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Conifold geometries, topological strings, and multimatrix models

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We study the open B-model representing D-branes on 2-cycles of local Calabi-Yau geometries. To this end we work out a reduction technique linking D-brane partition functions and multimatrix models in the case of conifold geometries so that the matrix potential is related to the complex moduli of the conifold. We study the geometric engineering of the multimatrix models and focus on two-matrix models with bilinear couplings. We show how to solve this models in an exact way, without resorting to the customary saddle point/large N approximation. The method consists of solving the quantum equations of motion and using the flow equations of the underlying integrable hierarchy to derive explicit expressions for correlators. Finally, we show how to incorporate in this formalism the description of several group of D-branes wrapped around different cycles.

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I. INTRODUCTION

Singular Calabi-Yau (CY) spaces are more and more frequently met in string compactifications. The reason is mostly the fact that compactifications on regular Calabi-Yau spaces do not seem to be able to describe crucial features of realistic physics. On the contrary, the presence of a conifold point [1] in a Calabi-Yau opens new prospects: in conjunction with fluxes and branes it may allow for warped compactifications, which in turn may create the conditions for large hierarchies of physical scales. On the other hand, singular Calabi-Yau's with conifold singularities seem to be necessary in order to realize low energy theory models with realistic cosmological features. The hallmark of a conifold is the possibility of resolving the conifold point in two different ways, by a 2-sphere (resolution) or a 3-sphere (deformation). This leads, from a physical point of view, to a geometric transition that establishes a duality relation between the theories defined by the two nonsingular geometries (gauge-gravity or open-closed string duality) [2,3]. In summary, conifold singularities are at the crossroads of many interesting recent developments in string theory. It has therefore become customary to study theories defined on conifolds, i.e. singular noncompact Calabi-Yau threefolds, as calculable and well-defined models to approximate more realistic situations.

Given the crucial role they play, it is of upmost importance to find methods of calculation for theories defined on conifold geometries. In this sense two main tools have been devised: topological field theories and matrix models. Topological field theories are truncations of full theories: one gives up the knowledge of the dynamical sectors of a given theory, drastically simplifies it by limiting it to the topological sector and ends up with a theory where very often many quantities (correlators) can be explicitly calculated. However, even topological field theories are sometimes not easily accessible to explicit calculations. Here come matrix models to the rescue. Sometimes, like in the examples of this paper, topological field theories can be shown to be equivalent to matrix models. This makes life easier, especially when the matrix models have couplings of a special type. In this case, one can rely on the integrable structure underlying the model (the Toda lattice hierarchy [4]) which usually provides algorithmic methods to obtain the desired results in an exact and controlled framework. The case of matrix models with more general couplings is more complex and represents a challenge that people have started to tackle only very recently.

In this paper we would like to elaborate on an idea that has recently received increasing attention: how data about the geometry of a local Calabi-Yau can be encoded, via a topological field theory, in a (multi)matrix model and how they can be efficiently calculated. The framework we consider is IIB string theory with spacetime filling D5-branes wrapped around two-dimensional cycles. This geometry defines a 4D gauge theory [5–8]. On the other hand, we can consider the open topological B-model representing the strings on the conifold. The latter has been shown by Witten long ago to be represented by a six-dimensional holomorphic Chern-Simons theory [9]. When reduced to the two-dimensional cycle this theory can be shown to boil down to a matrix model. In particular, if we wish to represent the most general deformations of the complex structure satisfying the Calabi-Yau condition, we end up with very general multimatrix models. This point of view was advocated in [10]. In this paper we concentrate on the topological string theory part of the story, and ignore both the 4D gauge theory part and the closed string theory side, which is attained by shrinking the resolved sphere to a point and passing to the deformed picture in which the singularity is replaced by a three-sphere (as would be possible at least for the cases corresponding to one- and two-matrix models).

As already pointed out, the general idea underlying our paper has already been developed in a number of papers [11–15]. Here we would like to concentrate on particular aspects that have not been stressed or have been left aside in the previous literature. The first question we concentrate on is the reduction from the six-dimensional holomorphic Chern-Simons theory to a two-dimensional field theory. We wish to understand what degree of arbitrariness this passage implies, so as to be able to assess whether the information we gather from the reduced theory is intrinsic or depends on the reduction process. Our conclusion is that the reduced theory does not depend on the reduction procedure.

The second point we deal with is whether there are limitations on the general form of the potential we find for the multimatrix model. We do find some conditions although rather mild ones. Finally, we concentrate on the subclass of matrix models represented by two-matrix models with bilinear coupling. In this case the functional integral can be explicitly carried out with the method of orthogonal polynomials. We show, using old results, how one can find explicit solutions: this is done by solving the *quantum equations of motion* and utilizing the recursiveness guaranteed by integrability. All the data turn out to be encoded in a Riemann surface (plane curve), which we call *quantum Riemann surface* in order to distinguish it from the Riemann surface of the standard saddle point approach. In particular, we are able to prove that the exact solutions found in this way are *more* in number than the ones found by the saddle point method.

The paper is organized as follows. In the next section we calculate the reduction of the B-model open string field theory corresponding to the wrapped D-branes on the 2-cycle of the conifolds and how it depends on its complex moduli. The result is given in terms of a family of multimatrix models. In Sec. III we discuss explicitly what multimatrix models we do get and we draw the geometrical engineering scheme for their realization with D-branes. In particular, we show how to obtain two-matrix models with bilinear couplings. In Sec. IV we review some general properties of the above class of 2-matrix models, their integrability and the relative genus expansion. The section is concerned with particular method of solving them which is based on the quantum equations of motion (EoMs) and the flow equations. Section V contains several explicit examples of models solved with this method. We compare these solutions with the ones obtained with the usual saddle point/large N expansion. All these solutions are interpreted as describing the physics of N D-branes wrapped around a 2-cycle. In Sec. VI we show how the physics of several groups of D-branes wrapped around different cycles can be incorporated in the exact scheme proposed in this paper. Finally, Sec. VII is devoted to some conclusions and open questions. The appendix extends the approach of Sec. II to local CY geometries around 4 cycles.

II. REDUCTION TO THE BRANES AND MULTIMATRIX MODELS

In this section we show in detail how the reduction of the topological open string field theory (B-model) to a 2-cycle in a local CY geometry is equivalent to a multimatrix model whose potential is parametrized by certain deformations of the complex structure of the noncompact CY space. Actually we will elaborate a more general framework. We consider three-dimensional Calabi-Yau geometries built around a generic Riemann surface Σ of any genus which is then the nontrivial 2-cycle we wrap the D-branes around. The normal bundle is specified by assigning a rank two holomorphic vector bundle V over Σ and the CY condition constrains the determinant line bundle to equal the canonical line bundle of Σ, while V is otherwise generic.

It is in this generic setup that we study the problem of the reduction of the holomorphic Chern-Simons action functional to the D-brane world-volume. This depends on the $(0, 1)$-part of a connection on a $U(N)$ gauge bundle E which we take to be trivial. Because of the nontriviality of the geometry of the normal bundle, in order to specify the reduction mechanism, we will need to choose a trivialization of the bundle by a reference nondegenerate bilinear structure K over it. We will show that choosing the reference connection to be the (generalized) Chern connection of the bilinear structure K makes the overall result actually independent both on the reference bilinear form K and on the base representative of the 2-cycle in the total CY space. The resulting reduced theory is a generalized holomorphic b-c (β-γ) system on Σ where the two bosonic fields span a section of V. These are minimally coupled to the reduced gauge connection.

After the above preliminary construction, we consider the effects of varying the complex structure of the total space. Actually we will study constrained variations leaving the complex structure on Σ fixed and preserving the CY condition. These can be seen to be parametrized by a set of geometric potential functions on the double intersections which are the Čech cohomology representatives of the complex moduli we are varying. These, in the spirit of Kodaira and Spencer, can be used to parametrize local singular coordinate changes which specify the variation of the complex structure.

In order to be able to explicitly deal with the moduli space of the conifold complex structures, we then limit ourselves to the genus zero case, i.e. $\Sigma = \mathbb{P}^1$. In this case, the cycle has a single complex structure and so the above analysis is enough to cover the full moduli space. Since the transverse fields are sections of the normal bundle V, the singular coordinate transformation defines the deformation of the reduced theory action in a well-defined way which is parametrized by the geometric potential at the intersection of the north and south pole. We calculate this explicitly in the generic case of CY deformations of the

$\mathcal{O}(n) \oplus \mathcal{O}(-n-2)$ reference complex structure on the conifold. The result we find is that the partition function generically reduces to an $(n+1)$-matrix model whose potential is obtained from the geometric potential in a specific way.

Let us stress that the above result with $\Sigma = \mathbb{P}^1$ and $n = 0$ was presented in [6] where it was suggested that it can be obtained by refining a sketchy calculation in [16]. The result presented here is a generalization thereof for Riemann surfaces of arbitrary genus and, for arbitrary n, on $\mathbb{P}^1$.

The same reduction method can be applied also to noncompact local CY geometries build around a 4-cycle. In such a case the CY condition specifies the normal line bundle to be the canonical line bundle on the base complex manifold. In the appendix, we elaborate it in a generic case and find the reduced holomorphic Chern-Simons theory in the form of an holomorphic BF model (see [17] for recent discussions on this model). It was recently suggested in [14] that it describes the topological open strings for D-branes wrapped around the above four cycles.

A. Reduction in the linear case

Let us consider noncompact six-dimensional geometries built around a Riemann surface Σ as the total space of a rank 2 holomorphic vector bundle V with $GL(2,\mathbb{C})$ structure group.

Any atlas $\{U_{(\alpha)}\}$ on Σ extends to an atlas on $CY(\Sigma, V)$ by $\hat{U}_{(\alpha)} = U_{(\alpha)} \times \mathbb{C}^2$. The complex manifold is defined by the overlapping conditions,

$$z_{(\alpha)} = f_{(\alpha)(\beta)}(z_{(\beta)}) \qquad w^i_{(\alpha)} = M^i_{j(\alpha)(\beta)}(z_{(\beta)}) w^j_{(\beta)} \tag{2.1}$$

in any double patch intersection $U_{(\alpha)} \cap U_{(\beta)}$.

Requiring the complex manifold to be of the Calabi-Yau type, restricts $\det V$ to be equal to the canonical line bundle $T^{(1,0)}(\Sigma)$ so that under this condition the total space of V is equipped with the holomorphic $(3,0)$-form $\Omega = dz \wedge dw^1 \wedge dw^2$, where z is a local coordinate system on Σ and w^i on the $\mathbb{C}^2$ fibers. This condition is just $\det M_{(\alpha)(\beta)} \times f'_{(\alpha)(\beta)} = 1$ and it is consistent with triple intersection conditions. We denote this manifold $CY(\Sigma, V)$.

Let us consider the topological open B-model on $CY(\Sigma, V)$, which can be obtained starting from open string field theory [9]. Because of the drastic reduction of the degrees of freedom due to the huge gauge symmetry present in the topological string, the string field theory is the holomorphic Chern-Simons (hCS) theory on $CY(\Sigma, V)$ for a (0,1)-form connection on a $U(N)$ bundle E, where N is the number of D-branes wrapped around Σ. For simplicity, we will restrict to the case in which E is trivial. The action of hCS is

$$S(\mathcal{A}) = \frac{1}{g_s} \int_{CY(\Sigma,V)} \mathcal{L},$$
$$\mathcal{L} = \Omega \wedge \mathrm{Tr}\left(\frac{1}{2}\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{1}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right), \tag{2.2}$$

where $\mathcal{A} \in T^{(0,1)}(CY(\Sigma, V))$.

The dynamics of D-branes wrapped around the 2-cycle Σ can be described by reducing the open string field theory from the total space manifold to the D-brane world-volume. Since the bundle V is nontrivial, in order to properly define the brane theory, the reduction of the Lagrangian has to be coherently prescribed patch by patch by a trivialization procedure in such a way that the end product is independent upon the particular trivialization we use.

As it is evident, the embedding equations for Σ in $CY(\Sigma, V)$ are just $w^i = 0$. In any local chart, the fibering structure defines a local notion of parallel and transverse directions along which we split $\mathcal{A} = \mathcal{A}_{\bar{z}} d\bar{z} + \mathcal{A}_{\bar{i}} d\bar{w}^i$. The parallel part $\mathcal{A}_{\bar{z}}$ glues on double patches intersections as an invariant (0,1)-form on Σ only when restricted to the base, while otherwise gets also a linear contribution in w due to the generic nontriviality of V The transverse coefficients $\mathcal{A}_{\bar{i}}$ glue[1] as a section of $\bar{V}^*$.

Because of this, since the reduction to the base has to be performed covariantly, we have to expand $\mathcal{A}_{\bar{z}} = A_{\bar{z}} - A_{\bar{k}} \Gamma^{\bar{k}}_{\bar{z}\bar{j}} \bar{w}^{\bar{j}}$ and $\mathcal{A}_{\bar{i}} = A_{\bar{i}}$, where $A_{\bar{z}} d\bar{z} \in T^{(0,1)}(\Sigma)$, $A_{\bar{i}} \in \bar{V}^*$, and $d\bar{z}\Gamma^{\bar{k}}_{\bar{z}\bar{j}}$ is the $(0,1)$ component of a reference connection of $\bar{V}$. The reduction process is defined by specifying the subfamily of $\mathcal{A}$ connections we limit our consideration to. Our prescription is that the matrix valued dynamical fields $(A_{\bar{z}}, A_{\bar{i}})$ that survive the reduction are those independent of the coordinates along $\mathbb{C}^2$. A direct calculation from the Lagrangian $\mathcal{L}$ in (2.2) for the above reduced configurations gives

$$L = \Omega \wedge Tr\left(\frac{1}{2}\{A_{\bar{i}} D_{\bar{z}} A_{\bar{j}} + A_{\bar{i}} \Gamma^{\bar{k}}_{\bar{z}\bar{j}} A_{\bar{k}}\}\right) dw^{\bar{i}} \wedge d\bar{z} \wedge dw^{\bar{j}}, \tag{2.3}$$

where $D_{\bar{z}}$ is the covariant derivative with respect to the gauge structure. Notice that the above does not depend on the base representative, that is on the values of w^i.

Another way of justifying the reduced Lagrangian (2.3) is the following. We start from (2.2) and replace the exterior differential $\bar{\partial}$ by the covariant differential $\bar{\mathcal{D}} = \bar{\partial} + \Gamma$ on $CY(\Sigma, V)$ and impose that $\mathcal{D}_{\bar{i}} \mathcal{A}_{\bar{j}} = 0 = \mathcal{D}_{\bar{i}} \mathcal{A}_{\bar{z}}$. The latter conditions are satisfied as follows. Because of the local product structure of $CY(\Sigma, V)$, we can suppose without loss of generality that the only nontrivial components of the connection relevant to the problem are $\Gamma^{\bar{k}}_{\bar{z}\bar{j}} = \Gamma^{\bar{k}}_{\bar{j}\bar{z}}$. Therefore $\mathcal{D}_{\bar{i}} \mathcal{A}_{\bar{j}} = 0$ simply means that $\mathcal{A}_{\bar{j}}$ does not

[1] We denote by V^* the dual vector bundle, glueing with $(M^{-1})^t$ and by $\bar{V}$ the complex conjugate one.

depend on w^i, while $\mathcal{D}_{\bar{i}}\mathcal{A}_{\bar{z}} = 0$ can be integrated and leads precisely to the expression for $\mathcal{A}_{\bar{z}}$ given above.

Equation (2.3) represents a six-form. Our purpose is to restrict it to a two-form defined on Σ. This has to be consistently prescribed patch by patch. That is, we have to couple the above reduction with the contraction of the differentials along the fiber directions to obtain a well-defined $(1, 1)$-form on Σ. To define this operation, let us consider a bilinear structure K in V, that is a local section $K \in \Gamma(V \otimes \bar{V})$, the components $K^{i\bar{j}}$ being an invertible complex matrix at any point.

The derivation of the basic $(1, 1)$-form is realized patch by patch with the help of K as a contraction of the hCS (3,3)-form Lagrangian by the two bivector fields $k = \frac{1}{2}\epsilon_{ij}K^{i\bar{l}}K^{j\bar{k}}\frac{\partial}{\partial\bar{w}^l}\frac{\partial}{\partial\bar{w}^k}$ and $\rho = \frac{1}{2}\epsilon^{ij}\frac{\partial}{\partial w^i}\frac{\partial}{\partial w^j}$. Notice that $k \in \det V$ and $\rho \in \det V^* = (\det V)^{-1}$ so that the combined application of the two is a globally well-defined operation. Calculating then the pullback Lagrangian, we obtain

$$\mathcal{L}_{\text{red}} = i_{\rho\wedge k}L = \frac{1}{2}dzd\bar{z}(\det K)\epsilon^{\bar{i}\bar{j}}\,\text{Tr}[A_{\bar{i}}D_{\bar{z}}A_{\bar{j}} + A_{\bar{i}}\Gamma^{\bar{k}}_{\bar{z}\,\bar{j}}A_{\bar{k}}]. \tag{2.4}$$

Our last step relates the reference connection and the reference bilinear structure in order to obtain a result which is independent upon the trivialization we used. Define the field components $\varphi^i = i_{V^i}A \in V$, where $V^i = K^{i\bar{j}}\frac{\partial}{\partial\bar{w}^j}$ and plug it in (2.4). One gets

$$\begin{aligned}\mathcal{L}_{\text{red}} = \frac{1}{2}dzd\bar{z}\,\text{Tr}[&\epsilon_{ij}\varphi^i D_{\bar{z}}\varphi^j \\ &+ (\det K)\varphi^m\varphi^n\epsilon^{\bar{i}\bar{j}}(K_{m\bar{i}}\partial_{\bar{z}}K_{n\bar{j}} + K_{m\bar{i}}K_{n\bar{k}}\Gamma^{\bar{k}}_{\bar{z}\,\bar{j}})],\end{aligned} \tag{2.5}$$

where $K_{\bar{i}j}$ are the components of the inverse bilinear structure, that is $K_{\bar{i}j}K^{j\bar{l}} = \delta^{\bar{l}}_{\bar{i}}$. In order to have a result which is independent on the trivialization, just set the reference connection to be the generalized Chern connection of the bilinear structure K, that is $\Gamma^{\bar{k}}_{\bar{z}\,\bar{j}} = K_{\bar{j}l}\partial_{\bar{z}}K^{l\bar{k}}$. Therefore, choosing our reference trivialization (Γ, K) data to satisfy this natural condition, we get

$$\mathcal{L}_{\text{red},} = \frac{1}{2}dzd\bar{z}\,\text{Tr}[\epsilon_{ij}\varphi^i D_{\bar{z}}\varphi^j] \tag{2.6}$$

which is a well-defined $(1, 1)$-form on Σ. Hence the action for the reduced theory is given by

$$S_{\text{red}} = \frac{1}{g_s}\int_\Sigma \mathcal{L}_{\text{red}} = \frac{1}{g_s}\frac{1}{2}\int_\Sigma dzd\bar{z}\,\text{Tr}[\epsilon_{ij}\varphi^i D_{\bar{z}}\varphi^j].$$

B. Deformations of the complex structure

Let us now discuss certain variations of the complex structures of the manifold $\text{CY}(\Sigma, V)$ following the approach of Kodaira [18].

A general complex structure variation of the vector bundle structure (2.1) is given by the deformed patching conditions

$$\begin{aligned}\xi_{(\alpha)} &= f_{(\alpha)(\beta)}(\xi_{(\beta)}) + \delta_{(\alpha)(\beta)}(\xi_{(\beta)}, \omega_{(\beta)}) \\ \omega^i_{(\alpha)} &= M^i_{j(\alpha)(\beta)}(\xi_{(\beta)})[\omega^j_{(\beta)} + \Psi^j_{(\alpha)(\beta)}(\xi_{(\beta)}, \omega_{(\beta)})],\end{aligned} \tag{2.7}$$

where α and β label the two local charts, and δ and Ψ^i are analytic functions on double patch intersections. The variation is trivial if it can be reabsorbed via an analytic change of coordinates. Notice that, in the general case, the deformation functions are constrained by the chain rules of multiple patch intersection.

In the following, we will consider variations leaving invariant the complex structure on Σ. It is obvious that this coincides with the general case if the moduli space of complex structures of Σ is a point. Then, from now on, we will restrict to variations of the form

$$\begin{aligned}z_{(\alpha)} &= f_{(\alpha)(\beta)}(z_{(\beta)}) \\ \omega^i_{(\alpha)} &= M^i_{j(\alpha)(\beta)}(z_{(\beta)})[\omega^j_{(\beta)} + \Psi^j_{(\alpha)(\beta)}(z_{(\beta)}, \omega_{(\beta)})].\end{aligned} \tag{2.8}$$

Notice that the deformed complex structure preserves the CY condition if in any $U_{(\alpha)} \cap U_{(\beta)}$ we have $\det(1 + \partial\Psi) = 1$, where $(1 + \partial\Psi)^i_j = \delta^i_j + \partial_j\Psi^i$.

The solution of the above CY condition can be easily given in terms of a set of potential functions (one for each double patch intersection modulo triple intersection identities) which generates the deformation, as

$$\epsilon_{ij}w^i_{(\alpha)(\beta)}dw^j_{(\alpha)(\beta)} = \epsilon_{ij}\omega^i_{(\alpha)}d\omega^j_{(\alpha)} - dX^{(\alpha)(\beta)},$$

where we defined $w^i_{(\alpha)(\beta)} = \omega^i_{(\alpha)} + \Psi^i_{(\alpha)(\beta)}(z^{(\beta)}, \omega_{(\beta)})$.

For later application, let us specify the previous general construction for $\Sigma = \mathbb{P}^1$. The patching on the sphere allows a drastic simplification of the above formulas. In this case the moduli space of complex structure of the base Riemann surface is pointlike, so keeping it fixed is not a constraint. The sphere can be described by the standard charts $U_{S/N}$ around the north and south poles and the single intersection $U_S \cap U_N = \mathbb{C}^\times$ is the cylinder $\mathbb{C} \setminus \{0\}$.

As it is well known, by Grothendieck's theorem, any holomorphic vector bundle on $\mathbb{P}^1$ can be presented as a direct sum of line bundles. In our case therefore $V = \mathcal{O}(-n_1) \oplus \mathcal{O}(-n_2)$, where we denote by $\mathcal{O}(-n)$ the line bundle defined by the glueing rules

$$z_N = -z_S^{-1} \quad \text{and} \quad w_N = z_S^n w_S.$$

The CY condition for the total space $\text{CY}(\Sigma, V)$ is therefore $n_1 + n_2 = 2$.

The generic variation is

$$\omega^i_N = z_S^{n_i}[\omega^i_S + \Psi^i(z_S, \omega_S)] \tag{2.9}$$

where $\omega_S = (\omega^1_S, \omega^2_S)$. In Eq. (2.9), since there is one single double intersection and no triple ones, the functions Ψ^i are

just constrained to be analytic on $\mathbb{C}^\times \times \mathbb{C}^2$, that is are allowed to have poles of finite orders at 0 and ∞ and have to be analytic in ω_S. The relevant terms in Ψ^i, i.e. the ones representing true variations of the complex structure of the initial space, are the ones which cannot be reabsorbed by analytic reparametrizations of ω_S and ω_N. Moreover, the Calabi-Yau condition in the new complex structure is solved by a single potential function $X = X(z_S, \omega_S)$ such that

$$\epsilon_{ij} w^i dw^j = \epsilon_{ij} \omega^i d\omega^j - dX, \tag{2.10}$$

where, as before, $w^i = \omega^i + \Psi^i(z_S, \omega)$.

Let us note now that we can, as it is usually done in Kodaira-Spencer theory, relate the deformed and the original complex structures by a singular change of coordinates. For the case at hand, it is enough to do it along the fibers above the south pole patch, namely,

$$w^i_N = \omega^i_N, \qquad \text{and} \quad w^i_S = \omega^i_S + \Psi^i(z_S, \omega_S) \tag{2.11}$$

In the singular coordinates (z, w^i) the patching rule is the original linear one. Therefore, Eq. (2.11) defines naturally the transformation rule for generic sections in the deformed complex structure from the singular to the nonsingular coordinate system.

C. Reduction over $\Sigma = \mathbb{P}^1$ in the deformed case

Let us now perform the reduction to the brane of the open string field theory action on a Calabi-Yau deformation of $\mathrm{CY}(\Sigma, V)$ with $\Sigma = \mathbb{P}^1$ and $V = \mathcal{O}(-n_1) \oplus \mathcal{O}(-n_2)$ with $n_1 + n_2 = 2$. Actually, from the perspective we adopted so far, it turns out that performing it is not crucially different from the linear case. That is because we can proceed by performing the reduction of the hCS theory in the singular coordinates (2.11) following the prescription proper to the linear undeformed case and then implement the variation of the complex structure by passing to the nonsingular variables by the proper field redefinition.

Let us start for simplicity with the reduction in *the Abelian* U(1) *case*. In this case the cubic term in the hCS Lagrangian is absent and the reduction is almost straightforward. In the singular coordinates we obtain

$$\mathcal{L}_{\mathrm{red}} = \frac{1}{2} \epsilon_{ij} \varphi^i \partial_{\bar z} \varphi^j dz d\bar z \tag{2.12}$$

in both the north and south charts. The coordinate change for the fields φ^i in terms of the ones corresponding to the deformed complex structure is induced by (2.11). Let us recall that the functions Ψ^i defining the deformation are built out from the potential X as in Eq. (2.10). This expresses exactly our Lagrangian terms (patch by patch):

$$\epsilon_{ij} \varphi^i \partial_{\bar z} \varphi^j = \epsilon_{ij} \phi^i \partial_{\bar z} \phi^j - \partial_{\bar z} X, \tag{2.13}$$

where $X_N = 0$ and X_S is an arbitrary analytic function of the ϕ's in $\mathbb{C}^2$ and of z in $\mathbb{C}^\times$ (φ^i are akin to the coordinate singular coordinate w^i of the previous subsection, while ϕ stem from ω^i).

The above potential term X gives the deformation of the action due to the deformation of the complex structure. Specifically, we have

$$S_{\mathrm{red}} = \frac{1}{g_s} \left[\int_{U_S} \chi_S (\mathcal{L}_{\mathrm{red}})_S + \int_{U_N} \chi_N (\mathcal{L}_{\mathrm{red}})_N \right], \tag{2.14}$$

where we explicitly indicated the resolution of the unity on the sphere $1 = \chi_S + \chi_N$. For simplicity we choose the χ's to be simply step functions on the two hemispheres. Substituting (2.12) and (2.13) in (2.14), we then obtain

$$S_{\mathrm{red}} = \frac{1}{2 g_s} \left[\int_{P^1} \epsilon_{ij} \phi^i \partial_{\bar z} \phi^j dz d\bar z - \int_D \partial_{\bar z} X(z, \phi) dz d\bar z \right], \tag{2.15}$$

where D is the unit disk (south hemisphere). The disk integral can be reduced by the Stokes theorem, leaving finally

$$S_{\mathrm{red}} = \frac{1}{2 g_s} \left[\int_{P^1} \epsilon_{ij} \phi^i \partial_{\bar z} \phi^j dz d\bar z + \oint X(z, \phi) dz \right], \tag{2.16}$$

where $\oint$ is a contour integral along the equator (we understand the factor $1/2pii$).

Therefore, we see that the reduced theory gives a b-c (β-γ) system on the two hemispheres with a junction interaction along the equator and the identifications (2.9) on the fields.

The non-Abelian case is a bit more complicated than the Abelian one because of the tensoring with the (trivial) gauge bundle. This promotes the vector bundle sections to matrices and therefore unambiguously defining the potential function X in the general case is not immediate. In the following, we show where the difficulty arises and what further constraint to the deformation of the complex structure is needed in order to suitably deal with the non-Abelian case.

To see this let us perform the reduction on $\mathbb{P}^1$ of the non-Abelian hCS (2.2), as we did in the Abelian case. Let us work in the singular coordinates and obtain again the pullback Lagrangian we got in the linear case. Now, in order to pass to the nonsingular coordinates we have to promote to a matrix equation the change of variables (2.13). This can be done by specifying a prescription for matrix ordering. Suppose we choose a specific ordering and denote it by $\hat P$. Then our change of variable is

$$\mathrm{Tr}\left[\epsilon_{ij} \varphi^i \partial_{\bar z} \varphi^j\right] = \mathrm{Tr}\left[\epsilon_{ij} \phi^i \partial_{\bar z} \phi^j\right] - \partial_{\bar z} \mathrm{Tr} X^{\hat P} \tag{2.17}$$

while the cubic term gives

$$\mathrm{Tr}\left[A_{\bar z} \epsilon_{ij} (\phi^i + \Psi^{i\hat P})(\phi^j + \Psi^{j\hat P})\right]. \tag{2.18}$$

It appears immediately that our result is complicated and seems to depend quite nontrivially on the matrix ordering prescription. Otherwise, it is well defined.

The easiest way to avoid matrix ordering prescriptions is to have to do with only one matrix. Henceforth we restrict to the case in which $X(z, \omega)$ does not depend, say, on ω_2 and we proceed further.

In this case the deformation formulas simplify considerably. Equation (2.10) is solved by $\Psi^1 = 0$ and Ψ^2 is determined by the potential by

$$\partial_{\omega^1}\left(\frac{\Psi^2}{\omega^1}\right) = -\frac{\partial_{\omega^1} X}{(\omega^1)^2}. \tag{2.19}$$

This condition can be written also as

$$2\Psi^2 = \partial_{\omega^1}[\omega^1\Psi^2 + X].$$

As far as the reduction is concerned, Eq. (2.17) is unchanged, since we do not need any prescription $\hat{P}$; while Eq. (2.18) simplifies to

$$\mathrm{Tr}\,[A_{\bar{z}}\epsilon_{ij}(\phi^i + \Psi^i)(\phi^j + \Psi^j)] = \mathrm{Tr}(A_{\bar{z}}[\phi^1, \phi^2]) \tag{2.20}$$

(where we used $[\phi^1, \Psi^2] = 0$) which, as in the linear case, is the contribution needed to complete the covariant derivative. The last operation to obtain our final result is an integration by part in the derivative term. As

$$\epsilon_{ij}\phi^i\partial_{\bar{z}}\phi^j = -2\phi^2\partial_{\bar{z}}\phi^1 + \partial_{\bar{z}}(\phi^1\phi^2)$$

in both the north and south charts, from the last term we get an additional contribution to the equator contour integral, that is $\frac{1}{g_s}\frac{1}{2}\oint \mathrm{Tr}\phi^1\Psi^2$. Adding it to the previously found term we get $\frac{1}{g_s}\frac{1}{2}\oint \mathrm{Tr}(X + \phi^1\Psi^2)$. This, by (2.19) can be written just as $\frac{1}{g_s}\oint \mathrm{Tr}B$, where $\partial^1_\omega B = \Psi^2$.

Therefore, summarizing, we find that in the non-Abelian case on the Riemann sphere we are able to treat the deformations of the type

$$\begin{aligned} &\omega^1_N = z_S^{-n}\omega^1_S, \\ &\text{and} \quad \omega^2_N = z_S^{2+n}[\omega^2_S + \partial_{\omega^1}B(z_S, \omega^1_S)], \end{aligned} \tag{2.21}$$

which corresponds to the choice $n_1 = -n$. This geometry has been introduced (in the matrix planar limit) by [10]. These geometries are CY for any potential B analytic in $\mathbb{C}^\times \times \mathbb{C}$. The relevant reduced theory action is given by

$$S_{\mathrm{red}} = \frac{1}{g_s}\left[\int_{P^1} -\mathrm{Tr}(\phi^2 D_{\bar{z}}\phi^1)dzd\bar{z} + \oint \mathrm{Tr}\,B(z, \phi^1)dz\right]. \tag{2.22}$$

D. Reduction to matrix models

For completeness, let us generalize to our case the argument of [19] to show the reduction of the previous action to the matrix models. In calculating the partition function of the open strings attached to the D-branes, [2] we can easily integrate out the gauge connection $A_{\bar{z}}$ which implies the constraint $[\phi^1, \phi^2] = 0$ and then ϕ^2 which implies the constraint $\partial_{\bar{z}}\phi^1 = 0$. As a result, [3] we get that the partition function

$$\begin{aligned} Z_{\mathrm{red}} &= \int D[A_{\bar{z}}, \phi^1, \phi^2]e^{-S_{\mathrm{red}}} \\ &\propto \int D[\phi^1]\delta(\partial_{\bar{z}}\phi^1)e^{-(1/g_s)\oint B}. \end{aligned} \tag{2.23}$$

The condition such that the equation $\partial_{\bar{z}}\phi^1 = 0$ admits solutions is $n \geq 0$ in (2.21). In such a case we have $n + 1$ independent solutions which in the south patch are the linear span of $\{z^i_S\}_{i=0\ldots n}$.

Therefore, expanding $\phi^1_S = \sum_{i=0}^n X_i z^i_S$ in (2.23), we are left with the multimatrix model partition function,

$$Z_{\mathrm{red}} = \int \prod_i dX_i e^{-W(X_0,\ldots X_n)}, \tag{2.24}$$

where the potential is given by

$$W(X_0, \ldots X_n) = \oint dz B\left(z, \sum_{i=0}^n X_i z^i\right). \tag{2.25}$$

This coincides with the one introduced by [10] in the matrix planar limit.

The original Dijkgraaf-Vafa case is reproduced for $n = 0$. Then, the only nontrivial complex structure deformation in (2.21) is with $B = \frac{1}{z}W(\omega_1)$ (since any other dependence in z can be reabsorbed by analytic reparametrizations) and hence we get the one-matrix model with potential W.

The above formula can be also inferred by just generalizing another conformal field theory (CFT) argument by Dijkgraaf-Vafa to the geometry (2.21). To this end, let us consider again the two-dimensional theory defined by the action

$$S = \frac{1}{g_s}\int_{\mathbb{P}^1} \mathrm{Tr}(\phi_2 D_{\bar{z}}\phi_1), \tag{2.26}$$

where $D_{\bar{z}} = \partial_{\bar{z}} + [A_{\bar{z}}, \cdot]$. This is a gauged chiral conformal field theory: a gauged b-c (β-γ) system in which ϕ_1 and ϕ_2 are conformal fields of dimensions $-n/2$ and $1 + n/2$, respectively. For any n, the fields ϕ_1 and ϕ_2 are canonically conjugated and on the plane they satisfy the usual OPE

$$\phi_1(z)\phi_2(w) \sim \frac{g_s}{z - w}. \tag{2.27}$$

In Hamiltonian formalism, that is in the radial quantization of the CFT, the partition function is given as

$$Z = \langle \mathrm{out}|\mathrm{in}\rangle. \tag{2.28}$$

The deformed transformation

[2] Notice that the field redefinition $\varphi \to \phi$ has unit Jacobian because of the CY condition.

[3] See [12] for more details about how the ghost contribution in the maximal Abelian gauge compensates the $\det^{-1} ad_{\phi^1}$ term.

$$\phi_2' = z^{n+2}(\phi_2 + \partial_{\phi_1} B(z, \phi_1)) \tag{2.29}$$

is given on the cylinder $z = e^w$ as

$$(\phi_2')_{\text{cyl}} = (\phi_2)_{\text{cyl}} + \frac{\partial B(z, \phi_1)}{\partial(\phi_1)_{\text{cyl}}} \tag{2.30}$$

and is implemented by the operator

$$U = \exp\left(\text{Tr} \oint \frac{\mathrm{d}z}{2\mathrm{i}\pi} B(z, \phi_1)\right). \tag{2.31}$$

Therefore the new partition function is

$$Z = \langle \text{out}|U|\text{in}\rangle, \tag{2.32}$$

which is our result.

We remark that this is an *a posteriori* argument, it is a consistency check but does not explain the dynamical origin of the matrix model from the string theory describing the brane dynamics.

III. ENGINEERING MATRIX MODELS

Once the link between D-brane configurations and multimatrix models is established, the next natural question to ask is what kind of matrix models we get in this way. In this section we single out the most general type of multimatrix model we can engineer by deforming D-branes on 2-cycles in the above way, and we produce some examples.

The geometric potential $B(z, \omega)$ is a general holomorphic function on $\mathbb{C}^* \times \mathbb{C}$ but the terms actually contributing to a change in the complex structure and giving a nonzero matrix potential are of the form

$$B(z, \omega) = \sum_{d=1}^{\infty} \sum_{k=0}^{d\cdot n} t_d^{(k)} z^{-k-1} \omega^d, \tag{3.1}$$

where $t_d^{(k)}$ are the *times* of the potential and ω is to be identified with the coordinate ω_1 of the previous section. It can be easily proven that other terms in the expansion can be reabsorbed by an analytic change of coordinates in the geometry. Consistently with the geometric theory of deformations [20], they do not contribute to the matrix potential.

The *degree* of the potential B is the maximum d such that $t_d^{(k)}$ is nonzero for some k in (3.1) and corresponds to the degree of the matrix potential, obtained as

$$W(X_0, \ldots, X_n) = \oint \frac{\mathrm{d}z}{2\mathrm{i}\pi} B\left(z, \sum_{j=0}^{n} X_i z^i\right). \tag{3.2}$$

Since this operation is linear, from (3.1) and (3.2) one gets a matrix potential of the form

$$W(X_0, \ldots, X_n) = \sum_{d=0}^{\infty} \sum_{k=0}^{d\cdot n} t_d^{(k)} W_d^{(k)}(X_0, \ldots, X_n), \tag{3.3}$$

where each term

$$W_d^{(k)}(X_0, \ldots, X_n) = \sum_{\substack{i_1,\ldots i_d=0 \\ i_1+\cdots+i_d=k}}^{n} X_{i_1} \ldots X_{i_d} \tag{3.4}$$

corresponds to $B_d^{(k)}(z, \omega) = z^{-k-1}\omega^d$ for $0 \le d \le +\infty$ and $0 \le k \le d \cdot n$. Note that these are directly obtained in completely symmetric ordered form with respect to the indices $i_1, \ldots, i_d$ labeling the different matrix variables. In the following we will sometimes write simply the polynomial W for c-number variables $W(x_0, \ldots, x_n)$, understanding the total symmetrization when matrices are plugged in.

As it was already anticipated in the previous section, the *one-matrix models* correspond directly to the Dijkgraaf-Vafa case with $n = 0$ and therefore $B = \frac{1}{z} W(\omega)$.

Two-matrix models are obtained by considering the case $n = 1$. Some of them have been derived in [10]. In this case, it is possible to engineer a general function for two commuting variables. In fact,

$$B(z, \omega) = z^{-k-1}\omega^{k+j} \rightarrow W(x) = \binom{k+j}{k} x_0^k x_1^j, \tag{3.5}$$

and the matrix potential reads

$$W(x_0, x_1) = \sum_{d=1}^{\infty} \sum_{k=0}^{d} t_d^{(k)} \binom{d}{k} x_1^k x_0^{d-k}, \tag{3.6}$$

which is, upon varying the possible couplings, a generic analytic potential in the two variables x_0 and x_1. The only constraint is the matrix ordering which is always the symmetric one. In particular, it is easy to engineer a two-matrix model with bilinear coupling. This is achieved by choosing, for $n = 1$, the geometric potential to be

$$B(z, \omega) = \frac{1}{z}\left[V(\omega) + U\left(\frac{\omega}{z}\right)\right] + \frac{c}{2z^2}\omega^2, \tag{3.7}$$

which generates the matrix potential

$$W(x_0, x_1) = V(x_0) + U(x_1) + c x_0 x_1. \tag{3.8}$$

In general, the *multimatrix models* one can engineer are not of arbitrary form. Actually, on top of the fact that we can generate only matrix potentials with symmetric ordering, there are also constraints between possibly different couplings. This can be inferred from the fact that a polynomial function in $n + 1$ variables of total maximal degree d is specified by many more coefficients than the ones we have at our disposal. (As an example, if $n = 3$ and $d = 3$ we would need 10 coefficients, while we have only 4 at our disposal.)

To end this section, we would like to remark that some deformations can connect cases with different values of n. The geometric equivalence of seemingly different complex structures becomes in fact explicit at the matrix model level. As an example, let us consider the case $n = 2$ and a geometric potential of the form

$$B(z,\omega) = -\frac{1}{2}z^{-4}\omega^2 + z^{-3}F(\omega).$$

Out of this, one obtains

$$W(x_0, x_1, x_2) = (F'(x_0) - x_1)x_2 + \frac{1}{2}(F''(x_0)x_1^2).$$

After integration of x_2, which appears linearly, this theory is equivalent to a one-matrix model with potential

$$V(x_0) = \frac{1}{2}F''F'^2(x_0),$$

which is equivalent to $n = 0$ and $B(z,\omega) = \frac{1}{z}V(\omega)$. As a matter of fact, the geometry with $n = 2$ and $B = -\frac{1}{2}z^{-4}\omega^2$ is equal, upon diagonalization, to the geometry $n = 0$ and $B = 0$.

IV. GENERAL PROPERTIES OF TWO-MATRIX MODELS

In the second part of the paper, we concentrate on a subclass of matrix models: the two-matrix models with bilinear coupling between the two matrices but arbitrary self-coupling of each matrix. Our purpose is to find *exact quantum* solutions. For this reason we will solve them with the method of orthogonal polynomials. This method allows one to explicitly perform the path integration, so that one is left with quantum equations. The two basic ingredients are the quantum equations of motion and the integrable linear systems. The latter, in particular, uncover the integrable nature of two-matrix models, which stems from the Toda lattice hierarchy [4] underlying all of them. Our approach for solving two-matrix models consists in solving the quantum equations of motion and, then, using the recursiveness intrinsic to integrability (the flow equations), finding explicit expressions for the correlators. An alternative method is based on the W-constraints on the functional integral. We do not use it here, but one can find definitions, applications, and comparisons with the other methods in [21–24].

For general reviews on matrix models applied to string theory, see [25–27]. For general application of matrix models, see [28,29]. Early literature on two-matrix models is contained in Refs. [30–43]. The method used in the present paper, although implicit in the early literature, is, quite incomprehensibly, seldom utilized. Different methods (from the saddle point to loop equations) are often preferred, see Refs. [44–55].

A. Review of old formulas

The model of two Hermitian $N \times N$ matrices M_1 and M_2 with bilinear coupling [see (3.7) and (3.8)] is introduced in terms of the partition function

$$Z_N(t,c) = \int dM_1 dM_2 e^{\mathrm{tr}\, W}, \qquad W = V_1 + V_2 + cM_1M_2 \tag{4.1}$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} \bar{t}_{\alpha,r} M_\alpha^r \qquad \alpha = 1, 2. \tag{4.2}$$

where p_α are finite numbers. These potentials define the model. We denote by $\mathcal{M}_{p_1,p_2}$ the corresponding two-matrix model.

We are interested in computing correlation functions (CF's) of the operators

$$\tau_k = \mathrm{tr}\, M_1^k, \qquad \sigma_k = \mathrm{tr}\, M_2^k, \quad \forall\ k.$$

For this reason we complete the above model by replacing (4.2) with the more general potentials,

$$V_\alpha = \sum_{r=1}^{\infty} t_{\alpha,r} M_\alpha^r, \qquad \alpha = 1, 2, \tag{4.3}$$

where $t_{\alpha,r} \equiv \bar{t}_{\alpha,r}$ for $r \le p_\alpha$. The CF's are defined by

$$\langle \tau_{r_1} \dots \tau_{r_n} \sigma_{s_1} \dots \sigma_{s_m} \rangle = \frac{\partial^{n+m}}{\partial t_{1,r_1} \dots \partial t_{1,r_n} \partial t_{2,s_1} \dots \partial t_{2,s_m}} \times \ln Z_N(t,g), \tag{4.4}$$

where, in the right-hand side, all the $t_{\alpha,r}$ are set equal to $\bar{t}_{\alpha,r}$ for $r \le p_\alpha$ and the remaining are set to zero. The unusual + sign at the exponent of the integrand in (4.1) is because we want to use a uniform notation for physical couplings $\bar{t}_{\alpha,r}$ and sources $t_{\alpha,r}$ (for the convergence of the integrals, see below). From now on, we will not distinguish between $t_{\alpha,r}$ and $\bar{t}_{\alpha,r}$ and use throughout only $t_{\alpha,r}$. We hope the context will always make clear what we are referring to.

We recall that the ordinary procedure to calculate the partition function consists of three steps [56–58]: (i) one integrates out the angular part so that only the integrations over the eigenvalues are left; (ii) one introduces the orthogonal monic polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers},$$
$$\eta_n(\lambda_2) = \lambda_2^n + \text{lower powers},$$

which satisfy the orthogonality relations

$$\int d\lambda_1 d\lambda_2 \xi_n(\lambda_1) e^{V_1(\lambda_1) + V_2(\lambda_2) + c\lambda_1\lambda_2} \eta_m(\lambda_2) = h_n(t,c)\delta_{nm}; \tag{4.5}$$

(iii) using the orthogonality relation (4.5) and the properties of the Vandermonde determinants, one can easily calculate the partition function

$$Z_N(t,c) = \mathrm{const}\, N! \prod_{i=0}^{N-1} h_i, \tag{4.6}$$

whereby we see that knowing the partition function means knowing the coefficients $h_n(t, c)$.

The crucial point is that the information concerning the latter can be encoded in (1) a suitable linear system subject to certain (2) equations of motion (coupling conditions), together with (3) relations that allows us to reconstruct Z_N.

Let us introduce some convenient notations. We will meet infinite matrices M_{ij} with $0 \leq i, j < \infty$. For any such matrix M, we define

$$\mathcal{M} = H^{-1}MH, \qquad H_{ij} = h_i\delta_{ij}, \qquad \tilde{M}_{ji} = M_{ji},$$
$$M_l(j) \equiv M_{j,j-l}.$$

We represent such matrices in the lower right quadrant of the (i, j) plane. They all have a band structure, with nonzero elements belonging to a band of lines parallel to the main descending diagonal. We will write $M \in [a, b]$, if all its nonzero lines are between the ath and the bth ones, setting $a = 0$ for the main diagonal. Moreover, M_+ will denote the upper triangular part of M (including the main diagonal), while $M_- = M - M_+$. We will write

$$\mathrm{Tr}\,(M) = \sum_{i=0}^{N-1} M_{ii}.$$

Next we pass from the basis of orthogonal polynomials to the basis of orthogonal functions

$$\Psi_n(\lambda_1) = e^{V_1(\lambda_1)}\xi_n(\lambda_1), \qquad \Phi_n(\lambda_2) = e^{V_2(\lambda_2)}\eta_n(\lambda_2).$$

The orthogonality relation (4.5) becomes

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1) e^{c\lambda_1\lambda_2}\Phi_m(\lambda_2) = \delta_{nm}h_n(t, c). \tag{4.7}$$

We will denote by Ψ the semi-infinite column vector $(\Psi_0, \Psi_1, \Psi_2, \ldots)^t$ and by Φ the vector $(\Phi_0, \Phi_1, \Phi_2, \ldots)^t$. Then we introduce the following Q-type matrices

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1)\lambda_\alpha e^{c\lambda_1\lambda_2}\Phi_m(\lambda_2)$$
$$\equiv Q_{nm}(\alpha)h_m = \tilde{\mathcal{Q}}_{mn}(\alpha)h_n, \qquad \alpha = 1, 2. \tag{4.8}$$

Beside these Q matrices, we will need two P-type matrices, defined by

$$\int d\lambda_1 d\lambda_2 \left(\frac{\partial}{\partial\lambda_1}\Psi_n(\lambda_1)\right) e^{c\lambda_1\lambda_2}\Phi_m(\lambda_2) \equiv P_{nm}(1)h_m, \tag{4.9}$$

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1) e^{c\lambda_1\lambda_2}\left(\frac{\partial}{\partial\lambda_2}\Phi_m(\lambda_2)\right) \equiv P_{mn}(2)h_n. \tag{4.10}$$

For later use we also introduce

$$\int d\lambda_1 d\lambda_2 \left(\frac{\partial}{\partial\lambda_1}\xi_n(\lambda_1)\right) e^{V_1(\lambda_1)+V_2(\lambda_2)+c\lambda_1\lambda_2}\eta_m(\lambda_2)$$
$$\equiv P^{\mathrm{o}}_{nm}(1)h_m, \tag{4.11}$$

$$\int d\lambda_1 d\lambda_2 \xi_n(\lambda_1) e^{V_1(\lambda_1)+V_2(\lambda_2)+c\lambda_1\lambda_2}\left(\frac{\partial}{\partial\lambda_2}\eta_m(\lambda_2)\right)$$
$$\equiv P^{\mathrm{o}}_{mn}(2)h_n. \tag{4.12}$$

Let us come now to the three elements announced above.

(1) *Quantum equations of motion*. The two matrices (4.8) we introduced above are not independent. More precisely both $Q(\alpha)$'s can be expressed in terms of only one of them and one matrix P. Expressing the trivial fact that the integral of the total derivative of the integrand in Eq. (4.7) with respect to λ_1 and λ_2 vanishes, we can easily derive the constraints or *coupling conditions, or quantum equations of motion* ,

$$P^{\mathrm{o}}(1) + V_1' + cQ(2) = 0, \qquad cQ(1) + V_2' + \tilde{\mathcal{P}}^{\mathrm{o}}(2) = 0. \tag{4.13}$$

These may be considered the quantum analog of the classical equations of motion. The difference with the classical equations of motion of the original matrix model is that, instead of the $N \times N$ matrices M_1 and M_2, here we have infinite $Q(1)$ and $Q(2)$ matrices together with the quantum deformation terms given by $P^{\mathrm{o}}(1)$ and $\mathcal{P}^{\mathrm{o}}(2)$, respectively. From the coupling conditions it follows at once that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \qquad \alpha = 1, 2,$$

where

$$m_1 = p_2 - 1, \qquad m_2 = 1, \qquad n_1 = 1,$$
$$n_2 = p_1 - 1,$$

where p_α, $\alpha = 1, 2$ is the highest order of the interacting part of the potential V_α [see (4.2)].

(2) *The associated linear systems.* The derivation of the linear systems associated to our matrix model is very simple. We take the derivatives of Eq. (4.7) with respect to the time parameters $t_{\alpha,r}$, and use Eqs. (4.8). We get in this way the time evolution of Ψ and Φ, which can be represented in two different ways:

Discrete Linear System I:

$$\begin{cases} Q(1)\Psi(\lambda_1) = \lambda_1\Psi(\lambda_1), \\ \frac{\partial}{\partial t_{1,k}}\Psi(\lambda_1) = Q^k(1)_+\Psi(\lambda_1), \\ \frac{\partial}{\partial t_{2,k}}\Psi(\lambda_1) = -Q^k(2)_-\Psi(\lambda_1), \\ \frac{\partial}{\partial\lambda}\Psi(\lambda_1) = P(1)\Psi(\lambda_1). \end{cases} \tag{4.14}$$

The corresponding consistency conditions are

$$[Q(1), P(1)] = 1, \tag{4.15a}$$

$$\frac{\partial}{\partial t_{\alpha,k}} Q(1) = [Q(1), Q^k(\alpha)_-], \qquad \alpha = 1, 2. \tag{4.15b}$$

In a similar way we can get the time evolution of Φ via a *discrete linear system II*, whose consistency conditions are

$$[\tilde{Q}(2), P(2)] = 1, \tag{4.16a}$$

$$\frac{\partial}{\partial t_{\alpha,k}} Q(2) = [Q^k(\alpha)_+, Q(2)]. \tag{4.16b}$$

One can write down flows for $P(1)$ and $P(2)$ as well, but we will not need them in the sequel.

(3) *Reconstruction formulas*. The third element announced above is the link between the quantities that appear in the linear system and in the quantum equations of motion with the original partition function. We have

$$\frac{\partial}{\partial t_{\alpha,r}} \ln Z_N(t, c) = \mathrm{Tr}(Q^r(\alpha)), \qquad \alpha = 1, 2. \tag{4.17}$$

It is evident that, by using Eqs. (4.15b) and (4.16b) above, we can express all the derivatives of Z_N in terms of the elements of the Q matrices. For example,

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} \ln Z_N(t, c) = (Q^r(\alpha))_{N,N-1}, \qquad \alpha = 1, 2, \tag{4.18}$$

and so on. We recall that the derivatives of $F(N, t, c) = \ln Z_N(t, c)$ at prescribed values of the coupling are nothing but the correlation functions of the model.

The above derivation is rigorous when, for example, highest negative even couplings guarantee that the measure in (4.5) is square integrable and decreases more then polynomially at infinity. But for generic values of the couplings it is heuristic. Nevertheless we notice that the consistency and quantum equations of motion make sense for any value of the couplings, and also when the couplings are infinite in number. In the latter case, Eqs. (4.15b) and (4.16b) form nothing but a very well-known discrete integrable hierarchy, the Toda lattice hierarchy (see [4]).

From these considerations it is clearly very convenient to refer to the integrable system formulation rather than to the original path integral formulation of our problem. This allows us not only to extend our problem to a larger region of the parameter space, but also to make full use of integrability.

To end this section, we collect a few formulas we will need later on. First, we will be using the following choice of coordinates of the Jacobi matrices:

$$\begin{aligned} Q(1) &= I_+ + \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i,i-l}, \\ \tilde{Q}(2) &= I_+ + \sum_i \sum_{l=0}^{m_2} b_l(i) E_{i,i-l}, \end{aligned} \tag{4.19}$$

where $I_+ = \sum_{i=0} E_{i,i+1}$ and $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$. One can immediately see that

$$\begin{aligned} (Q_+(1))_{ij} &= \delta_{j,i+1} + a_0(i)\delta_{i,j}, \\ (Q_-(2))_{ij} &= R(i)\delta_{j,i-1}, \end{aligned} \tag{4.20}$$

where $R(i + 1) \equiv h_{i+1}/h_i$. As a consequence of this choice of coordinates, Eq. (4.18) gives, in particular, the two important relations,

$$\frac{\partial^2}{\partial t_{1,1}^2} F(N, t, c) = a_1(N), \tag{4.21}$$

and

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{2,1}} F(N, t, c) = R(N). \tag{4.22}$$

For reasons of brevity we do not even touch on the subject of W-constraints. The latter are constraints on the partition function under the form of algebraic structure (see [21,24], for instance). They are obtained by putting together quantum equations of motion and flow equations. W-constraints (which are also called loop equations or Schwinger-Dyson equations) can be used to solve matrix models, but such a procedure is less efficient than the one used in the sequel.

B. Homogeneity and genus expansion

The CF's we compute are genus expanded. The genus expansion is strictly connected with the homogeneity properties of the CF's. The contribution pertinent to any genus is a homogeneous function of the couplings (and N) with respect to appropriate degrees assigned to all the involved quantities. Precisely, we assign to the couplings the following degrees:

$$\begin{aligned} \deg(\,) \equiv [\,], \qquad & [t_{\alpha,k}] = x(1 - k), \\ [N] = x, \qquad & [c] = -x, \end{aligned} \tag{4.23}$$

where x is an arbitrary positive number. Here and in the following, N is treated as a coupling $t_{1,0} = t_{2,0}$. If we rescale the couplings as follows,

$$t_{\alpha,k} \to \lambda^{[t_{\alpha,k}]} t_{\alpha,k},$$

we expect the free energy to scale like

$$F \to \sum_{h=0}^{\infty} \lambda^{x(2-2h)} F_h, \tag{4.24}$$

where h is the genus. In other words,

$$[F_h] = 2x(1 - h). \tag{4.25}$$

F_h is interpreted as the result of summing the open string partition function at fixed h over all the boundaries.

The CF's will be expanded accordingly. Such expectation, based on a path integral analysis, remains true in our

setup due to the fact that the homogeneity properties carry over to the Toda lattice hierarchy. To this end, we have simply to consider a genus expansion for all the coordinate fields that appear in $Q(1)$ and $Q(2)$ [see (4.19) and (4.20)]. The Toda lattice hierarchy splits accordingly. In genus 0 the following assignments,

$$[a_l^{(0)}] = (l+1)x, \qquad [b_l^{(0)}] = (l+1)x, \qquad [R^{(0)}] = 2x, \tag{4.26}$$

correspond exactly to the assignments (4.23) and $[F_0] = 2x$.

V. SOLVING TWO-MATRIX MODELS

As already pointed out in the previous section, a way to solve a two-matrix model is to solve the coupling conditions (quantum equations of motion). This allows us to determine the "fields" $a_i(n)$, $b_i(n)$ and $R(n)$. Once these are known we can compute all the correlation functions starting from (4.17) by repeated use of Eqs. (4.15a), (4.15b), (4.16a), and (4.16b), which form the flows of the Toda lattice hierarchy. As for the free energy $F(t, N, c)$, see for instance [22]. In [24], using this method, the bi-Gaussian model $\mathcal{M}_{2,2}$ was solved. This is of course a simple model. However, it is useful to check the coincidence of the results obtained in this way in the decoupling $c = 0$ case with the available results obtained by the traditional method based on eigenvalue density and resolvent for the Gaussian one-matrix model.

A very interesting case is the model $\mathcal{M}_{0,0}$, i.e. the limiting model when only the c parameter is different from zero. As a path integral this model does not make much sense. However, as we saw above, it does have sense as an integrable system to which the appropriate coupling conditions are applied. It turns out that this model describes $c = 1$ string theory at the self-dual radius, as was shown in [22,23].

A. Solving the quantum EoMs: $\mathcal{M}_{3,2}$ model

The next model in order of complexity is the $\mathcal{M}_{3,2}$ model [23]. The relevant quantum equations of motion are

$$P^{o}(1) + 3t_3 Q(1)^2 + 2t_2 Q(1) + t_1 + cQ(2) = 0, \tag{5.1}$$

$$\tilde{\mathcal{P}}^{o}(2) + 2s_2 Q(2) + s_1 + cQ(1) = 0. \tag{5.2}$$

Using the choice of coordinates (4.19) and (4.20) they produce the following equations for the fields $a_l(n)$, $b_l(n)$, $R(n)$:

$$\begin{gathered} cb_2(n) + 3t_3 R(n)R(n-1) = 0 \\ 2t_2 R(n) + cb_1(n) + 3t_3 R(n)(a_0(n) + a_0(n-1)) = 0 \\ 3t_3(a_0(n)^2 + a_1(n) + a_1(n+1)) + 2t_2 a_0(n) + t_1 + cb_0(n) = 0 \\ n + 3t_3 a_1(n)(a_0(n) + a_0(n-1)) + 2t_2 a_1(n) + cR(n) = 0 \\ 2s_2 R(n) + ca_1(n) = 0 \qquad 2s_2 b_0(n) + s_1 + ca_0(n) = 0 \qquad n + 2s_2 b_1(n) + cR(n) = 0, \end{gathered} \tag{5.3}$$

where we have introduced the simplified notation

$$t_{1,k} \equiv t_k, \qquad t_{2,k} \equiv s_k.$$

One easily realizes that the second, fourth, fifth, and seventh equations are linearly dependent. The remaining equations determine the lattice fields $a_0, a_1, b_0, b_1, b_2, R$ completely. The $\mathcal{M}_{3,2}$ model, even though comparatively simple is already rather complex due to the large number of involved fields. Therefore, for pedagogical purposes, let us further simplify it, by setting $c = 0$ and considering only the a_0, a_1 fields. This corresponds to the one-matrix model with cubic interaction. In Eqs. (5.3), for consistency, we have to set also $R = 0$. The relevant equations are

$$\begin{aligned} &3t_3(a_0(n)^2 + a_1(n) + a_1(n+1)) + 2t_2 a_0(n) + t_1 = 0 \\ &n + 3t_3 a_1(n)(a_0(n) + a_0(n-1)) + 2t_2 a_1(n) = 0. \end{aligned} \tag{5.4}$$

One can derive a_1 from the second equation in terms of a_0 and replace it into the first. One gets in this way a cubic algebraic recursive equation for a_0. We solve it with a genus by genus approach. The first step is to start with genus 0. To do so, one simply ignores the increments ± 1 on the n entry. In this way we get an ordinary cubic algebraic equation in the unknown a_0:

$$a_0^3 + \frac{t_2}{t_3} a_0^2 + \frac{2}{9}\left(\frac{t_2}{t_3}\right)^2 a_0 - \frac{n}{3t_3} = 0, \tag{5.5}$$

$$a_1 = -\frac{1}{2} a_0^2 - \frac{1}{3}\frac{t_2}{t_3} a_0, \tag{5.6}$$

where, for simplicity and without loss of generality, we have set $t_1 = 0$. In the large N limit, it is convenient to shift to the continuous formalism, by defining the continuous variable $x = \frac{n}{N}$. It is also useful to make contact with Sec. 4 of [56] for a comparison. So, also in order to simplify a bit further the notation, we set $t_2 = -\frac{N}{2}$ and $t_3 = -Ng$, where g is the cubic coupling constant there. Moreover, we denote $f = 3g a_0$. Then Eq. (5.5) becomes

$$18g^2 x + f(1+f)(1+2f) = 0. \tag{5.7}$$

It is easy to find the three solutions, which for small x take the form

$$f_1 = -18g^2x - 972g^4x^2 - 93\,312g^6x^3 - 11\,022\,480g^8x^4 + O(x^5), \tag{5.8}$$

$$f_2 = -1 - 18g^2x + 972g^4x^2 - 93\,312g^6x^3 + 11\,022\,480g^8x^4 + O(x^5), \tag{5.9}$$

$$f_3 = -\frac{1}{2} + 36g^2x + 186\,624g^6x^3 + O(x^5). \tag{5.10}$$

From them we can easily write the fields a_0 and a_1 in terms of g and x. They therefore lead to three different solutions for the correlators. Later on, we will show how to compute the latter. But, first, let us discuss the meaning of these three solutions. To start with, comparing this with [56], we see that the first solution corresponds to the unique solution found there provided we set $x = 1$. It corresponds to the minimum of the classical potential. In fact the correspondence with [56] can be made very precise: one can easily verify that Eqs. (46) there are nothing but Eqs. (5.5) and (5.6) provided we make the identifications: $a + b = a_0$ and $(b - a)^2 = 4ga_1$ with $x = 1$. In [56] the interval $(2a, 2b)$ represents the cut in the eigenvalue distribution function. This cut therefore has to be found in our formalism in the (a_0, a_1) plane, once we forget the dependence of the latter on x.

The classical potential for the continuous eigenvalue function $\lambda(x)$ (which is $\lambda_n/\sqrt{N}$ in the large N limit), is $V_{cl} = \frac{1}{2}\lambda^2 + g\lambda^3$. It has extrema at $\lambda = 0$ and $\lambda = -1/3g$. To find the classical limit we have to drop the last term in Eq. (5.5). Remember that dropping the latter (with $c = 0$) is equivalent to writing Eq. (5.1) as $V_1'(Q(1)) = 0$. In other words the latter is the equation that identifies the extrema of the quantum potential V_1. They are three, $f = 0, -1, -1/2$, which corresponds to $a_0 = 0, -1/3g, -1/6g$, not two as in the classical case. So we see that a_0 approaches, in the classical limit, the classical eigenvalue function. Moreover, $f = 0$ corresponds to the minimum of the potential, $f = -1$ corresponds to the maximum, and $f = -1/2$ to the vanishing of the second derivative. The first two cases (the classical extrema) are characterized by the fact that $a_1 = 0$, while the third corresponds to a nonvanishing a_1.

From this simple example we learn three important pieces of informations.

(i) The number of solutions of the quantum problem [i.e. the number of solutions to Eq. (5.7)] is larger than the number of the extrema of the classical potential.
(ii) The field a_0 can be regarded as the quantum version of the classical eigenvalue function.
(iii) The classical extrema are obtained by setting, together with $n = 0$ in Eq. (5.5), also $a_1 = 0$.

As we shall see, the last condition, in the most general case, must be replaced by all fields $a_1, a_2, \ldots$ present in the problem being set to zero (except a_0).

What is the meaning of the third solution, $f = -1/2$? It is a nonperturbative solution. It cannot be seen in the saddle point approximation (more on this later).

It is now easy to extend the analysis to the full $\mathcal{M}_{3,2}$ model. In genus 0 (i.e. disregarding the ± 1 increments on n) it leads to the following set of equations (see [24]):

$$\begin{aligned} a_1(n) &= -\frac{2s_2}{c}R(n), \\ b_0(n) &= -\frac{ca_0(n)}{2s_2}\, b_1(n) = -\frac{n + cR(n)}{2s_2}, \\ b_2(n) &= -\frac{3t_3}{c}R(n)^2 \end{aligned} \tag{5.11}$$

and the recursion relations

$$2a_0(n) = -\frac{2t_2}{3t_3} + \frac{c}{6s_2t_3}\left(c + \frac{n}{R(n)}\right), \tag{5.12}$$

$$2R(n) = \frac{c}{2s_2}a_0(n)^2 + \left(\frac{2ct_2}{6s_2t_3} - \frac{c^3}{12s_2^2t_3}\right)a_0(n). \tag{5.13}$$

As expected the last two equations lead to the same cubic equation for a_0 as (5.7) with modified coefficients. It is interesting to find the classical extrema of this model. According to the above recipe, we must set $a_1 = b_1 = b_2 = 0$, and drop the first term in the third and sixth of Eqs. (5.3). In the genus 0 version of the latter this leads to $R = 0$ and to

$$3t_3a_0^2 + 2t_2a_0 + cb_0 = 0 \qquad 2s_2b_0 + ca_0 = 0. \tag{5.14}$$

The extrema correspond therefore to

$$a = 0, \qquad \text{and} \quad a_0 = \frac{c^2 - 4s_2t_2}{6s_2t_3}, \tag{5.15}$$

which is what one expects by completing the quadratures in the original classical potential. Of course, as above, in this way we find only two extrema out of three.

The solutions of the $\mathcal{M}_{3,2}$ model therefore can be found in the same way as in the simplified one-matrix model above.

B. Solving the quantum EoMs: $\mathcal{M}_{4,2}$ model

As a further example we briefly analyze the $\mathcal{M}_{4,2}$ model with, for simplicity, only the t_4, t_2, c, and s_2 couplings switched on. The quantum equations of motion are

$$P^{\circ}(1) + 4t_4Q(1)^3 + 2t_2Q(1) + cQ(2) = 0, \tag{5.16}$$

$$\tilde{\mathcal{P}}^{\circ}(2) + 2s_2Q(2) + cQ(1) = 0. \tag{5.17}$$

The matrices $Q(1) \in [-1, 1]$ and $Q(2) \in [-1, 3]$. Using the choice of coordinates (4.19) and (4.20) they produce the

following equations for the fields $a_l(n)$, $b_l(n)$, $R(n)$, which we write down in the genus 0 version:

$$
\begin{gathered}
n + 12t_4(a_1^2 + a_0^2 a_1) + 2t_2 a_1 + cR = 0 \\
4t_4(a_0^3 + 6a_0 a_1) + 2t_2 a_0 + cb_0 = 0 \\
12t_4(a_0^2 + a_1) + 2t_2 + c\frac{b_1}{R} = 0 \qquad 12t_4 a_0 + c\frac{b_2}{R^2} = 0 \\
4t_4 + c\frac{b_3}{R^3} = 0 \qquad n + 2s_2 b_1 + cR = 0 \\
2s_2 b_0 + ca_0 = 0 \qquad 2s_2 R + ca_1 = 0.
\end{gathered} \tag{5.18}
$$

Now let us proceed as in the $\mathcal{M}_{3,2}$ and set $c = 0$. We obtain two decoupled one-matrix model systems, a Gaussian one on the right and a quartic one on the left. We are interested in the latter. The relevant equations are

$$
\begin{gathered}
n + 12t_4(a_1^2 + a_0^2 a_1) + 2t_2 a_1 = 0 \\
4t_4(a_0^3 + 6a_0 a_1) + 2t_2 a_0 = 0.
\end{gathered} \tag{5.19}
$$

Now, the second equation admits the solution $a_0 = 0$. Replacing it into the first we obtain

$$
n + 12t_4 a_1^2 + 2t_2 a_1 = 0. \tag{5.20}
$$

If $a_0 \neq 0$, we can derive a_1 from the second equation and replace the result into the first, obtaining a biquadratic equation for a_0:

$$
12\frac{t_4}{t_2}\frac{n}{t_2} - 20\left(\frac{t_4}{t_2}\right)^2 a_0^4 - 12\frac{t_4}{t_2}a_0^2 - 1 = 0. \tag{5.21}
$$

Both (5.20) and (5.21) can be solved exactly. They give rise to six distinct (in general complex) solutions. The classical potential for the eigenvalue function in the large N limit is $V_4(\lambda) \sim t_2\lambda^2 + t_4\lambda^4$. This potential has one or three real solutions depending on whether t_2 and t_4 have the same or opposite sign: $\lambda = 0$ and $\lambda^2 = -[t_2/(2t_4)]$. In order to single out among the above six the solutions that correspond to those in the classical limit, we follow the above given recipe. We drop the first term in the first Eq. (5.19) and set $a_1 = 0$ in both. We are left with

$$
4t_4 a_0^3 + 2t_2 a_0 = 0. \tag{5.22}
$$

This gives exactly the expected classical extrema for a_0. Once these are determined, we can easily find the corresponding quantum solutions either in exact form or in series of $x = n/N$. Following the example of the previous subsection, we can also determine the solutions of the complete $\mathcal{M}_{4,2}$ model.

C. Solving the quantum EoMs: $\mathcal{M}_{3,3}$ model

We study the model in the case $t_1 = s_1 = 0$ and limit ourselves to writing down the genus 0 quantum equations of motion:

$$
\begin{gathered}
3t_3 c a_0^2 + 2t_2 c a_0 - 36 s_3 t_3 b_0 R + c^2 b_0 - 12 s_2 t_3 R = 0 \\
3s_3 c b_0^2 + 2s_2 c b_0 - 36 s_3 t_3 a_0 R - 12 s_3 t_2 R + a_0 c^2 = 0 \\
nc + Rc^2 - 18 s_3 t_3 R^2 - 36 s_3 t_3 a_0 b_0 R - 12 s_2 t_3 a_0 R - 12 t_2 s_3 b_0 R - 4 s_2 t_2 R = 0, \\
a_1 = -\frac{6s_3}{g} b_0 R - \frac{2s_2}{g} R, \qquad a_2 = -\frac{3s_3}{g} R^2, \qquad b_1 = -\frac{6t_3}{g} a_0 R - \frac{2t_2}{g} R, \qquad b_2 = -\frac{3t_3}{g} R^2.
\end{gathered} \tag{5.23}
$$

For simplicity we compute only the classical vacua. To this end we drop the first term in the third equation and solve the resulting system. Then we set $a_2 = b_2 = a_1 = b_1 = 0$ as well as $R = 0$. In this branch we therefore have

$$
3t_3 a_0^2 + 2t_2 a_0 + cb_0 = 0, \tag{5.24}
$$

$$
3s_3 b_0^2 + 2s_2 b_0 + ca_0 = 0. \tag{5.25}
$$

From the first we can get $b_0 = -\frac{1}{c}(3t_3 a_0^2 + 2t_2 a_0)$, whence we get either $a_0 = 0$ or the cubic equation

$$
27 s_3 t_3^2 a_0^3 + 36 t_2 s_3 t_3 a_0^2 + (12 s_3 t_2^2 - 6 c s_2 t_3) a_0 + c(c^2 - 4 s_2 t_2) = 0.
$$

Therefore in general we have four classical extrema, with nonvanishing a_0 and b_0 while all the other fields vanish. A series expansion about these solutions is easy to find. For instance, around the vacuum $a_0 = b_0 = R = 0$ we have

$$
\begin{aligned}
a_0 &= \frac{12(-cs_3 t_2 + 2s_2^2 t_3)x}{(c^2 - 4s_2 t_2)^2} + \frac{648[-8cs_3^3 t_2^4 + c^2 s_3^2 t_2 (c^2 + 8s_2 t_2) t_3 - 4cs_2^2 s_3 (c^2 + 2s_2 t_2) t_3^2 + 16 s_2^5 t_3^3] x^2}{(c^2 - 4s_2 t_2)^5} + \mathcal{O}(x^3), \\
b_0 &= -\frac{12(-2s_3 t_2^2 + cs_2 t_3)x}{(c^2 - 4s2t2)^2} + \frac{648[16 s_3^3 t_2^5 - 4cs_3^2 t_2^2 (c^2 + 2s_2 t_2) t_3 + c^2 s_2 s_3 (c^2 + 8s_2 t_2) t_3^2 - 8cs_2^4 t_3^3] x^2}{(c^2 - 4s_2 t_2)^5} + \mathcal{O}(x^3), \\
R &= -\frac{cx}{c^2 - 4s_2 t_2} + \frac{18c[-16 s_3^2 t_2^3 + cs_3 (c^2 + 12 s_2 t_2) t_3 - 16 s_2^3 t_3^2] x^2}{(c^2 - 4s_2 t_2)^4} + \mathcal{O}(x^3).
\end{aligned}
$$

For reasons of space we have limited the expansion in $x = n/N$ to the quadratic order. From (5.23) one can easily compute the expansions for a_1, b_1, a_2, b_2.

D. The $\mathcal{M}_{p_1,p_2}$ model

In the general case the matrix rank for $Q(1)$ and $Q(2)$ was given in the previous section and the quantum EoMs become of course very complicated. It is however simple to write down the equations that identify the extrema with classical analog. They are

$$V_1'(a_0) + cb_0 = 0, \qquad V_2'(b_0) + ca_0 = 0, \tag{5.26}$$

while all the other fields are set to zero. We have in general $(p_1 - 1)(p_2 - 1)$ solutions of this type in perfect correspondence with the classical analysis. The simplest solution is $a_0 = b_0 = 0$. Other solutions may be hard or even impossible to determine explicitly. Anyhow, once one such solution is known it is possible to find explicit expressions for the fields around it in terms of $x = n/N$.

E. Calculating the correlators

Once we know the fields a_i, b_j, R in a given model, there exists an algorithmic procedure to determine the correlators. This in turn is due to the integrability underlying the Toda lattice hierarchy. In this subsection we give a few examples of exact correlators. The general scheme is known; it was already presented in [21,24]. A few explicit examples were worked out for the $\mathcal{M}_{0,0}$ model in [22,23]. In these references, one can find explicit calculations of correlators for finite N and for any genus. In this subsection we limit ourselves to large N genus 0 correlators. To start, let us briefly review the continuous versions of the quantum equations of motion and the flow equations in this case (which are known as the dispersionless Toda lattice hierarchy flows).

We proceed as in [21]. First we define the continuum quantities in the following way:

$$x = \frac{n}{N}, \qquad t_k^{\text{ren}} = \frac{t_k}{N}, \qquad s_k^{\text{ren}} = \frac{s_k}{N}, \qquad c^{\text{ren}} = \frac{c}{N}$$

in the large N limit. They are the *renormalized* coupling constants. In the following however we will understand the superscript ren. Further we define

$$F_0(x) = \lim_{N\to\infty} \frac{F_N}{N^2}, \qquad \zeta = \lim_{N\to\infty} I_+,$$

where F_0 is the genus zero free energy. The second equation is merely symbolic and simply means that ζ is the continuum counterpart of I_+. If we define a matrix $\rho = \sum_n nE_{n,n}$, it is easy to see that we have

$$[I_+, \rho] = I_+, \tag{5.27}$$

The continuous counterpart gives the following basic Poisson bracket:

$$\{\zeta, x\} = \zeta. \tag{5.28}$$

This allows us to establish the following correspondence:

$$N[\ ,\ ] \Rightarrow \{\ ,\ \} \tag{5.29}$$

and, similarly,

$$\frac{1}{N}\mathrm{Tr} = \frac{1}{N}\sum_0^{N-1} \Rightarrow \int_0^x dx \tag{5.30}$$

together with the replacements

$$Q(1) \to L, \qquad Q(2) \to \tilde{L},$$

where

$$L = \zeta + \sum_{l=0}^{\infty} a_l \zeta^{-l}, \qquad \tilde{L} = \frac{R}{\zeta} + \sum_{l=0}^{\infty} \frac{b_l}{R^l}\zeta^l. \tag{5.31}$$

Here a_l and b_l are the continuum fields that replace the lattice fields $a_l(i)$ and $b_l(i)$ of Eq. (4.19).

We stress that the above replacements holds only in genus 0.

Now let us come to the flow equations: *The dispersionless limit of the extended 2-dimensional Toda lattice integrable hierarchy* (4.15b) and (4.16b) *is*

$$\frac{\partial L}{\partial t_k} = \{L, (L^k)_-\}, \qquad \frac{\partial L}{\partial s_k} = \{L, (\tilde{L}^k)_-\}, \tag{5.32a}$$

$$\frac{\partial \tilde{L}}{\partial t_k} = \{(L^k)_+, \tilde{L}\}, \qquad \frac{\partial \tilde{L}}{\partial s_k} = \{(\tilde{L}^k)_+, \tilde{L}\} \tag{5.32b}$$

Here the subscript + denotes the part containing nonnegative powers of ζ, while $-$ indicates the complementary part.

Next, the continuum version of (4.17) provides *the link between the free energy and Lax operators*, i.e.

$$\frac{\partial}{\partial t_k}F = \int_0^x (L^k)_{(0)}(y)dy \qquad \frac{\partial}{\partial s_k}F = \int_0^x (\tilde{L}^k)_{(0)}(y)dy, \tag{5.33}$$

where subscript "$_{(0)}$" means that we select the coefficient of the zeroth power term of ζ. This formula opens the way to calculate CF's by simply differentiating both sides with respect to the appropriate coupling constants and use Eqs. (5.32a) and (5.32b). Therefore this equation together with the integrable hierarchy and the quantum equations of motion completely determines the genus zero correlators. For instance,

$$\begin{aligned} \langle \tau_k \tau_l \rangle &= \frac{\partial^2 F}{\partial t_k \partial t_l} = \oint (L^l)_- dL^k, \\ \langle \tau_k \sigma_l \rangle &= \frac{\partial^2 F}{\partial t_k \partial s_l} = \oint (\tilde{L}^l)_- dL^k, \end{aligned} \tag{5.34}$$

and

$$\langle \sigma_k \sigma_l \rangle = \frac{\partial^2 F}{\partial s_k \partial s_l} = -\oint (\tilde{L}^l)_{\geq 1} d\tilde{L}^k, \tag{5.35}$$

where $\oint$ represents the residue at the simple pole in ζ.

It is now very easy to extract explicit expressions for correlators in various models. Here we limit ourselves for simplicity to a simple example, the $\mathcal{M}_{3,2}$ model in the decoupling limit studied in Sec. II A. In this case we have two fields a_0 and a_1 and explicit expressions for them. One starts from Eqs. (5.8), (5.9), and (5.10), then computes $a_0 = -\frac{f}{3g}$ and finally a_1 from Eq. (5.6). The Lax operator is given in this case by

$$L = \zeta + a_0 + \frac{a_1}{\zeta}. \tag{5.36}$$

Inserting it in the previous formulas we get

$$\langle \tau_k \rangle = \sum_{2l=0}^{k} \frac{k!}{(k-2l)!l!l!} \int_0^x a_0(y)^{k-2l} a_1(y)^l dy \tag{5.37}$$

and

$$\langle \tau_k \tau_r \rangle = \sum_{l=0}^{r} \sum_{p=0}^{[(l-1)/2]} \sum_{n=0}^{k-1} \sum_{q=0}^{n} \frac{k!}{(k-1-n)!(n-q)!q!} a_0^{r-l+k-n-1} a_1^{l-p+n-q} [\delta_{2p+2q-l-n,-1} - a_1 \delta_{2p+2q-l-n,1}], \tag{5.38}$$

where $[\alpha]$ denotes the integral part of α. Replacing Eqs. (5.8), (5.9), and (5.10), as indicated above, we find explicit expressions for the correlators in terms of x and g.

Similarly one can compute the three point functions (referred to generically as Yukawa couplings)

$$\begin{aligned}\langle \tau_{l_1} \tau_{l_2} \tau_{l_3} \rangle = \prod_{i=1}^{3} \left(\sum_{k_i=0}^{l_i} \sum_{p_i=0}^{k_i} \frac{l_i!}{(l_i-k_i)!(k_i-p_i)!p_i!} \right) \delta_{\sum_{i=1}^3 k_i, 2\sum_{i=1}^3 p_i} \Big[& a_0^{l_1+l_2-k_1-k_2} a_1^{p_1+p_2} \frac{d}{dx} (a_0^{l_3-k_3} a_1^{p_3})(k_1-2p_1)(k_2-2p_2) \\ & \times \theta(k_2-2p_2)\theta(2p_3-k_3) - a_0^{l_1+l_3-k_1-k_3} a_1^{p_1+p_3} \frac{d}{dx} (a_0^{l_2-k_2} a_1^{p_2})(k_1-2p_1)(k_3-2p_3)\theta(k_1-2p_1) \\ & \times \theta(2p_3-k_3) + a_0^{l_2+l_3-k_2-k_3} a_1^{p_2+p_3} \frac{d}{dx} (a_0^{l_1-k_1} a_1^{p_1})(k_2-2p_2)(k_3-2p_3)\theta(2p_2-k_2)\theta(2p_3-k_3) \Big], \end{aligned} \tag{5.39}$$

where $\theta(x)$ means 0 for $x \leq 0$ and 1 otherwise.

F. Higher genus

We would like to introduce in this subsection a few basic notions concerning higher genus correlators. They are introduced here in order to render the subsequent discussion as self-contained as possible. For a more complete treatment see [22–24].

Let us consider the first example above, the decoupled $\mathcal{M}_{3,2}$ model, whose general solutions are characterized by Eq. (5.4) and genus 0 ones are explicitly given in Eqs. (5.8), (5.9), and (5.10). Let us start from a given genus 0 solution, specified by $a_0 = r_0$ and $a_1 = s_0$, where $r_0/3g$ is any one of the three solutions (5.8), (5.9), and (5.10). Then we expand the full solution in series of $\epsilon = 1/N$ as follows:

$$a_0 = \sum_{n=0} \epsilon^n r_n(x), \qquad a_1 = \sum_{n=0} \epsilon^n s_n(x). \tag{5.40}$$

Moreover, $f(n \pm 1)$ is replaced by $e^{\pm \epsilon \partial_x} f(x)$ for any lattice function $f(n)$. Inserting this into (5.4) we obtain the genus 0 equations and an infinite series of relations for the higher order terms, which can be recursively solved. For instance, the next to leading equations are

$$\begin{aligned} 6t_3 r_0 r_1 + 3t_3(s_0' + 2s_1) + 2t_2 r_1 &= 0 \\ 6t_3(s_1 r_0 + s_0 r_1) - 3t_3 s_0 r_0' + 2t_2 s_1 &= 0, \end{aligned} \tag{5.41}$$

where a prime denotes derivative with respect to x. They can be easily solved and lead to

$$\begin{aligned} r_1 &= -\frac{3}{2} t_3 \frac{3t_3 s_0 r_0' + t_2 s_0' + 3t_3 r_0 s_0'}{t_2^2 + 6t_2 t_3 r_0 + 9t_3^2(r_0^2 - s_0)} \\ s_1 &= \frac{3}{2} t_3 s_0 \frac{3t_3(s_0' + r_0 r_0') + t_2 r_0'}{t_2^2 + 6r_0 t_2 t_3 + 9(r_0^2 - s_0)t_3^2}. \end{aligned}$$

Similarly, for the second order we get

$$\begin{aligned} r_2 &= -\frac{3}{4} t_3 \frac{t_2(2r_1^2 + s_0'' + 2s_1') + 3t_3(2r_1' s_0 - r_0'' s_0 + 2r_0' s_1 - 4r_1 s_1 + r_0(2r_1^2 + s_0^2 + 2s_1'))}{t_2^2 + 6r_0 t_2 t_3 + 9t_3^2(r_0^2 - s_0)} \\ s_2 &= \frac{3}{4} t_3 \frac{-3s_0 t_3(2r_1^2 + s_0^2 + 2s_1') + (t_2 + 3r_0 t_3)(r_0'' s_0 - 2r_1' s_0 - 2r_0' s_1 + 4r_1 s_1)}{9s_0 t_3^2 - (t_2 + 3r_0 t_3)^2}, \end{aligned}$$

and so on. Replacing these expressions into the appropriate formulas for the correlators, we can write down their explicit genus expansions. To this end, one should recall that the appropriate expansion for correlators is given by a power series in ϵ, as one can infer from the free energy expansion,

$$F(x,\epsilon)=\sum_{h=0}^{\infty}F_h(x)\epsilon^{2h}, \tag{5.42}$$

where h is the genus.

We give, as an example, the genus 1 contribution to $\langle\tau_k\rangle$, which is the coefficient of ϵ^2 in the ϵ expansion:

$$\begin{aligned}\langle\tau_k\rangle_1=\int_0^x dy\Big[&3\binom{n}{4}(A^{n-4}_{(-2)}r_0'^2+2A^{n-4}_{(-1)}r_0's_0'+A^{n-4}_{(0)}(s_0'^2-2r_0r_0'^2)-4A^{n-4}_{(1)}s_0r_0's_0'+A^{n-4}_{(2)}(s_0^2r_0'^2-2s_0s_0'^2)\\&+2A^{n-4}_{(3)}s_0^2r_0's_0'+A^{n-4}_{(4)}s_0^2s_0'^2)+\binom{n}{3}(A^{n-3}_{(-1)}r_0''+A^{n-3}_{(0)}(r_0'^2+s_0''+3r_0'r_1)+A^{n-3}_{(1)}(2r_0's_0'-s_0r_0''-s_0'r_0'+3r_0's_1\\&+3s_0'r_1)+A^{n-3}_{(2)}(s_0'^2-s_0s_0''-s_0'^2+3s_0's_1))+\binom{n}{2}(A^{n-2}_{(-2)}r_1'+A^{n-2}_{(-1)}\Big(\frac{1}{2}r_0''+s_1'\Big)+A^{n-2}_{(0)}(r_1^2-2s_0r_1')\\&+A^{n-2}_{(1)}\Big(\frac{1}{2}s_0r_0''+\frac{1}{2}s_0''-s_1r_0'-2s_0s_1'+2r_1s_1\Big)+A^{n-2}_{(2)}(r_1's_0^2+s_1^2-s_1s_0')+A^{n-2}_{(3)}s_0^2s_1'\Big)\\&+\binom{n}{1}(A^{n-1}_{(0)}r_2+A^{n-1}_{(1)}s_2)\Big],\end{aligned} \tag{5.43}$$

where $r_0, s_0, r_1, s_1, r_2, s_2$ are functions of y and a prime denotes differentiation with respect to y. Moreover,

$$A^n_{(k)}=\sum_{2p+k=0}^{n}\frac{n!}{(n-2p-k)!(p+k)!p!}r_0^{n-2p-k}s_0^p. \tag{5.44}$$

VI. TWO-MATRIX MODELS AND MULTIPLE BRANE CONFIGURATIONS

All of the examples of the previous sections represent, according to the geometric description of Sec. II, the physics of N D-branes wrapped around the two-dimensional sphere located in one of the vacua. A related problem is to describe a more complex situation with N_1 D-branes at one vacuum and $N_2=N-N_1$ at another. There may of course be even more complicated configurations with several groups of D-branes in different vacua. Let us call them *multiple brane configurations*. However, the example with two groups of D-branes will be sufficient to illustrate the salient features of the problem. Let us consider once again the $\mathcal{M}_{3,2}$ model in the decoupling limit so that we can work with explicit formulas. We refer, in particular, to Eq. (5.7), which we rewrite here in the form

$$18g^2x+z(1+z)(1+2z)=0. \tag{6.1}$$

This can be solved exactly for z and gives the three solutions

$$z_1=-\frac{1}{2}+\frac{1}{2I(x)}+\frac{I(x)}{6}, \tag{6.2}$$

$$z_2=-\frac{1}{2}+\frac{1+i\sqrt{3}}{4I(x)}+\frac{1-i\sqrt{3}\,I(x)}{12}, \tag{6.3}$$

$$z_3=-\frac{1}{2}+\frac{1-i\sqrt{3}}{4I(x)}+\frac{1+i\sqrt{3}\,I(x)}{12}, \tag{6.4}$$

where

$$I(x)=3^{1/3}(-324g^2x+\sqrt{3}\sqrt{-1+34\,992g^4x^2})^{1/3}. \tag{6.5}$$

When expanded for small x they give rise to the series (5.8), (5.9), and (5.10), respectively. However, this is not a very illuminating way of studying Eq. (6.1). The best way is to consider it a plane curve [59] in the complex z, x plane. Then it represents a genus 0 Riemann surface with punctures at $x=0$ and $x=\infty$. It is made of three sheets joined through cuts running from $z=-1/(\sqrt{3}108g^2)$ to $z=1/(108\sqrt{3}\,g^2)$. The solutions (5.8), (5.9), and (5.10) correspond to the values z takes near $x=0$, away from the cuts. Therefore we can pass from one solution to another by crossing the cuts. We call the Riemann surface so constructed the *quantum Riemann surface associated to the model*. This Riemann surface picture is the clue to understanding the solutions with multiple brane configurations. Let us see how.

Let us start from $\mathcal{M}_{3,2}$, set $c=0$, and concentrate on the cubic interaction part. The relevant equations are (setting $t_1=0$)

$$\begin{aligned}&3t_3(a_0(n)^2+a_1(n)+a_1(n+1))+2t_2a_0(n)=0\\&n+3t_3a_1(n)(a_0(n)+a_0(n-1))+2t_2a_1(n)=0.\end{aligned} \tag{6.6}$$

This equation in genus 0 has three solutions. Let us denote by $a_0 = r_0$ one such genus 0 solution and $a_1 = s_0$ the corresponding a_1. Similarly, we pick another solution and we denote it $a_0 = u_0$ and $a_1 = v_0$. They represent the lowest order contribution of expansions like those considered in Sec. V F:

$$\begin{aligned} R(n) &\to R(x) = \sum_{n=0}^{\infty} \epsilon^n r_n(x), \\ S(n) &\to S(x) = \sum_{n=0}^{\infty} \epsilon^n s_n(x), \end{aligned} \tag{6.7}$$

$$\begin{aligned} U(n) &\to U(x) = \sum_{n=0}^{\infty} \epsilon^n u_n(x), \\ V(n) &\to V(x) = \sum_{n=0}^{\infty} \epsilon^n v_n(x), \end{aligned} \tag{6.8}$$

where we have indicated the large N expansion. $R(n)$, $S(n)$ and $U(n)$, $V(n)$ form, separately, two couples of solutions of (6.6). We recall that in the analogous problem formulated in the familiar saddle point approach one sets N_1 eigenvalues λ_i in one vacuum and N_2 in another. Considering the analogy between the field a_0 in genus 0 and the classical eigenvalues, we are led to pose the following problem: does there exist a solution of (6.6) that corresponds to $R(n)$, $S(n)$ for $0 \le n \le N_1 - 1$ and to $U(n)$, $V(n)$ for $N_1 \le n \le N - 1$? This means that (6.6) must hold with $a_0(n)$, $a_1(n)$ replaced by $R(n)$, $S(n)$ for $0 \le n \le N_1 - 1$ and by $U(n)$, $V(n)$ for $n \ge N_1$, respectively. However, in addition, we have the boundary equations,

$$\begin{aligned} &3t_3(R(N_1-1)^2 + S(N_1-1) + V(N_1)) + 2t_2 R(N_1-1) = 0 \\ &N_1 + 3t_3 V(N_1)(U(N_1) + R(N_1-1)) + 2t_2 V(N_1) = 0. \end{aligned} \tag{6.9}$$

While all the other equations are the same as in the previous section, these two equations represent the real novelty: they mix two different solutions. They are obviously satisfied if it makes sense to identify

$$V(N_1) = S(N_1), \qquad R(N_1 - 1) = U(N_1 - 1). \tag{6.10}$$

In the discrete formalism this is not easy to check. Therefore we shift to the continuous formalism. Recalling what we did in Secs. V E and V F, in the large N limit we set $N_2 = \alpha N_1$, $N = (1 + \alpha)N_1$, and $N_1/N = 1/(1 + \alpha) = \beta$. We also define $n/N = x$ for $0 \le n \le N_1 - 1$ and $n/N = N_1/N + (n - N_1)/N = \beta + y$. In this formalism Eqs. (6.10) read

$$V(\beta) = \lim_{\epsilon \to 0} S(\beta + \epsilon), \qquad R(\beta) = \lim_{\epsilon \to 0} U(\beta - \epsilon). \tag{6.11}$$

This is nothing but the statement that at $x = \beta$ we are crossing the cut that separates the two solutions. In hindsight this is quite obvious: the only way to satisfy Eq. (6.6) with two different solutions is to cross the cut that joins the corresponding sheets in the Riemann surface introduced above.

With this recognition in mind, one can now set out to calculate correlators in a theory with two sets of N_1 and N_2 D-branes in two different vacua. Leaving a more complete treatment for another occasion we can easily exhibit as an example the two point correlators analogous to (5.38). The correlator is given by the sum of two terms, each one equal to the right-hand side of (5.38): in the first a_0, a_1 are replaced by r_0, s_0 evaluated in x, while in the second they are replaced by u_0, v_0 evaluated in $\beta + y$.

The conclusion is therefore that quantum solutions to (5.4) that correspond to two groups of D-branes do exist. Moreover, due to the structure of the quantum EoMs which we have explored in the previous section, it is easy to generalize this conclusion. Every model will be characterized by a quantum Riemann surface with cuts that separate different solutions. It is therefore possible to construct multiple brane configurations, as quantum solutions of the matrix model, by means of solutions of the quantum EoMs on different sheets that join to one another across the appropriate cuts.

VII. CONCLUSIONS

In this paper we considered B-model D-branes on 2-cycles of local Calabi-Yau geometries. The theory describing these objects is given by the reduction to the D-brane world-volume of the open string theory with Dirichlet boundary conditions on it. We have described a precise dimensional reduction scheme for the holomorphic Chern-Simons theory, that is the B-model open string field theory, to the 2-cycle. This has been done for generic local CY geometries modeled around an arbitrary Riemann surface. In the case of the conifold geometry, i.e. when the 2-cycle is a $\mathbb{P}^1$, we have considered the relevant effective theory and found that it is given by a multimatrix model. The number of matrices involved depends on the reference complex structure about which we calculate the coupling to the Calabi-Yau complex moduli. The multimatrix potential is fixed by the complex moduli in a well-defined and simple way. The various allowed couplings turn out to be in correspondence with the projective parameters of complex structure deformation. The matrix models we have obtained are of generic type if they involve one or two matrices, while we found relevant constraints within their parameters for more than two matrices.

We have studied the geometric engineering of the multimatrix models and provided both a general reduction scheme and some examples. Actually, some multimatrix models are reducible to models involving less matrices. This phenomenon has a clear counterpart on the geometrical side, corresponding to the fact that different reference conifold complex structures can be connected via specific deformations.

In the second part of the paper, we have focused on two-matrix models with bilinear couplings. We have illustrated a general method to *exactly* solve these models. It consists in solving the quantum equations of motion and making subsequent use of the integrable flow equations. We have exhibited several examples of solutions in genus 0 as well as in higher order approximations. We have also discussed the relation of our method to the more popular saddle point approximation. One of the relevant differences is that our method leads in general to more solutions than the saddle point one.

Finally we have discussed the brane interpretation of our results. They represent the deformations of the complex structure generated by the strings attached to N D-branes wrapped around a 2-cycle in a local Calabi-Yau geometry. However it is possible to obtain more general solutions: group of D-branes localized near different vacua. This is due to a remarkable property of our approach: all the data of all the solutions of a given model are encoded in a Riemann surface (a plane curve), which we call the *quantum Riemann surface* of the model; different solutions lie on different sheets; there is room for solutions representing D-branes localized near different vacua by crossing the cuts that connect the sheets.

To conclude the paper we would like to list a few open questions. The first concerns geometric transitions and gauge duals. Laufer's theorem [60] implies that only few smooth geometries can be interpreted as resolution of the singular conifold. These, as already observed in [10], correspond to asymptotically free gauge theory duals. Then, for the non-Laufer's geometries, one should formulate a definite gauge dual. Among these, one should find the nonasymptotically free theories. It seems natural to guess the gauge dual of the geometries $\mathcal{O}(-n) \oplus \mathcal{O}(n-2)$ to be then given by $\mathcal{N} = 1$ SYM with $n+1$ Wess-Zumino multiplets in the adjoint representation of the U(N) gauge group and with superpotential given by the corresponding multimatrix model one. Unfortunately, we do not have convincing arguments to push further this hypothesis.

Secondly, we elaborated a scheme which can be applied to Calabi-Yau manifolds of a more general type than conifolds. It would be interesting to work out the effective coupling to the CY complex moduli of the reduced theory in such more general settings.

The last comment concerns multimatrix models. It is apparent that we have a method to solve any kind of two-matrix or multimatrix model with bilinear couplings. The next important step is to find analogous powerful tools to solve matrix models with more complicated couplings.

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APPENDIX

In this appendix we extend the method for the reduction of the holomorphic CS functional to the noncompact CY geometry around a four cycle. This geometry, once one fixes the complex structure on the four manifold M describing the cycle, is fully determined to be total space of the canonical line bundle K_M, that is the bundle of the top holomorphic forms on M. We denote this space as $X_M = \mathrm{tot}(K_M)$.

Any atlas $\{U_{(\alpha)}\}$ on M extends to an atlas on X_M by $\hat{U}_{(\alpha)} = U_{(\alpha)} \times \mathbb{C}$. The complex manifold is defined by the patching conditions

$$\mathbf{z}_{(\alpha)} = \mathbf{f}_{(\alpha)(\beta)}(\mathbf{z}_{(\beta)}), \qquad p_{(\alpha)} = [\det \mathbf{X}_{(\alpha)(\beta)}]^{-1} p_{(\beta)}, \quad \text{where} \quad [\mathbf{X}_{(\alpha)(\beta)}] = \partial_{\mathbf{z}_{(\beta)}} \mathbf{f}_{(\alpha)(\beta)} \tag{A1}$$

in any double patch intersection $U_{(\alpha)} \cap U_{(\beta)}$. In (A1) and in the following, $\mathbf{z} = (z^1, z^2)$ denotes the two complex coordinates. The holomorphic $(3,0)$-form on X_M is $\Omega = dz^1 \wedge dz^2 \wedge dp$.

Let us consider the topological B-model on X_M. In this case, D-branes can wrap the 4-cycle M and the theory describing the dynamics of these objects is obtained then by reducing the hCS functional to the D-brane world-volume. We consider here again only the case in which the gauge bundle E is trivial.

The action of hCS is

$$S(\mathcal{A}) = \frac{1}{g_s} \int_{X_M} \mathcal{L}, \qquad \mathcal{L} = \Omega \wedge \mathrm{Tr}\left(\frac{1}{2} \mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{1}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right), \tag{A2}$$

where $\mathcal{A} \in T^{(0,1)}(X_M)$.

We split $\mathcal{A} = \mathcal{A}_{\bar{\mathbf{z}}} d\bar{\mathbf{z}} + \mathcal{A}_{\bar{p}} d\bar{p}$ and we set, because of the glueing prescriptions for the parallel and transverse components, $\mathcal{A}_{\bar{\mathbf{z}}} d\bar{\mathbf{z}} = A_{\bar{\mathbf{z}}} d\bar{\mathbf{z}} - A_{\bar{p}} \Gamma_{\bar{\mathbf{z}}} \bar{p} d\bar{\mathbf{z}}$ and $\mathcal{A}_{\bar{p}} = A_{\bar{p}}$, where $A = A_{\bar{\mathbf{z}}} d\bar{\mathbf{z}} \in T^{(0,1)}(M)$ is an antiholomorphic 1-form on M, $A_{\bar{p}} \in \Gamma(\bar{K}_M^{-1})$ is a section of the inverse anticanonical line bundle, and $\Gamma_{\bar{\mathbf{z}}} d\mathbf{z}$ is the $(0,1)$ component of a reference connection on $\bar{K}_M$.

The reduction prescription is that the matrix valued dynamical fields $(A_{\bar{\mathbf{z}}}, A_{\bar{p}})$ are independent on the coordinate p along the fiber $\mathbb{C}$.

Direct calculation of the Lagrangian $\mathcal{L}$ in (A2) for the above reduced configurations gives

$$L = \mathcal{L}_{\text{red}} = \Omega \wedge \text{Tr}\Big(\frac{1}{2}\{A \wedge \bar{D}A_{\bar{p}} + A_{\bar{p}}F^{(0,2)} + A \wedge \Gamma A_{\bar{p}}\}\Big)d\bar{p}, \tag{A3}$$

where $\bar{D}$ is the covariant derivative with respect to A. Notice that the above result does not depend on p.

Introducing now a reference section $K \in \Gamma(K_M \otimes \bar{K}_M)$, we define $\phi^{(2,0)} = KA_{\bar{p}} \in \Gamma(K_M)$ and fix the reference connection to be $\Gamma = K^{-1}\bar{\partial}K$. This way we get

$$L = \mathcal{L}_{\text{red}} = \Omega K^{-1} \wedge \text{Tr}\Big(\frac{1}{2}\{A \wedge \bar{D}\phi^{(2,0)} + \phi^{(2,0)}F^{(0,2)}\}\Big)d\bar{p}. \tag{A4}$$

To reduce to a 4-form, we saturate the reduced Lagrangian with $K\partial_p \wedge \partial_{\bar{p}}$ so that the reduced hCS functional becomes just

$$\text{hCS}_{\text{red}} = \frac{1}{g_s}\int_M \text{Tr}\Big(\frac{1}{2}\{A \wedge \bar{D}\phi^{(2,0)} + \phi^{(2,0)}F^{(0,2)}\}\Big) = \frac{1}{g_s}\int_M \text{Tr}(\phi^{(2,0)}F^{(0,2)}), \tag{A5}$$

which is the form used in [14].

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