. \tag{16}$$

Substituting into (15), we derive two relations between the coefficients of (8),

$$M_3 = \frac{M_1}{8M_0^2}(4M_0 M_2 - M_1^2) \tag{17}$$

and

$$M_4 = \frac{(4M_0 M_2 - M_1^2)^2}{64 M_0^3}. \tag{18}$$

If we square (17) and divide by (18), then, for $M_1 \neq 0$, we find

$$M_4 = M_0 \frac{M_3^2}{M_1^2}. \tag{19}$$

The relation (17) was derived in Ref. [10] by an alternative method.

Under the conditions (17) and (18), the case of two positive definite factors is still permitted. In order to remove this possibility, we apply an additional condition:

Condition 2.—There is a nonzero covector $q = q_i dx^i$ such that

$$\mathcal{G}^{ijkl} q_i q_j q_k q_l = 0. \tag{20}$$

Equivalently, the roots of the quadratic polynomial $q_0^2 + a q_0 + b$ have to be real and of opposite signs. Consequently, we have an inequality

$$b < 0, \tag{21}$$

which is equivalent to

$$4 M_0 M_2 < M_1^2. \tag{22}$$

This condition also guarantees that the roots of the polynomial are real. Indeed, if the roots are complex, they are necessarily conjugate; hence, their product is positive. Observe that, for nonzero M_1, M_3, the inequality (22) is equivalent to

$$M_1 M_3 < 0. \tag{23}$$

Consequently, the relations (17) and (18) together with the inequality (23) guarantee uniqueness of the light cone, i.e., the absence of the birefringence effect.

Substituting (16) into (13), we derive

$$q_0^2 + \frac{M^a}{2M} q_0 q_a - \frac{1}{8M^2}(M^a M^b - 4MM^{ab}) q_a q_b = 0, \tag{24}$$

where we use the notations

$$M = \mathcal{G}^{0000}, \qquad M^a = 4\mathcal{G}^{000a}, \qquad M^{ab} = 6\mathcal{G}^{00ab}. \tag{25}$$

Consequently, we come to a metric tensor g^{ij} with the components

$$g^{00} = 1, \qquad g^{0a} = \frac{M^a}{4M}, \qquad g^{ab} = -\frac{1}{8M^2}(M^a M^b - 4MM^{ab}), \tag{26}$$

which coincides with Ref. [10]. The Lorentz nature of this metric is clear from (22). Since the metric has been derived from the light cone structure, it is determined only up to a scalar factor. The metric tensor (26) can be treated as a square root of the positive definite tensor $\mathcal{G}_{ijkl}$. Indeed,

$$\mathcal{G}^{ijkl} q_i q_j q_k q_l = M(g^{ij} q_i q_j)(g^{kl} q_k q_l). \tag{27}$$

It is straightforward to check that this equation holds when (26) is substituted.

Let us look for what values of the coefficients M_i Eq. (8) gives a unique light cone which is symmetric under a change of the time direction $t \longrightarrow -t$. It means that the incoming (past) light cone and the outgoing (future) light cone have the same angle. For this, (8) has to have two real solutions of the same absolute value and of opposite signs, i.e., the parameter $a = 0$. From (16), we have

$$M_1 = M_3 = 0, \qquad M_4 = \frac{M_2^2}{4M_0}. \tag{28}$$

The additional condition (22) takes the form

$$M_0 M_2 < 0. \tag{29}$$

The metric tensor components (26) are simplified to

$$g^{00} = 1, \qquad g^{0a} = 0, \qquad g^{ab} = \frac{M^{ab}}{2M}. \tag{30}$$

Since the light cone is defined only up to a scalar factor, the time symmetric light element can be taken as

$$ds^2 = M_0 dt^2 - \tfrac{1}{2} M_{ab} dx^a dx^b, \tag{31}$$

where the matrix M_{ab} is the inverse of M^{ab}.

As a result, we have derived two covariant conditions on the Fresnel surface to represent a unique light cone without birefringence.

In Ref. [12], the Lorentz signature of the metric is determined by a set of physically motivated conditions. In particular, we require the Dufay law (repulsion of like charges and attraction of opposite ones), the Lenz rule (the relative sign in Faraday's law), and the positive sign of the electromagnetic energy.

A connection between these requirements and the conditions derived in the current paper is now under consideration.

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